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MATHEMATICAL ANALYSIS OF PREY PREDATOR MODELS WITH HOLLING TYPE I FUNCTIONAL RESPONSES AND TIME DELAY

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Abstract. We examine two prey and one predator models with Holling type I functional behaviours in this paper. To demonstrate the system's permanence and boundedness, we used a discrete-time delay. Through the use of traditional mathematical techniques, the effects of random variations in the environment and time delay on the model's stability are analytically examined. The stability and Hopf-Bifurcation for the competition model are also described and shown. A few numerical computations are provided to demonstrate the efficacy of the theoretical findings.

Keywords: Routh Hurwitz criterion; time-delay; Hopf-bifurcation; stability analysis.

2020 AMS Subject Classification: 92D25, 34K18, 34K20.

1. Introduction

Mathematics plays a major role in biology with the help of a biological models [1] - [2]. All areas of ecology have seen a significant increase in mathematical developments in population biology, which have a long history of being created by mathematics research into the dynamical characteristics of population developments. Based on the existence and significance of predators

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and prey in nature, numerous authors have created mathematical models of the relationship between the two [3], [4]. The predator-prey interaction model is the main focus of this work. Predator-prey competition is based on interactions between two species and how they affect one another [5]. During the prey predator competition, there are various types of interactions between the species. Numerous mathematical models have been constructed to represent the dynamics of prey-predator systems as a result of substantial research. The functional response, which describes how the predator's feeding rate changes with regard to the prey density, is a crucial component of these models. The Holling Type I functional response is a prevalent and well-known type of functional response among the several functional response types. In our work, we study Holling type I functional response [6] to bring two prey and one predator into the conflict. A mathematical model with Holling type I functional response describes the connection between a predator's prey density and consumption rate. Assuming that the predator's consumption rate is directly proportional to the prey density up to a certain saturation point, it is one of the most fundamental functional response models. After this, even if the prey density rises further, the predator's consumption rate stays constant.

In population ecology, dynamics of predator-prey systems is crucial [7]. It establishes how various species are distributed within the environment and, in some cases, forecasts whether a particular species will flourish or go extinct. Time delay, in addition to functional response, has a considerable impact on the dynamics of prey-predator systems. Time lags can occur as a result of a variety of biological and environmental conditions, such as the time it takes the predator to seek for and capture prey after encountering it. These delays inject memory effects into the system, resulting in complicated dynamics that differ markedly from those reported in delay-free models. In order to represent and take into account the necessary reaction time, gestation period, feeding time, etc., delay differential equation DDEs have a long history of modelling prey-predator systems [8] [9], [10]. By considering multiple delays, Kundu and Maitra [11] developed a three-species predator-prey system with cooperation among the prey. They investigated how time delays affected the system and used time delays as the bifurcation parameters to derive the necessary conditions for the existence of Hopf bifurcation.

As a result, the mathematical analysis of prey-predator models with Holling Type I functional responses and time delay is the subject of this work. We intend to research the effect of time delay on the system's stability and bifurcation behaviour, as well as how it effects the coexistence or extinction of predator and prey populations. We aim to gain insights into the complicated dynamics shown by these models by using mathematical tools such as stability analysis, bifurcation theory, and numerical simulations. Understanding the behaviour of prey-predator systems with Holling Type I functional responses and time delay is of theoretical interest, but it also has practical relevance in ecology and conservation biology. It can help us better understand the repercussions of predator-prey interactions and contribute in the development of effective management and conservation measures. Overall, this study lays the foundation for further research into the mathematical properties and ecological implications of prey-predator models with Holling Type I functional responses and time delay, thereby improving our understanding of the dynamics of complex ecological systems and their conservation.

In this work, we investigate the dynamics of a two-prey one-predator delay differential model with Holling type I functional response. In section 3 and 4, we discuss about the positivity and boundedness of the model. We discuss about stability analysis without delay in section 5. Similary we discuss about stability analysis with delay in section 6. Finally, numerical simulations were performed to determine how the population of the species that competed changed dramatically in section 7.

2. Mathematical Model

Consider the following model,

$$
\begin{aligned}
u_1^{'} &= m_1\left(u_1 - \frac{u_1^2}{K_1}\right) - w_1 u_1 u_2 - \lambda_1 u_1 v,\\
u_2^{'} &= m_2\left(u_2 - \frac{u_2^2}{K_2}\right) - w_2 u_1 u_2 - \lambda_2 u_2 v,\\
v^{'} &= \alpha_1\lambda_1 u_1(t-\tau)v(t-\tau) + \alpha_2\lambda_2 u_2(t-\tau)v(t-\tau) - \delta v - \zeta v^2,
\end{aligned} \tag{1}
$$

with initial conditions

$$
u_1(0) > 0, u_2(0) > 0, v(0) > 0, \tag{2}
$$

where $u_1(t)$, $u_2(t)$ and $v(t)$ represent the density of prey 1, prey 2 and predator populations. m_1 and m_2 are the intrinsic growth rates of prey 1 and prey 2; The carrying capacities of prey 1 and prey 2 are represented by K_1 and K_2; w_1 denote the competition coefficient of prey 2 on 1 and w_2 denote the competition coefficient of prey 1 on 2; λ_1, λ_2 are rate of predation on prey 1 and prey 2; δ denote the death rate of predator; ζ denotes the predator's decreased rate as a result of intra-specific competition. Throughout this work, the time delay parameter is represented by τ.

3. POSITIVITY

Theorem 1. *For every solution of (1) with initial conditions (2) exists in* $[0,\infty)\ \forall\, t \geq 0$.

