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THREE-SPECIES FOOD CHAIN MODEL WITH CANNIBALISM IN THE SECOND LEVEL

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Abstract: This article suggests and explores a three-species food chain model that includes fear effects, refuges depending on predators, and cannibalism at the second level. The Holling type II functional response determines food consumption between stages of the food chain. This study examined the long-term behavior and impacts of the suggested model's essential elements. The model's solution properties were studied. The existence and stability of every probable equilibrium point were examined. The persistence needs of the system have been determined. It was discovered what conditions could lead to local bifurcation at equilibrium points. Appropriate Lyapunov functions are utilized to investigate the overall dynamics of the system. To support the analytical conclusions, numerical simulations were done to validate the model's inferred long-term behavior and to comprehend the implications of the model's significant parameters.

Keywords: prey-predator; refuge; fear; cannibalism; stability; bifurcation.

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1. INTRODUCTION

Food chains are significant environmental phenomena in several academic fields, including ecological science, applied mathematics, engineering, and economics. In a food chain model species, energy and resources flow in a single direction; however, food webs are complex because they comprise multiple food chains [1]. In a feeding chain, various trophic levels have been seen. Many types of organisms, including producers, consumers, and decomposers, can be found in the stimulation phases. On the other hand, a formation-wise lattice architecture is used in a food web. To describe the food chain as a system of differential equations, mathematical analysis, and modeling techniques could be employed. A food web is a conglomeration of food chains, although food chains are referred to as "food chains" in ecology [2-3].

Another intriguing aspect of the prey-predator relationship is cannibalism. When an animal consumes members of its own species, this behavior is known as cannibalism or intraspecific predation [4]. There has been a lot of discussion on how cannibalism affects environmental strategy for decades [5]. Cannibalism is influenced by a number of crucial variables, including population density, temperature, population size, developmental stage, and more [6]. Some researchers have looked into the mathematical representation of cannibalism, see for example [7-9]. It is intriguing to explore a prey-predator model with cannibalism because many animals in nature exhibit cannibalistic behaviors. Cannibalism has been observed in a wide range of animal species, including carnivore mammals, frogs, monkeys, spiders, fish, and insects, see [10]–[14].

In addition to cannibalism, the ecological term for the behavior of prey that hides after being trapped and attacked by predators is a refuge. Many prey species use the refuge strategy to ward off predators. Sea urchins conceal their young from crab predators in articulated coralline algae, while Daphnia hides its young from crab predators in shallow lakes in the Mediterranean [15-16]. In addition to prey's natural behavior, humans can help prey by creating conservation forests [17], natural areas, wildlife reserves, or even basic security. The mathematical model of prey-predator with prey refuge has also been the subject of many investigations [18–21].

Recent studies have shown that predators affect refuge prey populations in ways other than just killing the prey; they also instill fear in the prey, which reduces the prey birth rate [22-24]. Predator-induced fear keeps prey animals out of open settings, denying them the freedom to carry out regular activities like mating. As a result, their capacity for reproduction is decreased by their fear of predators. It is critical to consider the price of anxiety as a decrease in reproduction. Wang et al. published a prey-predator model that took into account the effect of fear on prey reproduction [22]. Additionally, it was explained how a high level of fear may stabilize the system by ruling out the possibility of periodic fixes. Panday et al. [23] also looked into the impacts of fear using a Holling type-II functional response and a tri-trophic food chain model. Since the system displays chaotic behavior for smaller values of both of these variables, they came to the conclusion that chaotic oscillations may be controlled by increasing the fear parameters. A prey refuge is a great way to reduce the possibility that predators may use their victim's biomass excessively. But Abdulghafour and Naji [24] constructed and investigated a mathematical model of a prey-predator system including infectious diseases in the prey population. They believed that prey serves as a constant refuge from predators' exploitation and hunting as a defense mechanism.

This research proposes and investigates a three-species food chain model with cannibalism at the second level while considering the aforementioned. The next section contains the model formulation. Section 3 addresses the solution's characteristics, nevertheless. The analysis of stability and persistence is examined in Section 4. Section 5 examines local bifurcation, while Section 6 provides a numerical simulation analysis of the system. Finally, the last section provided the conclusions.

2. MODEL FORMULATION

Recently, Andulghafour and Naji [20] proposed and studied a mathematical model of prey-predator incorporating fear cost, predator-dependent refuge, and cannibalism given by

$$\begin{aligned} \frac{dX}{dT} &= X\left(\frac{r}{1+fY} - d_1 - bX - \frac{a_1(1-cY)Y}{K_1+X(1-cY)}\right), \\ \frac{dY}{dT} &= Y\left(\frac{a_2X(1-cY)}{K_1+X(1-cY)} + a_3 - d_2 - \frac{e(1-m)Y}{K_2+(1-m)Y}\right), \end{aligned} \tag{1}$$

where $X(T)$ and $Y(T)$ are the population densities of the prey and the predator at the time T respectively. Since the environment contains many species that interact with each other in a food web and food chain forms. Therefore, in this section, system (1) will be extended so that it contains a top predator that represents their population density at time T by $Z(T)$ consumed the predator species in the system (1) according to Holling type II functional response. Hence the modified model that represents a food chain can be written as:

$$\begin{aligned}
\frac{dX}{dT} &= X\left(\frac{r}{1+fY} - d_1 - bX - \frac{a_1(1-cY)Y}{K_1+(1-cY)X}\right) = Xf_1(X,Y,Z), \\
\frac{dY}{dT} &= \mathrm{Y}\left(\frac{a_2(1-cY)X}{K_1+(1-cY)X} + a_3 - d_2 - \frac{e(1-m)Y}{K_2+(1-m)Y} - \frac{a_4(1-m)Z}{K_3+(1-m)Y}\right) = Yf_2(X,Y,Z), \\
\frac{dZ}{dT} &= Z\left(\frac{a_5(1-m)Y}{K_3+(1-m)Y} - d_3\right) = Zf_3(X,Y,Z),
\end{aligned} \tag{2}$$

where $X(0) \geq 0$, $Y(0) \geq 0$, and $Z(0) \geq 0$, and all the coefficients are non-negative constants and can be described in Table (1).

Table 1: parameters description.

Parameter	Description
r	The prey birth rate
d_1	The prey's natural death rate
b	The prey intraspecific competition
f	The prey's fear level, which is involved in the fear function $\frac{1}{1+fY}$.
a_1	The intermediate predator's attack rate
K_1	The middle predator's half-saturation constant.
$c \in [0,1]$	The prey's refuge rate; hence the refuge amount is cXY, which leaves $X(1-cY)$ of the prey available to be hunted by the predator
a_2	The conversion rate of prey biomass into middle predator biomass.
a_3	The conversion rate of cannibalism into middle predator birth
d_2	The middle predator's natural death rate
e	The cannibalism rate in the middle predator.
K_2	The half-saturation constant of cannibalism
$m \in [0,1]$	The middle predator's refuge rate
a_4	The middle predator's attack rate.
d_3	The top predator's natural death rate.
K_3	The top predator's half-saturation constant.
a_5	The conversion rate of middle predator's biomass into top predator biomass.

3. Properties of the Solution

Obviously, the interaction functions of the system (2) $F = (F_1, F_2, F_3) = (Xf_1, Yf_1, Zf_1)$ are continuous and have continuous partial derivatives on the domain $\mathbb{R}^3_+ = \{(X, Y, Z) \in \mathbb{R}^3 : X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0\}$. Therefore, by the fundamental theorem of existence and uniqueness, system (2) with a specific initial value has a unique solution in its domain. The system's positivity and boundedness in theoretical ecology establish its physiologically well-behaved form. The results that follow ensure the positivity and boundedness of the system's (2) solutions.

Theorem 1: The positive cone ($int. \mathbb{R}^3_+$) is invariant for the system (2).

Proof. By using a similar argument to that given in lemma (2.1) [25]. It is sufficient to prove that for all $T \in [0, \tau]$, $X(T) > 0$, $Y(T) > 0$, and $Z(T) > 0$, where τ is any positive real number. Hence, using a contradiction will yield that.

Suppose the opposite, then $X(T) > 0$, $Y(T) > 0$, $Z(T) > 0$ for all $T \in [0, \tau_0]$, and at least one of $X(\tau_0)$, $Y(\tau_0)$, and $Z(\tau_0)$ must vanish, where τ_0 exists with $0 < \tau_0 < \tau$. Therefore system (2) gives us

$$\left.\begin{array}{l} X(T) = X(0) \exp\left(\int_0^T f_1(X, Y, Z) dT\right) \\ Y(T) = Y(0) \exp\left(\int_0^T f_2(X, Y, Z) dT\right) \\ Z(T) = Z(0) \exp\left(\int_0^T f_3(X, Y, Z) dT\right) \end{array}\right\}$$

Since $f_i(X, Y, Z), i = 1,2,3$ are defined and continuous on $[0, \tau_0]$, then there exists $\mathrm{L} \geq 0$ such that for all $T \in [0, \tau_0]$:

$$\left.\begin{array}{l} X(T) = X(0) \exp\left(\int_0^T f_1(X, Y, Z) dT\right) \geq X(0) \exp(-\tau_0 \mathrm{L}) \\ Y(T) = Y(0) \exp\left(\int_0^T f_2(X, Y, Z) dT\right) \geq Y(0) \exp(-\tau_0 \mathrm{L}) \\ Z(T) = Z(0) \exp\left(\int_0^T f_3(X, Y, Z) dT\right) \geq Z(0) \exp(-\tau_0 \mathrm{L}) \end{array}\right\}$$

Therefore as $T \to \tau_0$, it is obtained

$$\left.\begin{array}{l} X(\tau_0) \geq X(0) \exp(-\tau_0 \mathrm{L}) \\ Y(\tau_0) \geq Y(0) \exp(-\tau_0 \mathrm{L}) \\ Z(\tau_0) \geq Z(0) \exp(-\tau_0 \mathrm{L}) \end{array}\right\}$$

This contradicts the fact that at least one of $X(\tau_0)$, $Y(\tau_0)$, and $Z(\tau_0)$ must die out. Hence for all $T \in [0, \tau]$, $X(t) > 0$, $Y(t) > 0$, and $Z(t) > 0$, which completes the proof.

Theorem 2: All system's (2) solutions are uniformly bounded.

Proof: According to the first equation of system (2), it is obtained that

$$\frac{dX}{dT} \leq rX - bX^2$$

Therefore, simple computation yields that: $X \leq \frac{r}{b}$ as $T \to \infty$. Now, define the $N = X + \frac{a_1}{a_2}Y + \frac{a_1 a_4}{a_2 a_5}Z$, then it is obtained that:

$$\frac{dN}{dT} \leq 2rX - rX - \frac{a_1}{a_2}(d_2 - a_3)\mathrm{Y} - \frac{a_1 a_4}{a_2 a_5} d_3 Z \leq 2\frac{r^2}{b} - \mu N,$$

where $\mu = min\{r, d_2 - a_3, d_3\}$. Thus, solving the above differential inequality it obtained that $N \leq \frac{2r^2}{\mu b}$, as $T \to \infty$. Therefore, the proof is complete.

4. STABILITY ANALYSIS AND PERSISTENCE

This section discusses the presence of every potential equilibrium point as well as the stability analysis of each one. The system's persistence constraints are then established (2). It has been determined that system (2) contains five potential equilibrium points, and the prerequisites for their existence and form are given below.

The vanishing equilibrium point denoted by $p_0 = (0,0,0)$ always exists.

The first axial equilibrium point, which is denoted by $p_1 = (\bar{X}, 0,0)$, where

$$\bar{X} = \frac{r-d_1}{b}, \tag{3}$$

exists always due to the prey survival condition, $r - d_1 > 0$.

The second axial equilibrium point denoted by $p_2 = (0, \bar{\bar{Y}}, 0)$ where

$$\bar{\bar{Y}} = \frac{K_2(a_3-d_2)}{(1-m)(e-a_3+d_2)}, \tag{4}$$

exists provided that the following conditions are satisfied.

$$0 < a_3 - d_2 < e. \tag{5}$$

The prey-free equilibrium point, which is represented by $p_3 = (0, \hat{Y}, \hat{Z})$, where

$$\left.\begin{aligned} \hat{Y} &= \frac{d_3 K_3}{(1-m)(a_5-d_3)} \\ \hat{Z} &= \frac{K_3 a_5[(a_3-d_2)(K_2(a_5-d_3)+d_3K_3)-ed_3K_3]}{(1-m)(a_5-d_3)a_4(K_2(a_5-d_3)+d_3K_3)} \end{aligned}\right\}, \tag{6}$$

exists in the positive quadrant of YZ −plane provided the following conditions are met.

$$d_3 < a_5, \tag{7}$$

$$ed_3K_3 < (a_3 - d_2)(K_2(a_5 - d_3) + d_3K_3). \tag{8}$$

The top predator-free equilibrium point denoted by $p_4 = (\check{X}, \check{Y}, 0)$, where

$$\check{X} = \frac{K_1[(1-m)\check{Y}(-e+a_3-d_2)+a_3K_2-d_2K_2]}{(1-c\check{Y})[(1-m)\check{Y}(e-a_2-a_3+d_2)-K_2(a_2+a_3-d_2)]}, \tag{9}$$

while $\check{Y}$ is a positive root of the following fifth-order polynomial equation

$$A_5Y^5 + A_4Y^4 + A_3Y^3 + A_2Y^2 + A_1Y + A_0 = 0, \tag{10}$$

where

$$A_5 = -(1-m)^2c^2fa_1[(e-a_2)-(a_3-d_2)]^2 < 0,$$

$$\begin{aligned} A_4 = &-c(1-m)^2a_1(c-2f)(e-a_2-a_3+d_2)^2 \\ &+2c^2K_2fa_1(1-m)(a_2+a_3-d_2)(e-a_2-a_3+d_2) \end{aligned},$$

$$\begin{aligned} A_3 = &(1-m)^2[a_1(2c-f)(e-a_2-a_3+d_2)^2 - cfa_2d_1K_1(e-a_2-a_3+d_2)] \\ &+2(1-m)ca_1K_2(c-2f)(e-a_2-a_3+d_2)(a_2+a_3-d_2) \\ &-c^2fa_1(a_2+a_3-d_2)^2K_2^2 \end{aligned},$$

$$\begin{aligned} A_2 = &(1-m)^2[-a_1(e-a_2)^2 + (e-a_2)(2a_1a_3 - 2a_1d_2 + cra_2K_1) - a_1(a_3-d_2)^2 \\ &-cra_2K_1(a_3-d_2) - bfa_2K_1^2(e-a_3+d_2)] \\ &-(1-m)^2(c-f)a_2d_1K_1[e-a_2-a_3+d_2] \\ &+(1-m)K_2[2a_1(2c-f)(a_2^2 - ea_3 + 2a_2 - 2a_2d_2 - 2a_3d_2) \\ &+cfa_2d_1K_1(a_2+a_3-d_2)] + 2a_1K_2(2c-f)[a_3^2(1-m) + ed_2(1-m) \\ &-ea_2 + d_2^2(1-m)] + ca_1(c-2f)[-a_2^2K_2^2 - 2a_2a_3 \\ &-a_3^2K_2^2 + 2a_2d_2K_2^2 + 2a_3d_2K_2^2 - d_2^2K_2^2] \end{aligned},$$

$$\begin{aligned} A_1 = &a_2K_1(1-m)^2[-r(e-a_2-a_3+d_2) + d_1(e-a_2-a_3+d_2) \\ &-bK_1(e-a_2-a_3+d_2)] + K_2(1-m)[2a_1a_2(e-a_2) \\ &+2a_1a_3(e-2a_2-a_3) - 2a_1d_2(e-2a_2-2a_3+d_2) \\ &-cra_2K_1(a_2+a_3-d_2) + bfa_2K_1^2(a_3-d_2)] \\ &+a_2d_1K_1K_2(c-f)(1-m)[a_2+a_3-d_2] \\ &+a_1K_2^2(2c-f)(a_2+a_3-d_2)^2 \end{aligned},$$

$$\begin{aligned} A_0 = &K_1K_2(1-m)(r-d_1)a_2(a_2+a_3-d_2) + K_1^2K_2ba_2(1-m)[a_3-d_2] \\ &-K_2^2a_1(a_2+a_3-d_2)^2 \end{aligned}.$$

Obviously, the point p_4 exists uniquely in the positive quadrant of the XY −plane provided that the following conditions are met

$$\left.\begin{aligned} (1-m)\check{Y}(-e+a_3-d_2) + a_3K_2 - d_2K_2 > 0 \\ (1-m)\check{Y}(e-a_2-a_3+d_2) - K_2(a_2+a_3-d_2) > 0 \end{aligned}\right\} \tag{11}$$

With one set of the following sets of conditions

$$\left.\begin{array}{l} A_4 < 0, A_3 < 0, A_2 < 0, A_1 < 0, A_0 > 0. \\ A_4 > 0, A_3 > 0, A_2 > 0, A_1 > 0, A_0 > 0. \\ A_4 < 0, A_3 < 0, A_2 < 0, A_1 \neq 0, A_0 > 0. \\ A_4 < 0, A_3 < 0, A_2 \neq 0, A_1 > 0, A_0 > 0. \\ A_4 < 0, A_3 \neq 0, A_2 > 0, A_1 > 0, A_0 > 0. \\ A_4 \neq 0, A_3 > 0, A_2 > 0, A_1 > 0, A_0 > 0. \end{array}\right\} \quad (12)$$

The positive equilibrium point denoted by $p_5 = (\tilde{X}, \tilde{Y}, \tilde{Z})$, where

$$\left.\begin{array}{c} \tilde{Y} = \frac{d_3 K_3}{(1-m)(a_5 - d_3)}, \\ \tilde{z} = \frac{K_3 + (1-m)\tilde{Y}}{a_4(1-m)}\left[\frac{a_2(1-c\tilde{Y})\tilde{X}}{K_1 + (1-c\tilde{Y})\tilde{X}} + a_3 - d_2 - \frac{e(1-m)\tilde{Y}}{K_2 + (1-m)\tilde{Y}}\right] \end{array}\right\}, \quad (13)$$

while $\tilde{X}$ represents the unique positive root of the second-order polynomial equation

$$B_2 X^2 + B_1 X + B_0 = 0, \quad (14)$$

where

$$B_2 = b(1-m)^3(a_5 - d_3)^3 - b d_3 K_3 (1-m)^2 (c - f)(a_5 - d_3)^2 - bcf d_3^2 K_3^2 (1-m)(a_5 - d_3).$$

$$B_1 = -(1-m)^3(a_5 - d_3)^3[r - d_1 - bK_1] + d_3 K_3 (1-m)^2[(cr + bfK_1)(a_5 - d_3)^2 - a_5^2 d_1 (c - f)] + 2a_5 d_1 d_3^2 K_3 (c - f)(1-m)^2 - d_1 d_3^3 K_3 (c - f)(1-m)^2 - cf d_1 d_3^2 K_3^2 (1-m)(a_5 - d_3).$$

$$B_0 = -K_1(1-m)^3(r - d_1)(a_5 - d_3)^3 - cf a_1 d_3^3 K_3^3 + K_3(1-m)^2 d_3 (a_5 - d_3)^2 [a_1 + f d_1 K_1] - a_1 d_3^2 K_3^2 (c - f)(1-m)(a_5 - d_3).$$

