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THE GLOBAL STABILITY OF FRACTIONAL EPIDEMIOLOGICAL MODEL WITH N STRAIN "ALL CORONAVIRUS MUTATIONS"

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Abstract. In this work, we have constructed a new system of differential equations which mathematically models infectious diseases with several mutations. (such as covid 19 disease and their mutations). Therefore, we are interested in studying the asymptotic stability of our new system.

Keywords: Mittag-Leffler; global stability; Lyapunov function.

2020 AMS Subject Classification: 33E12, 34D23, 37L45.

1. INTRODUCTION

After the appearance of the disease Covid 19 and after a small period of time, several mutations of the disease it appeared because of a change on the ADN of the virus. Most mutations have little or no effect on the properties of the virus, see [6]. However, some mutations can affect the properties of the virus and influence, for example, the ease with which it spreads, the severity of the disease it causes, or the effectiveness of vaccines, drugs, diagnostic tools or other social and public health measures.

The latest variants of concern have largely supplanted other SARS-CoV-2 variants that were circulating at the same time. The Delta variant accounted for nearly 90of all viral sequences

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submitted in October 2021, and Omicron is currently the dominant variant worldwide, accounting for $> 98\%$ of viral sequences after February 2022. As transmission of these variants of concern has persisted, there has been significant internal evolution. Since being designated as a variant of concern by WHO on November 26, 2021, viruses within the Omicron complex have continued to evolve and their descendant lineages exhibit different constellations of genetic mutations. The constellations may or may not differ in terms of the risk they pose to public health, and further research on each lineage with substitutions at key sites may be needed to determine whether or not its characteristics differ from those that define the variant of concern from which it originated.

The main goal of mathematical models in epidemiology is to understand the behavior of a particular infectious disease, such as the prevalence and the duration of the epidemic, and its impact in the population.

Mathematical models are used to describe reality, but usually they are simplifications because it is almost impossible to make computations with a large set of input parameters.

Recently, fractional derivatives have been used to describe epidemiological models and they have proven to be more accurate in some cases, when compared to the classical ones. We find in the literature different models described by fractional derivatives [7].

There are several definitions for fractional derivatives; in this paper we choose to work with the Caputo fractional derivative. One of the advantages of such derivative is allowing us to consider classical initial conditions to be included in the formulation of the problem. Also, the Caputo fractional derivative of a constant is zero, which is not true for other fractional derivatives.

In this paper, we propose a fractional $SIQR_TR_{NT}$ model, where the spread of the disease is de scribed by a system of nonlinear fractional-order differential equations. It is worthwhile mentioning that fractional derivatives are non-local operators, and thus may be more suitable for modeling systems dependent on past history (memory). Also, since the fractional order can be any positive 2 real α, we can choose the one that better fits the data. Therefore, we can adjust the model to real data and, thus, better predict the evolution of the disease.

We are interested in clustering all Covid 19 mutations in an epidemiological model so that at the end we study the asymptotic stability (our approach is based on a functional Lyapunov) and give a threshold condition for the disease that disappears or spreads for the epidemic model as a function of the base reproduction rate R_0.

2. PRELIMINARIES

In this section we recall some fundamental concepts of fractional differential calculus where the derivative is in the Caputo sense.

Definition 1. *The Caputo fractional derivative of order $\alpha > 0$ for a funcion $f \in \mathscr{C}^n(\mathbb{R}^+;\mathbb{R})$ is defined as*

$$ {}_0^c\mathbf{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha-n+1}} ds. $$

Where n is positive integer such that $\alpha \in (n-1,n)$. Also, the corresponding fractional integral of order α with $Re(\alpha) > 0$ is given by

$$ \mathbf{I}_{[0,t]}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. $$

Where $\Gamma(.)$ is the Gamma function.