Proof. Using the initial conditions (2), for $t \geq 0$, we have

$$\begin{aligned} u_1^{'} &\geq u_1\left(\left(1-\frac{u_1}{K_1}\right)m_1 - w_1u_2 - \lambda_1 v\right), \\ u_1(t) &\geq u_1(0)exp\left\{\int_0^t\left(m_1\left(1-\frac{u_i(s)}{K_1}\right) - w_1u_2(s) - \lambda_1 v(s)ds\right)\right\}. \end{aligned} \tag{3}$$

Thus, $u_1(t) > 0\ \forall\, t \geq 0$. Similarly, we can prove $u_2(t) > 0$, $v(t) > 0\ \forall\, t \geq 0$. □

4. BOUNDEDNESS

Theorem 2. *All the solutions of system (1) with positive initial values are bounded.*

Proof. Let

$$A'(t) = u_1'(t) + u_2'(t) + v'(t),\ \Omega > 0 \text{ a constant.} \tag{4}$$

Then,

$$\begin{aligned} A'(t) + \Omega A &= u_1'(t) + u_2'(t) + v'(t) + \Omega A, \\ A'(t) + \Omega A &= u_1m_1 - \frac{m_1{u_1}^2}{k_1} - w_1u_1u_2 - \lambda_1u_1v + u_2m_2 - \frac{m_2{u_2}^2}{k_2} - w_2u_1u_2 - \lambda_2u_2v + \\ &\quad \alpha_1\lambda_1u_1(t-\tau)v(t-\tau) + \alpha_2\lambda_2u_2(t-\tau)v(t-\tau) - \delta v - \zeta v^2 + \Omega(u_1+u_2+v) \\ &= (m_1+\Omega)\,u_1 - \frac{m_1{u_1}^2}{k_1} - w_1u_1u_2 - \lambda_1u_1v + (m_2+\Omega)\,u_2 - \frac{m_2{u_2}^2}{k_2} \\ &\quad - w_2u_1u_2 - \lambda_2u_2v + \alpha_1\lambda_1u_1(t-\tau)v(t-\tau) + \alpha_2\lambda_2u_2(t-\tau)v(t-\tau) \\ &\quad + (\Omega-\delta)\,v - \zeta v^2. \end{aligned}$$

If $\lambda_1 \geq \alpha_1$ and $\lambda_2 \geq \alpha_2$,

$$\begin{aligned} A'(t)+\Omega A &\leq (m_1+\Omega)\,u_1 - \frac{m_1 {u_1}^2}{k_1} - w_1u_1u_2 - \lambda_1u_1v + (m_2+\Omega)\,u_2 - \frac{m_2{u_2}^2}{k_2} \\ &\quad - w_2u_1u_2 - \lambda_2u_2v + (\Omega-\delta)\,v \\ &\leq \frac{-m_1}{k_1}\left(u_1 - k_1\frac{(m_1+\Omega)}{2m_1}\right)^2 - \frac{m_2}{k_2}\left(u_2 - k_2\frac{(m_2+\Omega)}{2m_2}\right)^2 \\ &\quad + (\Omega-\delta)v - w_1u_1u_2 - w_2u_1u_2 - \zeta v^2 + \frac{k_1(m_1+\Omega)^2}{6m_1} + \frac{k_2(m_2+\Omega)^2}{6m_2} \end{aligned}$$

Let p = min(w_1, w_2) and so

$$\begin{aligned} A'(t)+\Omega A &\leq \frac{-m_1}{k_1}\left(u_1 - k_1\frac{(m_1+\Omega)}{2m_1}\right)^2 - \frac{m_2}{k_2}\left(u_2 - k_2\frac{(m_2+\Omega)}{2m_2}\right)^2 \\ &\quad + \frac{k_1(m_1+\Omega)^2}{6m_1} + \frac{k_2(m_2+\Omega)^2}{6m_2} + (\Omega-\delta)v - 2pu_1u_2 \\ &\leq \frac{k_1(m_1+\Omega)^2}{6m_1} + \frac{k_2(m_2+\Omega)^2}{6m_2} \\ &= \rho \end{aligned} \tag{5}$$

The solution of (5) is $A = \frac{\rho}{\Omega} + ce^{-\Omega t}$.
When $t=0$, we get $A\left(u_1(0), u_2(0)\right) = \frac{\rho}{\Omega} + c$ and then $c = A\left(u_1(0), u_2(0)\right) - \frac{\rho}{\Omega}$.
Thus, $A\left(u_1(t), u_2(t)\right) = \frac{\rho}{\Omega}\left(1-e^{-\Omega t}\right) + A\left(u_1(0), u_2(0)\right)e^{-\Omega t}$, where $0 < A\left(u_1(t), u_2(t)\right) \leq \frac{\rho}{\Omega}\left(1-e^{-\Omega t}\right) + A\left(u_1(0), u_2(0)\right)e^{-\Omega t}$. As $t \to \infty$ we get $0 < A(t) \leq \frac{\rho}{\Omega}$. Hence Proved. □

5. Stability Analysis Without Delay

5.1. Local Stability. The non - linear matrix of (1) which is evaluated at the interior equilibrium point is given by

$$\begin{pmatrix} m_1 - \frac{2m_1u_1}{k_1} - w_1u_2 - \lambda_1 v & w_1u_1 & \lambda_1u_1 \\ w_2u_2 & m_2 - \frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v & -\lambda_2u_2 \\ \alpha_1\lambda_1 v & \alpha_2\lambda_2 v & \alpha_1\lambda_1u_1 + \alpha_2\lambda_2u_2 - \delta - 2\zeta v \end{pmatrix} \tag{6}$$

Characteristic equation of (6) is,

$$\lambda^3 + \eta_1\lambda^2 + \eta_2\lambda + \eta_3 = 0 \tag{7}$$

where

$$\eta_1 = u_1\left(\frac{2m_1}{k_1} + w_2 - \alpha_1\lambda_1\right) + u_2\left(\frac{2m_2}{k_2} + w_1 - \alpha_2\lambda_2\right) + v(\lambda_1 + \lambda_2 + 2\zeta) - m_1 - m_2 + \delta,$$

$$\eta_2 = M_1M_2 + M_2M_3 + M_1M_3 + \lambda_2^2 u_2\alpha_2 v - \lambda_1^2 u_1\alpha_1 v - w_1u_1w_2u_2,$$

$$\eta_3 = M_1\left(M_2M_3 + \lambda_2^2 u_2\alpha_2 v\right) - w_1u_1\left(M_3w_2u_2 + \lambda_2u_2\alpha_1\lambda_1 v\right) + \lambda_1u_1\left(w_2u_2\alpha_2\lambda_2 v - \alpha_1\lambda_1 vM_2\right).$$

Here M_1, M_2 and M_3 are given by

$M_1 = m_1 - \frac{2m_1u_1}{k_1} - w_1u_2 - \lambda_1 v$

$M_2 = \frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v$

$M_3 = \alpha_1\lambda_1u_1 + \alpha_2\lambda_2u_2 - \delta - 2\zeta v$

By Routh Hurwitz Criterion, the system is locally asymptotically stable, if $\eta_1 > 0$, $\eta_3 > 0$ and $\eta_1\eta_2 - \eta_3 > 0$ are satisfied.

