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MATHEMATICAL MODELING AND OPTIMAL CONTROL STRATEGY FOR THE INFLUENZA (H5N1)

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Abstract. This study aims to examine the optimal control strategy for a continuous-time mathematical model of avian influenza. The model comprises eleven compartments, with our focus directed at five categories: potential groups, vulnerable groups, symptomatic virus carriers, asymptomatic virus carriers, virus carriers with severe complications, and virus carriers with complications of lesser severity. Our objective is to identify an effective strategy for diminishing the number of critically ill patients with avian flu and for treating carriers of the avian flu virus. We investigate three control approaches: awareness programs through education and information dissemination, treatment, and psychological support with ongoing monitoring. The Pontryagin's principle of continuous-time maximum is employed to delineate the optimal controls. The study employs MATLAB software for numerical simulations, and the results obtained validate the efficacy of the optimization strategy.

Keywords: avian influenza; optimal control; avian disease.

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1. INTRODUCTION

Avian influenza, caused by an RNA virus within the Orthomyxoviridae family [1], is a respiratory infection affecting birds and mammals. It is categorized into three main types (A, B, and C) based on variances in two crucial internal proteins [2]. Of these types, avian virus A is the most perilous and epidemiologically significant. It holds ecological and evolutionary interest due to its presence across diverse bird and mammal species, often undergoing notable changes in immunological properties.

According to [3], avian influenza A comprises three subtypes transmissible to birds and humans: A H5, A H9, and A H7. Transmission to humans occurs through airborne exposure to the virus or contact with contaminated surfaces. In humans, infection symptoms include coughing, fever, chills, and headache. Although the virus naturally circulates in birds, human infections stem from contact with infected poultry excrement. The transmission of avian flu to humans leads to severe consequences.

In 1998, 16 confirmed human cases and three suspected cases were reported [4]. According to [5], Indonesia saw 151 cases, resulting in 52 fatalities, while Vietnam recorded 119 cases, with 59 fatalities. In 2004, the avian virus was identified in migratory birds in Hong Kong, with no evidence of local poultry, pet birds, or leisure birds being infected [6]. On April 5, 2023, a 53-year-old man from northern Chile was reportedly infected, displaying no comorbidities or recent travel history. WHO reported a poultry farm employee in England contracting highly pathogenic avian influenza A(H5N1) viruses in mid-May. A second individual involved in slaughter operations at the farm also tested positive for the virus. Global concerns about avian flu's rapid spread persist.

Inadequate treatment of this disease can result in severe health complications and significant economic and societal burdens. Mathematical models have emerged as valuable tools for understanding avian virus dynamics. In a related study, Manach et al. [7] incorporated spatial factors and phases into a model, while Iwami [8] proposed a differential equation model. Gumel extended Iwami's model in [9], considering human-bird contact and isolating infected humans. Jung et al. [10] presented another optimal avian influenza strategy, while Vaidya et al. [11]

explored avian influenza dynamics in wild birds. Matcheve [12] introduced a model focusing on human suffering due to the virus.

Most research on avian virus complications has concentrated on discrete-time models expressed by differential equations. The complexity of avian virus complications varies, with some being treatable and others reaching an incurable critical stage. Our research stems from the debate surrounding the virus's origin and the availability of vaccines. Some attribute the virus to imported infected poultry, while others point to migratory birds as carriers during specific seasons.

This study introduces an eleven-compartment continuous-time optimal control model. In Section 2, elucidating avian virus dynamics and propagation. Section 3 formulates an optimal control problem for the model, characterizing optimal controls using Pontryagin's maximum principle. Section 4 presents numerical simulations performed in MATLAB. Finally, Section 5 concludes the article.

2. Model Formulation

In this section, we introduce a mathematical model denoted as $SEIAC_wVHRS_aI_a$, designed to depict the transmission dynamics of bird flu among individuals. The population is divided into two categories: humain categories $N_h = S+E+I+A+C_w+V+H+R$ and bird categories $N_h = S_a+I_a$. The visualization of the proposed mathematical model is presented in Figure 1.