Definition 2. *The Mittag-Leffler function of two parameters is given by*

$$ E_{\alpha,\beta}(Z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k+\beta),} Z \in \mathbb{C}. $$

where $\alpha, \beta > 0$, and $\mathbb{C}$ denote the complex plane. Note that, when $\alpha = \beta = 1$, the Mittag-Leffler function $E_{1,1}(Z)$ reduces to the exponential function $exp(Z)$. Also, the Mittag-Leffler functions satisfies the following useful equality:

$$ E_{\alpha,\beta}(Z) = ZE_{\alpha,\alpha+\beta}(Z) + \frac{1}{\Gamma(\beta)}, \ \alpha,\beta > 0. $$

Lemma 1. *For $a \in \mathbb{R}$ and $\alpha, \beta > 0$, we obtain*

$$ \mathscr{L}(t^{\alpha-1} E_{\alpha,\beta}(at)) = \frac{S^{\alpha-\beta}}{S^{\alpha}-a}. $$

Also,

$$ \mathscr{L}({}_0^c\mathbf{D}_t^{\alpha} h(t)) = S^{\alpha-1}\widehat{h}(s) - \sum_{k=0}^{n-1} h^{(k)}(0) S^{\alpha-k-1}. $$

Where, $\widehat{h}(s) = \mathscr{L}(h(t))$.

In the following, we give an elementary lemma proved in Vargas-De-León [4] , which describes the Volterra-type Lyapunov function for the fractional-order epidemic systems.

Lemma 2. *Let* $0 < \alpha < 1$, *and* $\zeta \in [0,T]$ *be positive valued function. Then, for all* $t \in [0,T]$, *one has*

$$ {}_0^c D_t^\alpha \left(\zeta(t) - \zeta^* - \zeta^* \ln \frac{\zeta(t)}{\zeta^*} \right) \leq \left(1 - \frac{\zeta^*}{\zeta(t)} \right)_0^c D_t^\alpha \zeta(t), $$

for all, $\zeta^* \in \mathbb{R}_+$.

Lemma 3. *(*[12].*) Suppose that* $f(t) \in [a,b]$ *and* ${}^cD^\alpha f(t) \in C[a,b]$, *for* $0 < \alpha \leq 1$.
If ${}^cD^\alpha f(t) \geq 0$, $\forall t \in [a,b]$, *then* $f(t)$ *is non-decreasing for each* $t \in [a,b]$. *If* ${}^cD^\alpha f(t) \leq 0$, $\forall t \in (a,b)$, *then* $f(t)$ *is non-decreasing for each* $t \in [a,b]$.

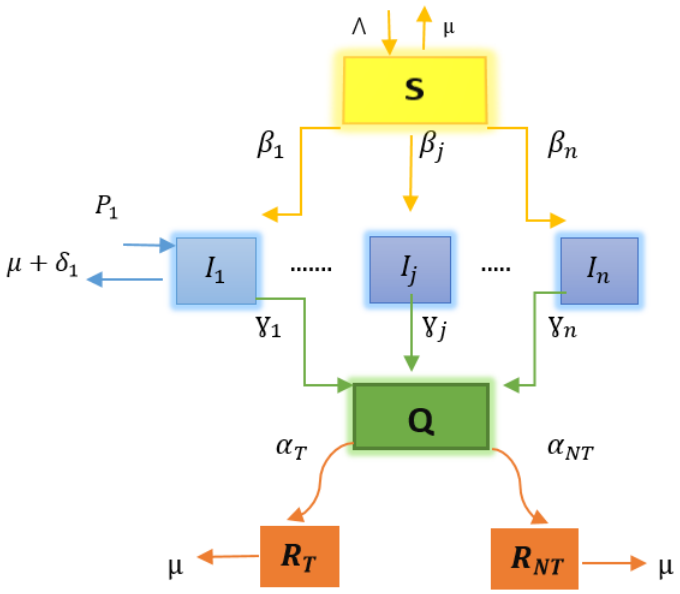
Lemma 4. *(*[12].*) Assume that the vector function* $f : \mathbb{R}^+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *satisfies the following conditions:*

(1) Function $f(t,x(t))$ *is Lebesgue measurable with respect to* $t \in \mathbb{R}^3$.
(2) Function $f(t,x(t))$ *is continuous with respect to* $X(t)$ *on* $\mathbb{R}^4$.
(3) $\frac{df(t,x)}{dx}$ *is continuous with respect to* $X(t)$ *on* $\mathbb{R}^4$.
(4) $||f(t,x)|| \leq \omega + \lambda||x||$, $\forall t \in \mathbb{R}^+$, $X \in \mathbb{R}^4$.