5.2. Global Stability. Consider the Lyapunov function [12] for demonstrating the global asymptotic stable behaviour.

$$\begin{aligned} Y(u_1,u_2) =& u_1 - u_1{}^* - u_1{}^* log\left(\frac{u_1}{u_1{}^*}\right) + a_1\left[u_2 - u_2{}^* - u_2{}^* log\left(\frac{u_2}{u_2{}^*}\right)\right] \\ &+ a_2\left[v - v^* - v^* log\left(\frac{v}{v^*}\right)\right] \\ =& u_1 - u_1{}^* - u_1{}^* log u_1 + u_1{}^* log u_1{}^* + a_1[u_2 - u_2{}^* - u_2{}^* log u_2 + u_2{}^* log u_2{}^*] \\ &+ a_2[v - v^* - v^* log v + v^* log v^*] \end{aligned}$$

Let

$$\frac{dY}{dt} = \frac{\partial Y}{\partial u_1}\frac{du_1}{dt} + \frac{\partial Y}{\partial u_2}\frac{du_2}{dt} + \frac{\partial Y}{\partial v}\frac{dv}{dt} \tag{8}$$

$$\begin{aligned} \frac{dY}{dt} =& \frac{u_1 - u_1{}^*}{u_1}\left[m_1\left(u_1 - \frac{u_1^2}{k_1}\right) - w_1u_1u_2 - \lambda_1u_1v\right] + a_1\frac{(u_2 - u_2{}^*)}{u_2}[m_2\left(u_2 - \frac{u_2^2}{k_2}\right) - \\ & w_2u_1u_2\lambda_2u_2v] + a_2\frac{(v - v^*)}{v}\left[\alpha_1\lambda_1u_1v + \alpha_2\lambda_2u_2v - \delta v - \zeta v^2\right] \\ =& (u_1 - u_1{}^*)\left[m_1 - \frac{m_1}{k_1}u_1 - w_1u_2 - \lambda_1 v\right] + a_1(u_2 - u_2{}^*)\left[m_2 - \frac{m_2}{k_2}u_2 - w_2u_1 - \lambda_2 v\right] \\ &+ a_2(v - v^*)[\alpha_1\lambda_1u_1 + \alpha_2\lambda_2u_2 - \delta - \zeta v] \\ =& \frac{-m_1}{k_1}(u_1 - u_1{}^*)^2 - \frac{m_2}{k_2}a_1(u_2 - u_2^*) - a_2(v - v^*)[\alpha_1\lambda_1(u_1 - u_1{}^*) + \alpha_2\lambda_1(u_2{}^* - u_2)] \end{aligned}$$

If $a_1 = \frac{w_1\alpha_1}{w_2\alpha_2}$ and $a_2 = \frac{\lambda_1\lambda_2}{\alpha_1\alpha_2}$ then,

$$\begin{aligned}\frac{dY}{dt} =& \frac{-m_1}{k_1}(u_1 - {u_1}^*)^2 - \frac{m_2}{k_2}\frac{\lambda_1\lambda_2}{\alpha_1\alpha_2}(u_2 - {u_2}^*) - \frac{w_1\alpha_1}{w_2\alpha_2}(v - v^*)[\alpha_1\lambda_1({u_1}^* - u_1) \\ & + \alpha_2\lambda_2({u_2}^* - u_2)] < 0.\end{aligned} \tag{9}$$

As a result, (1) is globally asymptotically stable near $E^*({u_1}^*, {u_2}^*, v^*)$.

6. Stability Analysis with Delay

The delayed model's characteristic equation (1) examined at E^* is

$$V(\Lambda) + e^{-\Lambda\tau}W(\Lambda) \tag{10}$$

where

$V(\Lambda) = \Lambda^3 + \Lambda^2 v_1 + \Lambda v_2 + v_3.$

$W(\Lambda) = \Lambda^2 w_1 + \Lambda w_2 + w_3.$

$$v_1 = -\left(m_1 - \frac{2m_1u_1}{k_1} - w_1u_2 - \lambda_1 v + m_2 - \frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v - \delta - 2\zeta v\right),$$

$$\begin{aligned} v_2 =& \left(m_2 - \frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v\right)\left(\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)e^{-\lambda\tau} - \delta - 2\zeta v\right) - \left(\alpha_2\lambda_2 e^{-\lambda\tau}\right)(-\lambda_2u_2) \\ & - \left(m_1 - \frac{2m_1u_1}{k_1} - w_1u_2 - \lambda_1 v\right)\left(\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)e^{-\lambda\tau} - \delta - 2\zeta v\right) - \left(\alpha_1\lambda_1 e^{-\lambda\tau}\right)(-\lambda_1u_1) \\ & + \left(m_1 - \frac{2m_1u_1}{k_1} - w_1u_2 - \lambda_1 v\right)\left(m_2 - \frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v\right) - w_1w_2u_1u_2, \end{aligned}$$

$$\begin{aligned} v_3 =& \left(m_1 - \frac{2m_1u_1}{k_1} - w_1u_2 - \lambda_1 v\right) \\ & \left[\left(m_2 - \frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v\right)\left(\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)e^{-\lambda\tau} - \delta - 2\zeta v\right) + \left(\alpha_2\lambda_2^2u_2\right)e^{-\lambda\tau}\right] \\ & + w_1u_1\left[-w_2u_2\left(\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)e^{-\lambda\tau} - \delta - 2\zeta v\right) + \alpha_1\lambda_1\lambda_2u_2e^{-\lambda\tau}\right] \\ & + (-\lambda_1u_1)\left[-w_2u_2\alpha_2\lambda_2e^{-\lambda\tau} - \left(\alpha_1\lambda_1e^{-\lambda\tau}\right)(m_2)\frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v\right], \end{aligned}$$