Obviously, the point p_5 exists uniquely in the positive cone $(int.\mathbb{R}_+^3)$ provided that the following conditions are met

$$\left.\begin{array}{c} a_5 - d_3 > 0 \\ \frac{a_2(1-c\tilde{Y})\tilde{X}}{K_1 + (1-c\tilde{Y})\tilde{X}} + a_3 - d_2 - \frac{e(1-m)\tilde{Y}}{K_2 + (1-m)\tilde{Y}} > 0 \end{array}\right\}, \quad (15)$$

with one set of the following sets of conditions

$$\left.\begin{array}{l} B_2 > 0, B_0 < 0 \\ B_2 < 0, B_0 > 0 \end{array}\right\}. \quad (16)$$

In the following, the linearization technique is used to study the stability of the system (2). Then the Jacobian matrix of the system (2) at the point (X, Y, Z) can be written

$$J = \begin{bmatrix} X\frac{\partial f_1}{\partial X} + f_1 & X\frac{\partial f_1}{\partial Y} & X\frac{\partial f_1}{\partial Z} \\ Y\frac{\partial f_2}{\partial X} & Y\frac{\partial f_2}{\partial Y} + f_2 & Y\frac{\partial f_2}{\partial Z} \\ Z\frac{\partial f_3}{\partial X} & Z\frac{\partial f_3}{\partial Y} & Z\frac{\partial f_3}{\partial Z} + f_3 \end{bmatrix} = [a_{ij}]_{3\times 3}, \tag{17}$$

where

$$a_{11} = \frac{r}{1+fY} - bX - d_1 - \frac{a_1(1-cY)Y}{K_1+(1-cY)X} + X\left(-b + \frac{a_1(1-cY)^2Y}{(K_1+(1-cY)X)^2}\right),$$

$$a_{12} = X\left[-\frac{fr}{(1+fY)^2} - \frac{cXY(1-cY)a_1}{(K_1+(1-cY)X)^2} + \frac{cYa_1}{K_1+(1-cY)X} - \frac{(1-cY)a_1}{K_1+(1-cY)X}\right],$$

$$a_{13} = 0,$$

$$a_{21} = Y\left[-\frac{X(1-cY)^2a_2}{(K_1+(1-cY)X)^2} + \frac{(1-cY)a_2}{K_1+(1-cY)X}\right],$$

$$\begin{aligned} a_{22} = {} & \frac{a_2(1-cY)X}{K_1+(1-cY)X} + a_3 - d_2 - \frac{e(1-m)Y}{K_2+(1-m)Y} - \frac{a_4(1-m)Z}{K_3+(1-m)Y} \\ & + Y\left[\frac{ca_2(1-cY)X^2}{(K_1+(1-cY)X)^2} - \frac{ca_2X}{K_1+X(1-cY)} + \frac{e(1-m)^2Y}{(K_2+(1-m)Y)^2}\right. \\ & \left. - \frac{e(1-m)}{K_2+(1-m)Y} + \frac{a_4(1-m)^2Z}{(K_3+(1-m)Y)^2}\right], \end{aligned}$$

$$a_{23} = -\frac{a_4(1-m)Y}{K_3+(1-m)Y},$$

$$a_{31} = 0,$$

$$a_{32} = Z\left[-\frac{(1-m)^2Ya_5}{(K_3+(1-m)Y)^2} + \frac{(1-m)a_5}{K_3+(1-m)Y}\right],$$

$$a_{33} = \frac{a_5(1-m)Y}{K_3+(1-m)Y} - d_3.$$

Thus, the Jacobian matrix at the equilibrium point p_0 can be written as

$$J(p_0) = \begin{bmatrix} r - d_1 & 0 & 0 \\ 0 & a_3 - d_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}. \tag{18}$$

Hence the eigenvalues of $J(p_0)$ are given by

$$\lambda_{01} = r - d_1,\ \lambda_{02} = a_3 - d_2, \text{ and } \lambda_{03} = -d_3. \tag{19}$$

Therefore, all the eigenvalues are negative and p_0 is stable node provided that the following conditions are met.

$$r < d_1. \tag{20}$$

$$a_3 < d_2. \tag{21}$$

Thus, the Jacobian matrix at the equilibrium point p_1 is determined as.

$$J(p_1)=\begin{bmatrix} -r+d_1 & \frac{(r-d_1)}{b}\left(-fr-\frac{ba_1}{r-d_1+K_1b}\right) & 0 \\ 0 & a_3-d_2+\frac{a_2(r-d_1)}{r-d_1+K_1b} & 0 \\ 0 & 0 & -d_3 \end{bmatrix}. \quad (22)$$

So, the eigenvalues of $J(p_1)$ will be written as

$$\lambda_{11}=-(r-d_1),\ \lambda_{12}=a_3-d_2+\frac{a_2(r-d_1)}{r-d_1+K_1b}, \text{ and } \lambda_{13}=-d_3. \quad (23)$$

Obviously, these eigenvalues are negative provided that the following condition holds.

$$a_3+\frac{a_2(r-d_1)}{r-d_1+K_1b}<d_2. \quad (24)$$

Clearly, when the equilibrium point p_0 is stable node the equilibrium point p_1 dose not exist. Now, the Jacobian matrix at the equilibrium point p_2 is determined as.

$$J(p_2)=\begin{bmatrix} \frac{r}{1+f\bar{\bar{Y}}}-d_1-\frac{a_1(1-c\bar{\bar{Y}})\bar{\bar{Y}}}{K_1} & 0 & 0 \\ \frac{a_2(1-c\bar{\bar{Y}})\bar{\bar{Y}}}{K_1} & \frac{e(1-m)^2\bar{\bar{Y}}^2}{(K_2+(1-m)\bar{\bar{Y}})^2}-\frac{e(1-m)\bar{\bar{Y}}}{K_2+(1-m)\bar{\bar{Y}}} & \frac{a_4(1-m)\bar{\bar{Y}}}{K_3+(1-m)\bar{\bar{Y}}} \\ 0 & 0 & \frac{a_5(1-m)\bar{\bar{Y}}}{K_3+(1-m)\bar{\bar{Y}}}-d_3 \end{bmatrix}. \quad (25)$$

Then the eigenvalues of $J(p_2)$ are given by:

$$\left.\begin{aligned} \lambda_{21}&=\frac{r}{1+f\bar{\bar{Y}}}-d_1-\frac{a_1(1-c\bar{\bar{Y}})\bar{\bar{Y}}}{K_1} \\ \lambda_{22}&=\frac{e(1-m)^2\bar{\bar{Y}}^2}{(K_2+(1-m)\bar{\bar{Y}})^2}-\frac{e(1-m)\bar{\bar{Y}}}{K_2+(1-m)\bar{\bar{Y}}} \\ \lambda_{32}&=\frac{a_5(1-m)\bar{\bar{Y}}}{K_3+(1-m)\bar{\bar{Y}}}-d_3 \end{aligned}\right\}. \quad (26)$$

Consequently, the eigenvalues of $J(p_2)$ are negative and hence p_2 is a stable node point provided that the following conditions are statisfied.

$$\frac{r}{1+f\bar{\bar{Y}}}<d_1+\frac{a_1(1-c\bar{\bar{Y}})\bar{\bar{Y}}}{K_1}. \quad (27)$$

$$\frac{e(1-m)^2\bar{\bar{Y}}^2}{(K_2+(1-m)\bar{\bar{Y}})^2}<\frac{e(1-m)\bar{\bar{Y}}}{K_2+(1-m)\bar{\bar{Y}}}. \quad (28)$$

$$\frac{a_5(1-m)\bar{\bar{Y}}}{K_3+(1-m)\bar{\bar{Y}}}<d_3. \quad (29)$$

The Jacobian matrix at the equilibrium point p_3 is computed as:

$$J(p_3)=\begin{bmatrix} \frac{r}{1+f\hat{Y}}-d_1-\frac{a_1(1-c\hat{Y})\hat{Y}}{K_1} & 0 & 0 \\ \frac{a_2(1-c\hat{Y})\hat{Y}}{K_1} & \hat{Y}\left(-\frac{e(1-m)K_2}{(K_2+(1-m)\hat{Y})^2}+\frac{a_4(1-m)^2\hat{Z}}{(K_3+(1-m)\hat{Y})^2}\right) & -\frac{a_4(1-m)\hat{Y}}{K_3+(1-m)\hat{Y}} \\ 0 & \hat{Z}\left[\frac{(1-m)a_5K_3}{(K_3+(1-m)\hat{Y})^2}\right] & 0 \end{bmatrix}. \quad (30)$$

The characteristic equation can be written as follows:

$$\left(\frac{r}{1+f\hat{Y}} - d_1 - \frac{a_1(1-c\hat{Y})\hat{Y}}{K_1} - \lambda\right)(\lambda^2 - T_1\lambda + D_1) = 0, \tag{31}$$

where

$$T_1 = \hat{Y}\left(-\frac{e(1-m)K_2}{(K_2+(1-m)\hat{Y})^2} + \frac{a_4(1-m)^2\hat{Z}}{(K_3+(1-m)\hat{Y})^2}\right),$$

$$D_1 = \frac{a_4(1-m)\hat{Y}}{K_3+(1-m)\hat{Y}}\hat{Z}\left[\frac{(1-m)a_5K_3}{(K_3+(1-m)\hat{Y})^2}\right].$$

Therefore, all the eigenvalues of $J(p_3)$ have negative real parts and hence p_3 is a stable point if the following conditions are satisfied.

$$\frac{r}{1+f\hat{Y}} < d_1 + \frac{\hat{Y}(1-c\hat{Y})a_1}{K_1}. \tag{32}$$

$$\frac{a_4(1-m)^2\hat{Z}}{(K_3+(1-m)\hat{Y})^2} < \frac{e(1-m)K_2}{(K_2+(1-m)\hat{Y})^2}. \tag{33}$$

Now, the Jacobian matrix at the equilibrium point p_4 is witten as:

$$J(p_4) = \begin{bmatrix} \check{X}\left(-b + \frac{a_1(1-c\check{Y})^2\check{Y}}{(K_1+\check{X}(1-c\check{Y}))^2}\right) & \check{X}\left[-\frac{fr}{(1+f\check{Y})^2} - \frac{a_1\check{X}(1-c\check{Y})^2+K_1(1-2c\check{Y})a_1}{(K_1+\check{X}(1-c\check{Y}))^2}\right] & 0 \\ \check{Y}\left[\frac{(1-c\check{Y})K_1a_2}{(K_1+\check{X}(1-c\check{Y}))^2}\right] & \check{Y}\left(-\frac{c\check{X}a_2K_1}{(K_1+\check{X}(1-c\check{Y}))^2} - \frac{e(1-m)K_2}{(K_2+(1-m)\check{Y})^2}\right) & -\frac{a_4(1-m)\check{Y}}{K_3+(1-m)\check{Y}} \\ 0 & 0 & -d_3 + \frac{a_5(1-m)\check{Y}}{K_3+(1-m)\check{Y}} \end{bmatrix}. \tag{34}$$

The characteristic equation of $J(p_4)$ can be written as

$$\left(-d_3 + \frac{a_5(1-m)\check{Y}}{K_3+(1-m)\check{Y}} - \lambda\right)(\lambda^2 - T_2\lambda + D_2) = 0, \tag{35}$$

where

$$T_2 = \check{X}\left(-b + \frac{a_1(1-c\check{Y})^2\check{Y}}{(K_1+\check{X}(1-c\check{Y}))^2}\right) - \check{Y}\left(\frac{ca_2K_1\check{X}}{(K_1+\check{X}(1-c\check{Y}))^2} + \frac{e(1-m)K_2}{(K_2+(1-m)\check{Y})^2}\right),$$

$$D_2 = \check{X}\check{Y}\left(-b + \frac{a_1(1-c\check{Y})^2\check{Y}}{(K_1+\check{X}(1-c\check{Y}))^2}\right)\left(-\frac{ca_2K_1\check{X}}{(K_1+\check{X}(1-c\check{Y}))^2} - \frac{e(1-m)K_2}{(K_2+(1-m)\check{Y})^2}\right)$$
$$+\check{X}\check{Y}\left[\frac{fr}{(1+f\check{Y})^2} + \frac{a_1(1-c\check{Y})^2\check{X}+a_1K_1(1-2c\check{Y})}{(K_1+\check{X}(1-c\check{Y}))^2}\right]\left[\frac{(1-c\check{Y})K_1a_2}{(K_1+\check{X}(1-c\check{Y}))^2}\right].$$

Consequently, all the eigenvalues of the $J(p_4)$ will have negative real parts and makes p_4 a stable point if the following conditions are met.

$$\frac{a_5(1-m)\check{Y}}{K_3+(1-m)\check{Y}} < d_3. \tag{36}$$

$$\frac{a_1(1-c\check{Y})^2\check{X}\check{Y}}{\left(K_1+\check{X}(1-c\check{Y})\right)^2} < b\check{X} + \frac{ca_2K_1\check{X}\check{Y}}{\left(K_1+\check{X}(1-c\check{Y})\right)^2} + \frac{e(1-m)K_2\check{Y}}{(K_2+(1-m)\check{Y})^2}. \tag{37}$$

$$\frac{2K_1ca_1\check{Y}}{(K_1+(1-c\check{Y})\check{X})^2} < \frac{fr}{(1+f\check{Y})^2} + \frac{a_1(1-c\check{Y})^2\check{X}+K_1a_1}{(K_1+(1-c\check{Y})\check{X})^2}. \tag{38}$$

The Jacobian matrix of the system (2) at the positive equilibrium point can be written as:

$$J(p_5)=[q_{ij}], \tag{39}$$

where

$$q_{11}=\tilde{X}\left(-b+\frac{a_1(1-c\tilde{Y})^2\tilde{Y}}{(K_1+(1-c\tilde{Y})\tilde{X})^2}\right),$$

$$q_{12}=\tilde{X}\left[-\frac{fr}{(1+f\tilde{Y})^2}-\frac{a_1(1-c\tilde{Y})^2\tilde{X}+(1-2c\tilde{Y})a_1K_1}{(K_1+(1-c\tilde{Y})\tilde{X})^2}\right],$$

$$q_{13}=0,$$

$$q_{21}=\tilde{Y}\left[\frac{(1-c\tilde{Y})a_2K_1}{(K_1+(1-c\tilde{Y})\tilde{X})^2}\right],$$

$$q_{22}=\tilde{Y}\left[-\frac{ca_2K_1\tilde{X}}{(K_1+(1-c\tilde{Y})\tilde{X})^2}-\frac{e(1-m)K_2}{(K_2+(1-m)\tilde{Y})^2}+\frac{a_4(1-m)^2\tilde{Z}}{(K_3+(1-m)\tilde{Y})^2}\right],$$

$$q_{23}=-\frac{a_4(1-m)\tilde{Y}}{K_3+(1-m)\tilde{Y}},$$

$$q_{31}=0,$$

$$q_{32}=\tilde{Z}\left[\frac{(1-m)a_5K_3}{(K_3+(1-m)\tilde{Y})^2}\right],$$

$$q_{33}=0.$$

Therefore, the characteristic equation of $J(p_5)$ can be written as

$$\lambda^3+G_1\lambda^2+G_2\lambda+G_3=0, \tag{40}$$

where

$$G_1=-(q_{11}+q_{22}),$$

$$G_2=q_{11}q_{22}-q_{12}q_{21}-q_{23}q_{32},$$

$$G_3=q_{11}q_{23}q_{32},$$

with

$$\Delta=G_1G_2-G_3=-(q_{11}+q_{22})[q_{11}q_{22}-q_{12}q_{21}]+q_{22}q_{23}q_{32}.$$

Accordingly, the stability conditions of p_5 can be determined through the following theorem.

Theorem 3: The positive equilibrium point p_5 of the system (2) is locally asymptotically stable provided the following sufficient conditions are met.

$$\frac{\tilde{Y}(1-c\tilde{Y})^2a_1}{(K_1+(1-c\tilde{Y})\tilde{X})^2}<b. \tag{41}$$

$$\frac{2a_1K_1c\tilde{Y}}{(K_1+(1-c\tilde{Y})\tilde{X})^2}<\frac{fr}{(1+f\tilde{Y})^2}+\frac{a_1(1-c\tilde{Y})^2\tilde{X}+a_1K_1}{(K_1+(1-c\tilde{Y})\tilde{X})^2}. \tag{42}$$

$$\frac{a_4(1-m)^2\tilde{Z}}{(K_3+(1-m)\tilde{Y})^2}<\frac{ca_2K_1\tilde{X}}{(K_1+(1-c\tilde{Y})\tilde{X})^2}+\frac{e(1-m)K_2}{(K_2+(1-m)\tilde{Y})^2}. \tag{43}$$

Proof. According to the Routh- Hurwitz criterion the proof follows if and only if $G_1 > 0$, $G_3 > 0$, and $\Delta > 0$ are met. Direct computation shows that these requirements are satisfied under the given conditions, and hence the proof is done.

While stable coexistence is the capacity of species to coexist forever in the absence of external perturbations, persistence can be defined as the length of time that a species remains in a community before local extinction takes place. It follows mathematically that there are no boundary attractors in the solution's omega limit set. Therefore, an investigation of the boundary plane dynamics is carried out in the following.