Here ω , λ *are two positive constants.*

3. Diagram Transmission of All Mutation COVID-19 Between Humans

Some of the emergence of Corona disease and the infection of many people in the whole world with it and for a short time, then a mutated appearance of this disease called 'British Variant, omecron, delta,...'. Moreover, the new idea in our article is to divide the people who are exposed to the disease into alot of categories, a category that has previously contracted the omicron anoder mutation like delta.



Description of biological parameters:

- S : The susceptible individuals of covid-19.
- I_i : The individuals infected by covid 19 mutations each strain i for one mutation (omecron delta,...).
- R : The individuals withdrawn (healed or dead).
- β_i The rate of individuals who become infected by the mutation of covid-19.
- γ_i : Infection rate of Q class.
- α_T, α_{NT} : Recovery rate.
- μ : Natural mortality rate.
- Λ_N : Birth rate.

This diagram can translated mathematically by the following system of differential equations:

$$
(1) \quad \begin{cases} {}_0^cD_t^\alpha S = \Lambda^\alpha - \sum_{i=1}^{n} \beta_i^\alpha S I_i + \sum_{i=1}^{n} \psi_i^\alpha I_i - \mu^\alpha S \\ {}_0^cD_t^\alpha I_i = \beta_i^\alpha S I_i - \gamma_i^\alpha I_i - \mu^\alpha I_i - \delta_i^\alpha I_i + P_i^\alpha I_i \\ {}_0^cD_t^\alpha Q = \sum_{i=1}^{n} \gamma_i^\alpha I_i Q - (\alpha_1^\alpha + \alpha_2^\alpha) Q - \mu^\alpha Q \\ {}_0^cD_t^\alpha R_T = \alpha_1^\alpha Q - \mu^\alpha R_T \\ {}_0^cD_t^\alpha R_{NT} = \alpha_2^\alpha Q - \mu^\alpha R_{NT} \end{cases}
$$

The system (1) is provided with the initial conditions:

$S(0) = S_0 > 0,\ I_i(0) = I_{i0} > 0,\ Q(0) = Q_0 > 0,\ R_T(0) = R_{T_0} > 0,\ R_{NT}(0) = R_{NT_0} > 0.$

And,

$$N = S_0 + I_{i0} + Q_0 + R_{T_0} + R_{NT_0}.$$

where ${}_0^cD_t^\alpha$ is the fractional Caputo derivative having order $0 < \alpha \leq 1$ in order to describe the memory effects in the prosed epidemic model. We assume that the functions $S(t), I_i(t), Q(t), R_T(t), R_{NT}(t)$ and their Caputo fractional derivatives of Order $0 < \alpha < 1$ are continuous functions.

4. Global Existence, Positivity and Limitation of the Solution

Proposition 1. *Given* $(S_0, I_{i0}, Q_0, R_{T_0}, R_{NT_0}) \in \mathbb{R}^n$, *there is a unique solution to the problem* (1) *defined on* $[0, +\infty)$ *and this solution rest non négative and bounded* $\forall t \geq 0$.