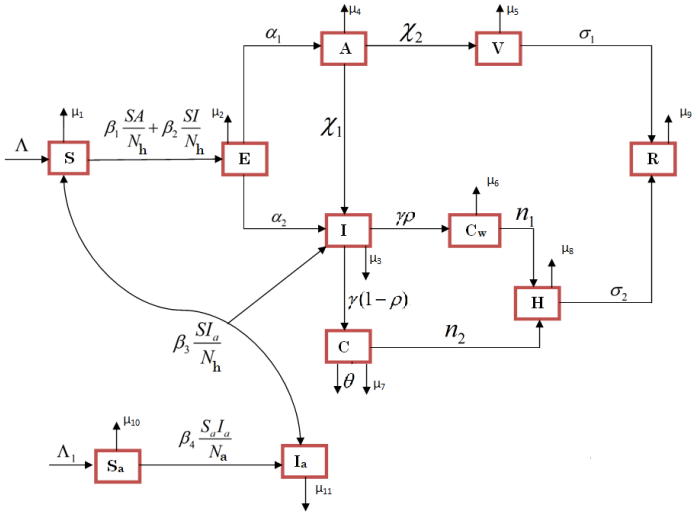

Figure 1. Model description

Compartment (S): this compartment is representing the number of susceptible is increasing by Λ (denote the incidence of susceptible. S is decreasing by μ_1 (natural mortality) and decreasing by $\beta_1 \frac{S(t)A(t)}{N_h}$ (the number of people who were infected with the virus by contact with the infected and asymptomatic people) and also decreasing by $\beta_2 \frac{S(t)I(t)}{N_h}$ (the number of people who were infected with the virus by contacting with the infected and symptomatic people) and decreasing by $\beta_3 \frac{S(t)I_a(t)}{N_h}$ (the number of people who were infected with the virus by contacting with the infected birds to disease).

Compartment (E): this compartment is representing the number of exposed, E is increasing by amounts $\beta_1 \frac{S(t)A(t)}{N_h}$ and $\beta_2 \frac{S(t)I(t)}{N_h}$,this compartment is decreasing by μ_2(natural mortality) and $\alpha_1 E(t)$ (the number of exposed become asymptomatic and infectious) and also by $\alpha_2 E(t)$ (the number of exposed become symptomatic and infectious).

Compartment (I): this compartment is representing the number of infected and symptomatic individuals, this compartment is increasing by $\alpha_2 E(t)$,$\chi_1 A(t)$ and $\beta_3 \frac{S(t)I_a(t)}{N_h}$.this compartment is decreasing by μ_3 (natural mortality).$I(t)$ is decreasing by $\gamma\rho I(t)$ (the number of people become infected whit complication and without chronic disease) and decreasing by $\gamma(1-\rho)I(t)$ (the number of people become infected whit complication and those whit chronic disease).

Compartment (A): This compartment signifies the count of individuals infected with asymptomatic bird flu. The quantity $A(t)$ experiences augmentation through $\alpha_1 E(t)$. The decrease of this compartment is governed by several factors: μ_4 (natural mortality), $\chi_1 A(t)$ (depicting the transition of infected and asymptomatic individuals to infected and symptomatic), and $\chi_2 A(t)$ (representing the subset of infected individuals undergoing treatment).

Compartment (C): This compartment represents the populace infected with complications or chronic conditions due to bird flu. The variable $C(t)$ undergoes augmentation via the term $\gamma(1-\rho)I(t)$, denoting the transition from the infected group I(t) to the complication-inflicted group $C(t)$. The diminishment of this compartment is influenced by various components: μ_5 (natural mortality), $n_2 C(t)$ (indicating the count of individuals with severe complications who are under medical observation), and $u'C(t)$ (indicating the mortality rate attributable to complications).

Compartment (Cw): This compartment pertains to individuals afflicted with complications but without underlying chronic diseases. The variable $Cw(t)$ experiences augmentation through

$\gamma\rho I(t)$, signifying the transition of infected individuals to this category. The decrease of this compartment is governed by μ_6 (natural mortality) and $n_1 Cw(t)$ (representing the count of individuals with severe complications who are being closely monitored).