$$w_1 = -\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right),$$

$$\begin{aligned} w_2 =& \left(m_2 - \frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v\right)\left(\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)e^{-\lambda\tau}\right) + \lambda_2^2u_2\alpha_2e^{-\lambda\tau} \\ & - \left(m_1 - \frac{2m_1u_1}{k_1} - w_1u_2 - \lambda_1 v\right)\left(\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)e^{-\lambda\tau}\right) + \lambda_1^2u_1\alpha_1e^{-\lambda\tau}, \end{aligned}$$

$$w_3 = \left(m_1 - \frac{2m_1u_1}{k_1} - w_1u_2 - \lambda_1 v\right)$$
$$\left[\left(m_2 - \frac{2m_2u_2}{k_2} - w_2u_1 - \lambda_2 v\right)\left(\left(\alpha_1\lambda_1 + \alpha_2\lambda_2\right)e^{-\lambda\tau}\right) - \alpha_2\lambda_2^2 u_2 e^{-\lambda\tau}\right]$$
$$- w_1u_1\left[w_2u_2e^{-\lambda\tau}(\alpha_1\lambda_1 + \alpha_2\lambda_2) + \alpha_1\lambda_1\lambda_2u_2e^{-\lambda\tau}\right]$$
$$+ (\lambda_1u_1)\left[-w_2u_2\alpha_2\lambda_2e^{-\lambda\tau} - \left(\alpha_1\lambda_1e^{-\lambda\tau}\right)\left(m_2 - \frac{2m_2u_2}{k_2} - w_2u_1 - \delta\right)\right].$$

Consider $\lambda = i\omega$ to be a root of (10), where ω is a real number. Substitute $\lambda = i\omega$ in (10) and separate real and imaginary terms. We get,

$$s_3 - \omega^2 s_1 = (\omega^2 r_1 - r_3)cos\omega\tau - \omega r_2 sin\omega\tau. \tag{11}$$

$$s_2\omega - \omega^3 = (r_3 - r_1\omega^2)sin\omega\tau - \omega r_2 cos\omega\tau. \tag{12}$$

Squaring and adding (11) and (12), we obtain,

$$\omega^6 + \omega^4 Z_1 + \omega^2 Z_2 + Z_3 = 0, \tag{13}$$

where

$$Z_1 = {s_1}^2 - 2s_2 - {\omega_1}^2 > 0$$
$$Z_2 = {s_2}^2 - 2s_3s_1 + 2r_1r_3 - {r_2}^2$$
$$Z_3 = {s_3}^2 - {r_3}^2.$$

According to Descartes' rule, if $Z_3 = 0$, then (13) has a unique positive root ω_0^2 and (10) has a pair of imaginary roots $\pm\omega_0^2$.

From (11) and (12), we get

$$cos\omega\tau = \frac{\omega r_2(\omega^3 - \omega s_2) - (s_3 - \omega^2 s_1)(r_3 - r_1\omega^2)}{(r_3 - \omega^2 r_1)^2 + (\omega r_2)^2} \tag{14}$$

Then τ_k, corresponding to $\omega = \omega_0$ is given by

$$\tau_k = \frac{1}{\omega_0}cos^{-1}\left[\frac{\omega r_2(\omega^3 - \omega s_2) - (s_3 - \omega^2 s_1)(r_3 - r_1\omega^2)}{(r_3 - \omega^2 r_1)^2 + (\omega r_2)^2}\right] + \frac{2k\pi}{\omega_0}, k = 0, 1, 2, ... \tag{15}$$

(1) is stable around E^* for $\tau < \tau_0$ according to Buttler's lemma.

Now, differentiate (10) with respect to τ,

$$S'(\Lambda)\frac{d\Lambda}{dt} + e^{-\Lambda\tau}R'(\Lambda)\frac{d\Lambda}{dt} + R(\Lambda)e^{-\Lambda\tau}(-\Lambda - \tau\frac{d\Lambda}{d\tau}) = 0. \tag{16}$$

Hence,

$$\left(\frac{d\Lambda}{d\tau}\right)^{-1} = \frac{S'(\Lambda)}{-\Lambda S(\lambda)} + \frac{R'(\Lambda)}{\Lambda R(\Lambda)} - \frac{\tau}{\Lambda}$$

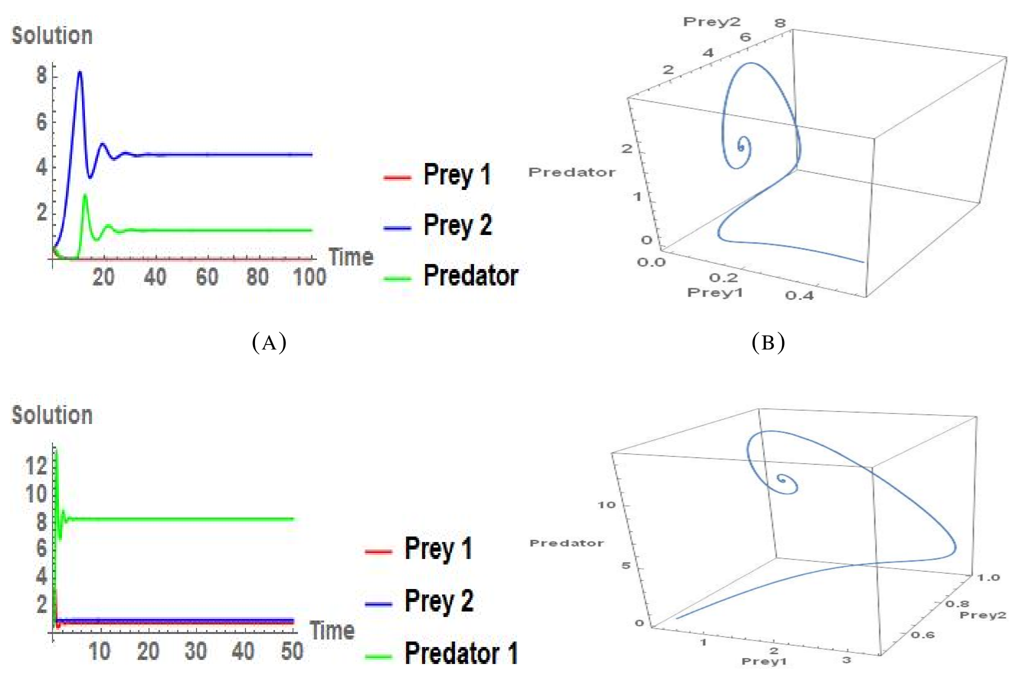
$$\left[\left(\frac{d\Lambda}{dt}\right)^{-1}\right]_{\lambda=i\omega_0} = \frac{\omega_0{}^2(3\omega_0{}^4 + (2s_1{}^2 - 4s_2)\omega_0{}^2 + s_2{}^2 - 2s_1s_3)}{(\omega_0{}^4 - s_2\omega_0{}^2)^2 + (s_3\omega_0 - f_1\omega_0{}^3)^2} - \frac{2r_1r_3\omega_0{}^2 - 2r_1{}^2\omega_0{}^4 - r_2{}^2\omega_0{}^2}{(r_2\omega_0{}^2)^2 + (r_3\omega_0 - r_1\omega_0{}^3)^2} - \frac{\tau}{i\omega_0} \tag{17}$$

