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ASYMPTOTIC COMPORTMENT OF A STOCHASTIC SIQR MODEL WITH MEAN-REVERTING INHOMOGENEOUS GEOMETRIC BROWNIAN MOTION

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Abstract. The object of this work is to analyze the dynamical behavior of an SIQR epidemic model incorporating the mean-reverting inhomogeneous geometric Brownian motion process (IGBM for short). As a first step, we prove that a global-in-time solution exists, and we show equally that it is unique and positive. Then, we find out an appropriate hypothetical framework leading to the existence of an ergodic stationary distribution. After that, we provide certain sufficient conditions for the disease's exponential extinction, and we show that they match those of the deterministic version in this case. Finally, we outline some numerical simulation examples to back up our theoretical outcomes.

Keywords: SIQR epidemic model; inhomogeneous geometric Brownian motion; extinction; stationary distribution.

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1. Introduction and Model Formulation

Over the past twenty years, numerous infectious diseases have forcefully reappeared as a result of the rampant phenomenon of overcrowding and the astounding rise in human migration.

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This remarkable reemergence of these diseases has prompted various academic disciplines to focus on understanding their dissemination dynamics. In this vein, mathematical epidemiology, and more precisely via the so-called compartmental models, has constituted a very imminent tool that may help us to take in a clear manner an overall view of any epidemic's behavior. One of the most famed compartmental models, is the SIR one, which was constructed by Kermack and Mckendrick's in 1927 [2]. In the latter, the population is divided into three classes of individuals: the susceptible ones (S), the infectious ones (I), and the permanently recovered ones (R). The transmission principle associated with this model is relatively easy to grasp: a person, presumed to be alive throughout the transition mechanism, moves from the susceptible compartment to the infectious one after contacting a sick individual and then enters the recovered compartment after gaining permanent immunity against the disease. However, quarantining the diseased individuals is a very efficient intervention strategy to lower the danger of infection. For instance, during the worldwide outbreak of COVID-19, the World Health Organization (WHO) issued many recommendations encouraging the governments and local authoroties to isolate all sick people in order to reduce the virus's spread. So, the analysis of an extended version of the SIR model, and that takes into consideration the effect of isolation or quarantine, becomes a necessary step to explore and learn more about this strategy's effect on the spread of infectious illnesses. For this reason, a lot of scholars and academics have proposed and studied many variants of the so-named SIQR model over the last half-century. Nevertheless, treatment and analysis of the SIQR model did not cease at all, and as of this writing, the foremost later literature is still full of multiple publications that are addressing this issue, but in different contexts and ways. One of the most intriguing recent works in this connection is Zhou and Jiang's paper [3]. In this latter, the authors have introduced the following SIQR model:

$$
\begin{cases}
\frac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \mu S - \frac{\beta SI}{N}, \\
\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta SI}{N} - (\mu + r_I + \nu + \mu_I)I, \\
\frac{\mathrm{d}Q}{\mathrm{d}t} = \nu I - (\mu + r_q + \mu_q)Q, \\
\frac{\mathrm{d}R}{\mathrm{d}t} = r_I I + r_q Q - \mu R.
\end{cases} \tag{1}
$$

As we can see, the overall population size $N := S+I+Q+R$ in the previous model was split into four compartments: the first is (S), and it contains susceptible individuals; the second is (I) and it contains infected ones; the third is (Q) and it contains quarantined ones; and finally (R) which contains recovered ones. Regarding the parameters appearing in this model, we have:

- Λ is the susceptible recruitment rate.
- β stands for the rate of contact between S and I.
- μ denotes the natural mortality rate.
- μ_I and μ_q are the disease-induced death rates associated respectively to the classes I and Q.
- r_I and r_q are the recovery or healing rates associated respectively to the classes I and Q.
- ν is the quarantine rate.

All the above-listed parameters are presumed to be in the positive real line $\mathbb{R}_+ := (0,+\infty)$. On the basis of the mathematical findings depicted in [3], the SIQR model (1) is well-posed, and its propagation comportment is governed by the basic reproductive ratio $\mathscr{R}_0$ given in this case by

$$\mathscr{R}_0 := \frac{\beta}{(\mu+\mu_I+\nu+r_I)}. \tag{2}$$

More explicitly, if $\mathscr{R}_0 < 1$, the model (1) possesses one illness-free equilibrium $E^\circ = \left(\frac{\Lambda}{\mu},0,0,0\right)$, and it is globally asymptotically stable in the invariant set Σ given by

$$\Sigma := \left\{(S,I,Q,R) \in \mathbb{R}_+^4 \mid S+I+Q+R \leqslant \frac{\Lambda}{\mu}\right\}.$$

The disease-free equilibrium E° is present even if $\mathscr{R}_0 > 1$, but it is unstable this time, and another equilibrium $E^\bullet = (S^\bullet, I^\bullet, Q^\bullet, R^\bullet)$, called the endemic equilibrium, appears and becomes globally asymptotically stable in the set Σ. In virtue of the well-famed limit region approach [3], the dimensionality of system (1) can be reduced. More detailedly, for any real constants α_1 and α_2, we have

$$\begin{aligned} &\frac{\mathrm{d}(N+\alpha_1 Q+\alpha_2 R)}{\mathrm{d}t} \\ &= \Lambda - \mu(N+\alpha_1 Q+\alpha_2 R) + \underbrace{(\alpha_1\nu-\mu_I+\alpha_2 r_I)}_{(i)} I + \underbrace{\big(\alpha_2 r_q-\mu_q-\alpha_1(\mu_q+r_q)\big)}_{(ii)} Q. \end{aligned} \tag{3}$$