Proof:

Firstly, we prove that $\forall (S(0), I_i(0), Q(0), R_T(0), R_{NT}(0)) \in \mathbb{R}^n_+$, system (1) has a unique solution. Obviously, vector function f of system (1) satisfies conditions 1 and 3 of Lemma 4. Following, we prove system

$$X(t) = \begin{pmatrix} S(t) \\ I_i(t) \\ Q(t) \\ R_T(t) \\ R_{NT}(t) \end{pmatrix} \quad K = \begin{pmatrix} \Lambda^\alpha \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -\mu^\alpha & \sum_{i=1}^n \psi_i^\alpha & 0 & 0 & 0 \\ 0 & -(\gamma_i^\alpha + \mu^\alpha + \delta_i^\alpha - P_i^\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & -\sum_{i=1}^{n}\beta_i^{\alpha} & 0 & 0 & 0 \\ 0 & \sum_{i=1}^{n}\beta_i^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \sum_{i=1}^{n}\gamma_i^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\alpha_1^{\alpha}+\alpha_2^{\alpha}+\mu^{\alpha}) & 0 \\ 0 & 0 & 0 & -\mu^{\alpha} & 0 \end{pmatrix} A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1^{\alpha} & -\mu^{\alpha} & 0 \\ 0 & 0 & \alpha_2^{\alpha} & 0 & -\mu^{\alpha} \end{pmatrix}.$$

$${}^{c}D^{\alpha}x(t) = A_1x(t) + S(t)A_2x(t) + Q(t)A_3x(t) + A_4x(t) + A_5x(t) + K.$$

$$||f(t,x(t))|| = ||A_1x(t) + S(t)A_2x(t) + Q(t)A_3x(t) + A_4x(t) + A_5x(t) + K||$$

$$\leq ||k|| + ||A_1||||x(t)|| + ||A_2||||x(t)|| + ||A_3||||x(t)|| + ||A_4||||x(t)|| + ||A_5||||x(t)||;$$

$$\leq \omega + (||A_1|| + ||A_2|| + ||A_3|| + ||A_4|| + ||A_5||)||x(t)||;$$

$$= \omega + \lambda||x(t)||.$$

By Lemma 4 system (1) has a unique solution.

Now, we show the positivity of the solution, we have:

$$\frac{dS(t)}{dt}|_{S=0} = \Lambda + \sum_{i=1}^{n}\psi_i I_i \geq 0$$
$$\frac{dI(t)}{dt}|_{I=0} = 0 \geq 0$$
$$\frac{dQ(t)}{dt}|_{Q=0} = 0 \geq 0$$
$$\frac{dR_T(t)}{dt}|_{R_T=0} = \alpha_1 Q \geq 0$$
$$\frac{dR_{NT}(t)}{dt}|_{R=0} = \alpha_2 Q \geq 0$$

and since the initials conditions are positives, then we deduce the positivity of the local solution.

Finally, we establish the boundary of the solution.

Proposition 2. *The Set* $\Omega = \{(S, \sum_{i=1}^{n} I_i, Q, R_T, R_{NT}) \in \mathscr{R}_+^n; 0 < S(t) + \sum_{i=1}^{n} I_i(t) + Q(t) + R_T(t) + R_{NT}(t) \leq \frac{\Lambda^\alpha}{\mu^\alpha}\}$ *is a positively invariant and attraction region for system (1).*

Proof:

We have

$$N(t) = S(t) + \sum_{i=1}^{n} I_i(t) + Q(t) + R_T(t) + R_{NT}(t).$$

Consequently, adding equations yields

$$ {}_0^c D_t^\alpha N(t) = \Lambda^\alpha - \mu^\alpha N(t). \ (*)$$

Taking the Laplace transform in equation (*), we get,

$$S^\alpha \widehat{N}(s) + \mu^\alpha \widehat{N}(s) = \frac{\Lambda^\alpha}{S} + S^{\alpha-1} N(0).$$

Hence,

$$\widehat{N}(s) = \Lambda^\alpha \frac{S^{\alpha-(1+\alpha)}}{S^\alpha + \mu^\alpha} + N(0) \frac{S^{\alpha-1}}{S^\alpha + \mu^\alpha}.$$