Using (13) we obtain,

$$Re\left[\left(\frac{d\Lambda}{d\tau}\right)^{-1}\right]_{\Lambda=i\omega_0} = \frac{3\omega_0{}^6 - 2r_1{}^2\omega_0{}^4 + (2s_1{}^2 + 2r_1r_3 - r_2{}^2 - 4s_2)\omega_0{}^2 + s_2{}^2 - 2s_1s_3}{\chi^2} \tag{18}$$

where, $\chi^2 = (\omega_0{}^4 - s_2\omega_0{}^2)^2 + (s_3\omega_0 - s_1\omega_0{}^3)^2 = (r_2\omega_0{}^2)^2 + (r_3\omega_0 - r_1\omega_0{}^3)^2$

If $2r_1{}^2\omega_0{}^4 < 0$ then $Re\left[\left(\frac{d\Lambda}{d\tau}\right)^{-1}\right]_{\Lambda=i\omega_0} > 0$. Hence, $\frac{d}{d\tau}(Re(\Lambda)) > 0$. Thus, the Hopf-bifurcation condition is satisfied, and the system exhibits periodic oscillations at $\tau > \tau_0$.



(A) (B)

(C) (D)

FIGURE 1. *Time evolution of all the population for the model (1) with $\tau = 0$.*

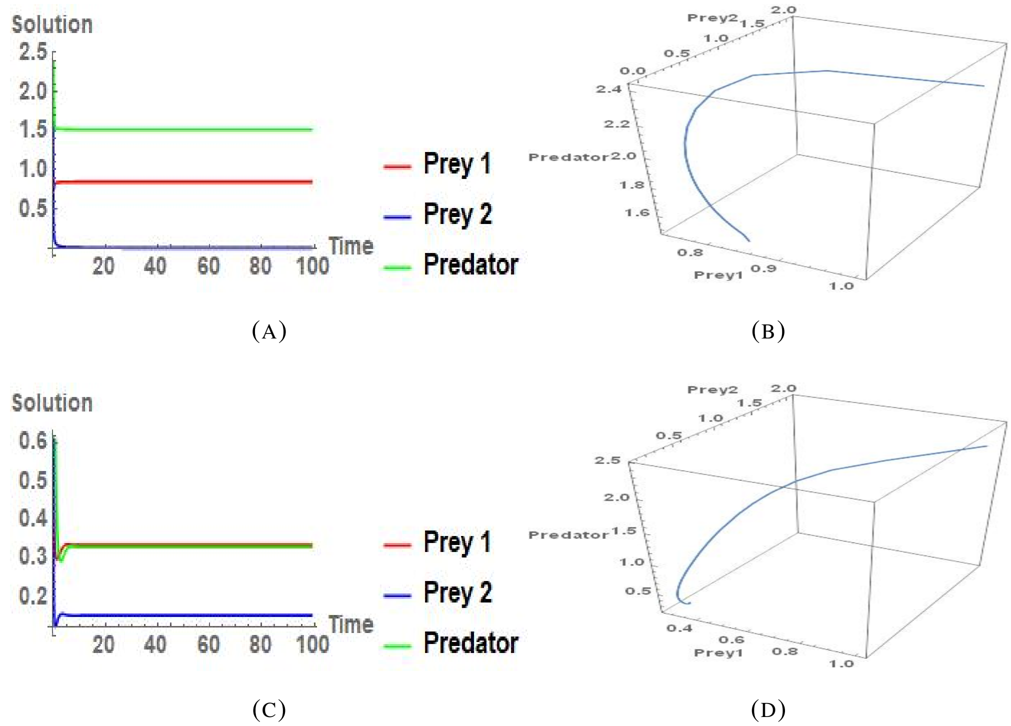


FIGURE 2. *Left panel demonstrates the deterministic system's time series evaluation with* $\tau = 0.01$. *Right panel demonstrates the dynamics of the prey 1, prey 2, and predator space-phase delay.*

7. RANDOM FLUCTATION ANALYSIS USING WHITE NOISE

We permit stochastic perturbations of the variables u_1, u_2, and v around E^* in this section if it is locally asymptotically stable. We consider white noise stochastic perturbations that are proportional to u_1, u_2, and v distances from u_1^*, u_2^*, and v^* . As a result, the stochastically perturbed system with t is given by

$$\begin{aligned} du_1 &= \left(m_1\left(u_1 - \frac{u_1^2}{K_1}\right) - w_1u_1u_2 - \lambda_1u_1v\right)dt + \upsilon_1\left(u_1 - u_1^*\right)d\kappa_t^1, \\ du_2 &= \left(m_2\left(u_2 - \frac{u_2^2}{K_2}\right) - w_2u_1u_2 - \lambda_2u_2v\right)dt + \upsilon_2\left(u_2 - u_2^*\right)d\kappa_t^2, \\ v' &= \left(\alpha_1\lambda_1u_1v + \alpha_2\lambda_2u_2v - \delta v - \zeta v^2\right)dt + \upsilon_3\left(v - v^*\right)d\kappa_t^3. \end{aligned} \tag{19}$$

where $\upsilon_i, i = 1,2,3$ are real constant and $\kappa_t^i, i = 1,2,3$ are standard Wiener processes that are independent.