Compartment (H): This compartment signifies the count of individuals under hospital lockdown with ongoing health monitoring. $H(t)$ is augmented by the sum of $n_1 Cw(t)$ and $n_2 C(t)$, representing individuals with complications being closely observed. The decrease in this compartment is influenced by μ_7 (natural mortality) and $\sigma_2 H(t)$, which represents the rate of individuals recovering from the virus.

Compartment (V): This compartment represents the number of individuals who have undergone vaccination and successfully recovered. $V(t)$ increases due to $\chi_2 A(t)$, depicting individuals transitioning from the infected asymptomatic group to the vaccinated group. The decrease of $V(t)$ is determined by both μ_8 (natural mortality) and $\sigma_1 V(t)$, signifying the rate of individuals recovering from the virus.

Compartment(R): This compartment pertains to the count of recovered individuals. $R(t)$ experiences augmentation through $\sigma_1 V(t)$ and $\sigma_2 H(t)$, representing those who have successfully recovered from vaccination and hospital lockdown. The decrease in $R(t)$ is governed by μ_9 (natural mortality).

Compartment (Sa): This compartment signifies the susceptible population exposed to the virus from birds. $S_a(t)$ is increased by Λ_1, indicating the incidence of susceptibility due to bird exposure. The decrease in $S_a(t)$ is driven by two factors: $\beta_4 \frac{Sa(t)Ia(t)}{N_a}$, which represents the rate of susceptible individuals becoming infected by contact with infected birds, and $\mu_1 0$ (natural mortality).

Compartment (Ia): This compartment represents infected birds. $I_a(t)$ experiences augmentation through the term $\beta_4 \frac{Sa(t)Ia(t)}{N_a}$, indicating the transition of susceptible individuals to the infected bird category. The decrease of $I_a(t)$ is influenced by μ_{11} (natural mortality) and $\beta_3 \frac{S(t)Ia(t)}{N_h}$, signifying the rate of infected birds being treated and recovering.

$$
(1)\qquad \begin{cases}
\dfrac{dS(t)}{dt} = \Lambda - \beta_1 \dfrac{S(t)A(t)}{N_h} - \beta_2 \dfrac{S(t)I(t)}{N_h} - \beta_3 \dfrac{S(t)I_a(t)}{N_h} - \mu_1 S(t) \\
\dfrac{dE(t)}{dt} = \beta_1 \dfrac{S(t)A(t)}{N_h} + \beta_2 \dfrac{S(t)I(t)}{N_h} - \mu_2 E(t) - \alpha_1 E(t) - \alpha_2 E(t) \\
\dfrac{dA(t)}{dt} = \alpha_1 E(t) - \mu_4 A(t) - \chi_1 A(t) - \chi_2 A(t) \\
\dfrac{dI(t)}{dt} = \alpha_2 E(t) - \mu_3 I(t) - \gamma\rho I(t) + \chi_1 A(t) - \gamma(1-\rho) I(t) + \beta_3 \dfrac{S(t)I_a(t)}{N_h} \\
\dfrac{dV(t)}{dt} = \chi_2 A(t) - \mu_5 V(t) - \sigma_1 V(t) \\
\dfrac{dC(t)}{dt} = \gamma(1-\rho) I(t) - (\theta + \mu_7) C(t) - n_2 C(t) \\
\dfrac{dC_\omega(t)}{dt} = \gamma\rho I(t) - \mu_6 C_\omega(t) - n_1 C_\omega(t) \\
\dfrac{dH(t)}{dt} = n_1 C_\omega(t) + n_2 C(t) - \mu_8 H(t) - \sigma_2 H(t) \\
\dfrac{dR(t)}{dt} = \sigma_1 V(t) + \sigma_2 H(t) - \mu_9 R(t) \\
\dfrac{dS_a(t)}{dt} = \Lambda_1 - \mu_{10} S_a(t) - \beta_4 \dfrac{S_a(t)I_a(t)}{N_a} \\
\dfrac{dI_a(t)}{dt} = \beta_4 \dfrac{S_a(t)I_a(t)}{N_a} - \mu_{11} I_a(t) - \beta_3 \dfrac{S(t)I_a(t)}{N_h}
\end{cases}
$$