By selecting α_1 and α_2 in a such way as to cancel the coefficients (i) and (ii) in (3), we get

$$\begin{cases} \alpha_1 = \dfrac{r_q\mu_I - r_I\mu_q}{\nu r_q + r_I(r_q+\mu_q)}, \\ \alpha_2 = \dfrac{\nu\mu_q + \mu_I(r_q+\mu_q)}{\nu r_q + r_I(r_q+\mu_q)}, \\ \mathrm{d}\big(N+\alpha_1 Q+\alpha_2 R\big) = \big(\Lambda - \mu(N+\alpha_1 Q+\alpha_2 R)\big)\mathrm{d}t. \end{cases} \tag{4}$$

From the third equation of (4), one can derive that for any $t \geqslant 0$

$$N(t)+\alpha_1 Q(t)+\alpha_2 R(t) = \frac{\Lambda}{\mu} + \left(N(0)+\alpha_1 Q(0)+\alpha_2 R(0) - \frac{\Lambda}{\mu}\right)\times \exp(-\mu t).$$

So

$$\begin{aligned} &\lim_{t\to\infty}\big(N(t)+\alpha_1 Q(t)+\alpha_2 R(t)\big) \\ &= \lim_{t\to\infty}\big(I(t)+S(t)+(\alpha_1+1)Q(t)+(\alpha_2+1)R(t)\big) \\ &= \lim_{t\to\infty}\left(\frac{\Lambda}{\mu} + \left(N(0)+\alpha_1 Q(0)+\alpha_2 R(0) - \frac{\Lambda}{\mu}\right)\times \exp(-\mu t)\right) = \frac{\Lambda}{\mu}. \end{aligned}$$

Hence, the limit region associated with the SIQR system (1) is

$$\mathscr{S}_1 = \left\{(S,I,Q,R)\in\mathbb{R}_+^4 \mid S+I+(\alpha_1+1)Q+(\alpha_2+1)R = \frac{\Lambda}{\mu}\right\}.$$

Consequently, one may use the following system to examine the dynamic features of model (4):

$$\begin{cases} \frac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{\beta I\times\left(\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R\right)}{\frac{\Lambda}{\mu} - \alpha_1 Q - \alpha_2 R} - (\mu + r_I + \nu + \mu_I)I, \\ \frac{\mathrm{d}Q}{\mathrm{d}t} = \nu\times I - (\mu + r_q + \mu_q)\times Q, \\ \frac{\mathrm{d}R}{\mathrm{d}t} = r_I\times I + r_q\times Q - \mu R. \end{cases} \tag{5}$$

In the most recent works, it has been emphasized that many principal parameters in the epidemic models oscillate around some average value due to the continuous effect of environmental noises. Thus, if this effect is taken into account while modeling contagious illnesses, the understanding of their spread behavior will be better. As a result, the decision-makers will have

the ability to reduce the severity of these diseases' dissemination by making sound choices regarding the control measures. Along this line, a large number of academics and scholars have taken an interest in investigating the stochastic epidemic models [15, 16, 17], and specially, those where the disease transmission parameter β is randomized by the Ornstein-Uhlenbeck process [3, 4, 11, 12, 13, 14]. The inclusion of the aleatory effects by means of the Orstein-Uhlembeck process is defended by the boundedness of its variance for any short time period $[0,t]$, which seems to be more rational and consistent with the stochastic noise's constant disturbance property (see [3] for more details). Usually, the Ornstein-Uhlenbeck process is defined and expressed via the stochastic differential equation below:

$$\mathrm{d}\beta(t) = \theta \times \left(\tilde{\beta} - \beta(t)\right)\mathrm{d}t + \sigma \mathrm{d}B(t). \tag{6}$$

Here, $\tilde{\beta} > 0$ and $\theta > 0$ designate respectively the long-term mean of the process, and the reversion speed to it, while $\sigma > 0$ denotes the instantaneous volatility of the random fluctuations that are modeled by a standard Brownian motion process $\big(B(t)\big)_{t\geq 0}$. The latter, and all the random variables that will be encountered in this article, are presumed to be defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $(F_t)_{t\geqslant 0}$ satisfying the usual conditions (it is increasing, right continuous and F_0 contains all the $\mathbb{P}-$null sets). Indeed, the transmission rate β must be positive. Therefore when randomizing β, this property should be kept in mind by adopting a positive process. Unfortunately, it is not at all the case for the Ornstein-Uhlebeck one. For this reason, we will provide an alternate perturbation approach resembling that of the Ornstein-Uhlebeck process, but taking into account the β's positivity. More explicitly, we will think about perturbing the parameter β using another process named mean-reverting inhomogeneous geometric brownian motion (IGBM for short) [8, 9, 10]. The latter, and which takes also the name of Brennan-schowartz process, is usually defined through the following stochastic differential equation:

$$\mathrm{d}\beta(t) = \theta \times \left(\tilde{\beta} - \beta(t)\right)\mathrm{d}t + \sigma\beta(t)\mathrm{d}B(t). \tag{7}$$

As mentioned in [5, 6, 7], the stochastic process $(\beta(t))_{t\geqslant 0}$ admits an ergodic stationary distribution π which is distributed following the inverse-gamma density with shape $\frac{\sigma^2+2\theta}{\sigma^2}$ and scale

$\frac{2\theta\tilde{\beta}}{\sigma^2}$. So, and for any $\pi-$integrable function f we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t f(\beta(s))\mathrm{d}s=\int_0^{+\infty}f(x)\pi(\mathrm{d}x).$$

In the light of what precedes, the system (5) can be rewriting as follow:

$$\begin{cases}\mathrm{d}\beta(t)=\theta\times(\tilde{\beta}-\beta(t))\mathrm{d}t+\sigma\beta(t)\mathrm{d}B(t),\\ \frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\beta I\left(\frac{\Lambda}{\mu}-I-(\alpha_1+1)Q-(\alpha_2+1)R\right)}{\frac{\Lambda}{\mu}-\alpha_1Q-\alpha_2R}-(\mu+r_I+\nu+\mu_I)I,\\ \frac{\mathrm{d}Q}{\mathrm{d}t}=\nu I-(\mu+r_q+\mu_q)Q,\\ \frac{\mathrm{d}R}{\mathrm{d}t}=r_II+r_qQ-\mu R,\end{cases}\tag{8}$$

and the latter is well defined on the domain:

$$\mathscr{S}_2=\left\{(\beta,I,Q,R)\in\mathbb{R}_+^4\mid I+(\alpha_1+1)Q+(\alpha_2+1)R\leqslant\frac{\Lambda}{\mu}\right\}.$$

