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## A FRACTIONAL DYNAMICS MODEL OF HEPATITIS B DISEASE SPREAD UNDER INFLUENCE OF CAMPAIGN AND TREATMENT

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**Abstract.** In this work, we present a fractional dynamic model to describe the spread of Hepatitis B disease in human population under influence of campaign and treatment parameters. It was shown that the stability of disease-free equilibrium and disease endemic equilibrium depend on the basic reproduction number. These results are in accordance with the epidemic theory. A numerical example is given to demonstrate the validity of the results. The results show that the media campaigns and treatment increase susceptible subpopulations, reduce infectious ones, and increase recovered subpopulations, thus the model gives adequate information about the spread of the Hepatitis B virus.

**Keywords:** Caputo fractional-order derivative; SIR model; basic reproduction number; equilibrium.

**2020 AMS Subject Classification:** 26A33, 34C60, 34D23.

### 1. INTRODUCTION

Hepatitis B is an inflammation of the liver that is caused by a variety of infectious viruses leading to a range of health problems, some of which can be fatal. Usually this disease transmits from one person by different ways to another, e.g., through semen, blood, and vaginal secretion

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etc.. Sexual transmission is also one of the dominant sources of hepatitis B virus transmission [1, 2].

Mathematical modeling is a method to better understand the dynamics of hepatitis B virus transmission and evaluate the effectiveness of various control and prevention strategies. Several studies on the use of mathematical models to study the spread of hepatitis B can be seen in [3, 4, 5, 6, 7, 8].

One of the well-known models of the spread of the hepatitis B virus is the SIR compartment model where the model is given in the form of a nonlinear differential equation, see [4, 6, 9] for a wide discussion. In this SIR model, the observed human population ( $N$ ) is divided into three epidemiological compartments denoted by susceptible  $S(t)$ , infectious  $I(t)$  and recovered individuals  $R(t)$ , thus the total population at the time  $t$  is given by  $N(t) = S(t) + I(t) + R(t)$ . The assumption made in developing this model can be found in [9] and the involve various parameters in (1) are described in Table 1. The dynamics of SIR model for hepatitis B spread

TABLE 1. Parameter with biological meaning occurring in the model (1).

Parameter	Biological meaning
$\Lambda$	Birth rate
$a$	Transmission rate of hepatitis B
$d_0$	Natural death rate
$d_1$	Disease induced death rate
$b$	Recovery rate
$\nu$	Vaccination rate
$c$	Saturation rate

in human population are governed by the following system of coupled nonlinear differential equation [9],

$$(1) \quad \begin{cases} \dot{S}(t) = \Lambda - \frac{aS(t)I(t)}{1+cI(t)} - (d_0 + \nu)S(t) \\ \dot{I}(t) = \frac{aS(t)I(t)}{1+cI(t)} - (d_0 + d_1 + b)I(t) \\ \dot{R}(t) = bI(t) + \nu S(t) - d_0R(t), \end{cases}$$

with the initial conditions  $S(0) = S_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0$ . In the model (1), quantity  $\frac{aS(t)I(t)}{1+cI(t)}$  is the saturated incidence rate, in which  $\frac{aI(t)}{1+cI(t)}$  reaches the saturation level whenever  $I$  increases [9].

Currently, several epidemiological model was formulated in the form of fractional order differential equations and widely discussed by many researchers, see [10, 11, 12, 13, 14, 15, 16]. It is known that fractional order derivatives are generalizations of integer order derivatives, so modeling using fractional differential equations is a powerful method for studying the overall spread of the disease.

Motivated by the current study, in this manuscript, we modified model (1) by replacing the first-order derivative with fractional-order derivatives and including the media campaign parameter (i.e. education about the threat of hepatitis B disease) ( $\mu_1$ ) and the hepatitis B treatment parameter to infected individuals ( $\mu_2$ ) into the model, with  $\mu_1, \mu_2 \in [0, 1)$ , such that the model (1) can be written as a following new model:

$$(2) \quad \begin{cases} \mathcal{D}^{(\gamma)}S(t) = \Lambda - \frac{aS(t)I(t)}{1+cI(t)}(1-\mu_1) - (d_0 + \nu)S(t) \\ \mathcal{D}^{(\gamma)}I(t) = \frac{aS(t)I(t)}{1+cI(t)}(1-\mu_1) - (d_0 + d_1 + b + \mu_2)I(t) \\ \mathcal{D}^{(\gamma)}R(t) = (b + \mu_2)I(t) + \nu S(t) - d_0R(t). \end{cases}$$

In this new model,  $\mathcal{D}^{(\gamma)}$  is the fractional-order derivative operator of Caputo type of order  $\gamma$  with  $0 < \gamma < 1$ . We study the influence of parameters  $\mu_1$  and  $\mu_2$  on each compartment by inspecting the stability behavior of the equilibrium points of the model (2). To the best of the author's knowledge, this issue has not been solved yet to date. Therefore the results of this research constitute a new contribution in the field of fractional-order epidemic dynamics.