It is clear that system (2) has two subsystems that fall in the positive quadrant of the YZ −plane and XY −plane respectively. At the same time, there is no subsystem in the XZ −plane. These subsystems can be described respectively:

$$\begin{aligned} \frac{dY}{dT} &= Y\left(a_3 - d_2 - \frac{e(1-m)Y}{K_2+(1-m)Y} - \frac{a_4(1-m)Z}{K_3+(1-m)Y}\right) = Yf_{11}(Y,Z), \\ \frac{dZ}{dT} &= Z\left(\frac{a_5(1-m)Y}{K_3+(1-m)Y} - d_3\right) = Zf_{12}(Y,Z), \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{dX}{dT} &= X\left(\frac{r}{1+fY} - d_1 - bX - \frac{a_1(1-cY)Y}{K_1+(1-cY)X}\right) = Xf_{21}(X,Y), \\ \frac{dY}{dT} &= Y\left(\frac{a_2(1-cY)X}{K_1+(1-cY)X} + a_3 - d_2 - \frac{e(1-m)Y}{K_2+(1-m)Y}\right) = Yf_{22}(X,Y), \end{aligned} \tag{45}$$

Straightforward computation shows that the subsystem (44) has the equilibrium points $p_{10} = (0,0)$, $p_{11} = (\bar{\bar{Y}}, 0)$, and $p_{12} = (\hat{Y}, \hat{Z})$. At the same time, the subsystem (45) has the equilibrium points $p_{20} = (0,0)$, $p_{21} = (\bar{X}, 0)$, and $p_{22} = (\breve{X}, \breve{Y})$. It is easy to verify that all these equilibrium points are simply projections of their corresponding equilibrium points of system (2) and have the same form with existing conditions.

Consequently, to investigate the persistence of the system (2), it is necessary to investigate the dynamics in the interior of positive quadrants of YZ −plane and XY −plane respectively.

Define the Dulac functions as $D_1(Y,Z) = \frac{1}{YZ}$ and $D_2(X,Y) = \frac{1}{XY}$. Clearly the Dulac functions $D_1(Y,Z) > 0$, $D_2(X,Y) > 0$, and they are C^1 functions in the $int.\mathbb{R}_+^2$ of the YZ −plane and XY −plane respectively. Furthermore, direct computation gives that

$$\Delta(Y,Z)=\frac{\partial(D_1 f_{11})}{\partial Y}+\frac{\partial(D_1 f_{12})}{\partial Z}=-\frac{eK_2(1-m)}{Z[K_2+(1-m)Y]^2}+\frac{a_4(1-m)^2}{[K_3+(1-m)Y]^2}.$$

$$\Delta(X,Y)=\frac{\partial(D_2 f_{21})}{\partial X}+\frac{\partial(D_2 f_{22})}{\partial Y}=-\frac{b}{Y}+\frac{a_1(1-cY)^2}{[K_1+(1-cY)X]^2}-\frac{a_2K_1c}{[K_1+(1-cY)X]^2}-\frac{eK_2(1-m)}{X[K_2+(1-m)Y]^2}.$$

Then the exprations $\Delta(Y,Z)$ and $\Delta(X,Y)$ do not identically zero in the $int.\mathbb{R}_+^2$ of the YZ −plane and XY −plane and they do not change sign under the following conditions:

$$\frac{a_4(1-m)^2Z}{[K_3+(1-m)Y]^2}<\frac{eK_2(1-m)}{[K_2+(1-m)Y]^2}. \tag{46}$$

$$\frac{a_1(1-cY)^2}{[K_1+(1-cY)X]^2}<\frac{b}{Y}+\frac{a_2K_1c}{[K_1+(1-cY)X]^2}+\frac{eK_2(1-m)}{X[K_2+(1-m)Y]^2}. \tag{47}$$

According to the Dulac-Bendixson criterion [28], for all trajectories meeting conditions (46)-(47), there is no closed curve lying in the $int.\mathbb{R}_+^2$ of the YZ −plane and XY −plane. Moreover, the unique equilibrium points in the $int.\mathbb{R}_+^2$ of the YZ −plane and XY −plane that is determined by p_{12} and p_{22} will therefore be globally asymptotically stable whenever they are locally asymptotically stable, according to the Poincare-Bendixon theorem [28].

Theorem 4*:* Assume that conditions (46)-(47) are satisfied and the following conditions are met then system (2) is uniformly persistent.

$$d_1<r. \tag{48}$$

$$d_2<a_3+\frac{a_2(r-d_1)}{r-d_1+K_1b}. \tag{49}$$

$$d_3<\min\left\{\frac{a_5(1-m)\bar{\bar{Y}}}{K_3+(1-m)\bar{\bar{Y}}},\frac{a_5(1-m)\check{Y}}{K_3+(1-m)\check{Y}}\right\}. \tag{50}$$

$$d_1+\frac{\hat{Y}(1-c\hat{Y})a_1}{K_1}<\frac{r}{1+f\hat{Y}}. \tag{51}$$

Proof*:* Define the function $\varphi(X,Y,Z)=X^{b_1}Y^{b_2}Z^{b_3}$, where b_j , $\forall j=1,2,3$ are positive constants. Obviously $\varphi(X,Y,Z)>0$ for all $(X,Y,Z)\in int.\mathbb{R}_3^+$ and $\varphi(X,Y,Z)\to 0$ when $X\to 0$ or $Y\to 0$ or $Z\to 0$. Then by utilizing the average Lyapunov method [26], it is obtained that:

$$\begin{aligned}\Omega(X,Y,Z)=\frac{\varphi'(X,Y,Z)}{\varphi(X,Y,Z)}&=b_1\left[\frac{r}{1+fY}-d_1-bX-\frac{a_1(1-cY)Y}{K_1+(1-cY)X}\right]\\&+b_2\left[\frac{a_2(1-cY)X}{K_1+(1-cY)X}+a_3-d_2-\frac{e(1-m)Y}{K_2+(1-m)Y}-\frac{a_4(1-m)Z}{K_3+(1-m)Y}\right].\\&+b_3\left[\frac{a_5(1-m)Y}{K_3+(1-m)Y}-d_3\right]\end{aligned}$$

Thus, the proof is done if $\Omega(E)>0$ for any boundary equilibrium point E, with suitable choice of constants $b_1>0$, $b_2>0$, and $b_3>0$.

$$\Omega(p_0) = b_1(r - d_1) + b_2(a_3 - d_2) + b_3(-d_3)\ .$$

$$\Omega(p_1) = b_2\left(\frac{a_2(r-d_1)}{r-d_1+K_1 b} + a_3 - d_2\right) + b_3(-d_3)\ .$$

$$\Omega(p_2) = b_1\left[\frac{r}{1+f\bar{\bar{Y}}} - d_1 - \frac{a_1(1-c\bar{\bar{Y}})\bar{\bar{Y}}}{K_1}\right] + \ b_3\left[\frac{a_5(1-m)\bar{\bar{Y}}}{K_3+(1-m)\bar{\bar{Y}}} - d_3\right].$$

$$\Omega(p_3) = b_1\left[\frac{r}{1+f\hat{Y}} - d_1 - \frac{a_1(1-c\hat{Y})\hat{Y}}{K_1}\right].$$

$$\Omega(p_4) = \ b_3\left[\frac{a_5(1-m)\check{Y}}{K_3+(1-m)\check{Y}} - d_3\right].$$

Clearly, by using the given conditions with suitable choice of the positive constants it is obtained that $\Omega(p_i) > 0$ for all $i = 0,1,\dots,4$. Hence the proof is complete.

In the following theorems, the global stability of the above mentioned equilibrium points is studied.

Theorem 5: The vanishing equilibrium point is a global asymptotic stable whenever it is locally asymptotically stable.

Proof. Consider the following candidate Lyapunov function

$$L_0(X,Y,Z) = X + Y + Z.$$

Clearly, $L_0\colon \mathbb{R}^3_+ \to \mathbb{R}$ is a C^1 positive definite function so that $L_0(0,0,0) = 0$ and $L_0(X,Y,Z) > 0$ for all $\{(X,Y,Z) \in \mathbb{R}^3_+ : X \geq 0, Y \geq 0, Z \geq 0\}$ with $(X,Y,Z) \neq (0,0,0)$. Direct computation shows that:

$$\frac{dL_0}{dT} = \frac{rX}{1+fY} - d_1X - bX^2 - \frac{(a_1-a_2)(1-cY)XY}{K_1+(1-cY)X} + (a_3 - d_2)Y$$
$$- \frac{e(1-m)Y^2}{K_2+(1-m)Y} - \frac{(a_4-a_5)(1-m)YZ}{K_3+(1-m)Y} - d_3Z.$$

Biologically, it is well known that $a_1 - a_2 > 0$, and $a_4 - a_5 > 0$, hence it is obtained that

$$\frac{dL_0}{dT} < -(d_1 - r)X - (d_2 - a_3)Y - d_3Z.$$

Therefore, under the local stability conditions (20)-(21), $\frac{dL_0}{dT}$ is negative definite. Hence, the vanishing equilibrium point is globally asymptotically stable.

Theorem 6: The first axial equilibrium point is a global asymptotically stable provided that the following condition is met.

$$a_3 + \frac{rfK_1+a_1}{K_1}\bar{X} < d_2. \tag{52}$$

Proof. Consider the following candidate Lyapunov function

$$L_1(X,Y,Z) = \left(X - \bar{X} - \bar{X}\ln\frac{X}{\bar{X}}\right) + Y + Z,$$

Clearly, $L_1: \mathbb{R}_+^3 \to \mathbb{R}$ is a C^1 positive definite function so that $L_1(\bar{X},0,0) = 0$ and $L_1(X,Y,Z) > 0$ for all $\{(X,Y,Z) \in \mathbb{R}_+^3: X > 0, Y \geq 0, Z \geq 0\}$ with $(X,Y,Z) \neq (\bar{X},0,0)$. Direct computation shows that:

$$\frac{dL_1}{dT} = -\frac{rfY}{1+fY}(X-\bar{X}) - b(X-\bar{X})^2 - \frac{(a_1-a_2)(1-cY)XY}{K_1+(1-cY)X} + \frac{a_1(1-cY)\bar{X}Y}{K_1+(1-cY)X}$$
$$+(a_3-d_2)Y - \frac{e(1-m)Y^2}{K_2+(1-m)Y} - \frac{(a_4-a_5)(1-m)YZ}{K_3+(1-m)Y} - d_3Z.$$

Then

$$\frac{dL_1}{dT} < -b(X-\bar{X})^2 - \left(d_2 - a_3 - rf\bar{X} - \frac{a_1\bar{X}}{K_1}\right)Y - d_3Z.$$

Obviously, condition (52) guarantees that $\frac{dL_1}{dT}$ is negative definite. Hence, the first axial equilibrium point is a globally asymptotically stable.

Theorem 7: The second axial equilibrium point is a global asymptotically stable provided that condition (20) and the following condition are met.

$$\frac{a_4(1-m)}{K_3}\bar{\bar{Y}} < d_3. \tag{53}$$

Proof. Consider the following candidate Lyapunov function

$$L_2(X,Y,Z) = X + \left(Y - \bar{\bar{Y}} - \bar{\bar{Y}}\ln\frac{Y}{\bar{\bar{Y}}}\right) + Z,$$

Clearly, $L_2: \mathbb{R}_+^3 \to \mathbb{R}$ is a C^1 positive definite function so that $L_2\left(0,\bar{\bar{Y}},0\right) = 0$ and $L_2(X,Y,Z) > 0$ for all $\{(X,Y,Z) \in \mathbb{R}_+^3: X \geq 0, Y > 0, Z \geq 0\}$ with $(X,Y,Z) \neq \left(0,\bar{\bar{Y}},0\right)$. Direct computation shows that:

$$\frac{dL_2}{dT} = \frac{rX}{1+fY} - d_1X - bX^2 - \frac{(a_1-a_2)(1-cY)XY}{K_1+(1-cY)X} - \frac{a_2(1-cY)X\bar{\bar{Y}}}{K_1+(1-cY)X} - \frac{(a_4-a_5)(1-m)YZ}{K_3+(1-m)Y}$$
$$-\frac{e(1-m)K_2\left(Y-\bar{\bar{Y}}\right)^2}{(K_2+(1-m)Y)\left(K_2+(1-m)\bar{\bar{Y}}\right)} + \frac{a_4(1-m)\bar{\bar{Y}}Z}{K_3+(1-m)Y} - d_3Z.$$

Then

$$\frac{dL_2}{dT} < -(d_1-r)X - \frac{e(1-m)K_2\left(Y-\bar{\bar{Y}}\right)^2}{(K_2+(1-m)Y)\left(K_2+(1-m)\bar{\bar{Y}}\right)} - \left(d_3 - \frac{a_4(1-m)\bar{\bar{Y}}}{K_3}\right)Z.$$

Obviously, conditions (20) and (53) guarantee that $\frac{dL_2}{dT}$ is negative definite. Hence, the second axial equilibrium point is globally asymptotically stable.

Theorem 8: The prey-free equilibrium point is a global asymptotically stable provided that condition (20) and the following condition are met.

$$\frac{a_4(1-m)^2\hat{Z}}{K_3(K_3+(1-m)\hat{Y})} < \frac{e(1-m)K_2}{(K_2+(1-m)Y_{max})(K_2+(1-m)\hat{Y})}, \tag{54}$$

where all the new symbols are given in the proof.

Proof. Consider the following candidate Lyapunov function

$$L_3(X,Y,Z) = X + \left(Y - \hat{Y} - \hat{Y}\ln\frac{Y}{\hat{Y}}\right) + n_1\left(Z - \hat{Z} - \hat{Z}\ln\frac{Z}{\hat{Z}}\right),$$

where n_1 is a positive constant to be determined. Clearly, $L_3: \mathbb{R}_+^3 \to \mathbb{R}$ is a C^1 positive definite function so that $L_3(0,\hat{Y},\hat{Z}) = 0$ and $L_3(X,Y,Z) > 0$ for all $\{(X,Y,Z) \in \mathbb{R}_+^3 : X \geq 0, Y > 0, Z > 0\}$ with $(X,Y,Z) \neq (0,\hat{Y},\hat{Z})$. Direct computation shows that:

$$\begin{aligned}\frac{dL_3}{dT} &= \frac{rX}{1+fY} - d_1X - bX^2 - \frac{(a_1-a_2)(1-cY)XY}{K_1+(1-cY)X} - \frac{a_2(1-cY)X\hat{Y}}{K_1+(1-cY)X} \\ &- \left[\frac{e(1-m)K_2}{(K_2+(1-m)Y)(K_2+(1-m)\hat{Y})} - \frac{a_4(1-m)^2\hat{Z}}{(K_3+(1-m)Y)(K_3+(1-m)\hat{Y})}\right]\left(Y-\hat{Y}\right)^2 \\ &- \frac{(1-m)[a_4(K_3+(1-m)\hat{Y}) - n_1a_5K_3]}{(K_3+(1-m)Y)(K_3+(1-m)\hat{Y})}\left(Y-\hat{Y}\right)\left(Z-\hat{Z}\right)\end{aligned}$$

Then, by choosing $n_1 = \frac{a_4(K_3+(1-m)\hat{Y})}{a_5K_3}$, and maximizing the right-hand side, it is obtained that

$$\frac{dL_3}{dT} < -(d_1-r)X - \left[\frac{e(1-m)K_2}{(K_2+(1-m)Y_{max})(K_2+(1-m)\hat{Y})} - \frac{a_4(1-m)^2\hat{Z}}{K_3(K_3+(1-m)\hat{Y})}\right]\left(Y-\hat{Y}\right)^2, \tag{55}$$

where Y_{max} represents the upper bound of the Y.

Obviously, conditions (20) and (54) guarantee that $\frac{dL_3}{dT}$ is negative semi definite, which leads to the prey-free equilibrium point is a stable point. Hence, the proof results from equation (55) and Lyapunov–Lasalle's invariance principle [27].

Theorem 9: The top predator-free equilibrium point is globally asymptotically stable if the following conditions are met.

$$\frac{a_2K_2\breve{Y}}{\breve{X}K_1(K_1+(1-c\breve{Y})\breve{X})} < \frac{a_2K_2b}{a_1(1-c\breve{Y})\breve{X}}, \tag{56}$$

$$\frac{a_4(1-m)\breve{Y}}{K_3} < d_3, \tag{57}$$

$${g_{12}}^2 < 4g_{11}g_{22}, \tag{58}$$

where all the new symbols are given in the proof.

Proof. Consider the following candidate Lyapunov function

$$L_4(X,Y,Z) = n_2\left(X - \hat{X} - \hat{X}\ln\frac{X}{\hat{X}}\right) + \left(Y - \hat{Y} - \hat{Y}\ln\frac{Y}{\hat{Y}}\right) + Z,$$

where n_2 is a positive constant to be determined. Clearly, $L_4: \mathbb{R}_+^3 \to \mathbb{R}$ is a C^1 positive definite function so that $L_4(\breve{X},\breve{Y},0) = 0$ and $L_4(X,Y,Z) > 0$ for all $\{(X,Y,Z) \in \mathbb{R}_+^3 : X > 0, Y > 0, Z \geq 0\}$ with $(X,Y,Z) \neq (\breve{X},\breve{Y},0)$. Direct computation shows that:

$$\frac{dL_4}{dT} = n_2\left(X - \breve{X}\right)\left(\frac{r}{1+fY} - d_1 - bX - \frac{a_1(1-cY)Y}{K_1+(1-cY)X}\right)$$
$$+\left(Y - \breve{Y}\right)\left(\frac{a_2(1-cY)X}{K_1+(1-cY)X} + a_3 - d_2 - \frac{e(1-m)Y}{K_2+(1-m)Y} - \frac{a_4(1-m)Z}{K_3+(1-m)Y}\right)$$
$$+Z\left(\frac{a_5(1-m)Y}{K_3+(1-m)Y} - d_3\right).$$

Using some mathematical mainupolation gives that

$$\frac{dL_4}{dT} = -\frac{n_2 rf}{(1+fY)(1+f\breve{Y})}\left(X - \breve{X}\right)\left(Y - \breve{Y}\right) - n_2 b\left(X - \breve{X}\right)^2 - \frac{n_2 a_1 K_1(1-cY-c\breve{Y})(X-\breve{X})(Y-\breve{Y})}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})}$$
$$-\frac{n_2 a_1 \breve{X}(1-cY)(1-c\breve{Y})(X-\breve{X})(Y-\breve{Y})}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})} + \frac{n_2 a_1 \breve{Y}(1-cY)(1-c\breve{Y})(X-\breve{X})^2}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})}$$
$$+\frac{a_2 K_1(1-cY)(X-\breve{X})(Y-\breve{Y})}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})} - \frac{a_2 c K_1 \breve{X}(Y-\breve{Y})^2}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})} - \frac{e(1-m)K_2(Y-\breve{Y})^2}{(K_2+(1-m)Y)(K_2+(1-m)\breve{Y})}$$
$$-\frac{a_4(1-m)YZ}{K_3+(1-m)Y} + \frac{a_4(1-m)\breve{Y}Z}{K_3+(1-m)Y} + \frac{a_5(1-m)YZ}{K_3+(1-m)Y} - d_3 Z$$

Then, by choosing $n_2 = \frac{a_2 K_2}{a_1(1-c\breve{Y})\breve{X}}$, and maximizing the right-hand side, it is obtained that

$$\frac{dL_4}{dT} < -\left[\frac{a_2 K_2 b}{a_1(1-c\breve{Y})\breve{X}} - \frac{a_2 K_2 \breve{Y}}{\breve{X}K_1(K_1+(1-c\breve{Y})\breve{X})}\right]\left(X - \breve{X}\right)^2 - \left[d_3 - \frac{a_4(1-m)\breve{Y}}{K_3}\right]Z$$
$$-\frac{a_2 K_2}{a_1(1-c\breve{Y})\breve{X}}\left[\frac{rf}{(1+fY)(1+f\breve{Y})} + \frac{a_1 K_1(1-cY-c\breve{Y})}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})}\right]\left(X - \breve{X}\right)\left(Y - \breve{Y}\right)$$
$$-\left[\frac{a_2 c K_1 \breve{X}}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})} + \frac{e(1-m)K_2}{(K_2+(1-m)Y)(K_2+(1-m)\breve{Y})}\right]\left(Y - \breve{Y}\right)^2$$

Using the conditions (56) and (58) lead to:

$$\frac{dL_4}{dT} < -\left[\sqrt{g_{11}}\left(X - \breve{X}\right) + \sqrt{g_{22}}\left(Y - \breve{Y}\right)\right]^2 - \left[d_3 - \frac{a_4(1-m)\breve{Y}}{K_3}\right]Z,$$

where

$$g_{11} = \frac{a_2 K_2 b}{a_1(1-c\breve{Y})\breve{X}} - \frac{a_2 K_2 \breve{Y}}{\breve{X}K_1(K_1+(1-c\breve{Y})\breve{X})},$$

$$g_{12} = \frac{a_2 K_2}{a_1(1-c\breve{Y})\breve{X}}\left[\frac{rf}{(1+fY)(1+f\breve{Y})} + \frac{a_1 K_1(1-cY-c\breve{Y})}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})}\right],$$

$$g_{22} = \frac{a_2 c K_1 \breve{X}}{(K_1+(1-cY)X)(K_1+(1-c\breve{Y})\breve{X})} + \frac{e(1-m)K_2}{(K_2+(1-m)Y)(K_2+(1-m)\breve{Y})}.$$

Obviously, condition (57) guarantees that $\frac{dL_4}{dT}$ is negative definite. Hence, the top predator-free equilibrium point is a globally asymptotically stable under the given conditions. Hence the proof is complete.