We consider the linear system of (19) around E^* in order to conduct the following analysis on E^* stochastic stability:

$$dx(t) = a(x(t))\,dt + b(x(t))\,d\kappa(t), \tag{20}$$

where

$x(t) = col(x_1(t), x_2(t), x_3(t))$,

$a(x(t)) = Jx(t)$,

$b(x) = \begin{bmatrix} \upsilon_1 x_1 & 0 & 0 \\ 0 & \upsilon_2 x_2 & 0 \\ 0 & 0 & \upsilon_3 x_3 \end{bmatrix}$; $d\kappa(t) = col(\kappa_1(t), \kappa_2(t))$, $x_1 = u_1 - u_1^*, x_2 = u_2 - u_2^*, x_3 = u_3 - u_3^*$.

Let $X = \{(t \geq t_0) \times R^n, t_0 \in R^+\}$ and $Y \in C_2^0(X)$. We have,

$$LY(t,x) = \frac{\partial Y(t,x)}{\partial t} + a^T(x)\frac{\partial Y(t,x)}{\partial x} + \frac{1}{2}Tr\left(b^T(x)\frac{\partial^2 Y(t,x)}{\partial x^2}b(x)\right) \tag{21}$$

where $\frac{\partial Y}{\partial x} = Col\left(\frac{\partial Y}{\partial x_1}, \frac{\partial Y}{\partial x_2}\right)$,

$\frac{\partial^2 Y(t,x)}{\partial x^2} = \frac{\partial^2 Y(t,x)}{\partial x_j \partial x_i}$, $i, j = 1, 2$ and T denotes transposition.

Theorem 3. *If there is a function $Y \in C_2^0(X)$ that satisfies the subsequent criteria,*

$$\mathbb{K}_1|x|^q \leq Y(t,x) \leq \mathbb{K}_2|x|^q; LY(t,x) \leq -\mathbb{K}_3|x|^q, \mathbb{K}_i > 0, q > 0. \tag{22}$$

for $t \geq 0$ the trivial solution of 20 is exponentially p-stable.

The trivial solution of 20 is globally asymptotically stable if $p = 2$ in (22). The proof is similar to the theorem in [13].

Theorem 4. *If* $\left(\frac{m_1 u_1^*}{K_1} - \frac{1}{2}\nu_1^2\right) > 0$, $\left(\frac{m_2 u_2^*}{K_2} - \frac{1}{2}\nu_2^2\right) > 0$ *and* $\left(\zeta v^* - -\frac{1}{2}\nu_3^2\right)$ *then the zero solution of (20) is asymptotically mean square stable.*

Proof. Consider the Lyapunov function

$$Y(x) = \frac{1}{2}\left(n_1 x_1^2 + n_2 x_2^2 + n_3 x_3^2\right), n_i > 0 \in R \tag{23}$$

For $p = 2$ inequalities in (22) are true.

$$\begin{aligned} LY(x) = n_1 & \left(\left(-\frac{m_1 u_1^*}{K_1} \right) x_1 - w_1 u_1^* x_2 - \lambda_1 u_1^* x_3 \right) \\ & + n_2 \left(\left(-\frac{m_2 u_2^*}{K_2} \right) x_2 - w_2 u_2^* x_1 - \lambda_2 u_2^* x_3 \right) \\ & + n_3 \left(\alpha_1 \lambda_1 x_1 v^* + \alpha_2 \lambda_2 x_2 v^* - \zeta v^* x_3 \right) x_2 + \frac{1}{2} Tr \left(b^T(x) \frac{\partial^2 Y}{\partial x^2} b(x) \right). \end{aligned} \tag{24}$$

It is clear that $\frac{\partial^2 Y}{\partial x^2} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3. \end{bmatrix}.$

Hence $b^T(x) \frac{\partial^2 Y}{\partial x^2} b(x) = \begin{bmatrix} n_1 \upsilon_1^2 x_1 & 0 & 0 \\ 0 & n_2 \upsilon_2^2 x_2 & 0 \\ 0 & 0 & n_3 \upsilon_3^2 x_3. \end{bmatrix}$ with

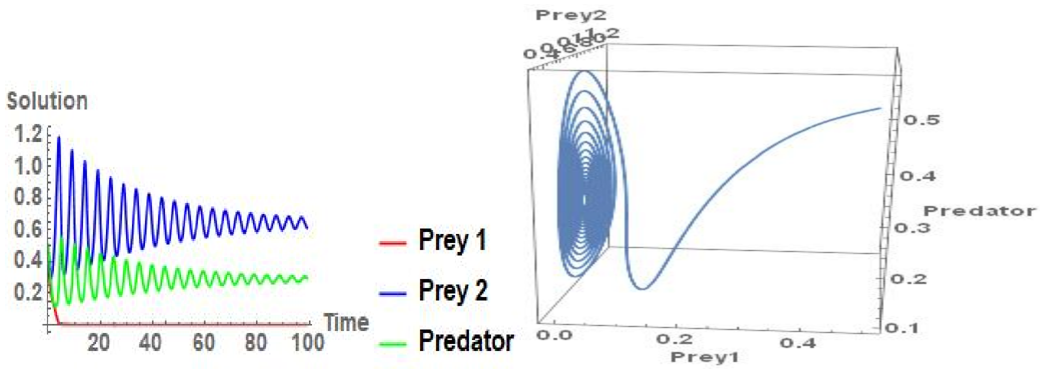
$$\frac{1}{2} Tr \left[b^T(x) \frac{\partial^2 Y}{\partial x^2} b(x) \right] = \frac{1}{2} \left[n_1 \upsilon_1^2 x_1^2 + n_2 \upsilon_2^2 x_2^2 + n_3 \upsilon_3^2 x_3^2 \right] \tag{25}$$

In (24), choose $n_1 \lambda_1 u_1^* = n_3 \alpha_1 v^*$, $n_2 \lambda_2 u_2^* = n_3 \alpha_2 v^*$.