## 2. SOME USEFUL RESULTS

In this section we recall several mathematical tools used in this study. Let  $\mathbf{y} : [0, \infty) \rightarrow \mathcal{R}^n$  is an integrable vector function and  $\gamma \in (k-1, k)$ ,  $k \in \mathcal{N}$ . The Caputo fractional-order derivative of order  $\gamma$  is defined by

$$(3) \quad \mathcal{D}^{(\gamma)}\mathbf{y}(t) = \frac{1}{\Gamma(k-\gamma)} \int_0^t (t-\tau)^{k-\gamma-1} \mathbf{y}^{(k)}(\tau) d\tau$$

where  $\Gamma(\cdot)$  is the Euler Gamma function [19]. Let us consider the general fractional-order dynamic system involving Caputo derivative

$$(4) \quad \mathcal{D}^{(\gamma)} \mathbf{y}(t) = \mathbf{g}(t, \mathbf{y}(t))$$

with suitable initial conditions  $\mathbf{y}(t_0) = \mathbf{y}_0$ , where  $\mathbf{y}(t)$  is the state at time  $t$  of the system (4),  $\mathbf{g} : [0, \infty) \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ . Note that the system (4) may be non-linear, or vice versa. If  $\mathbf{g}$  is linear, the system (4) can be written as

$$(5) \quad \mathcal{D}^{(\gamma)} \mathbf{y}(t) = A\mathbf{y}(t),$$

where  $A$  is a  $n$  by  $n$  matrix.

One important thing of the system (4) is stability of equilibrium point. When talking about stability, one is interested in the behavior of the solutions of (4) for  $t \rightarrow \infty$  [17, 18]. The point  $\mathbf{y}^*$  is said the equilibrium point of the system (4) if  $\mathbf{g}(t, \mathbf{y}^*) = \mathbf{0}$ . Note that the equilibrium point is a constant solution to the dynamic system (4).

**Definition 2.1.** [19, 20] *Let  $\mathbf{y}^*$  is an equilibrium point of the fractional-order system (4).*

- (1).  $\mathbf{y}^*$  is said to be stable if for  $\varepsilon > 0$ , there exists a  $\rho(\varepsilon) > 0$  such that  $\|\mathbf{y}(t_0) - \mathbf{y}^*\| < \rho(\varepsilon)$  implies  $\|\mathbf{y}(t) - \mathbf{y}^*\| < \varepsilon$  for  $t \geq t_0$ .
- (2).  $\mathbf{y}^*$  is said to be asymptotically stable if it is stable and  $\lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{y}^*$ .

**Theorem 2.2.** [19, 20] *The equilibrium point  $\mathbf{y}^*$  of the fractional-order linear system (5) with  $\gamma \in (0, 1)$  is asymptotically stable if*

$$(6) \quad |\arg(\beta_i)| > \frac{1}{2} \gamma \pi,$$

where  $\beta_i$ ,  $i = 1, 2, \dots, n$  are eigenvalues of the matrix  $A$ .

**Theorem 2.3.** [19, 20] *The equilibrium point  $\mathbf{y}^*$  of the the fractional-order nonlinear system (4) with  $\gamma \in (0, 1)$  is asymptotically stable if*

$$(7) \quad |\arg(\beta)| > \frac{1}{2} \gamma \pi,$$

for all roots  $\beta$  of the equation

$$(8) \quad |J_{\mathbf{y}^*} - \beta I| = 0$$

where  $J_{\mathbf{y}^*}$  is the Jacobian matrix of system (4) around the equilibrium  $\mathbf{y}^*$ .