According, we have:

$$N(t) = \Lambda^\alpha t^\alpha E_{\alpha,\alpha+1}(-\mu^\alpha t^\alpha) + N(0) E_{\alpha,1}(-\mu^\alpha t^\alpha).$$

$$N(t) = \frac{\Lambda^\alpha}{\mu^\alpha} - \frac{\Lambda^\alpha}{\mu^\alpha} E_{\alpha,1}(-\mu^\alpha t^\alpha) + N(0) E_{\alpha,1}(-\mu^\alpha t^\alpha).$$

Since, $0 \leq E_{\alpha,1}(-\mu^\alpha t^\alpha) \leq 1$ holds and, $N(0) \leq \frac{\Lambda^\alpha}{\mu^\alpha}$, then one obtains $N(t) \leq \frac{\Lambda^\alpha}{\mu^\alpha}$. Thus, Ω is a positively invariant set, and all initial solutions belong to Ω remain in Ω for all $t > 0$. And that is what we wanted to prove.

5. EQUILIBRIUM

5.1 Disease free equilibrium (DFE). We search $\overline{S} \geq 0$, $\overline{I_i} \geq 0$, $\overline{Q} \geq 0$, $\overline{R_T} \geq 0$ et $\overline{R_{NT}} \geq 0$ satisfying:

$$\begin{cases} 0 &= \Lambda^\alpha - \sum_{i=1}^{n} \beta_i^\alpha \overline{SI_i} + \sum_{i=1}^{n} \psi_i^\alpha \overline{I_i} - \mu^\alpha \overline{S} \\ 0 &= \beta_i^\alpha \overline{SI_i} - \gamma_i^\alpha \overline{I_i} - \mu^\alpha \overline{I_i} - \delta_i^\alpha \overline{I_i} + P_i^\alpha \overline{I_i} \\ 0 &= \sum_{i=1}^{n} \gamma_i^\alpha \overline{I_i Q} - (\alpha_1^\alpha + \alpha_2^\alpha) \overline{Q} - \mu^\alpha \overline{Q} \\ 0 &= \alpha_1^\alpha \overline{Q} - \mu^\alpha \overline{R_T} \\ 0 &= \alpha_2^\alpha \overline{Q} - \mu^\alpha \overline{R_{NT}} \end{cases}$$

With, $\overline{I_i} = 0$ We obtain: $\overline{S} = \frac{\Lambda^\alpha}{\mu^\alpha},\ \overline{Q} = 0,\ \overline{R_T} = 0$ et $\overline{R_{NT}} = 0$.

Therefore,

$$E_0 = (\frac{\Lambda^\alpha}{\mu^\alpha}, 0, 0, 0, 0).$$

5.2 Calcul of R_0: (Method of van den Driessche watmough): We denote by:

- $\mathscr{F}_j(S, I_i, Q, R_{NT}, R_T)$ the rate of newly infected in the compartment j.
- $\mathscr{V}_j(S, I_i, Q, R_{NT}, R_T)$ the transfer rate of an individual from one compartment to another everywhere average.

The matrices $\mathscr{F}$ and $\mathscr{V}$ are represented by:

$$\mathscr{F} = \begin{pmatrix} 0 \\ \beta_i^\alpha S I_i + P_i^\alpha I_i \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And,

$$\mathscr{V} = \begin{pmatrix} \Lambda^\alpha - \beta_i^\alpha S I_i + \psi_i^\alpha I_i - \mu^\alpha S \\ -(\gamma_i^\alpha + \mu^\alpha + \delta_i^\alpha) I_i \\ \gamma_i^\alpha I_i Q - (\alpha_1^\alpha + \alpha_2^\alpha) Q - \mu^\alpha Q \\ \alpha_1^\alpha Q - \mu^\alpha R_T \\ \alpha_2^\alpha Q - \mu^\alpha R_{NT} \end{pmatrix}$$