The next sections of this article are organized as follows: in Section 2, we establish the well-posedness of the model (8) in the sens that it has one and only one solution which is global-in-time and positive. In sections 3 and 4, we offer respectively some suitable conditions for the existence of a stationary distribution and for the exponential extinction. Then, we illustrate in Section 5 our theoretical findings by some simulations and numerical examples before arriving at the paper's key conclusions in Section 6.

2. Existence and Uniqueness of the Global Solution

Theorem 2.1. *For any initial state* $\big(\beta(0),I(0),Q(0),R(0)\big)\in\mathscr{S}_2$*, it corresponds a unique solution* $\big(\beta(t),I(t),Q(t),R(t)\big)$ *to the system* (8) *on* $t\in[0,+\infty)$*. Moreover, the latter will still in* $\mathscr{S}_2$ *for any* $t\geqslant 0$*.*

Proof. Let us consider the C^2-real valued function $\varphi:\mathbb{R}_+^4\to\mathbb{R}_+$ defined by

$$\begin{aligned}&\varphi(\beta,I,Q,R)\\&=[I-1-\ln I]+(\alpha_1+1)\times[Q-1-\ln Q]+(\alpha_2+1)\times[R-1-\ln R]+\theta^{-1}[\beta-1-\ln\beta]\\&\quad+\left[\left(\frac{\Lambda}{\mu}-I-(\alpha_1+1)Q-(\alpha_2+1)R\right)-1-\ln\left(\frac{\Lambda}{\mu}-I-(\alpha_1+1)Q-(\alpha_2+1)R\right)\right].\end{aligned}$$

By using the renowned Itô's formula, one obtains that

$$\begin{aligned}\mathscr{L}\varphi = \beta \times \frac{\left(\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R\right)}{\frac{\Lambda}{\mu} - \alpha_1 Q - \alpha_2 R} + (r_I + \nu + \mu + \mu_I) \\ - (\alpha_1+1) \times \frac{\nu I}{Q} + (\alpha_1+1)(r_q + \mu + \mu_q) - (\alpha_2+1) \times \frac{r_I I}{R} - (\alpha_2+1) \times \frac{r_q Q}{R} \\ + (\alpha_2+1)\mu + \frac{\beta I}{\frac{\Lambda}{\mu} - \alpha_1 Q - \alpha_2 R} - \mu \frac{I + (\alpha_1+1)Q + (\alpha_2+1)R}{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R} \\ + (\tilde{\beta} - \beta) - \frac{\tilde{\beta}}{\beta} + 1 + \frac{\sigma^2}{2\theta} + \underbrace{\big(-(r_I + \nu + \mu_I) + (\alpha_1+1)\nu + (\alpha_2+1)r_I\big)}_{=0 \text{ (see the first equation of (4))}} I \\ + \underbrace{\big(-(\alpha_1+1)(r_q+\mu_q) + (\alpha_2+1)r_q\big)}_{=0 \text{ (see the second equation of (4))}} Q.\end{aligned}$$

Therefore, we get

$$\mathscr{L}\varphi \leqslant \underbrace{(r_I + \nu + \mu + \mu_q) + (\alpha_1+1)(r_q+\mu+\mu_q) + (\alpha_2+1)\mu + \tilde{\beta} + \frac{\sigma^2}{2\theta} + 1}_{:=A},$$

where A is a positive constant that is not depending on the initial values $\beta(0), I(0), Q(0)$ and $R(0)$. The rest of the demonstration follows the same lines as the proof of Theorem 3.1 in [18], so we skip it here for the sake of brevity. □

3. Existence of a Stationary Distribution

When dealing with infectious illnesses models, we are principally interested in two situations: the first is when the disease dominates and persists, and the second is when it vanishes and disappears. In this section, we aim to offer some sufficient conditions that guarantee the existence and ergodicity of a stationary distribution to system. We mention here, that this last property permits to deduce the disease persistence in the long run (see Remark 2). Until further notice, we designates from now on by $\mathscr{R}_0^{\blacklozenge}$ the following quantity:

$$\mathscr{R}_0^{\blacklozenge} = \tilde{\beta} \times \left(\mu + r_I + \nu + \mu_I + \frac{1}{2}\int_0^{+\infty} |\tilde{\beta} - x| \pi(x) \mathrm{d}x\right)^{-1}.$$

Theorem 3.1. *Let $(\beta(0), I(0), Q(0), R(0)) \in \mathscr{S}_2$ be a given departure point. If the quantity $\mathscr{R}_0^{\blacklozenge}$ outnumbers one (i.e. $\mathscr{R}_0^{\blacklozenge} > 1$), then the system* (8) *will admit at least one ergodic stationary distribution ζ on $\mathscr{S}_2$.*

Proof. For the reader's convenience, we start our proof by defining the following C^2-real valued functions:

- $V_1 = -\ln(I) + \overbrace{\left(\frac{\tilde{\beta}\left[\mu(\alpha_1+1)+r_q(\alpha_2+1)\right]}{\Lambda(r_q+\mu+\mu_q)}\right)}^{:=e} \times Q + \overbrace{\left(\frac{\tilde{\beta}(\alpha_2+1)}{\Lambda}\right)}^{:=f} \times R,$
- $V_2 = -\ln(Q) - \ln(R) - \ln\left(\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R\right),$
- $V_3 = \beta - \ln\beta,$
- $V = C_1V_1 + V_2 + C_2V_3,$

where C_1 and C_2 are two positive constants to be chosen suitably later. According to the well known Itô's formula, applied to the function V_1, we obtain

$$\begin{aligned}
\mathscr{L}V_1 &= -\beta \times \frac{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R}{\frac{\Lambda}{\mu} - \alpha_1 Q - \alpha_2 R} + (r_I + \nu + \mu + \mu_I) \\
&\quad + e(\nu I - (r_q+\mu+\mu_q)Q) + f \times (r_I I + r_q Q - \mu R) \\
&\leqslant -\tilde{\beta}\frac{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R}{\frac{\Lambda}{\mu}} + (\tilde{\beta}-\beta)^+ + (r_I + \nu + \mu + \mu_I) \\
&\quad + e(\nu I - (r_q+\mu+\mu_q)Q) + f \times (r_I I + r_q Q - \mu R) \qquad (9)
\end{aligned}$$

So

$$\begin{aligned}
\mathscr{L}V_1 &\leqslant -\tilde{\beta} + (\tilde{\beta}-\beta)^+ + (r_I + \nu + \mu + \mu_I) + \left(\frac{\tilde{\beta}\mu}{\Lambda} + e\nu + f \times r_I\right) I \\
&\quad + \underbrace{\left(\frac{\tilde{\beta}\mu}{\Lambda}(\alpha_1+1) + f r_q - e(r_q+\mu+\mu_q)\right)}_{=0} Q + \underbrace{\left(\frac{\tilde{\beta}\mu}{\Lambda}(\alpha_2+1) - \mu f\right)}_{=0} R \\
&\quad . \leqslant -(\mathscr{R}_0^{\blacklozenge} - 1)\left[(r_I + \nu + \mu + \mu_I) + \frac{1}{2}\int_0^{+\infty} |\tilde{\beta} - x|\pi(x)\mathrm{d}x\right] \\
&\quad + \left(\frac{\tilde{\beta}\mu}{\Lambda} + e\nu + f \times r_I\right) I + (\tilde{\beta}-\beta)^+ - \frac{1}{2}\int_0^{+\infty} |\tilde{\beta} - x|\pi(x)\mathrm{d}x. \qquad (10)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathscr{L}V_2 &= r_q + 2\mu + \mu_q - \nu\frac{I}{Q} - r_I\frac{I}{R} - r_q\frac{Q}{R} + \frac{1}{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R} \\ &\quad \times \left[\frac{\beta I\left(\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R\right)}{\frac{\Lambda}{\mu} - \alpha_1 Q - \alpha_2 R} + \Bigg(\underbrace{(\alpha_1+1)\nu + (\alpha_2+1)r_I - (r_I+\nu+\mu_I)}_{=0 \text{ (see (4))}} - \mu\Bigg) I\right. \\ &\quad \left. + \Bigg(\underbrace{(\alpha_2+1)r_q - (\alpha_1+1)(r_q+\mu_q)}_{=0 \text{ (see (4))}} - \mu(\alpha_1+1)\Bigg) Q - \mu(\alpha_2+1)R\right] \\ &= r_q + 2\mu + \mu_q - \nu \times \frac{I}{Q} - r_I \times \frac{I}{R} - r_q \times \frac{Q}{R} + \frac{\beta I}{\frac{\Lambda}{\mu} - \alpha_1 Q - \alpha_2 R} + \frac{-\mu I - \mu(\alpha_1+1)Q - \mu(\alpha_2+1)R}{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R}\end{aligned}$$

(11)

$$\leqslant -\mu \times \frac{I + (\alpha_1+1)Q + (\alpha_2+1)R}{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R} - \nu \times \frac{I}{Q} - r_I \times \frac{I}{R} + (r_q + 2\mu + \mu_q) + \tilde{\beta} + (\beta - \tilde{\beta}).$$

Combining (10) and (11) with the fact that $\mathscr{L}V_3 = \theta(\tilde{\beta} - \beta) - \theta\frac{\tilde{\beta}}{\beta} + \theta + \frac{\sigma^2}{2}$ leads us to

$$\begin{aligned}\mathscr{L}V &\leqslant -C_1(\mathscr{R}_0^{\blacklozenge} - 1)\left[(r_I + \nu + \mu + \mu_I) + \frac{1}{2}\int_0^{+\infty} |\tilde{\beta} - x|\pi(x)\mathrm{d}x\right] - \nu \times \frac{I}{Q} - r_I \times \frac{I}{R} \\ &\quad + C_1\left(\mu\frac{\tilde{\beta}}{\Lambda} + e\mu + f \times r_I\right) I - \mu \times \frac{I + (\alpha_1+1)Q + (\alpha_2+1)R}{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R} + (\beta - \tilde{\beta})(1 - C_2\theta) \\ &\quad + C_2\left(\theta + \frac{\sigma^2}{2} - \theta\frac{\tilde{\beta}}{\beta}\right) + (r_q + 2\mu + \mu_q) + C_1\left[(\tilde{\beta} - \beta)^+ - \frac{1}{2}\int_0^{+\infty} |\tilde{\beta} - x|\pi(x)\mathrm{d}x\right] + \tilde{\beta}.\end{aligned}$$

So

$$\begin{aligned}\mathscr{L}V \leqslant &\overbrace{\left(-\lambda C_1 + C + C_1\left(\mu\frac{\tilde{\beta}}{\Lambda} + e\mu + f \times r_I\right) I - \frac{\nu I}{Q} - \frac{r_I I}{R} + (1 - C_2\theta)\beta - \theta C_2\frac{\tilde{\beta}}{\beta} - \frac{\mu I}{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R}\right)}^{:= \Upsilon(\beta, I, Q, R)} \\ &+ C_1\left[(\tilde{\beta} - \beta)^+ - \frac{1}{2}\int_0^{+\infty} |\tilde{\beta} - x|\pi(x)\mathrm{d}x\right], \qquad (12)\end{aligned}$$

where

$$\begin{cases}\lambda = (\mathscr{R}_0^{\blacklozenge} - 1)\left[(r_I + \nu + \mu + \mu_I) + \frac{1}{2}\int_0^{+\infty} |\tilde{\beta} - x|\pi(x)dx\right], \\ C = \tilde{\beta}C_2\theta + C_2(\theta + \frac{\sigma^2}{2}) + (r_q + 2\mu + \mu_q).\end{cases}$$