### 3. ASYMPTOTIC STABILITY OF THE EQUILIBRIA

By following the procedure in [3], it is easy to show that the solution of the model under consideration is restricted to the feasible region given by

$$\mathcal{U} = \left\{ (S, I, R) \in \mathcal{R}_+^3 : 0 \leq N(t) \leq N(0) \right\}$$

if the initial conditions  $S(0) = S_0 \geq 0$ ,  $I(0) = I_0 \geq 0$ ,  $R(0) = R_0 \geq 0$ . It is well-known in epidemiology that the dynamical behavior of the model (4) depends on the basic reproductive number. By using the next generation method, the basic reproduction number for the model (2) is given by

$$(9) \quad \mathcal{R}_0 = \frac{a\Lambda}{(d_0 + \nu)(d_0 + d_1 + b + \mu_2)}.$$

In order to find the equilibrium point of the model (2), we must solve the following equations:

$$\mathcal{D}^{(\gamma)}S(t) = \mathcal{D}^{(\gamma)}I(t) = \mathcal{D}^{(\gamma)}R(t) = 0.$$

By assuming  $I = 0$ , one finds the disease-free equilibrium, denoted by  $\mathcal{E}_0$ , of the fractional order Hepatitis B model (2), that is

$$\mathcal{E}_0 = \left( \frac{\Lambda}{d_0 + \nu}, 0, \frac{\Lambda\nu}{d_0(d_0 + \nu)} \right).$$

We will analyze the stability of this free disease equilibrium point. First of all, the Jacobian matrix of the vector field corresponding to model (2) around  $\mathcal{E}_0$  is

$$J_{\mathcal{E}_0} = \begin{bmatrix} -\frac{aI^0}{1+cI^0}(1-\mu_1) - (d_0 + \nu) & -\frac{aS^0}{(1+cI^0)^2}(1-\mu_1) & 0 \\ -\frac{aI^0}{1+cI^0}(1-\mu_1) & \frac{aS^0}{(1+cI^0)^2}(1-\mu_1) - (d_0 + d_1 + b + \mu_2) & 0 \\ \nu & b + \mu_2 & -d_0 \end{bmatrix}$$

$$= \begin{bmatrix} -(d_0 + \nu) & -\frac{a\Lambda}{d_0 + \nu}(1-\mu_1) & 0 \\ 0 & \frac{a\Lambda}{d_0 + \nu}(1-\mu_1) - (d_0 + d_1 + b + \mu_2) & 0 \\ \nu & b + \mu_2 & -d_0 \end{bmatrix}.$$

The stability of the free disease equilibrium point  $\mathcal{E}_0$  is given in the following theorem.

**Theorem 3.1.** *The free disease equilibrium point  $\mathcal{E}_0$  is asymptotically stable if  $\mathcal{R}_0 < 1$ , and if  $\mathcal{R}_0 > 1$  then it becomes unstable.*

*Proof.* Clearly  $J_{\mathcal{E}_0}$  has the following three eigenvalues given by

$$\lambda_1 = -(d_0 + \nu),$$

$$\lambda_2 = -d_0,$$

$$\lambda_3 = -(d_0 + d_1 + b + \mu_2)(1 - \mathcal{R}_0).$$

One can see that all eigenvalues of  $J_{\mathcal{E}_0}$  satisfy  $|\arg(\beta_i)| > \frac{\gamma\pi}{2}$  if  $\mathcal{R}_0 < 1$  for  $i = 1, 2, 3$ , and one eigenvalue satisfy  $|\arg(\beta_3)| < \frac{\gamma\pi}{2}$  when  $\mathcal{R}_0 > 1$ . Hence,  $\mathcal{E}_0$  is asymptotically stable if  $\mathcal{R}_0 < 1$  and becomes unstable if  $\mathcal{R}_0 > 1$ .  $\square$

To find the disease endemic equilibrium point (denoted by  $\mathcal{E}^*$ ) of the fractional-order of hepatitis model (2), we solve the model (2) at steady state for  $S, I$  and  $R$ . One can observe that  $\mathcal{E}^* = (S^*, I^*, R^*)$ , where

$$(10) \quad S^* = \frac{1}{a(1 - \mu_1)}(d_0 + d_1 + b + \mu_2)(1 + cI^*),$$

$$(11) \quad I^* = \frac{d_0 + \nu}{d_0 + \nu + 1}(\mathcal{R}_0 - 1),$$

$$(12) \quad R^* = \frac{1}{d_0}((b + \mu_2)I^* + \nu S^*),$$

is the disease endemic equilibrium point of hepatitis model (2). The stability of the disease endemic equilibrium  $\mathcal{E}^*$  is given in the following theorem.