The calculation of their resprctive Jacobian at the disease free equilibrium point $E_0 = (\frac{\Lambda^\alpha}{\mu^\alpha}, 0, 0, 0, 0)$ given:

$$\mathscr{F}(E_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_i^\alpha \Lambda^\alpha}{\mu^\alpha} + P_i^\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathscr{V}(E_0) = \begin{pmatrix} -\mu^\alpha & -\frac{\beta_i^\alpha \Lambda^\alpha}{\mu^\alpha} + \psi_i^\alpha & 0 & 0 & 0 \\ 0 & -(\gamma_i^\alpha + \mu^\alpha + \delta_i^\alpha) I_i & 0 & 0 & 0 \\ 0 & 0 & -(\alpha_1^\alpha + \alpha_2^\alpha - \mu^\alpha) & 0 & 0 \\ 0 & 0 & \alpha_1^\alpha & -\mu^\alpha & 0 \\ 0 & 0 & \alpha_2^\alpha & 0 & -\mu^\alpha \end{pmatrix}$$

The basic reproduction rate is the spectral radius of the matrix $-FV^{-1}$ the calculation given:

$$R_0 = \frac{\beta_i^\alpha \Lambda^\alpha + P_i^\alpha \mu^\alpha}{\mu^\alpha(\alpha_1^\alpha + \alpha_2^\alpha + \mu^\alpha)}.$$

5.3 Endemic equilibrium (EE). The system (1) has n endemic equilibrium.

Be $(S, I_1, ..., Q, R_T, R_{NT}), ..., (S, ..., I_n, Q, R_T, R_{NT})$.

For $\overline{I_i} > 0$. Using the second equation of the system we get:

$$\overline{S} = \frac{\gamma_i^\alpha + \mu^\alpha + \delta_i^\alpha - P_i^\alpha}{\beta_i^\alpha}.$$

By considering the first equation of the system we obtain:

$$\overline{I_i} = \frac{\mu^\alpha(\gamma_i^\alpha + \mu^\alpha + \delta_i^\alpha - P_i^\alpha) - \Lambda^\alpha \beta_i^\alpha}{\beta_i^\alpha(\mu_i^\alpha - \gamma_i^\alpha - \mu^\alpha - \delta_i^\alpha + P_i^\alpha)}.$$

Using the third equation of the system we get:

$$\overline{Q} = 0.$$

According to the fourth equations we obtain:

$$\overline{R_T} = 0.$$

According to the 6 equation of the system we obtain:

$$\overline{R_{NT}} = 0.$$

Therefore, the endemic equilibrium point given by:

$$E = \left(\frac{\gamma_i^\alpha + \mu^\alpha + \delta_i^\alpha - P_i^\alpha}{\beta_i^\alpha}, \frac{\mu^\alpha(\gamma_i^\alpha + \mu^\alpha + \delta_i^\alpha - P_i^\alpha) - \Lambda^\alpha \beta_i^\alpha}{\beta_i^\alpha(-\gamma_i^\alpha - \mu^\alpha - \delta_i^\alpha + P_i^\alpha)}, 0, 0, 0 \right).$$

6. GLOBAL STABILITY ANALYSIS

We know that, the basic reproduction rate is a dimensionless quantity which measures the ability of an infectious agent to spread infection through a given population immediately after without introductions, In addition, from a mathematical point of view it allows under certain conditions to establish the stability either local or global of a point of equilibrium of a dynamic system.

The basic reproduction number for the complete system is the maximum of all the basic reproduction numbers taken individually

$$R_0 = \max_{i=0,\dots,n} R_{0,i}$$

In our epidemiological model we have n strain so it is difficult to control the disease by the basic reproduction rate, and it is not the basic reproduction rate that determines the destion of a strain,this is why we associate with each strain i which models such a mutation of covid 19 a threshold which depends on the personal parameter for strain i, which helps us to properly control the spread of each strain i. Moreover, the strain that maximizes its threshold wins the competition.

Thus, we can speak of a maximization of a single but not of a maximization of basic reproduction rate.