Let ε be positive number, and denote by $\mathscr{K}$ the following set:

$$\mathscr{K} = \left\{(\beta, I, Q, R) \in \mathscr{S}_2 \mid I \geq \varepsilon, Q \geq \varepsilon^2, R \geq \varepsilon^2, \frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R \geq \varepsilon^2, \varepsilon \leqslant \beta \leqslant \frac{1}{\varepsilon}\right\}.$$

Needless to say, $\mathscr{S}_2 \setminus \mathscr{K} = \bigcup_{i=1}^{6} \mathscr{K}_i$ where

- $\mathscr{K}_1 = \{(\beta, I, Q, R) \in \mathscr{S}_2 \mid I \in (0, \varepsilon)\}$.
- $\mathscr{K}_2 = \left\{(\beta, I, Q, R) \in \mathscr{S}_2 \mid Q \in (0, \varepsilon^2), I \in [\varepsilon, +\infty)\right\}$.
- $\mathscr{K}_3 = \left\{(\beta, I, Q, R) \in \mathscr{S}_2 \mid R \in (0, \varepsilon^2), I \in [\varepsilon, +\infty)\right\}$.
- $\mathscr{K}_4 = \left\{(\beta, I, Q, R) \in \mathscr{S}_2 \mid \left(\frac{\Lambda}{\mu} - I - (\alpha_1 + 1)Q - (\alpha_2 + 1)R\right) \in (0, \varepsilon^2), I \in [\varepsilon, +\infty)\right\}$.
- $\mathscr{K}_5 = \{(\beta, I, Q, R) \in \mathscr{S}_2 \mid \beta \in (0, \varepsilon)\}$.
- $\mathscr{K}_6 = \left\{(\beta, I, Q, R) \in \mathscr{S}_2 \mid \beta \in \left(\frac{1}{\varepsilon}, +\infty\right)\right\}$.

By denoting $A = \left(\frac{C_1 \Lambda}{\mu} \times (\mu \frac{\tilde{\beta}}{\Lambda} + e\mu + f \times r_I) - 2\right)$ and choosing $\varepsilon = \frac{1}{C_1^2}$ with $C_2 = C_1^4$ and C_1 is large enough to make the following inequalities true:

$$\begin{cases} -\lambda C_1 \leqslant -2, \\ -2 + C_1(\mu \dfrac{\tilde{\beta}}{\Lambda} + e\mu + f \times r_I)\varepsilon \leqslant -1, \\ -\dfrac{\min(\nu, r_I, \theta C_2 \tilde{\beta}, \mu)}{\varepsilon} + A \leqslant -1, \\ (1 - C_2\theta)\varepsilon + A \leqslant -1, \end{cases}$$

we find that for any x of $\mathscr{S}_2 \setminus \mathscr{K} = \bigcup_{i=1}^{6} \mathscr{K}_i$, we will perforce have one of the following cases:

<u>1^{st} **case**</u> : Whenever $(\beta, I, Q, R) \in \mathscr{K}_1$, we have

$$\Upsilon(\beta, I, Q, R) \leqslant -2 + C_1 \left(\mu \frac{\tilde{\beta}}{\Lambda} + e\mu + f \times r_I\right) \varepsilon \leqslant -1.$$

<u>2^{nd} **case**</u> : Whenever $(\beta, I, Q, R) \in \mathscr{K}_2$, we have

$$\begin{aligned} \Upsilon(\beta, I, Q, R) &\leqslant -2 + C_1 \left(\mu \frac{\tilde{\beta}}{\Lambda} + e\mu + f \times r_I\right) \frac{\Lambda}{\mu} - \frac{\nu}{\varepsilon} - 2 \\ &\leqslant -\frac{\min(\nu, r_I, \theta C_2 \tilde{\beta}, \mu)}{\varepsilon} + A \leqslant -1. \end{aligned}$$

<u>3^{rd} **case**</u> : Whenever $(\beta, I, Q, R) \in \mathscr{K}_3$, we have

$$\begin{aligned} \Upsilon(\beta, I, Q, R) &\leqslant -2 + C_1 \left(\mu \frac{\tilde{\beta}}{\Lambda} + e\mu + f \times r_I\right) \frac{\Lambda}{\mu} - \frac{r_I}{\varepsilon} - 2 \\ &\leqslant -\frac{\min(\nu, r_I, \theta C_2 \tilde{\beta}, \mu)}{\varepsilon} + A \leqslant -1. \end{aligned}$$

$\underline{4^{th}\ \textbf{case}}$: Whenever $(\beta,I,Q,R)\in\mathscr{K}_4$, we have

$$\begin{aligned}\Upsilon(\beta,I,Q,R)&\leqslant -2+C_1\left(\mu\frac{\tilde{\beta}}{\Lambda}+e\mu+f\times r_I\right)\frac{\Lambda}{\mu}-\frac{\mu}{\varepsilon}-2\\&\leqslant -\frac{\min(\nu,r_I,\theta C_2\tilde{\beta},\mu)}{\varepsilon}+A\leqslant -1.\end{aligned}$$