**Theorem 3.2.** *If  $\mathcal{R}_0 > 1$ , then the disease endemic equilibrium  $\mathcal{E}^*$  is asymptotically stable and becomes unstable when  $\mathcal{R}_0 < 1$ .*

*Proof.* The Jacobian matrix of (2) around  $\mathcal{E}^*$  is

$$J_{\mathcal{E}^*} = \begin{bmatrix} -\frac{aq_0 I^*}{1 + cI^*} - q_1 & -\frac{aq_0 S^*}{(1 + cI^*)^2} & 0 \\ -\frac{aq_0 I^*}{1 + cI^*} & \frac{aq_0 S^*}{(1 + cI^*)^2} - q_2 & 0 \\ \nu & b + \mu_2 & -d_0 \end{bmatrix}$$

where  $q_0 = 1 - \mu_1$ ,  $q_1 = (d_0 + \nu)$ ,  $q_2 = (d_0 + d_1 + b + \mu_2)$ . It is obvious that  $\beta_1 = -d_0$  constitutes an eigenvalue of  $J_{\mathcal{E}^*}$  that have negative real part. In order to find the remaining, we take the following matrix

$$K = \begin{bmatrix} -\frac{aq_0 I^*}{1 + cI^*} - q_1 & -\frac{aq_0 S^*}{(1 + cI^*)^2} \\ -\frac{aq_0 I^*}{1 + cI^*} & \frac{aq_0 S^*}{(1 + cI^*)^2} - q_2 \end{bmatrix}.$$

The eigenvalues of the matrix  $K$  are negative if  $\text{trace}(K) < 0$  and  $\det(K) > 0$ . Observe that

$$(13) \quad \text{trace}(K) = -\left(\frac{aq_0 I^*}{1 + cI^*} + q_1\right) \left(\frac{aq_0 S^*}{(1 + cI^*)^2} - q_2\right)$$

$$(14) \quad \det(K) = \text{trace}(K) - \left(\frac{aq_0 S^*}{(1 + cI^*)^2}\right) \left(\frac{aq_0 I^*}{1 + cI^*}\right).$$

By substituting (10), (11) into (13) and (14), and using the condition  $\mathcal{R}_0 > 1$ , one get  $\text{trace}(K) < 0$  and  $\det(K) > 0$ . It is easy to check that the negativity of all eigenvalues of  $J_{\mathcal{E}^*}$  implies  $|\arg(\beta_i)| > \frac{\gamma\pi}{2}$ , for  $i = 1, 2, 3$ . Hence,  $\mathcal{E}^*$  is asymptotically stable if  $\mathcal{R}_0 > 1$  and becomes unstable if  $\mathcal{R}_0 < 1$ .  $\square$

In order to show the validity of the results, let us consider the following numerical example. For the model (2), let us assume  $\Lambda = 0.088 \text{ day}^{-1}$ ,  $c = 0.09 \text{ day}^{-1}$ ,  $\nu = 0.3 \text{ day}^{-1}$ ,  $b = 0.03 \text{ day}^{-1}$ ,  $a = 0.24 \text{ day}^{-1}$ ,  $d_0 = 0.002 \text{ day}^{-1}$ ,  $d_1 = 0.001 \text{ day}^{-1}$ ,  $\mu_1 = 0.02$ ,  $\mu_2 = 0.03$  and  $N = 270$  individuals. The initial conditions are  $S_0 = 100$ ,  $I_0 = 90$  and  $R_0 = 80$ . Based on these parameter values, we find the basic reproduction number  $\mathcal{R}_0 = 1.1101$  which shows that the endemic equilibrium is asymptotic stable.

Graphs of the susceptible subpopulation, infectious subpopulation, and recovered subpopulation under the effect of media campaign/education and the treatment for several fractional-order  $\gamma$ , respectively, are given in Figure 1, Figure 2, and Figure 3. The graphs show that the media campaigns (education) and treatment increase susceptible subpopulations, reduce infectious ones, and increase recovered subpopulations.