Theorem 10: The positive equilibrium point is a globally asymptotically stable if the following conditions are met.

$$\frac{a_1}{K_1\tilde{B}_1}\left(1-c\tilde{Y}\right)\tilde{X} < b, \tag{59}$$

$$\frac{a_4(1-m)^2\tilde{Z}}{K_3\tilde{B}_3} < \frac{a_2cK_1\tilde{X}}{(K_1+X_{max})\tilde{B}_1} + \frac{e(1-m)K_2}{(K_2+(1-m)Y_{max})\tilde{B}_2}, \tag{60}$$

$$h_{12}{}^2 < 4h_{11}h_{22}, \tag{61}$$

where all the new symbols are given in the proof.

Proof. Consider the following candidate Lyapunov function

$$L_5(X,Y,Z) = \gamma_1\left(X-\tilde{X}-\tilde{X}\ln\frac{X}{\tilde{X}}\right) + \left(Y-\tilde{Y}-\tilde{Y}\ln\frac{Y}{\tilde{Y}}\right) + \gamma_2\left(Z-\tilde{Z}-\tilde{Z}\ln\frac{Z}{\tilde{Z}}\right),$$

where γ_1 and γ_2 are positive constants to be determined. Clearly, $L_5: \mathbb{R}_+^3 \to \mathbb{R}$ is a C^1 positive definite function so that $L_4(\tilde{X},\tilde{Y},\tilde{Z}) = 0$ and $L_4(X,Y,Z) > 0$ for all $\{(X,Y,Z) \in \mathbb{R}_+^3 : X > 0, Y > 0, Z > 0\}$ with $(X,Y,Z) \neq (\tilde{X},\tilde{Y},\tilde{Z})$. Direct computation shows that:

$$\begin{aligned}\frac{dL_5}{dT} = &-\left[\frac{\gamma_1 rf}{B_0\tilde{B}_0} + \frac{\gamma_1 a_1}{B_1\tilde{B}_1}\left(K_1 - cK_1Y - cK_1\tilde{Y} + (1-cY)(1-c\tilde{Y})\tilde{X}\right)\right](X-\tilde{X})(Y-\tilde{Y})\\ &-\gamma_1\left[b - \frac{a_1}{B_1\tilde{B}_1}(1-cY)(1-c\tilde{Y})\tilde{X}\right](X-\tilde{X})^2 + \frac{a_2K_1}{B_1\tilde{B}_1}(1-cY)(X-\tilde{X})(Y-\tilde{Y})\\ &-\left[\frac{a_2cK_1\tilde{X}}{B_1\tilde{B}_1} + \frac{e(1-m)K_2}{B_2\tilde{B}_2} - \frac{a_4(1-m)^2\tilde{Z}}{B_3\tilde{B}_3}\right](Y-\tilde{Y})^2 - \frac{a_4(1-m)\tilde{B}_3}{B_3\tilde{B}_3}(Y-\tilde{Y})(Z-\tilde{Z})\\ &+\frac{\gamma_2 a_5(1-m)K_3}{B_3\tilde{B}_3}(Y-\tilde{Y})(Z-\tilde{Z}),\end{aligned}$$

where $B_0 = 1 + fY$, $\tilde{B}_0 = 1 + f\tilde{Y}$, $B_1 = K_1 + (1-cY)X$, $\tilde{B}_1 = K_1 + (1-c\tilde{Y})\tilde{X}$, $B_2 = K_2 + (1-m)Y$, $\tilde{B}_2 = K_2 + (1-m)\tilde{Y}$, $B_3 = K_3 + (1-m)Y$, and $\tilde{B}_3 = K_3 + (1-m)\tilde{Y}$.

Thus, by choosing $\gamma_1 = \frac{a_2}{a_1}$, and $\gamma_2 = \frac{a_4\tilde{B}_3}{a_5K_3}$ with maximizing the right-hand side, it is obtained that

$$\begin{aligned}\frac{dL_5}{dT} < &-\left[\frac{a_2 rf}{a_1B_0\tilde{B}_0} + \frac{\gamma_1 a_2}{B_1\tilde{B}_1}\left(-cK_1\tilde{Y} + (1-cY)(1-c\tilde{Y})\tilde{X}\right)\right](X-\tilde{X})(Y-\tilde{Y})\\ &-\frac{a_2}{a_1}\left[b - \frac{a_1}{K_1\tilde{B}_1}\left(1-c\tilde{Y}\right)\tilde{X}\right](X-\tilde{X})^2\\ &-\left[\frac{a_2cK_1\tilde{X}}{(K_1+X_{max})\tilde{B}_1} + \frac{e(1-m)K_2}{(K_2+(1-m)Y_{max})\tilde{B}_2} - \frac{a_4(1-m)^2\tilde{Z}}{K_3\tilde{B}_3}\right](Y-\tilde{Y})^2,\end{aligned}$$

where X_{max} and Y_{max} represent the upper bound of the X and Y respectively. Using the conditions (59) - (61) lead to:

$$\frac{dL_5}{dT} < -\left[\sqrt{h_{11}}(X-\tilde{X}) + \sqrt{h_{22}}(Y-\tilde{Y})\right]^2, \tag{62}$$

where

$$h_{11} = \frac{a_2}{a_1}\left[b - \frac{a_1}{K_1\tilde{B}_1}(1-c\tilde{Y})\tilde{X}\right],$$

$$h_{22} = \frac{a_2 c K_1 \tilde{X}}{(K_1+X_{max})\tilde{B}_1} + \frac{e(1-m)K_2}{(K_2+(1-m)Y_{max})\tilde{B}_2} - \frac{a_4(1-m)^2\tilde{Z}}{K_3\tilde{B}_3},$$

$$h_{12} = \frac{a_2 r f}{a_1 B_0 \tilde{B}_0} + \frac{\gamma_1 a_2}{B_1\tilde{B}_1}\left(-cK_1\tilde{Y} + (1-cY)(1-c\tilde{Y})\tilde{X}\right).$$

Obviously, $\frac{dL_5}{dT}$ is negative sime definite, which leads to that, the positive equilibrium point is a stable point. Hence, the proof results from equation (62) and Lyapunov–Lasalle's invariance principle [27].

5. LOCAL BIFURCATION

The present section investigates the influence of the varying parameters on the qualitative dynamics of the system (2) near the non-hyperbolic. An application to the Sotomayor theorem [27] for local bifurcation is performed.

Rewrite the system (2) as the following vector norm

$$\frac{d\mathbf{X}}{dT} = \mathbf{F}(\mathbf{X},\vartheta), \ \boldsymbol{X} = (X,Y,Z)^{\mathrm{T}}, \ \mathbf{F} = \left(Xf_1(\mathbf{X},\vartheta), Yf_2(\mathbf{X},\vartheta), Zf_3(\mathbf{X},\vartheta)\right)^{\mathrm{T}}. \tag{63}$$

where $\vartheta \in \mathbb{R}$ is the bifurcation parameter and $f_i(\boldsymbol{X},\vartheta)$ for all $i = 1,2,3$ are given in the system (2). Therefore, for any vector of the form $\mathbf{V} = (v_1, v_2, v_3)^{\mathrm{T}}$, the following expressions can be determined

$$D^2\mathbf{F}(\mathbf{X},\vartheta)(\mathbf{V},\mathbf{V}) = [c_{i1}]_{3\times1}, \tag{64}$$

where

$$c_{11} = -\frac{2\left[-bX^3(-1+cY)^3 + (-1+cY)^2(3bX^2 - Ya_1)K_1 - 3bX(-1+cY)K_1^2 + bK_1^3\right]v_1^2}{(X(1-cY)+K_1)^3} + 2\left(-\frac{fr}{(1+fY)^2} - \frac{a_1K_1(X - cXY + K_1 - 2cYK_1)}{(X(1-cY)+K_1)^3}\right)v_1v_2 + 2X\left(\frac{f^2r}{(1+fY)^3} + \frac{ca_1K_1(X+K_1)}{(X(1-cY)+K_1)^3}\right)v_2^2,$$

$$c_{21} = -\frac{2a_2K_1\left[Y(-1+cY)^2v_1^2 + [X(-1+cY) + (-1+2cY)K_1]v_1v_2 + cX(X+K_1)v_2^2\right]}{(X(1-cY)+K_1)^3} - 2(1-m)v_2\left(\frac{eK_2^2v_2}{(Y-mY+K_2)^3} - \frac{a_4K_3((-1+m)Zv_2 + (Y-mY+K_3)v_3)}{((-1+m)Y-K_3)^3}\right),$$

$$c_{31} = \frac{2(-1+m)a_5K_3v_2[(-1+m)Zv_2+(Y-mY+K_3)v_3]}{((-1+m)Y-K_3)^3}.$$

While

$$D^3\mathbf{F}(\mathbf{X},\vartheta)(\mathbf{V},\mathbf{V},\mathbf{V}) = [d_{i1}]_{3\times 1}, \tag{65}$$

where

$$\begin{aligned} d_{11} &= \frac{6f^2rv_2^2((1+fY)v_1-fXv_2)}{(1+fY)^4} \\ &+ \frac{6a_1K_1[Y(-1+cY)^3v_1^3+(-1+cY)(X(-1+cY)+(-1+3cY)K_1)v_1^2v_2]}{(X(1-cY)+K_1)^4}, \\ &+ \frac{6a_1K_1[c(X^2(-1+cY)+2cXYK_1+K_1^2)v_1v_2^2+c^2X^2(X+K_1)v_2^3]}{(X(1-cY)+K_1)^4} \\ d_{21} &= \frac{6a_2K_1[-Y(-1+cY)^3v_1^3-(-1+cY)[X(-1+cY)+(-1+3cY)K_1]v_1^2v_2]}{(X(1-cY)+K_1)^4} \\ &+ \frac{6a_2K_1[-c(X^2(-1+cY)+2cXYK_1+K_1^2)v_1v_2^2-c^2X^2(X+K_1)v_2^3]}{(X(1-cY)+K_1)^4}, \\ &+6(-1+m)^2v_2^2\left[\frac{eK_2^2v_2}{((1-m)Y+K_2)^4}+\frac{a_4K_3((-1+m)Zv_2+(Y-mY+K_3)v_3)}{((1-m)Y+K_3)^4}\right] \\ d_{31} &= -\frac{6(-1+m)^2a_5K_3v_2^2[(-1+m)Zv_2+(Y-mY+K_3)v_3]}{((1-m)Y+K_3)^4}. \end{aligned}$$

Theorem 11: Assume that condition (21) holds, then when the parameter r passes through $r = d_1 \equiv (r^*)$, the system (2) undergoes a transcritical bifurcation at the vanishing equilibrium point.

Proof. When $r = d_1 \equiv (r^*)$ the Jacobian matrix becomes

$$J_0 = J(p_0, r^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_3 - d_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}.$$

Clearly, J_0 has the eigenvalues $\lambda_{01} = 0$, $\lambda_{02} = a_3 - d_2$, and $\lambda_{03} = -d_3$. Clearly, condition (21) guarantees that $\lambda_{02} < 0$. Hence, the eigenvectors of J_0 and ${J_0}^{\mathrm{T}}$ corresponding $\lambda_{01} = 0$ can be written as $\mathbf{V}_0 = (v_{01}, v_{02}, v_{03})^{\mathrm{T}}$ and $\mathbf{U}_0 = (u_{01}, u_{02}, u_{03})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_0 = (1,0,0)^{\mathrm{T}},\ \mathbf{U}_0 = (1,0,0)^{\mathrm{T}}.$$

Moreover, with the use of equation (64), it is obtained that

$$\mathbf{F}_r = \begin{pmatrix} \frac{X}{1+fY} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_r(p_0, r^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow {\mathbf{U}_0}^{\mathrm{T}}\mathbf{F}_r(p_0, r^*) = 0$$

$$D\mathbf{F}_r(p_0, r^*).\mathbf{V}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow {\mathbf{U}_0}^{\mathrm{T}}[D\mathbf{F}_r(p_0, r^*).\mathbf{V}_0] = 1$$

$$D^2\mathbf{F}(p_0,r^*)(\mathbf{V}_0,\mathbf{V}_0)=\begin{pmatrix}-2b\\0\\0\end{pmatrix}\Rightarrow \mathbf{U}_0{}^{\mathrm{T}}[D^2\mathbf{F}(p_0,r^*)(\mathbf{V}_0,\mathbf{V}_0)]=-2b$$

Then, as the parameter r crosses through r^*, the Sotomayor theorem makes the system (2) undergo a transcritical bifurcation at the equilibrium point p_0.

Theorem 12: When the parameter d_2 passes through $d_2=a_3+\frac{a_2(r-d_1)}{r-d_1+K_1b}\equiv(d_2^*)$, the system (2) undergoes a transcritical bifurcation at the first axial equilibrium point.

Proof. When $d_2=d_2^*$ the Jacobian matrix becomes

$$J_1=J(p_1,d_2^*)=\begin{bmatrix}-r+d_1 & \frac{(r-d_1)}{b}\left(-fr-\frac{ba_1}{r-d_1+K_1b}\right) & 0\\ 0 & 0 & 0\\ 0 & 0 & -d_3\end{bmatrix}.$$

Clearly, J_1 has the eigenvalues $\lambda_{11}=-r+d_1<0$, $\lambda_{12}=0$, and $\lambda_{13}=-d_3$. Hence, the eigenvectors of J_1 and $J_1{}^{\mathrm{T}}$ corresponding $\lambda_{12}=0$ can be written as $\mathbf{V}_1=(v_{11},v_{12},v_{13})^{\mathrm{T}}$ and $\mathbf{U}_1=(u_{11},u_{12},u_{13})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_1=\left(-\left(\frac{fr}{b}+\frac{a_1}{r-d_1+K_1b}\right),1,0\right)^{\mathrm{T}}=(\rho_1,1,0)^{\mathrm{T}},\ \mathbf{U}_1=(0,1,0)^{\mathrm{T}}.$$

Moreover, with the use of equation (64), it is obtained that

$$\mathbf{F}_{d_2}=\begin{pmatrix}0\\-Y\\0\end{pmatrix}\Rightarrow \mathbf{F}_{d_2}(p_1,d_2^*)=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow \mathbf{U}_1{}^{\mathrm{T}}\mathbf{F}_{d_2}(p_1,d_2^*)=0$$

$$D\mathbf{F}_{d_2}(p_1,d_2^*).\mathbf{V}_1=\begin{pmatrix}0\\-1\\0\end{pmatrix}\Rightarrow \mathbf{U}_1{}^{\mathrm{T}}\left[D\mathbf{F}_{d_2}(p_1,d_2^*).\mathbf{V}_1\right]=-1$$

$$D^2\mathbf{F}(p_1,d_2^*)(\mathbf{V}_1,\mathbf{V}_1)=\begin{pmatrix}-2b\rho_1^2-2\left(fr+\frac{a_1K_1}{(\bar{X}+K_1)^2}\right)\rho_1+2\bar{X}\left(f^2r+\frac{ca_1K_1}{(\bar{X}+K_1)^2}\right)\\ -\frac{2a_2K_1(-\rho_1+c\bar{X})}{(\bar{X}+K_1)^2}-2(1-m)\frac{e}{K_2}\\ 0\end{pmatrix}$$

Therefore, it is obtained that:

$$\mathbf{U}_1{}^{\mathrm{T}}[D^2\mathbf{F}(p_1,d_2^*)(\mathbf{V}_1,\mathbf{V}_1)]=-\frac{2a_2K_1(-\rho_1+c\bar{X})}{(\bar{X}+K_1)^2}-\frac{2e(1-m)}{K_2}\neq 0$$

Then, as the parameter d_2 crosses through d_2^*, the Sotomayor theorem makes the system (2) undergo a transcritical bifurcation at the equilibrium point p_1.

Theorem 13: Assume that conditions (27)-(28) are staisfied, then when the parameter d_3 passes through $d_3=\frac{a_5(1-m)\bar{\bar{Y}}}{K_3+(1-m)\bar{\bar{Y}}}\equiv(d_3^*)$, the system (2) undergoes a transcritical bifurcation at the second axial equilibrium point.