Border balance $(\overline{S}, 0, ..., \overline{I_i}, 0...0, R_T, R_{NT})$ is in Ω if and only if

$$\xi_{0,i} = \frac{\mu^\alpha(\gamma_i^\alpha + \mu^\alpha + \delta_i^\alpha - P_i^\alpha)}{\beta_i^\alpha \Lambda^\alpha}.$$

It is clear that

$$\xi_{0,i} > 1 \Leftrightarrow R_{0,i} > 1.$$

Without loss of generality, let's remember that, $R_0 = \max_{i=0,\dots,n} R_{0,i}$ and $\xi_0 = \max_{i=0,\dots,n} \xi_{0,i}$, moreover, we have seen that, $\xi_0 \Leftrightarrow R_0$.

Since each strain in our system has its own threshold which depends on the rate concerned, it is preferable to use the threshold of each strain ξ_0, to properly study the overall stability of our system.

6.1 GLOBAL STABILITY OF THE DISEASE FREE EQUILIBRIUM:

In this section, we investigate the global stability of the disease free equilibrium DFE and endemic equilibrium EE for system (1) by constructing proper lyapunov functions.

Let us define a function $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, giving by

$$\Phi(\zeta(t)) = \zeta(t) - \zeta^* - \zeta^* \ln \frac{\zeta(t)}{\zeta^*}$$

for all $t > 0$. Note that $\Phi(\zeta)$ non-negative function for any $\zeta > 0$ that allains a global minimum at $\zeta = 1$. Moreover, we define

$$\Omega = \{(S, I_i, Q, R_T, R_{NT}) \in \mathbb{R}^n, S > 0, I_i > 0, Q > 0, R_T > 0, R_{NT} > 0\}$$

Theorem 5. *If $\xi_{0,i} \leq 1$, the DFE is globally asymptotically stable in the positive orthant. if $\xi_{0,i} > 1$, the DFE est instable.*

Proof:

Consider the following Lyapunov function:

$$V(t) = \frac{1}{\mu^\alpha}\Phi(S(t)) + \Phi(I(t))$$

Applying the Caputo fractional derivative on equations of system (1), we obtain,

$$ {}_0^c D_t^\alpha V(t) = \frac{1}{\mu^\alpha} {}_0^c D_t^\alpha \Phi(S(t)) + \frac{\beta_i^\alpha \Lambda^\alpha}{\mu^\alpha}\Phi(I(t))$$

$$\leq \frac{1}{\mu^\alpha}\left(1 - \frac{S^0}{S}\right) {}_0^c D_t^\alpha S(t) + \frac{\beta_i^\alpha \Lambda^\alpha}{\mu^\alpha}\left(1 - \frac{I_i^0}{I}\right) {}_0^c D_t^\alpha I_i(t)$$

$$\leq \frac{-\psi_i}{\mu^\alpha S(t)}(S - S^0)^2 - \frac{\beta_i^\alpha \Lambda^\alpha}{\mu^\alpha}(\xi_{0,i} - 1)I(t)$$

Therefore, $R_0 < 1$ ensures for all $(S(t), I_i(t), Q(t), R_T(t), R_{NT}(t))$ that,

$$ {}_0^c D_t^\alpha V(t) \leq 0,$$

for all $t \geq 0$.

In addition, it is easy to verify that ${}_0^c D_t^\alpha V(t) = 0$ if and only if $S(t) = S^0$, $I(t) = 0$, $Q(t) = 0$, $R_T(t) = 0$, and $R_{NT}(t) = 0$.

6.2 GLOBAL STABILITY OF THE ENDEMIC EQUILIBRIUM (EE):

We assume that $R_0 > 1$ or equivalently $\xi_0 > 1$.