$\underline{5^{th}\ \textbf{case}}$: Whenever $(\beta,I,Q,R)\in\mathscr{K}_5$, we have

$$\begin{aligned}\Upsilon(\beta,I,Q,R)&\leqslant -2+C_1\left(\mu\frac{\tilde{\beta}}{\Lambda}+e\mu+f\times r_I\right)\frac{\Lambda}{\mu}-\frac{C_2\theta\tilde{\beta}}{\varepsilon}-2\\&\leqslant -\frac{\min(\nu,r_I,\theta C_2\tilde{\beta},\mu)}{\varepsilon}+A\leqslant -1.\end{aligned}$$

$\underline{6^{th}\ \textbf{case}}$: Whenever $(\beta,I,Q,R)\in\mathscr{K}_6$, we have

$$\begin{aligned}\Upsilon(\beta,I,Q,R)&\leqslant -2+C_1\left(\mu\frac{\tilde{\beta}}{\Lambda}+e\mu+f\times r_I\right)\frac{\Lambda}{\mu}+(1-C_2\theta)\varepsilon\\&\leqslant (1-C_2\theta)\varepsilon+A\leqslant -1.\end{aligned}$$

Summarizing the six cases depicted above, one can deduce that $\Upsilon(\beta,I,Q,R)\leqslant -1$ for all $(\beta,I,Q,R)\in\mathscr{S}_2\setminus\mathscr{K}$. In addition, Υ is a continuous function on the compact set $\mathscr{K}$, so it is bounded on this set and

$k_0=\sup\limits_{(\beta,I,Q,R)\in\mathscr{K}}\Upsilon(\beta,I,Q,R)<+\infty$. Therefore, and for all $(\beta,I,Q,R)\in\mathscr{S}_2$, we have

$$\Upsilon(\beta,I,Q,R)\leqslant \underbrace{k:=\left(\max\left\{k_0,-1\right\}+1\right)}_{\in\mathbb{R}_+}<+\infty. \tag{13}$$

Since $\Upsilon(\beta,I,Q,R)$ goes to $+\infty$ as $||(\beta,I,Q,R)||$ tends to $+\infty$ or approaches the frontier of $\mathscr{S}_2$, we can guarantee that there exists an interior point (β^0,I^0,Q^0,R^0) of $\mathscr{S}_2$ such that $\Upsilon(\beta^0,I^0,Q^0,R^0)=\inf\limits_{(\beta,I,Q,R)\in\mathscr{K}}\Upsilon(\beta,I,Q,R)$. So, one can introduce the non-negative C_2-function $V_f(\beta,I,Q,R)$ constructed as:

$$V_f(\beta,I,Q,R)=\Upsilon(\beta,I,Q,R)-\Upsilon(\beta^0,I^0,Q^0,R^0).$$

According to (12), we get

$$\mathscr{L}V_f\leqslant\Upsilon(\beta,I,Q,R)+C_1\left[(\tilde{\beta}-\beta)^+-\frac{1}{2}\int_0^{+\infty}|\tilde{\beta}-x|\pi(x)\mathrm{d}x\right].$$

By taking the expectation, then integrating from 0 to t and dividing by t on both sides of the last inequality, we get

$$\begin{aligned}
0 &\leqslant \frac{\mathbb{E}\big(V_f(\beta(t),I(t),Q(t),R(t)\big)}{t} \\
&= \frac{\mathbb{E}\big(V_f(\beta(0),I(0),Q(0),R(0)\big)}{t} + \frac{1}{t}\int_0^t \mathbb{E}\big(V_f(\beta(s),I(s),Q(s),R(s)\big)\mathrm{d}s \\
&\leqslant \frac{\mathbb{E}\big(V_f(\beta(0),I(0),Q(0),R(0)))}{t} + \frac{1}{t}\int_0^t \mathbb{E}(\Upsilon(\beta(s),I(s),Q(s),R(s)\big)\mathrm{d}s \\
&\quad + C_1\mathbb{E}\left[\frac{1}{t}\int_0^t(\tilde{\beta}-\beta(s))^+\mathrm{d}s\right] - \frac{C_1}{2}\int_0^{+\infty}|\tilde{\beta}-x|\pi(x)\mathrm{d}x. \qquad (14)
\end{aligned}$$

By introducing the inferior limit on both sides of (14), and using the fact that

$$\lim_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t(\tilde{\beta}-\beta(s))^+\mathrm{d}s\right] - \frac{1}{2}\int_0^{+\infty}|\tilde{\beta}-x|\pi(x)\mathrm{d}x = 0 \text{ a.s.} \qquad (15)$$

we get

$$0 \leqslant \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}(\Upsilon(\beta(s),I(s),Q(s),R(s)))\mathrm{d}s \text{ a.s.}$$

Hence

$$\begin{aligned}
&\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}(\Upsilon(\beta(\tau),I(\tau),Q(\tau),R(\tau)))\mathrm{d}\tau \\
&= \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}(\Upsilon(\beta(\tau),I(\tau),Q(\tau),R(\tau)))\mathbb{I}_{\mathscr{K}^c}\mathrm{d}\tau \\
&\quad + \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}(\Upsilon(\beta(\tau),I(\tau),Q(\tau),R(\tau)))\mathbb{I}_{\mathscr{K}}\mathrm{d}\tau \\
&\leqslant k\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{I}_{\mathscr{K}}\mathrm{d}\tau - \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{I}_{\mathscr{K}^c}\mathrm{d}\tau \\
&\leqslant -1 + (k+1)\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{I}_{\mathscr{K}}\mathrm{d}\tau.
\end{aligned}$$

Which implies that

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{I}_{\mathscr{K}}d\tau \geqslant \frac{1}{k+1} > 0 \quad \text{a.s.} \qquad (16)$$

Using Fatou's lemma and the definition of probability event, we get

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{P}(\Upsilon(\beta(\tau),I(\tau),Q(\tau),R(\tau)),\mathscr{K})\mathrm{d}\tau \geqslant \frac{1}{k+1} \quad \text{a.s.} \qquad (17)$$

where $\mathbb{P}(\beta(\tau),I(\tau),Q(\tau),R(\tau),\mathscr{S}_2)$ stands for the transition probability of $(\beta(\tau),I(\tau),Q(\tau),R(\tau))$ into the set $\mathscr{S}_2$. Thus, we conclude that system (8) admits at least one stationary distribution π on $\mathscr{S}_2$, and it has the Feller and ergodic property. □

Remark 3.1. *As stated by the aforementioned theorem, once $\mathscr{R}_0^{\blacklozenge} > 1$, the system* (8) *will have a stationary distribution, and this implies that the illness will persist over the long run.*

4. EXPONENTIAL EXTINCTION

After having demonstrated the well-posedness of our system and found the conditions for the existence of a stationary distribution, our objective in this section is to find the conditions leading to the exponential extinction of system (8).