Proof. As $d_3 = d_3^*$ the Jacobian matrix becomes

$$J_2 = J(p_2, d_3^*) = \begin{bmatrix} \frac{r}{1+f\bar{\bar{Y}}} - d_1 - \frac{a_1(1-c\bar{\bar{Y}})\bar{\bar{Y}}}{K_1} & 0 & 0 \\ \frac{a_2(1-c\bar{\bar{Y}})\bar{\bar{Y}}}{K_1} & \frac{e(1-m)^2\bar{\bar{Y}}^2}{(K_2+(1-m)\bar{\bar{Y}})^2} - \frac{e(1-m)\bar{\bar{Y}}}{K_2+(1-m)\bar{\bar{Y}}} & \frac{a_4(1-m)\bar{\bar{Y}}}{K_3+(1-m)\bar{\bar{Y}}} \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, J_2 has the eigenvalues $\lambda_{21} = \frac{r}{1+f\bar{\bar{Y}}} - d_1 - \frac{a_1(1-c\bar{\bar{Y}})\bar{\bar{Y}}}{K_1}$, $\lambda_{22} = \frac{e(1-m)^2\bar{\bar{Y}}^2}{(K_2+(1-m)\bar{\bar{Y}})^2} - \frac{e(1-m)\bar{\bar{Y}}}{K_2+(1-m)\bar{\bar{Y}}}$, are negative due to conditions (27)-(28). While $\lambda_{23} = 0$. Hence, the eigenvectors of J_2 and J_2^{T} corresponding $\lambda_{23} = 0$ can be written as $\mathbf{V}_2 = (v_{21}, v_{22}, v_{23})^{\mathrm{T}}$ and $\mathbf{U}_2 = (u_{21}, u_{22}, u_{23})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_2 = \left(0, \frac{a_4(K_2+(1-m)\bar{\bar{Y}})^2}{eK_2(K_3+(1-m)\bar{\bar{Y}})}, 1\right)^{\mathrm{T}} = (0, \rho_2, 1)^{\mathrm{T}},\ \mathbf{U}_2 = (0,0,1)^{\mathrm{T}}.$$

Moreover, with the use of equation (64), it is obtained that

$$\mathbf{F}_{d_3} = \begin{pmatrix} 0 \\ 0 \\ -Z \end{pmatrix} \Rightarrow \mathbf{F}_{d_3}(p_2, d_3^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{U}_2^{\mathrm{T}} \mathbf{F}_{d_3}(p_2, d_3^*) = 0$$

$$D\mathbf{F}_{d_3}(p_2, d_3^*).\mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \mathbf{U}_2^{\mathrm{T}}\left[D\mathbf{F}_{d_3}(p_2, d_3^*).\mathbf{V}_2\right] = -1$$

$$D^2\mathbf{F}(p_2, d_3^*)(\mathbf{V}_2, \mathbf{V}_2) = \begin{pmatrix} 0 \\ -2(1-m)\rho_2\left(\frac{eK_2^2\rho_2}{(\bar{\bar{Y}}-m\bar{\bar{Y}}+K_2)^3} + \frac{a_4K_3}{((1-m)\bar{\bar{Y}}+K_3)^2}\right) \\ \frac{2(1-m)a_5K_3\rho_2}{((1-m)\bar{\bar{Y}}+K_3)^2} \end{pmatrix}$$

Therefore, it is obtained that:

$$\mathbf{U}_2^{\mathrm{T}}[D^2\mathbf{F}(p_2, d_3^*)(\mathbf{V}_2, \mathbf{V}_2)] = \frac{2(1-m)a_5K_3\rho_2}{((1-m)\bar{\bar{Y}}+K_3)^2} \neq 0$$

Then, as the parameter d_3 crosses through d_3^*, the Sotomayor theorem makes the system (2) undergo a transcritical bifurcation at the equilibrium point p_2.

Theorem 14: Assume that condition (33) is staisfied, then when the parameter d_1 passes through $d_1 = \frac{r}{1+f\hat{Y}} - \frac{a_1(1-c\hat{Y})\hat{Y}}{K_1} \equiv (d_1^*)$, the system (2) undergoes a transcritical bifurcation at the prey-free equilibrium point provided that the following condition is met.

$$-a_1(1-c\hat{Y})^2\hat{Y} + bK_1^2 \neq 0. \tag{66}$$

Otherwise, a Pitchfork bifurcation takes place.

Proof. As $d_1 = d_1^*$ the Jacobian matrix becomes

$$J_3 = J(p_3, d_1^*) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{a_2(1-c\hat{Y})\hat{Y}}{K_1} & \hat{Y}\left(-\frac{e(1-m)K_2}{(K_2+(1-m)\hat{Y})^2} + \frac{a_4(1-m)^2\hat{Z}}{(K_3+(1-m)\hat{Y})^2}\right) & -\frac{a_4(1-m)\hat{Y}}{K_3+(1-m)\hat{Y}} \\ 0 & \hat{Z}\left[\frac{(1-m)a_5K_3}{(K_3+(1-m)\hat{Y})^2}\right] & 0 \end{bmatrix}.$$

Clearly, J_3 has the eigenvalues $\lambda_{31} = 0$, while $\lambda_{32} = \frac{T_1}{2} + \frac{1}{2}\sqrt{{T_1}^2 - 4D_1}$, and $\lambda_{33} = \frac{T_1}{2} - \frac{1}{2}\sqrt{{T_1}^2 - 4D_1}$ have negative real parts due to condition (33), where T_1 and D_1 are given in equation (31). Hence, the eigenvectors of J_3 and ${J_3}^{\mathrm{T}}$ corresponding $\lambda_{31} = 0$ can be written as $\mathbf{V}_3 = (v_{31}, v_{32}, v_{33})^{\mathrm{T}}$ and $\mathbf{U}_3 = (u_{31}, u_{32}, u_{33})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_3 = \left(1,0,\frac{a_2(1-c\hat{Y})(K_3+(1-m)\hat{Y})}{K_1a_4(1-m)}\right)^{\mathrm{T}} = (1,0,\rho_3)^{\mathrm{T}},\ \mathbf{U}_3 = (1,0,0)^{\mathrm{T}}.$$

Moreover, with the use of equation (64), it is obtained that

$$\mathbf{F}_{d_1} = \begin{pmatrix} -X \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_{d_1}(p_3, d_1^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow {\mathbf{U}_3}^{\mathrm{T}}\mathbf{F}_{d_1}(p_3, d_1^*) = 0$$

$$D\mathbf{F}_{d_1}(p_3, d_1^*).\mathbf{V}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow {\mathbf{U}_3}^{\mathrm{T}}\left[D\mathbf{F}_{d_1}(p_3, d_1^*).\mathbf{V}_3\right] = -1$$

$$D^2\mathbf{F}(p_3, d_1^*)(\mathbf{V}_3, \mathbf{V}_3) = \begin{pmatrix} -\frac{2\left[-a_1(1-c\hat{Y})^2\hat{Y} + bK_1^2\right]}{K_1^2} \\ -\frac{2a_2(1-c\hat{Y})^2\hat{Y}}{K_1^2} \\ 0 \end{pmatrix}$$

Therefore, by using condition (66), it is obtained that:

$${\mathbf{U}_3}^{\mathrm{T}}[D^2\mathbf{F}(p_3, d_1^*)(\mathbf{V}_3, \mathbf{V}_3)] = -\frac{2\left[-(1-c\hat{Y})^2\hat{Y}a_1 + bK_1^2\right]}{K_1^2} \neq 0$$

Then, as the parameter d_1 crosses through d_1^*, the Sotomayor theorem makes the system (2) undergo a transcritical bifurcation at the equilibrium point p_3. Otherwise, it is obtained with the help of equation (65) that

$$D^3\mathbf{F}(p_3, d_1^*)(\mathbf{V}_3, \mathbf{V}_3, \mathbf{V}_3) = \begin{pmatrix} -\frac{6a_1\hat{Y}(1-c\hat{Y})^3}{K_1^3} \\ \frac{6a_2\hat{Y}(1-c\hat{Y})^3}{K_1^3} \\ 0 \end{pmatrix}$$

Therefore, it is obtained that:

$$\mathbf{U}_3{}^{\mathrm{T}}[D^3\mathbf{F}(p_3,d_1^*)(\mathbf{V}_3,\mathbf{V}_3,\mathbf{V}_3)] = -\frac{6a_1\hat{Y}(1-c\hat{Y})^3}{K_1^3} \neq 0$$

Hence, a pitchfork bifurcation takes place in the sence of Sotomayor theorem and the proof is done.

Theorem 15: Assume that conditions (37)-(38) are staisfied, then when the parameter a_5 passes through $a_5 = \frac{d_3(K_3+(1-m)\check{Y})}{(1-m)\check{Y}} \equiv (a_5^*)$, the system (2) undergoes a transcritical bifurcation at the top predator-free equilibrium point.

Proof. As $a_5 = a_5^*$ the Jacobian matrix becomes

$$J_4 = J(p_4,a_5^*) = \begin{bmatrix} \check{X}\left(-b+\frac{a_1(1-c\check{Y})^2\check{Y}}{(K_1+\check{X}(1-c\check{Y}))^2}\right) & \check{X}\left[-\frac{fr}{(1+f\check{Y})^2}-\frac{a_1\check{X}(1-c\check{Y})^2+K_1(1-2c\check{Y})a_1}{(K_1+\check{X}(1-c\check{Y}))^2}\right] & 0 \\ \check{Y}\left[\frac{(1-c\check{Y})K_1a_2}{(K_1+\check{X}(1-c\check{Y}))^2}\right] & \check{Y}\left(-\frac{c\check{X}a_2K_1}{(K_1+\check{X}(1-c\check{Y}))^2}-\frac{e(1-m)K_2}{(K_2+(1-m)\check{Y})^2}\right) & -\frac{a_4(1-m)\check{Y}}{K_3+(1-m)\check{Y}} \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $J_4 = (b_{ij})_{3\times3}$ has the eigenvalues $\lambda_{41} = \frac{T_2}{2}+\frac{1}{2}\sqrt{T_2{}^2-4D_2}$, and $\lambda_{42} = \frac{T_2}{2}-\frac{1}{2}\sqrt{T_2{}^2-4D_2}$, which have negative real parts due to conditions (37)-(38), where T_2 and D_2 are given in equation (35). while $\lambda_{43} = 0$. Hence, the eigenvectors of J_4 and $J_4{}^{\mathrm{T}}$ corresponding $\lambda_{43} = 0$ can be written as $\mathbf{V}_4 = (v_{41},v_{42},v_{43})^{\mathrm{T}}$ and $\mathbf{U}_4 = (u_{41},u_{42},u_{43})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_4 = (\rho_4,\rho_5,1)^{\mathrm{T}},\ \mathbf{U}_4 = (0,0,1)^{\mathrm{T}},$$

where

$$\rho_4 = \frac{b_{12}b_{23}}{b_{11}b_{22}-b_{12}b_{21}} = \frac{b_{12}b_{23}}{D_2} > 0.$$

$$\rho_5 = -\frac{b_{11}b_{23}}{b_{11}b_{22}-b_{12}b_{21}} = -\frac{b_{11}b_{23}}{D_2} < 0.$$

Moreover, with the use of equation (64), it is obtained that

$$\mathbf{F}_{a_5} = \begin{pmatrix} 0 \\ 0 \\ \frac{(1-m)YZ}{K_3+(1-m)Y} \end{pmatrix} \Rightarrow \mathbf{F}_{a_5}(p_4,a_5^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{U}_4{}^{\mathrm{T}}\mathbf{F}_{a_5}(p_4,a_5^*) = 0$$

$$D\mathbf{F}_{a_5}(p_4,a_5^*).\mathbf{V}_4 = \begin{pmatrix} 0 \\ 0 \\ \frac{(1-m)\check{Y}}{K_3+(1-m)\check{Y}} \end{pmatrix} \Rightarrow \mathbf{U}_4{}^{\mathrm{T}}\left[D\mathbf{F}_{a_5}(p_4,a_5^*).\mathbf{V}_4\right] = \frac{(1-m)\check{Y}}{K_3+(1-m)\check{Y}}$$

$$D^2\mathbf{F}(p_4,a_5^*)(\mathbf{V}_4,\mathbf{V}_4) = [c_{i1}(p_4,a_5^*)]_{3\times1} ,$$

where

$$c_{11}(p_4,a_5^*)=-\frac{2\left[b\check{X}^3(1-c\check{Y})^3+(1-cY)^2\left(3b\check{X}^2-\check{Y}a_1\right)K_1+3b\check{X}(1-c\check{Y})K_1^2+bK_1^3\right]\rho_4^2}{(\check{X}(1-c\check{Y})+K_1)^3}$$
$$+2\left(-\frac{fr}{(1+f\check{Y})^2}-\frac{a_1K_1(\check{X}-c\check{X}\check{Y}+K_1-2c\check{Y}K_1)}{(\check{X}(1-c\check{Y})+K_1)^3}\right)\rho_4\rho_5 ,$$
$$+2\check{X}\left(\frac{f^2r}{(1+f\check{Y})^3}+\frac{ca_1K_1(\check{X}+K_1)}{(\check{X}(1-c\check{Y})+K_1)^3}\right)\rho_5^2$$

$$c_{21}(p_4,a_5^*)=-\frac{2a_2K_1\left[\check{Y}(1-c\check{Y})^2\rho_4^2-[\check{X}(1-c\check{Y})+(1-2c\check{Y})K_1]\rho_4\rho_5+c\check{X}(\check{X}+K_1)\rho_5^2\right]}{(\check{X}(1-c\check{Y})+K_1)^3} ,$$
$$-2(1-m)\rho_5\left(\frac{eK_2^2\rho_5}{((1-m)\check{Y}+K_2)^3}+\frac{a_4K_3}{((1-m)\check{Y}+K_3)^2}\right)$$

$$c_{31}(p_4,a_5^*)=\frac{2(1-m)a_5K_3\rho_5}{((1-m)\check{Y}+K_3)^2}.$$

Therefore, it is obtained that:

$$\mathbf{U}_4{}^{\mathrm{T}}[D^2\mathbf{F}(p_4,a_5^*)(\mathbf{V}_4,\mathbf{V}_4)]=\frac{2(1-m)a_5K_3\rho_5}{((1-m)\check{Y}+K_3)^2}\neq 0$$

Then, as the parameter a_5 passes through a_5^*, the Sotomayor theorem makes the system (2) undergo a transcritical bifurcation at the equilibrium point p_4.

Theorem 16: Assume that conditions (42)-(43) are satisfied, then when the parameter b passes through $b=\frac{a_1(1-c\check{Y})^2\check{Y}}{(K_1+(1-c\check{Y})\check{X})^2}\equiv(b^*)$, the system (2) undergoes a saddle-node bifurcation at the positive equilibrium point.

Proof. As $b=b^*$ the Jacobian matrix becomes

$$J_5=J(p_5,b^*)=\begin{bmatrix}0 & q_{12} & 0\\ q_{21} & q_{22} & q_{23}\\ 0 & q_{32} & 0\end{bmatrix},$$

where $q_{ij};i,j=1,2,3$ are given in equation (39). Obviously, by using equation (40), J_5 has the eigenvalues

$$\lambda_{51}=0,\ \lambda_{52}=\frac{G_1}{2}+\frac{1}{2}\sqrt{{G_1}^2-4G_2},\ \lambda_{53}=\frac{G_1}{2}-\frac{1}{2}\sqrt{{G_1}^2-4G_2},$$

where $G_1=-q_{22}$, and $G_2=-q_{12}q_{21}-q_{23}q_{32}$ are positive under the conditions (42)-(43). Hence the eigenvalues λ_{52}, and λ_{53} have negative real parts. Moreover, the eigenvectors of J_5 and ${J_5}^{\mathrm{T}}$ corresponding $\lambda_{51}=0$ can be written as $\mathbf{V}_5=(v_{51},v_{52},v_{53})^{\mathrm{T}}$ and $\mathbf{U}_5=(u_{51},u_{52},u_{53})^{\mathrm{T}}$ respectively, where

$$\mathbf{V}_5=(1,0,\rho_6)^{\mathrm{T}},\ \mathbf{U}_4=(1,0,\rho_7)^{\mathrm{T}},$$

where $\rho_6=-\frac{q_{21}}{q_{23}}>0$ and $\rho_7=-\frac{q_{12}}{q_{32}}>0.$

Moreover, with the use of equation (64), it is obtained that

$$\mathbf{F}_b = \begin{pmatrix} -X^2 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_b(p_5, b^*) = \begin{pmatrix} -\tilde{X}^2 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{U}_5{}^{\mathrm{T}}\mathbf{F}_b(p_5, b^*) = -\tilde{X}^2 \neq 0$$

$$D^2\mathbf{F}_b(p_5, b^*)(\mathbf{V}_5, \mathbf{V}_5) = \begin{pmatrix} -\frac{2b^*\left[(1-c\tilde{Y})^3\tilde{X}^3+2K_1(1-c\tilde{Y})^2\tilde{X}^2+K_1^2(1-c\tilde{Y})\tilde{X}\right]}{(\tilde{X}(1-c\tilde{Y})+K_1)^3} \\ -\frac{2a_2K_1\tilde{Y}(1-c\tilde{Y})^2}{(\tilde{X}(1-c\tilde{Y})+K_1)^3} \\ 0 \end{pmatrix},$$

Therefore, it is obtained that:

$$\mathbf{U}_5{}^{\mathrm{T}}[D^2\mathbf{F}_b(p_5, b^*)(\mathbf{V}_5, \mathbf{V}_5)] = -\frac{2b^*\left[(1-c\tilde{Y})^3\tilde{X}^3+2K_1(1-c\tilde{Y})^2\tilde{X}^2+K_1^2(1-c\tilde{Y})\tilde{X}\right]}{(\tilde{X}(1-c\tilde{Y})+K_1)^3} \neq 0$$

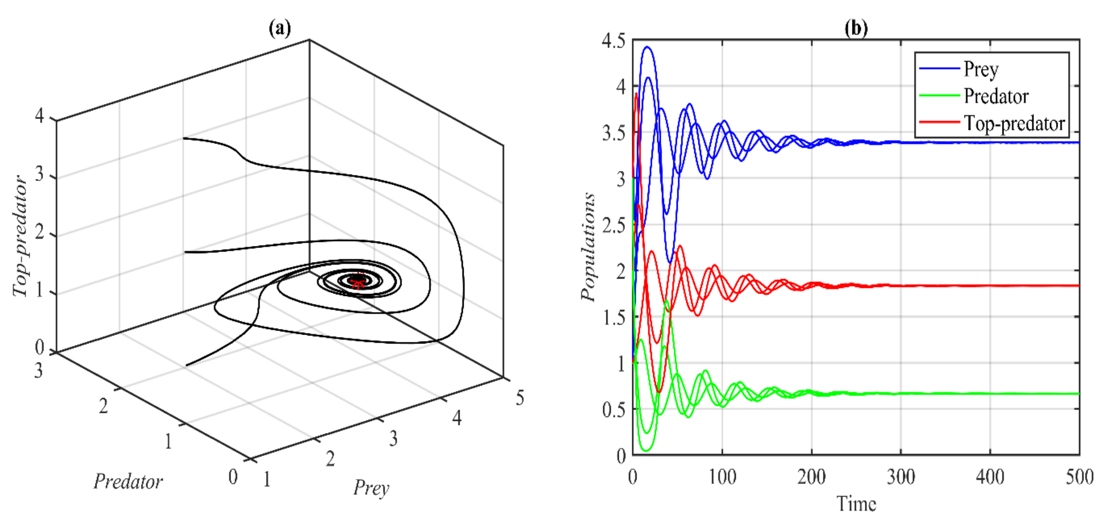
Then, as the parameter b passes through b^*, the Sotomayor theorem makes the system (2) undergo a saddle-node bifurcation at the equilibrium point p_5.