From (25), we have

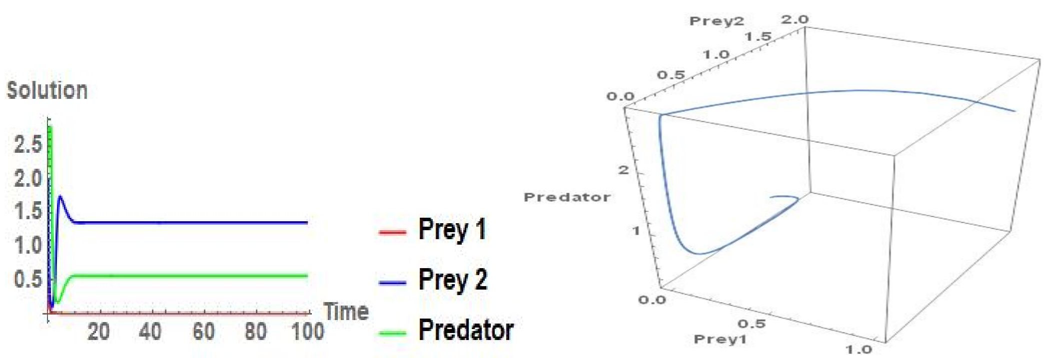
$LY(x) = -n_1 \left(\frac{m_1 u_1^*}{K_1} - \frac{1}{2} \nu_1^2 \right) x_1^2 - n_2 \left(\frac{m_2 u_2^*}{K_2} - \frac{1}{2} \nu_2^2 \right) x_2^2 - n_3 \left(\zeta v^* - \frac{1}{2} \nu_3^2 \right) x_3^2$

By Theorem 3, the proof is complete. □

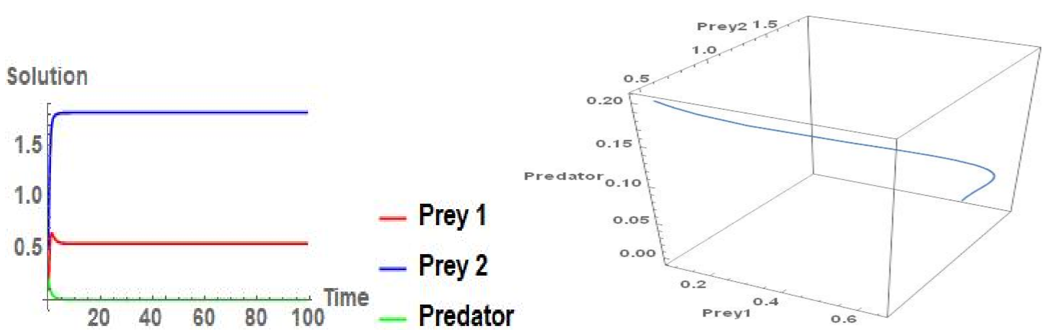


(A) (B)

FIGURE 3. *Time evolution of all the population for the model (1) with $\tau = 0.04$.*

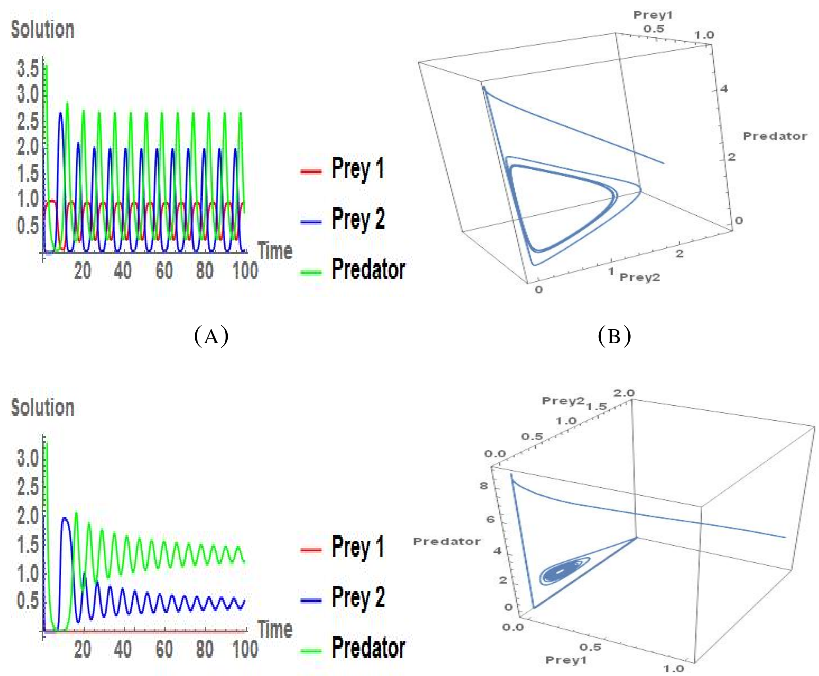


(A) (B)



(C) (D)

FIGURE 4. *Fig (a) depicts the time evolution of all the population in two dimension when $\tau = 1$. Fig (c) shows the associated three-dimensional phase diagram.*



(A) (B)

(C) (D)

FIGURE 5. *Left panel shows the time series evaluation of the deterministic system with random fluctuations when $\tau = 1$. Right panel shows the space phase delay dynamics between prey1, prey2 and predator.*

8. NUMERICAL ANALYSIS

By randomly choosing appropriate and suitable sets of parameters, we evaluated the conditions, particularly the stability and impact of white noise, that were carried out in the preceding sections. Here, we use Mathematica to run numerical simulations to validate our analytical results for system (1).