Theorem 4.1. *Let $\big(\beta(t),I(t),Q(t),R(t)\big)$ be the solution of system* (8) *that begins from a certain initial data $\big(\beta(0),I(0),Q(0),R(0)\big) \in \mathscr{S}_2$. Then,*

$$\limsup_{t\to+\infty} \frac{\ln I(t)}{t} \leqslant (\mathscr{R}_0 - 1)(r_I + \nu + \mu + \mu_I) \text{ a.s.} \tag{18}$$

Particularly, when $\mathscr{R}_0 < 1$, the disease will vanish exponentially almost surely.

Proof. On the basis of Itô's formula we have

$$\begin{aligned}\mathscr{L}(\ln I(t)) &= \beta \frac{\frac{\Lambda}{\mu} - I - (\alpha_1+1)Q - (\alpha_2+1)R}{\frac{\Lambda}{\mu} - \alpha_1 Q - \alpha_2 R} - (r_I + \nu + \mu + \mu_I) \\ &\leqslant \beta - (r_I + \nu + \mu + \mu_I).\end{aligned}$$

Integrating on $[0,t]$ and dividing by t both sides of the last inequality, leads to

$$\frac{\ln(t) - \ln(0)}{t} \leqslant \frac{1}{t}\int_0^t \beta(s)ds - (r_I + \nu + \mu + \mu_I).$$

By letting t tends to $+\infty$, we find that

$$\begin{aligned}\limsup_{t\to+\infty} \frac{\ln(t)}{t} &\leqslant \tilde{\beta} - (r_I + \nu + \mu + \mu_I) \\ &\leqslant (\mathscr{R}_0 - 1)(r_I + \nu + \mu + \mu_I),\end{aligned}$$

which is precisely what the theorem asserts. Furthermore, it is evident that once $\mathscr{R}_0 < 1$, the sickness would vanish exponentially. Consequently, the proof is finished. □

Remark 4.1. *As we can clearly observe in this case and unlike usual, the extinction threshold of the stochastic model corresponds exactly to that of the deterministic one.*

5. Model Simulation and Results

Our aim in this section is to provide some numerical simulation examples in order to back up our theoretical results. By adopting the initial datum $(\beta(0), I(0), Q(0), R(0)) = (0.9, 0.9, 0.9, 0.9) \in \mathscr{S}_2$ and the parameters' values as presented respectively in the second and the third column of the table bellow (for the parameters description we refer the reader to the page 2)

Parameters	persistence case	Extinction case
Λ	0.6	0.6
$\tilde{\beta}$	0.85	0.78
r_I	0.2	0.4
r_q	0.2	0.2
μ	0.1	0.1
μ_I	0.3	0.1
μ_q	0.1	0.1
ν	0.2	0.2
σ	0.1	0.7
θ	0.7	0.6

TABLE 1. list of parameter values to simulate system 8.

we find that:

- In the first case when we take the numerical values as depicted in the second column of table 5, the condition $\mathscr{R}_0^{\blacklozenge} > 1$ holds ($\mathscr{R}_0^{\blacklozenge} = 1.0257 > 1$). So, and by the virtue of Theorem 3.1 and remark 3.1 the illness will persist almost surely, and this goes well with the curves presented in Figure 1.
- In the second case when we pick the numerical values as listed in the third column of table 5, the condition $\mathscr{R}_0 < 1$ becomes true ($\mathscr{R}_0 = 0.7895 < 1$). So, and by the virtue

of Theorem 4.1, the disease vanishes exponentially almost surely, which is precisely depicted in Figure 2.

Remark 5.1. *The integral* $\int_0^{+\infty} |\tilde{\beta} - x|\pi(x)\mathrm{d}x$ *appearing in the expression of* $\mathscr{R}_0^{\blacklozenge}$*, in the persistence case, is approximated using the well known Monte-Carlo Method. More precisely, we have generated 100000 values of X distributed following the inverse gamma law of shape* $1+\frac{2\theta}{\sigma^2}$ *and scale* $\frac{2\theta\tilde{\beta}}{\sigma^2}$*, and then we approximate the said integral by the expectation of* $|\tilde{\beta} - X|$*.*