6. Numerical Simulation

In the following, an investigation of the system's dynamics (2) is carried out using numerical simulation depending on the next set of hypothetical parameter values. The objective is to validate the theoretical finding and understand the influence of the varying parameter values on the system's dynamics.

$$\left.\begin{aligned} r = 1, f = 0.2, d_1 = 0.1, b = 0.2, a_1 = 0.75, c = 0.4, K_1 = 1, a_2 = 0.5, a_3 = 0.15, \\ d_2 = 0.1, e = 0.25, m = 0.5, a_4 = 0.5, K_2 = 1, K_3 = 1, a_5 = 0.4, d_3 = 0.1 \end{aligned}\right\} \quad (67)$$

System (2) has an asymptotically stable positive equilibrium point for the set of data (67), as depicted in Figure (1). As you can see from the figures, the red star symbolizes the point of attraction for which system (2)'s solution is intended.



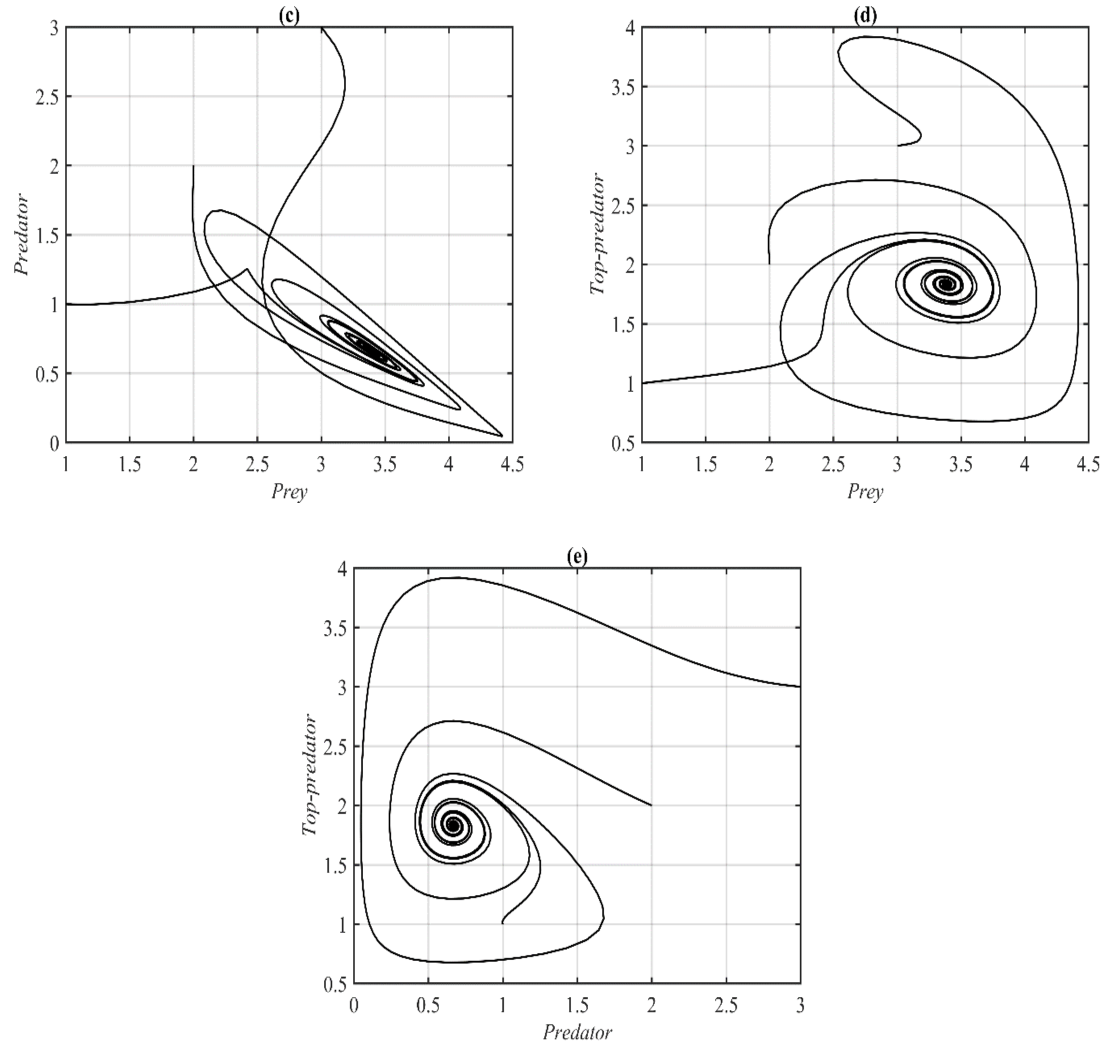
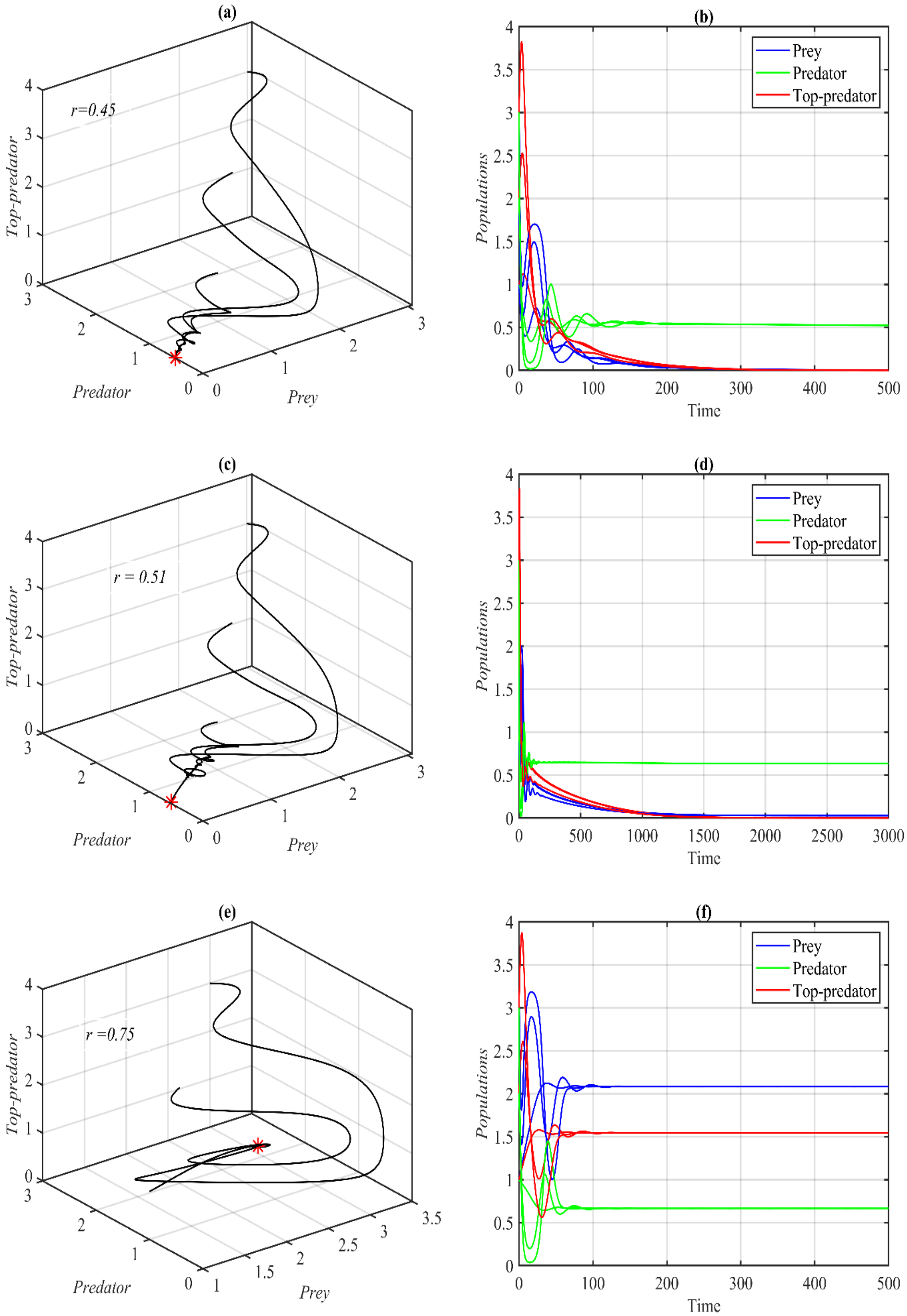


Fig. 1. The trajectories of the system (2), utilizing the set of data (67), starting from different initial points. (a) Phase portrait that approached to $p_5 = (3.38, 0.66, 1.83)$. (b) Time series. (c) Projection of phase portrait on XY −plane. (d) Projection of phase portrait on XZ −plane. (e) Projection of phase portrait on YZ −plane.

In the following, depending on the result shown in figure (1), the influence of varying the parameter values on the stability of the positive equilibrium point is investigated. For the parameter r in the ranges $r \in (0,0.46]$, $r \in (0.46,0.52]$, $r \in (0.52,1.34]$, and $r > 1.34$ the solution of system (2) approaches asymptotically to p_2, p_4, p_5, and periodic dynamics respectively, see figure (2) for the selected values of r.


(a)
r=0.45
Top-predator
Predator
Prey
(b)
Prey
Predator
Top-predator
Populations
Time
(c)
r = 0.51
Top-predator
Predator
Prey
(d)
Prey
Predator
Top-predator
Populations
Time
(e)
r =0.75
Top-predator
Predator
Prey
(f)
Prey
Predator
Top-predator
Populations
Time

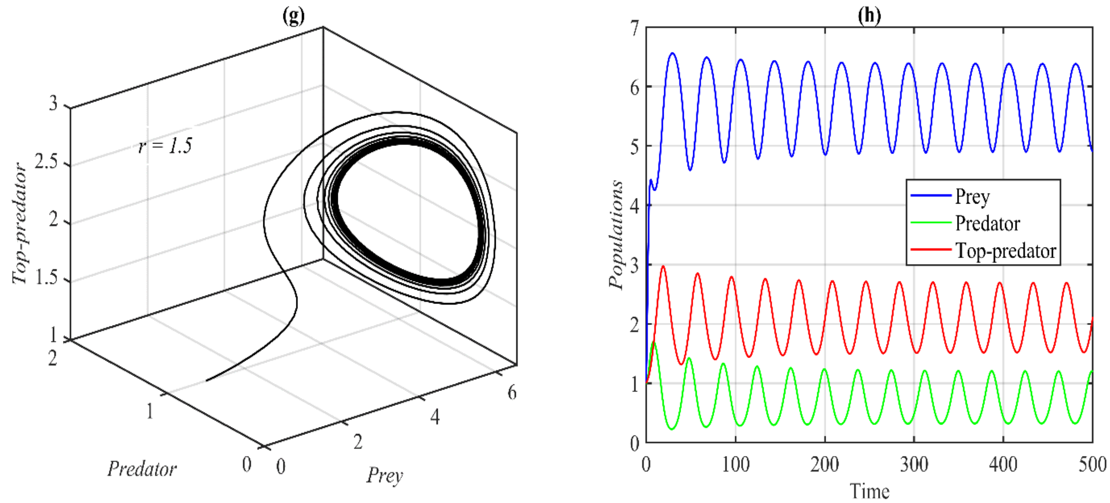
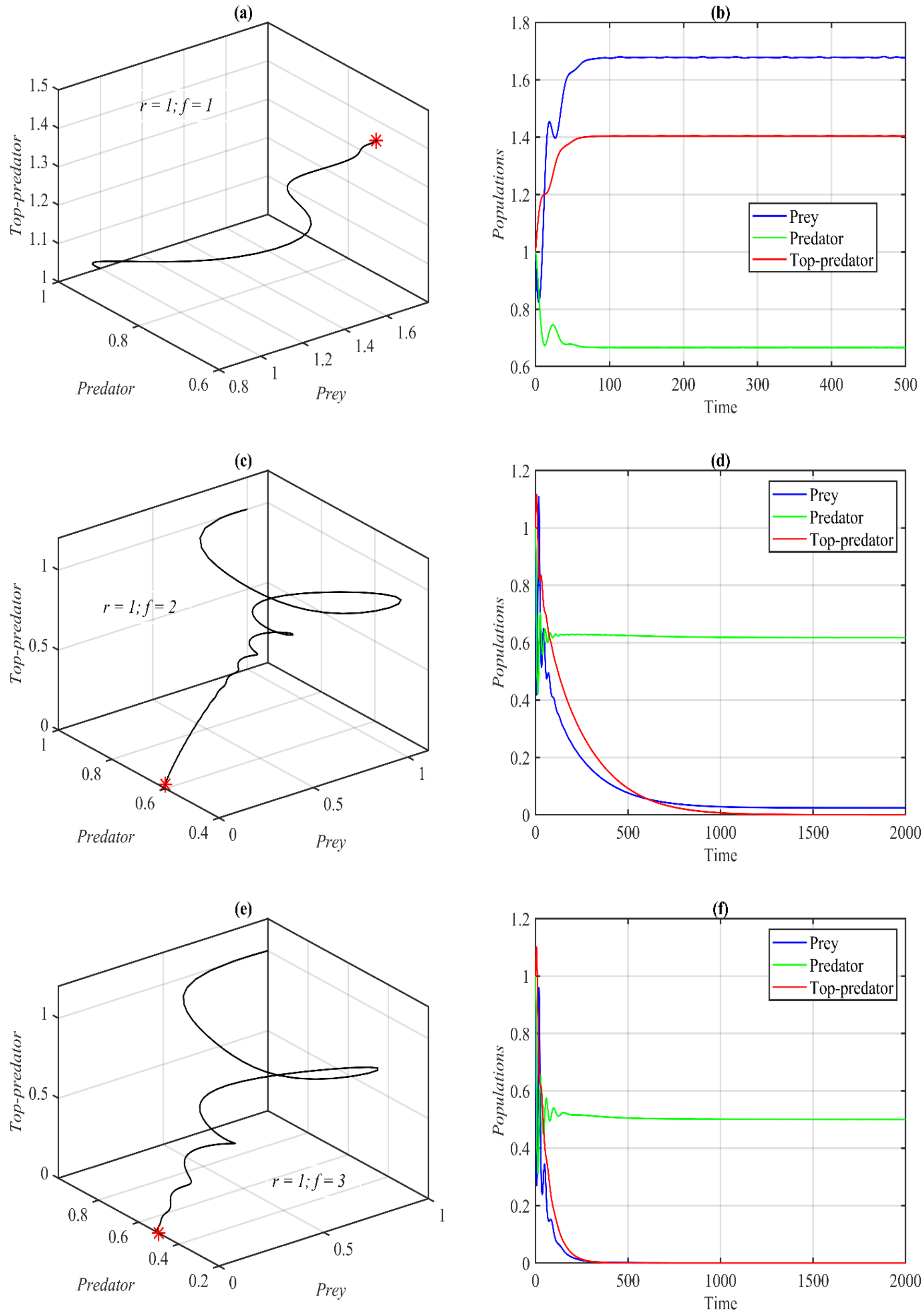


Fig. 2. The trajectories of the system (2), utilizing the set of data (67) with different values of r, starting from different initial points. (a) Phase portrait for $r = 0.45$ that approached to $p_2 = (0,0.52,0)$. (b) Time series for $r = 0.45$. (c) Phase portrait for $r = 0.51$ that approached to $p_4 = (0.03,0.63,0)$. (d) Time series for $r = 0.51$. (e) Phase portrait for $r = 0.75$ that approached to $p_5 = (2.08,0.66,1.54)$. (f) Time series for $r = 0.75$. (g) Stable limit cycle for $r = 1.5$. (h) Time series for $r = 1.5$.

Now, the influence of varying the parameter f on the system's dynamics (2) is studied in two cases, first when the system (2) has a stable coexistence point and second when the system has a stable limit cycle. In the first case, it is observed that, for $f \in [0,1.76]$, $f \in (1.76,2.51]$, and $f > 2.52$ the solution of system (2) approaches asymptotically to p_5, p_4, and p_2, respectively. On the other hand, in the second case, it is observed that rising the value of f stabilizes the system so that the solution approaches asymptotically the p_5. Moreover, rising the parameter further leads to extinction in top predators first and then extinction in prey species, and then the system's (2) solution stabilized at p_2, see figure (3) for an explanation of the selected values of f.