Case 1: Simulation in the absence of delay

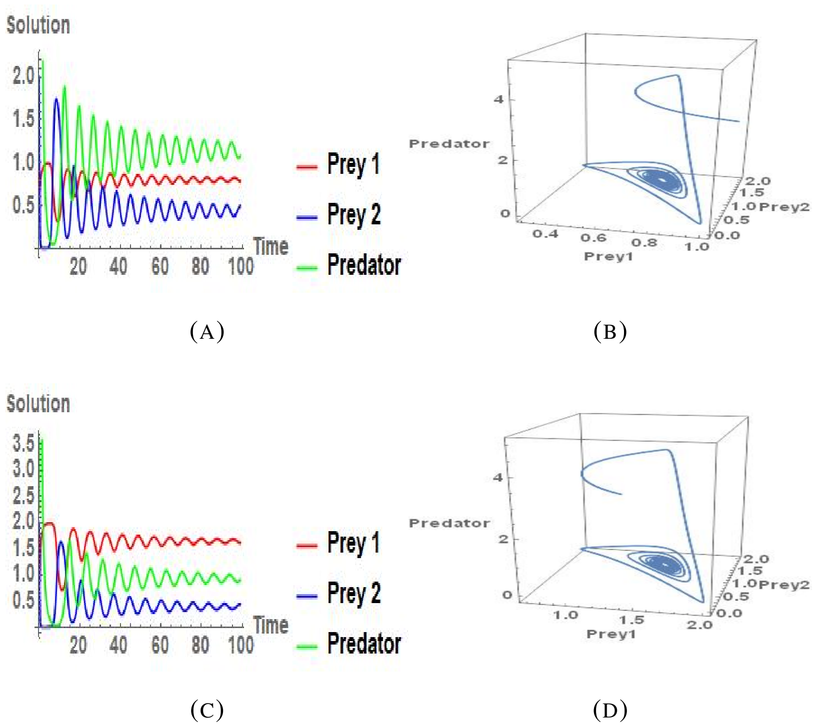
Here we considered (1) without time delay. Numerical simulations were performed to validate the analytical results with $\tau = 0$. Choose $m_1 = 0.0001$, $K_1 = 0.5$, $w_1 = 0.3$, $\lambda_1 = 1.1$, $m_2 = 0.5$, $K_2 = 11$, $w_2 = 0.5$, $\lambda_2 = 0.23$, $\alpha_1 = 3$, $\alpha_2 = 1.6$, $\delta = 1.5$, $\zeta = 0.15$ in appropriate units to illustrate the results numerically.

Upper panel depics that only the second prey and predator populations are alive, while the first prey population has gone extinct. One can easily identify that the system of equations which is free from the time delay terms is always stable which is shown in Lower panel.

Case 2: Simulation in the presence of delay

Here we considered (1) with time delay. When time delay increased to 0.01, Figure 2 a) shows that only the first prey and predator populations are alive, whereas the second prey population has gone extinct. In Figure 2 b) all three populations coexist simultaneously. Further when time delay is increased to 0.04, a periodic solution occurs between prey 2 and the predator while prey 1 remains at zero level and vanishes.

When $\tau = 1$, in Figure 4a), prey population will become extinct while a stable behaviour exist between prey 2 and the predator. Similarly, the two prey populations exist in Figure 4 (b), but the predator population has vanished. Figure 5 and Figure 6, exhibits a periodic solution between all the three populations.



(A) (B) (C) (D)

FIGURE 6. *Time evolution of all the population for the model (1) with $\tau = 1$.*

9. Conclusion

The interaction of two prey and one predator in an ecosystem with a discrete-time delay and a Holling type I functional response has been investigated. We examined the well-posedness of the system, such as positive invariance and boundedness. The stability analysis was conducted both locally and globally, with and without a time delay. Descartes' rule and Buttler's lemma are also used to describe and prove the Hopf - bifurcation characteristics. Finally, numerical simulations were run to determine how the population of the species that competed changed dramatically.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] M. Aakash, C. Gunasundari, Q.M. Al-Mdallal, Mathematical modeling and simulation of SEIR model for COVID-19 outbreak: A case study of Trivandrum, Front. Appl. Math. Stat. 9 (2023), 1124897. https://doi.org/10.3389/fams.2023.1124897.

[2] M. Aakash, C. Gunasundari, Effect of partially and fully vaccinated individuals in some regions of India: A mathematical study on COVID-19 outbreak, Commun. Math. Biol. Neurosci. 2023 (2023), 25. https://doi.org/10.28919/cmbn/7825.

[3] G. Santhosh Kumar, C. Gunasundari, Turing instability of a diffusive predator-prey model along with an Allee effect on a predator, Commun. Math. Biol. Neurosci. 2022 (2022), 40. https://doi.org/10.28919/cmbn/7361.

[4] N.B. Sharmila, C. Gunasundari, Travelling wave solutions for a diffusive prey-predator model with one predator and two preys, Int. J. Appl. Math. 35 (2022), 46-67. https://doi.org/10.12732/ijam.v35i5.3.

[5] V. Volterra, Variazioni e fluttuazioni del numero d'individui in specie conviventi, Mem. Accad. Lincei, 2 (1926), 31–113.

[6] D. Savitri, A. Suryanto, W.M. Kusumawinahyu, Abadi, Dynamics of two preys - one predator system with competition between preys, J. Phys.: Conf. Ser. 1562 (2020), 012010. https://doi.org/10.1088/1742-6596/1562/1/012010.

[7] N.B. Sharmila, C. Gunasundari, Stability analysis of a fractional order prey-predator model with disease in prey, Math. Appl. 50 (2023), 287-302. https://doi.org/10.14708/ma.v50i2.7142.

[8] F. Chen, M. You, Permanence for an integrodifferential model of mutualism, Appl. Math. Comput. 186 (2007), 30-34. https://doi.org/10.1016/j.amc.2006.07.085.

[9] J.F. Zhang, Bifurcation analysis of a modified Holling–Tanner predator–prey model with time delay, Appl. Math. Model. 36 (2012), 1219-1231. https://doi.org/10.1016/j.apm.2011.07.071.

[10] G.A. Bocharov, F.A. Rihan, Numerical modelling in biosciences using delay differential equations, J. Comput. Appl. Math. 125 (2000), 183-199. https://doi.org/10.1016/s0377-0427(00)00468-4.

[11] S. Kundu, S. Maitra, Dynamical behaviour of a delayed three species predator–prey model with cooperation among the prey species, Nonlinear Dyn. 92 (2018), 627-643. https://doi.org/10.1007/s11071-018-4079-3.

[12] Y.S. Kumar, N.S. Rao, B.V.A. Rao, Time delay model for a predator and two species with mutualism interaction, ARPN J. Eng. Appl. Sci. 13 (2018), 8664-8677.

[13] V.N. Afanas'ev, V.B. Kolmanowski, V.R. Nosov, Mathematical theory of global systems design, Kluwer Academic, Dordrecht, (1996).