Theorem 6. *If $\xi_{0,i} \geq 1$, the EE is globally asymptotically stable . if $\xi_{0,i} < 1$, the EE is instable.*

Proof:

Consider the following Lyapunov function:

$$V(t) = \frac{\beta_i^{\alpha}}{\mu^{\alpha} + \delta_i^{\alpha} + \gamma_i^{\alpha}} \Phi(S(t)) + \frac{\Lambda^{\alpha}}{\psi_i^{\alpha}} \Phi(I(t))$$

Applying the Caputo fractional derivative on equations of system (1), we obtain,

$$ {}_0^c D_t^{\alpha} V(t) = \frac{\beta_i^{\alpha}}{\mu^{\alpha} + \delta_i^{\alpha} + \gamma_i^{\alpha}} {}_0^c D_t^{\alpha} \Phi(S(t)) + \frac{\Lambda^{\alpha}}{\psi_i^{\alpha}} \Phi(I(t))$$

$$\leq \frac{\beta_i^{\alpha}}{\mu^{\alpha} + \delta_i^{\alpha} + \gamma_i^{\alpha}} \left(1 - \frac{S^0}{S}\right) {}_0^c D_t^{\alpha} S(t) + \frac{\Lambda^{\alpha}}{\psi_i^{\alpha}} \left(1 - \frac{I_i^0}{I}\right) {}_0^c D_t^{\alpha} I_i(t)$$

$$\leq \frac{-\psi_i}{\mu^{\alpha} S(t)} (S - S^0)^2 + \frac{\beta_i^{\alpha} \Lambda^{\alpha}}{\mu^{\alpha}} (\xi_{0,i} - 1) I(t)$$

Therefore, $R_0 < 1$ ensures for all $(S(t), I_i(t), Q(t), R_T(t), R_{NT}(t))$ that,

$$ {}_0^c D_t^{\alpha} V(t) \leq 0,$$

for all $t \geq 0$.

The equality holds only at the endemic equilibrium point EE.Fuethermore, the largest invariant set of $\Omega_2 = \{S(t), \sum_{i=1}^{n} I_i, Q, R_T, R_{NT} : {}_0^c D_t^{\alpha} V(t) = 0\}$ in the singleton $\{E\}$. By LaSalle's invariance principle, E is globally asymptotically stable .

7. NUMERICAL SIMULATIONS

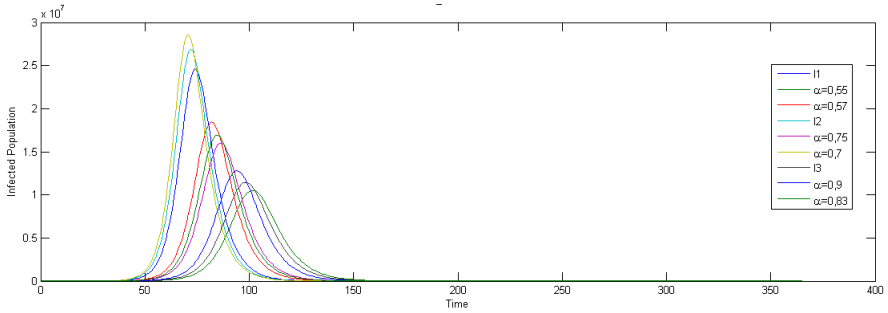
For the numerical simulation of our model, we make the reconstruction only on three strains, the first strain noted I_1 for the Omecron mutation, the second strain for the English Variant mutation noted I_2 and finally the last strain I_3 for the Delta mutation.

To illustrate the theoretical results obtained in the previous sections, we provide a numerical scheme for finding the solution of the fractional-order. First,we have chosen the following rates

for the first strain I_1, $R_0 = 2.2$; $N = 47000000$; $\mu = 0.05$; $\beta_1 = 0.073$; $\beta_2 = 0.076$; $\beta_3 = 0.079$; $\gamma_1 = 0.062$; $\gamma_2 = 0.064$; $\gamma_3 = 0.068$; In this case; According to the theorem the equilibrem points is globally symptotically stable.

Figure 1: The initial condition is set to be $I_1(0) = 0$ for $\alpha = 0,55$ $\alpha = 0,57$, for the second strain I_2 the initial condition is set to be $I_2(0) = 0$ for $\alpha = 0,75$ $\alpha = 0,7$, for the third strain I_3 the initial condition is set to be $I_3(0) = 0$ for $\alpha = 0,83$ $\alpha = 0,9$.

Infected population by the british variant covid-19 converges asymtotically to R.



The infected poeple for different α's

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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