(a)
r = 1; f = 1
Top-predator
1.5
1.4
1.3
1.2
1.1
1
1
0.8
Predator
0.6
0.8
1
1.2
1.4
1.6
Prey
(b)
Populations
1.8
1.6
1.4
1.2
1
0.8
0.6
0
100
200
300
400
500
Time
Prey
Predator
Top-predator
(c)
r = 1; f = 2
Top-predator
1
0.5
0
1
0.8
0.6
Predator
0.4
0
0.5
1
Prey
(d)
Populations
1.2
1
0.8
0.6
0.4
0.2
0
0
500
1000
1500
2000
Time
Prey
Predator
Top-predator
(e)
r = 1; f = 3
Top-predator
1
0.5
0
0.8
0.6
0.4
Predator
0.2
0
0.5
1
Prey
(f)
Populations
1.2
1
0.8
0.6
0.4
0.2
0
0
500
1000
1500
2000
Time
Prey
Predator
Top-predator

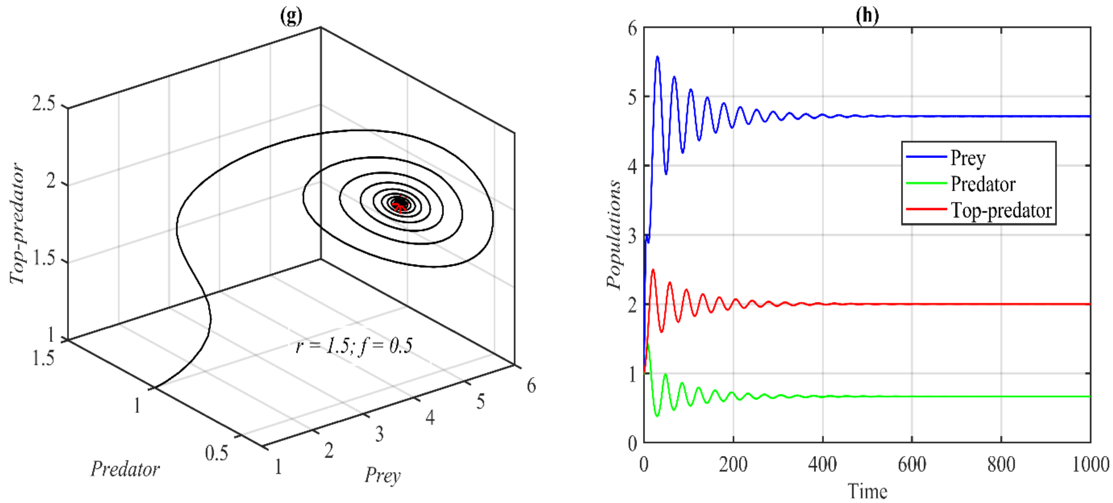


Fig. 3. The trajectory of the system (2), utilizing the set of data (67) with different values of f, when $r = 1$ and $r = 1.5$. (a) Phase portrait for $r = 1$; $f = 1$ that approached to $p_5 = (1.67,0.66,1.4)$. (b) Time series for $r = 1$; $f = 1$. (c) Phase portrait for $r = 1$; $f = 2$ that approached to $p_4 = (0.02,0.61,0)$. (d) Time series for $r = 1$; $f = 2$. (e) Phase portrait for $r = 1$; $f = 3$ that approached to $p_2 = (0,0.5,0)$. (f) Time series for $r = 1$; $f = 3$. (g) Phase portrait for $r = 1.5$; $f = 0.5$ that approached to $p_5 = (4.71,0.66,2)$. (h) Time series for $r = 1.5$; $f = 0.5$.

It is observed that, when the system indergoes a periodic dynamics as for $r = 1.5$ in figure (2g) increasing f in the ranges $f \in (0,0.3)$, $f \in [0.3,3.4]$, $f \in [3.4, 5]$, and $f > 5$ the solution of system (2) approaches to stable limit cycle, p_5, p_4, and p_2, respectively, as shown in figure (3g) for $f = 0.5$. For the parameters d_1, and a_1, they have a similar influence on the system's (2) solution as that obtained for f in the first case. Now, the influence of varying the parameter b on the system's (2) dynamics is studied in figure (4) below at a selected values. It is obtained that for the ranges $b \in (0,0.13)$, and $b > 0.13$ the solution approaches a stable limit cycle, and p_5 respectively.

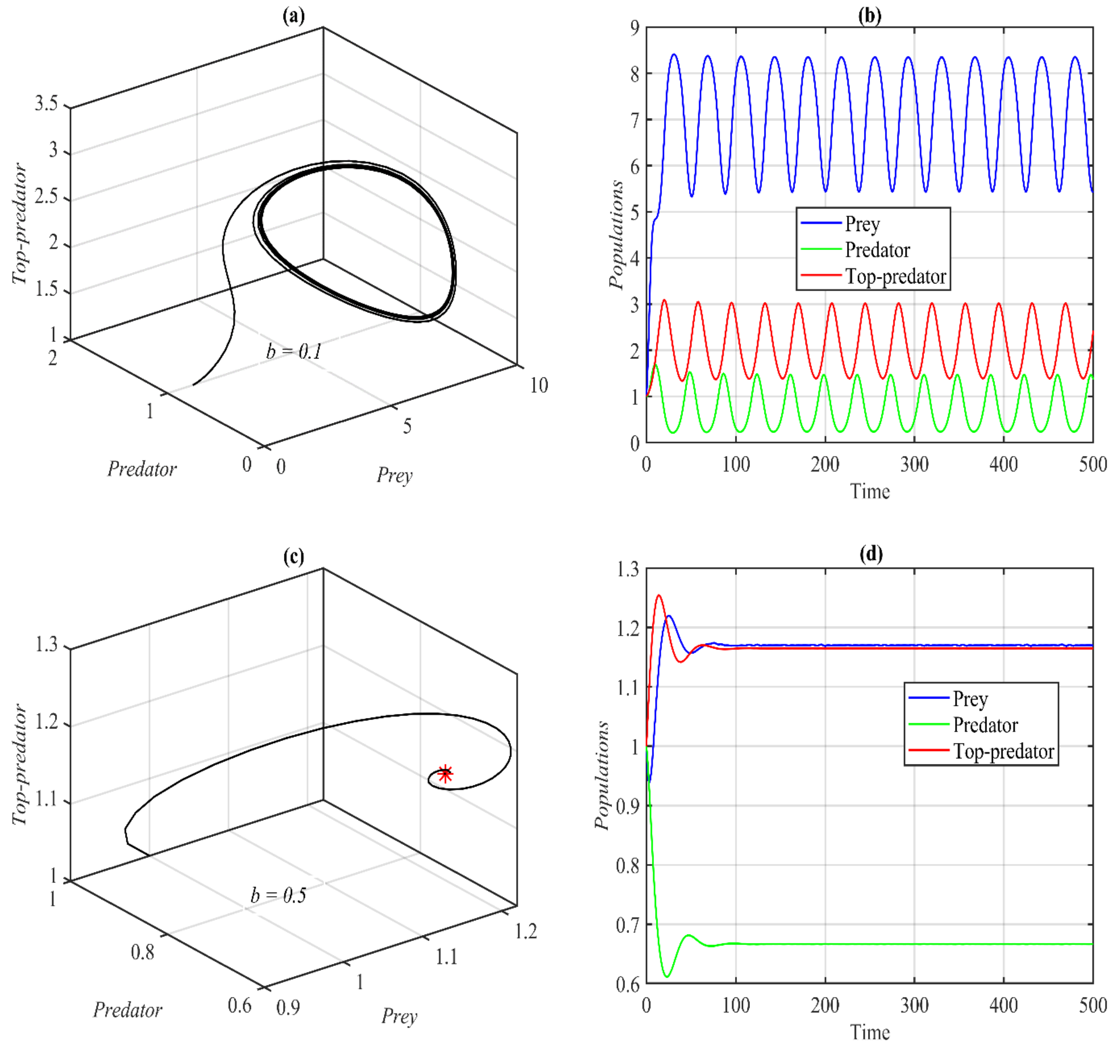


Fig. 4. The trajectory of the system (2), utilizing the set of data (67) with different values of b. (a) Stable limit cycle for $b = 0.1$. (b) Time series for $b = 0.1$. (c) Phase portrait for $b = 0.5$ that approached to $p_5 = (1.17, 0.66, 1.16)$. (d) Time series for $b = 0.5$.

Note that, a similar impact on the system's (2) dynamics, as shown by the parameter b, is obtained when the parameter value c varies. For the parameter K_1 in the ranges $K_1 \in (0, 0.14]$, $K_1 \in (0.14, 0.53]$, $K_1 > 0.53$, it is observed that the system's (2) solution approaches asymptotically to p_2, stable limit cycle, and p_5 respectively, as shown in figure (5) for the selected parameter values.

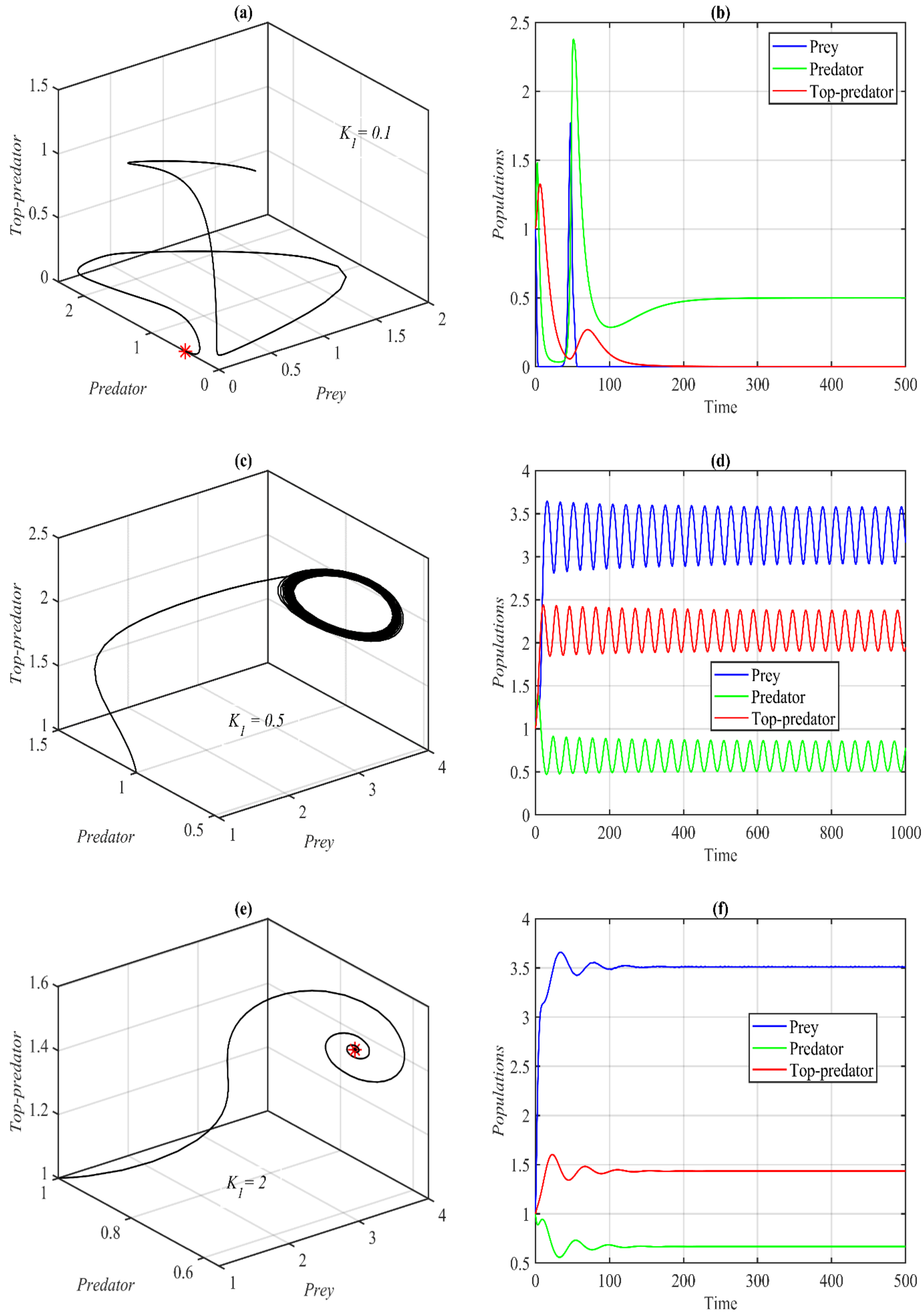


Fig. 5. The trajectory of the system (2), utilizing the set of data (67) with different values of K_1.

(a) Phase portrait for $K_1 = 0.1$ that approached to $p_2 = (0,0.5,0)$. (b) Time series for $K_1 = 0.1$. (c) Stable limit cycle for $K_1 = 0.5$. (d) Time series for $K_1 = 0.5$. (e) Phase portrait for $K_1 = 2$ that approached to $p_5 = (3.51,0.66,1.43)$. (f) Time series for $K_1 = 2$.

For the parameter a_3 in the ranges $a_3 \in (0,0.26]$, and $a_3 > 0.26$ the system's (2) solution approaches asymptotically to p_5 and stable limit cycle respectively, as shown in figure (6) at the selected values of a_3.

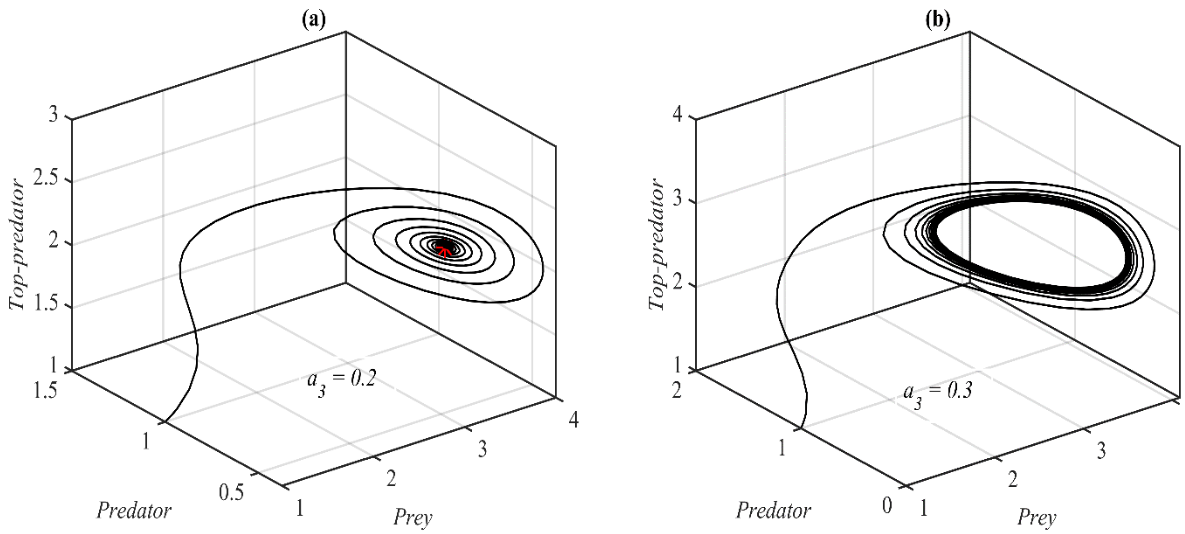


Fig. 6. The trajectory of the system (2), utilizing the set of data (67) with different values of a_3. (a) Phase portrait for $a_3 = 0.2$ that approached to $p_5 = (3.38,0.66,2.1)$. (b) Stable limit cycle for $a_3 = 0.3$.

Note that, a similar impact on the system's (2) dynamics, as shown by the parameter a_3, is obtained when the parameter value K_2 varies. For the parameter d_2 in the ranges $d_2 \in (0,0.44]$, $d_2 \in (0.44,0.55]$, $d_2 > 0.55$, it is observed that the system's (2) solution approaches asymptotically to p_5, p_4, and p_1 respectively, as shown in figure (7) for the selected parameter values.

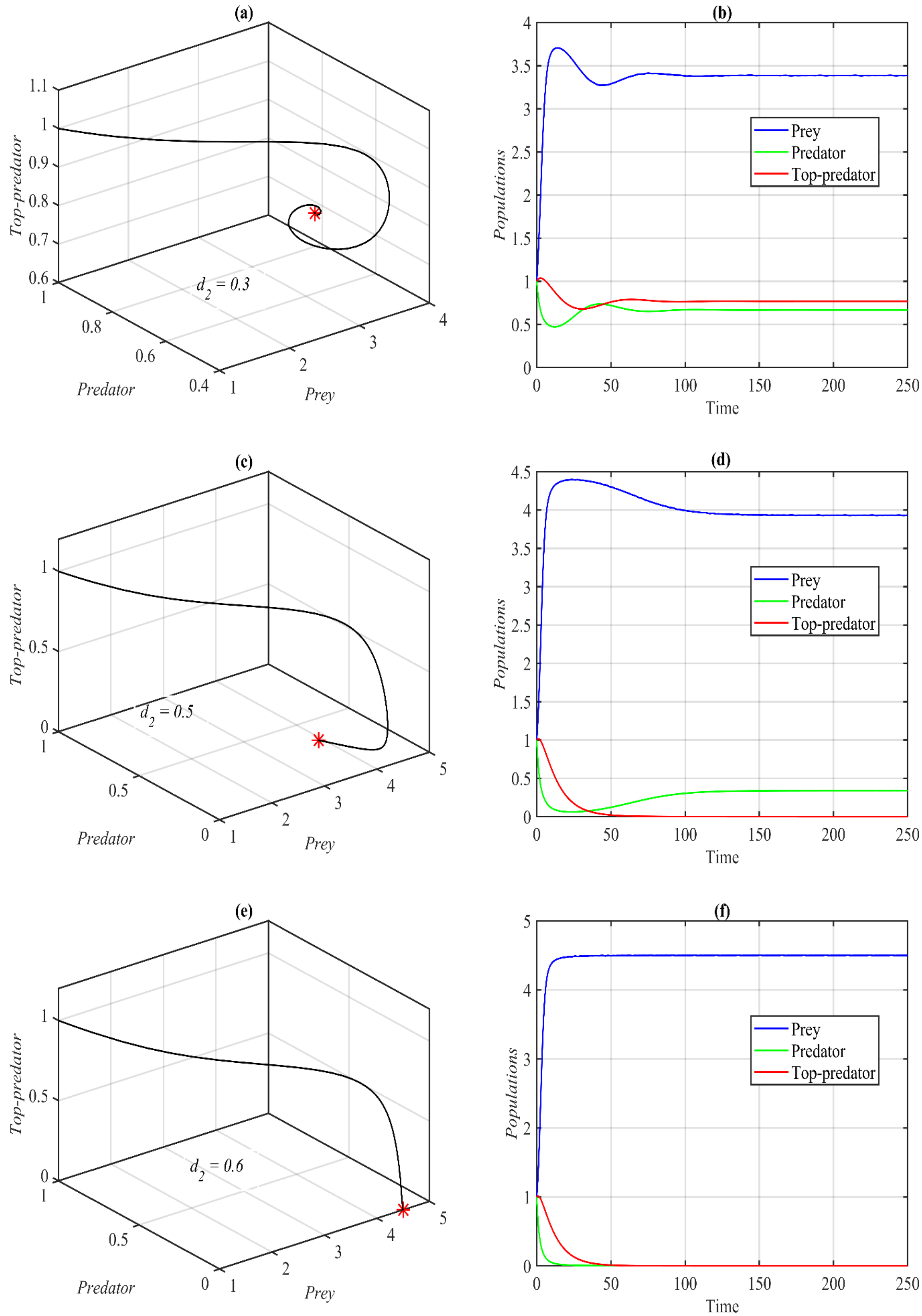
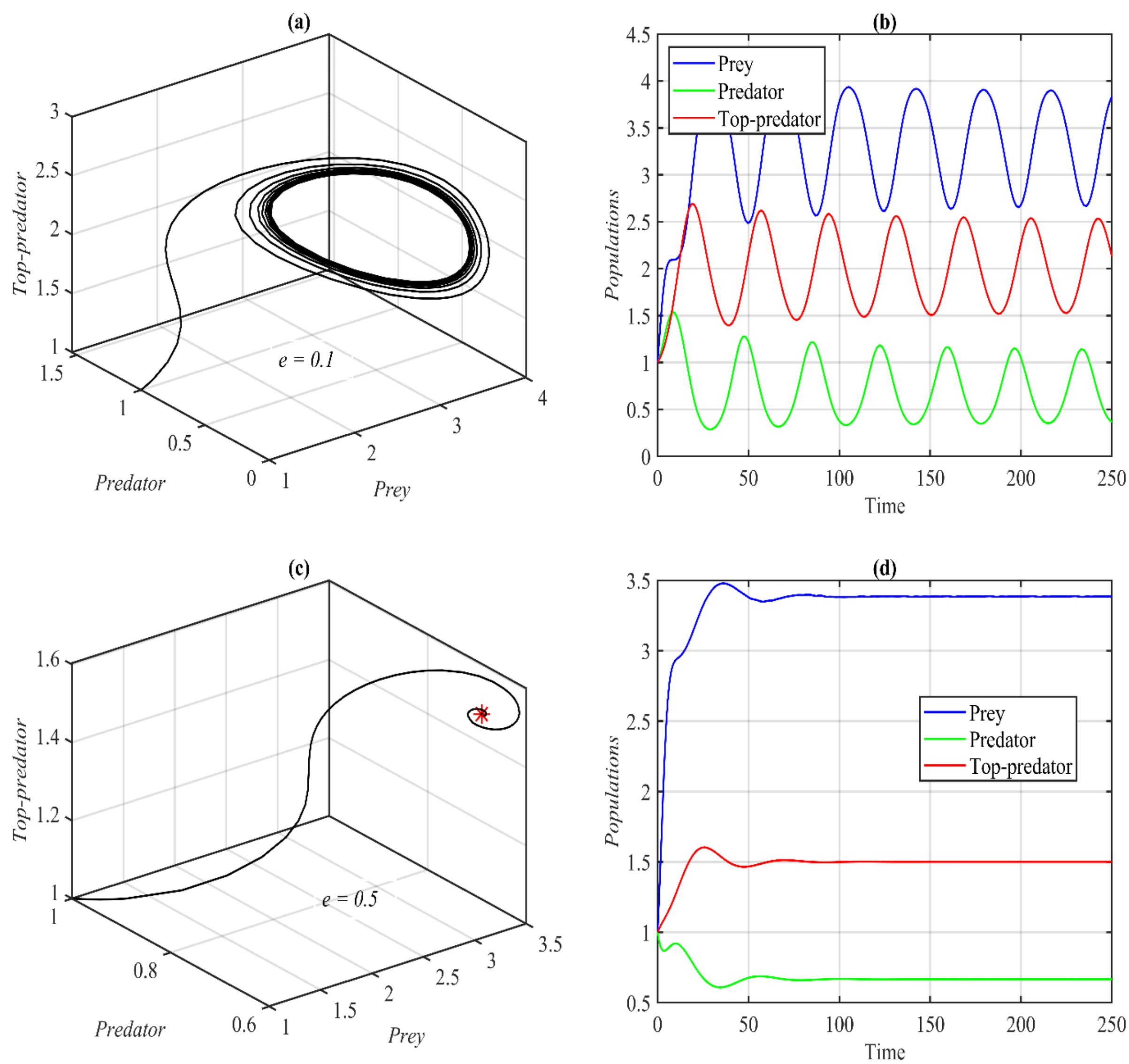