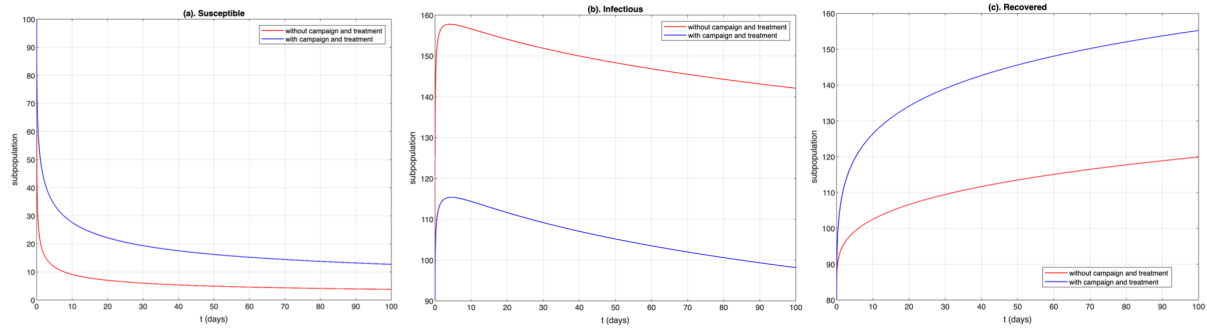


FIGURE 1. The curves of Susceptible, Infectious, Recovered for  $\gamma = 0.4$

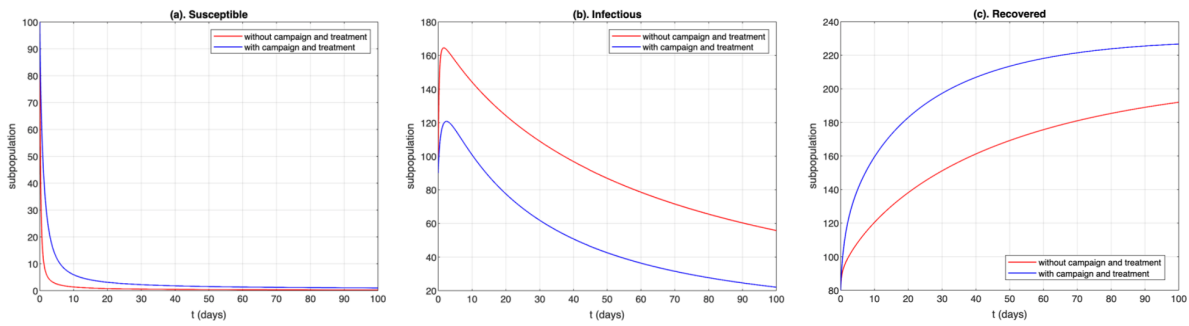


FIGURE 2. The curves of Susceptible, Infectious, Recovered for  $\gamma = 0.75$

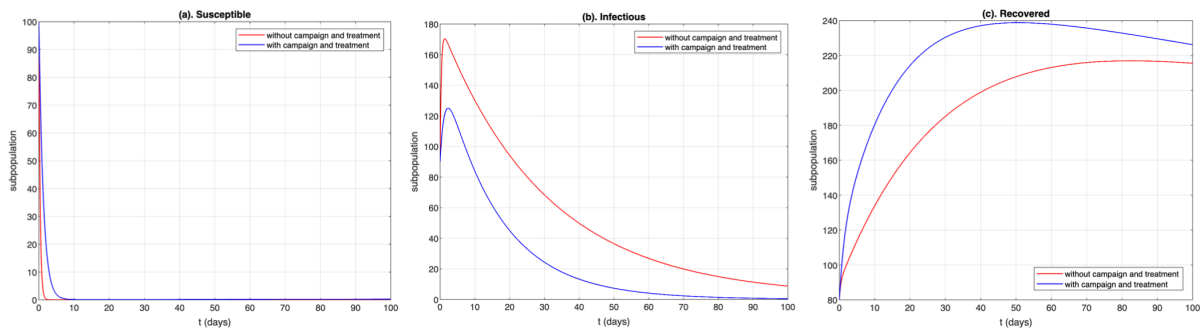


FIGURE 3. The curves of Susceptible, Infectious, Recovered for  $\gamma = 0.95$

#### 4. CONCLUSION

We have found the fractional SIR model for the dynamic of Hepatitis B virus spread. An example that illustrates the result has been presented. The analysis shows that the media campaigns (education) and treatment increase susceptible subpopulations, reduce infectious ones,



and increase recovered subpopulations, thus the SIR model gives adequate information about the spread of the Hepatitis B virus.

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## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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