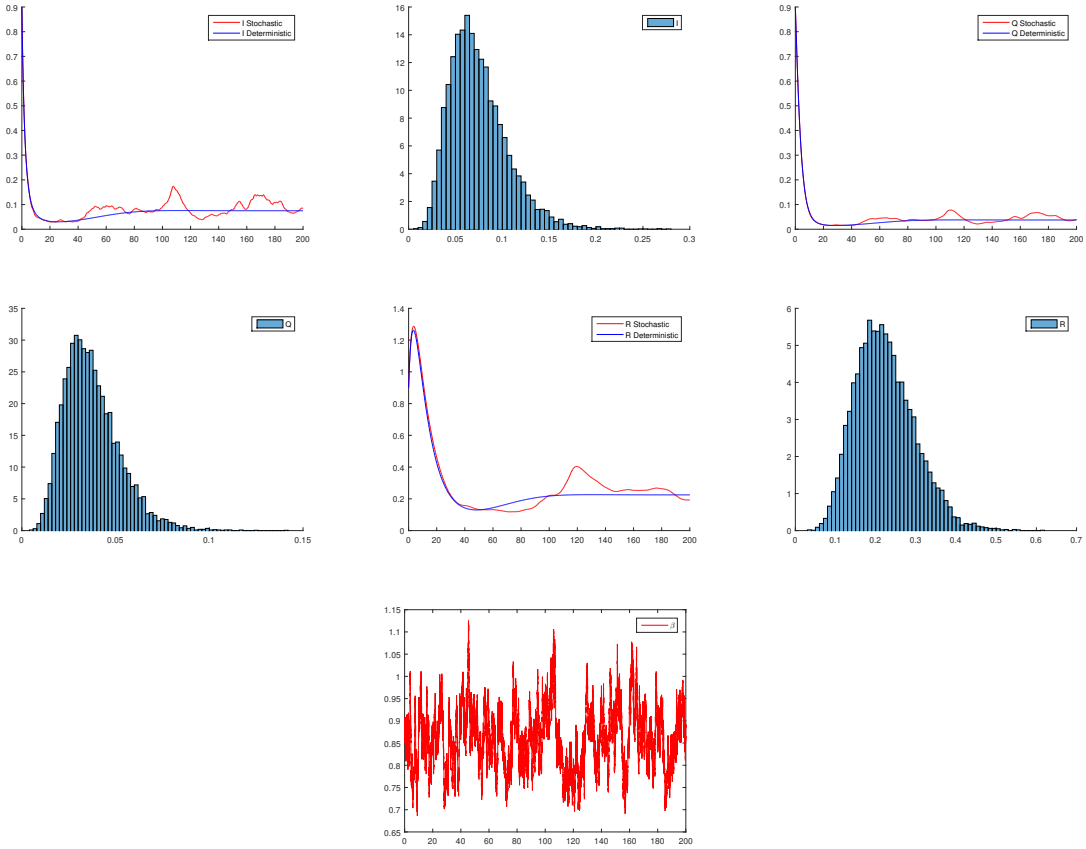

FIGURE 1. The paths and frequency histograms of system 8 initialized by $(\beta(0), I(0), Q(0), R(0)) = (0.9, 0.9, 0.9, 0.9)$, and with numerical values chosen as the second column of Table 5. The red curves show the behavior under IGBM process in the persistence case, while the corresponding deterministic system trajectories are illustrated in blue. We set, on the right, the corresponding histograms of I, Q and R associated with the stochastic system ($\mathscr{R}_0^{\blacklozenge} = 1.0257 > 1$).

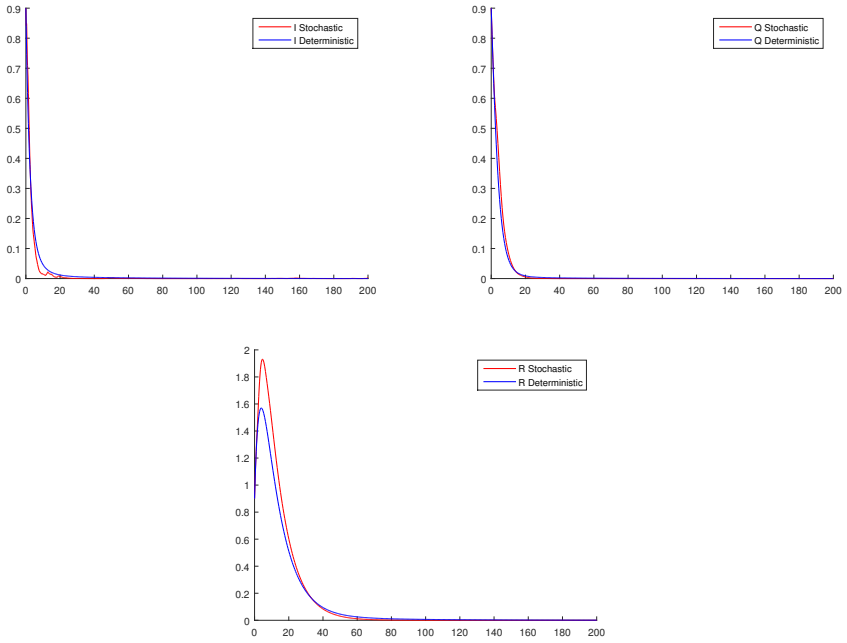

FIGURE 2. The extinction of system 8 initialized by $(\beta(0), I(0), Q(0), R(0)) = (0.9, 0.9, 0.9, 0.9)$ with the parameters values as in the third column of Table 5. The red curves, show the behavior under IGBM effect while the extinction of the corresponding deterministic system is illustrated in blue lines. ($\mathscr{R}_0 = 0.7859 < 1$).

6. CONCLUSION

In this work, a stochastic SIQR epidemic model with dimensional reduction is investigated in order to find out its spread behavior. In contrast to considerations of Zhou and Jiang in [3], we have taken into account the positivity of the transmission parameter β in our model by perturbing it using the Mean-reverting inhomogeneous geometric Brownian motion process instead of the Ornstein-Uhlembeck one. In light of this consideration, we have found that there is a unique and global positive solution of our system. Also, we employed a suitable C^2-function to prove that the model admits, at least, one stationary distribution under the assumption of

$$\mathscr{R}_0^{\blacklozenge} = \frac{\tilde{\beta}}{(\mu + r_I + \nu + \mu_I) + \frac{1}{2}\int_0^{+\infty} |\tilde{\beta} - x|\pi(x)dx} > 1.$$

Moreover, we have demonstrated that the sill to get the extinction is the same in the deterministic case as in the stochastic one. The remaining posed question in light of our mathematical analysis is to reveal how is it the dynamical comportment of the SIQR system (8) in the case when $\mathscr{R}_0 \leqslant 1 \leqslant \mathscr{R}_0^{\blacklozenge}$?.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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