Fig. 7. The trajectory of the system (2), utilizing the set of data (67) with different values of d_2.

(a) Phase portrait for $d_2 = 0.3$ that approached to $p_5 = (3.38,0.66,0.76)$. (b) Time series for $d_2 = 0.3$. (c) Phase portrait for $d_2 = 0.5$ that approached to $p_4 = (3.93,0.34,0)$. (d) Time series for $d_2 = 0.5$. (e) Phase portrait for $d_2 = 0.6$ that approached to $p_1 = (4.5,0,0)$. (f) Time series for $d_2 = 0.6$.

For the parameter e in the ranges $e \in (0,0.13]$, $e \in (0.13,1.61]$, and $e > 1.61$ the system's (2) solution approaches asymptotically to stable limit cycle, p_5, and p_4 respectively, see figure (8) for the selected values of the paramere e.



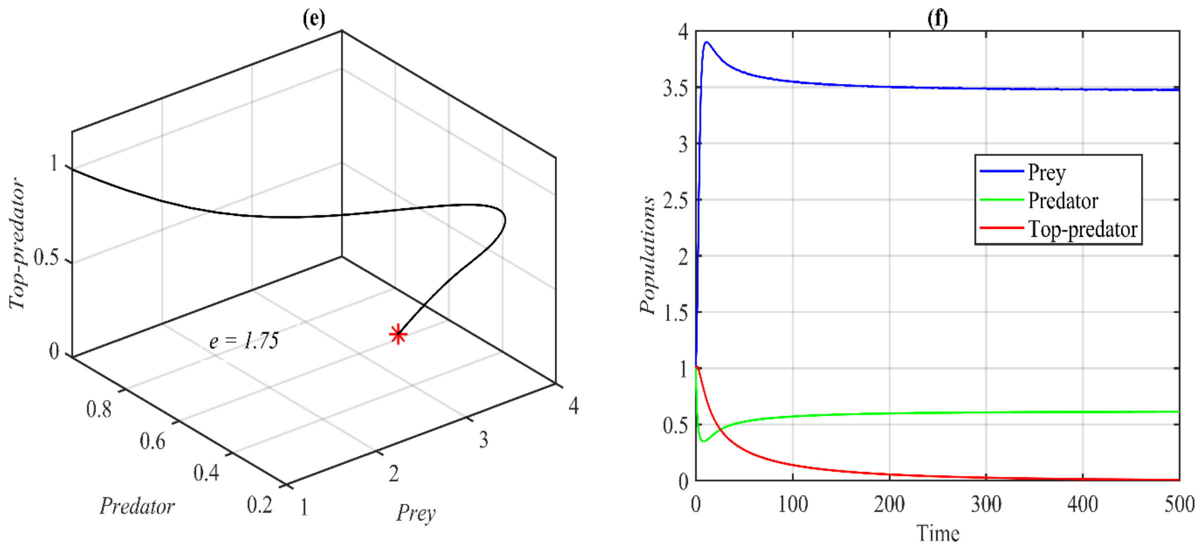
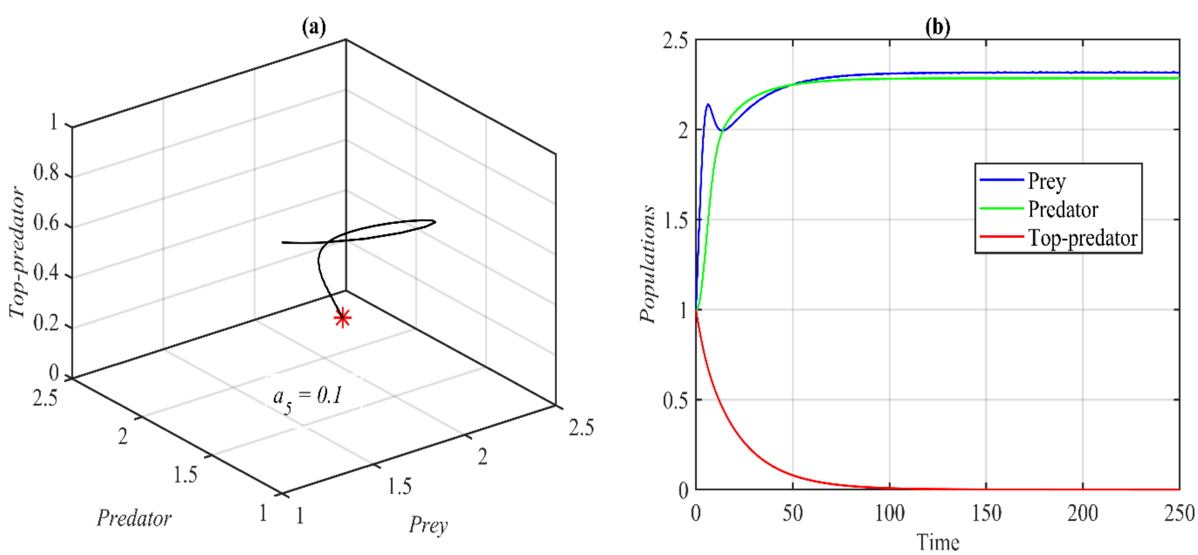


Fig. 8. The trajectory of the system (2), utilizing the set of data (67) with different values of e. (a) Stable limit cycle for $e = 0.1$. (b) Time series for $e = 0.1$. (c) Phase portrait for $e = 0.5$ that approached to $p_5 = (3.38, 0.66, 1.5)$. (d) Time series for $e = 0.5$. (e) Phase portrait for $e = 1.75$ that approached to $p_4 = (3.47, 0.61, 0)$. (f) Time series for $e = 1.75$.

For the parameters K_3, d_3, and m, they have a similar influence on the system's (2) solution as that obtained for e in the first case. Now, the influence of varying the parameter a_5 on the system's (2) dynamics is studied in figure (9) below at a selected values of a_5, so that for the ranges $a_5 \in (0, 0.18]$, and $b \in (0.18, 0.5]$ the solution approaches p_4, and p_5 respectively.



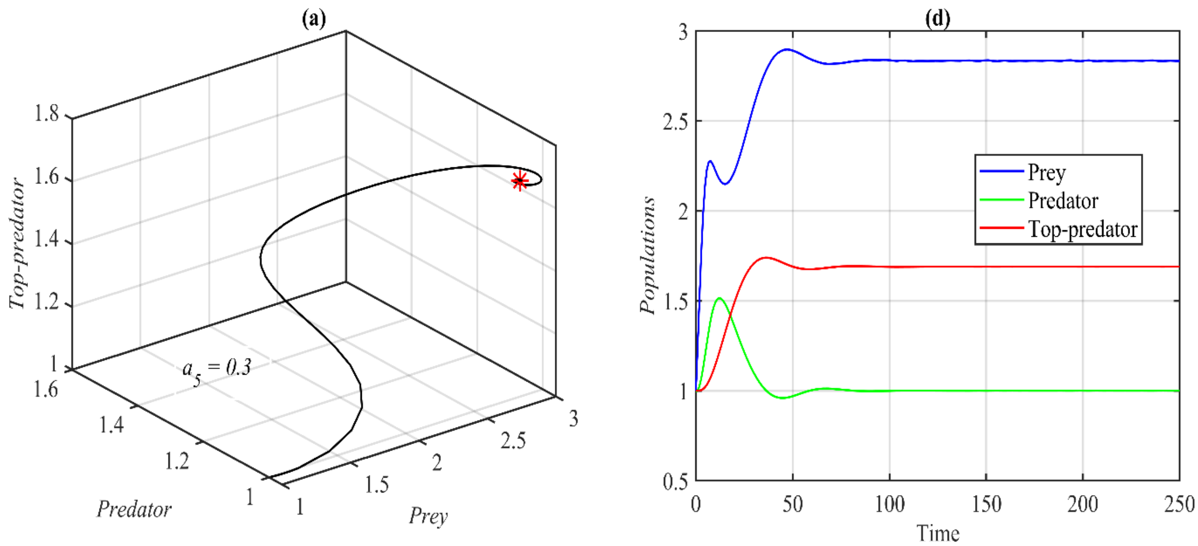


Fig. 9. The trajectory of the system (2), utilizing the set of data (67) with different values of a_5. (a) Phase portrait for $a_5 = 0.1$ that approached to $p_4 = (2.31,2.28,0)$. (b) Time series for $a_5 = 0.1$. (c) Phase portrait for $a_5 = 0.3$ that approached to $p_5 = (2.83,1,1.68)$. (d) Time series for $a_5 = 0.3$.

The influence of the rest of the parameter values on the dynamics of the system (2) using the data (67) is summarized in table (2) below. However, for the data (67) with $r = 0.09$, and $a_3 = 0.09$ it is observed that, the system (2) approaches asymptotically to $p_0 = (0,0,0)$ as shown in figure (10).

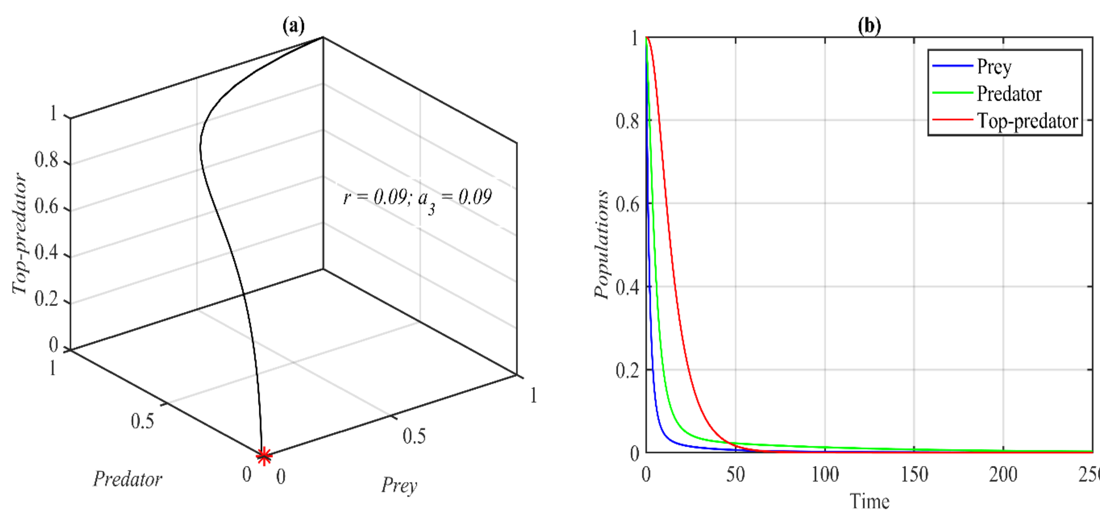


Fig. 10. The trajectory of the system (2), utilizing the set of data (67) with $r = 0.09$, and $a_3 = 0.09$. (a) Phase portrait approached to p_0. (b) Time series.

Clearly, for the data used in figure (10), the conditions (20)-(21) are satisfied and hence the stability of p_0 is confirmed.

Table 2: The dynamics of system (2) as a function of the specific parameter with rest of parameters as given in (67)

Parameter	Range	The dynamics
d_1	$d_1 \in (0,0.52]$	The system (2) approaches asymptotically to p_5
	$d_1 \in (0.52,0.58]$	The system (2) approaches asymptotically to p_4
	$d_1 > 0.58$	The system (2) approaches asymptotically to p_2
a_1	$a_1 \in (0,2.01]$	The system (2) approaches asymptotically to p_5
	$a_1 > 2.01$	The system (2) approaches asymptotically to p_2
c	$c \in (0,0.17]$	The system (2) approaches asymptotically to limit cycle
	$c \in (0.17,1]$	The system (2) approaches asymptotically to p_5
a_2	$a_2 \in (0, a_1]$	The system (2) approaches asymptotically to p_5
a_4	$a_4 > 0$	The system (2) approaches asymptotically to p_5
m	$m \in (0,0.32]$	The system (2) approaches asymptotically to limit cycle
	$m \in (0.32,0.86]$	The system (2) approaches asymptotically to p_5
	$m \in (0.86,1]$	The system (2) approaches asymptotically to p_4
K_2	$K_2 \in (0,2.47]$	The system (2) approaches asymptotically to p_5
	$K_2 > 2.47$	The system (2) approaches asymptotically to limit cycle
K_3	$K_3 \in (0,0.81]$	The system (2) approaches asymptotically to limit cycle
	$K_3 \in (0.81,3.42]$	The system (2) approaches asymptotically to p_5
	$K_3 > 3.43$	The system (2) approaches asymptotically to p_4
d_3	$d_3 \in (0,0.06]$	The system (2) approaches asymptotically to limit cycle
	$d_3 \in (0.06,0.21]$	The system (2) approaches asymptotically to p_5
	$d_3 > 0.22$	The system (2) approaches asymptotically to p_4

7. CONCLUSIONS

This work proposes and investigates a three-species food chain model including fear cost, predator-dependent refuge, and cannibalism at the second level. Food consumption between stages of the food chain is designed using the Holling type II functional response. The solution's entire collection of characteristics was investigated. It is noted that the system has six nonnegative equilibrium points. Each one's stability analysis is looked into locally. The system's persistence requirements have been identified. The transcritical bifurcation of system (2) is demonstrated to occur close to the boundary equilibrium point, with the pitchfork bifurcation occurring possibly also at the prey-free equilibrium point. Saddle-node bifurcation is, nevertheless, discovered close to the positive equilibrium point. Finally, the model is investigated numerically using a hypothetical set of parameter values to confirm the obtained finding and understand the impact of varying the parameters on the system's (2) dynamics. The following results were obtained numerically depending on the parameter values (67).

- The prey birth rate has three bifurcation points. As its value increases, the system (2) loses its stability at the positive equilibrium point and transfers to periodic dynamics through Hopf bifurcation. On the other hand, decreasing its value leads to extinction in the top predator first and then in the prey so that the solution approaches the second axial equilibrium point through the top predator-free equilibrium point.
- The prey's fear level (similarly the prey's natural death rate and the intermediate predator's attack rate) causes extinction in the top predator first and then in the prey when its value exceeds a specific value. On the other hand, when the system undergoes periodic dynamics, it is observed that increasing the prey's fear level stabilizes the system at the positive equilibrium point.
- The prey intraspecific competition (similarly the prey's refuge rate and the middle predator's half-saturation constant) has a stabilizing effect on the system's dynamics.
- The conversion rate of cannibalism into middle predator birth (similarly the half-saturation constant of cannibalism) has a destabilizing effect on the system's dynamics.
- The middle predator's natural death rate causes extinction in the system and the solution ultimately approaches the first axial equilibrium point.
- The cannibalism rate in the middle predator (similarly the middle predator's refuge rate, the top predator's natural death rate, and the top predator's half-saturation constant) has a stabilizing effect on the system's dynamics up to a threshold value and then the persistence of the system (2) is lost through extinction in top predator.

- The conversion rate of middle predator's biomass into top predator biomass makes the system persist at the positive equilibrium point.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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