Hence, we present the spread of bird flu mathematical model in the country of Chile is governed by the following system of differential equation.

where $S(0) \geq 0$, $E(0) \geq 0$, $A(0) \geq 0$, $I(0) \geq 0$, $V(0) \geq 0$, $C(0) \geq 0$, $C_\omega(0) \geq 0$, $H(0) \geq 0$, $R(0) \geq 0$, $S_a(0) \geq 0$, $I_a(0) \geq 0$ are the initial rate.

2.2. Model Basic Properties

2.2.1. Positivity of Solutions.

Theorem 1. if $S(0) \geq 0$, $E(0) \geq 0$, $A(0) \geq 0$, $I(0) \geq 0$, $V(0) \geq 0$, $C(0) \geq 0$, $C_\omega(0) \geq 0$, $H(0) \geq 0$, $R(0) \geq 0$, $S_a(0) \geq 0$, $I_a(0) \geq 0$ are the initial rate $t \geq 0$. the solution of system are positive for all $t \geq 0$

Proof. It follows from the first equation of system (1) that

$$
\frac{dS(t)}{dt} = \Lambda + \left(-\beta_1 \frac{A(t)}{N_h} - \beta_2 \frac{I(t)}{N_h} - \beta_3 \frac{I_a(t)}{N_h} - \mu_1 \right) S(t) \;\geq\; -\left(\beta_1 \frac{A(t)}{N_h} + \beta_2 \frac{I(t)}{N_h} + \beta_3 \frac{I_a(t)}{N_h} + \mu_1 \right) S(t)
$$

$$
\frac{dS(t)}{dt} + \left(\beta_1 \frac{A(t)}{N_h} + \beta_2 \frac{I(t)}{N_h} + \beta_3 \frac{I_a(t)}{N_h} + \mu_1 \right) S(t) \;\geq\; 0
$$

where

$$F(t) = \beta_1 \frac{A(t)}{N_h} + \beta_2 \frac{I(t)}{N_h} + \beta_3 \frac{I_a(t)}{N_h} + \mu_1$$

$$\frac{dS(t)}{dt} + F(t)S(t) \geq 0$$

The both sides in last inequality are multiplied by $\exp\left(\int_0^t F(s)ds\right)$ We obtain

$$\exp\left(\int_0^t F(s)ds\right)\frac{dS(t)}{dt} + F(t)\exp\left(\int_0^t F(s)ds\right)S(t) \geq 0$$

$$\frac{d}{dt}\left(S(t)\exp\left(\int_0^t F(s)ds\right)\right) \geq 0$$

Integrating this inequality from 0 to t gives:

$$\int_0^t \left(\frac{d}{ds}\left(S(t)\exp\left(\int_0^t F(s)ds\right)\right)\right)ds \geq 0$$

then $S(t) \geq S(0)\exp\left(\int_0^t F(s)ds\right)$

$\Rightarrow S(t) \geq 0$ similarly, we prove that

$S(t) \geq 0$, $E(t) \geq 0$, $A(t) \geq 0$, $I(t) \geq 0$, $V(t) \geq 0$, $C(t) \geq 0$,

$C_W(t) \geq 0$, $H(t) \geq 0$, $I_a(t) \geq 0, S_a(t) \geq 0$ and $S(t) \geq 0$.

2.2.2. BOUDEDNESS OF THE SOLUTIONS.

Theorem 2. The set

$$\begin{cases} \Omega_h = \left\{(S,E,I,A,H,C_W,C,V,R) \in \Re_+^9 / 0 \leq S+E+I+A+C_W+C+V+H+R \leq \frac{\Lambda}{\mu_h}\right\} \\ \Omega_a = \left\{(S_a,I_a) \in \Re_+^2 / 0 \leq S_a + I_a \leq \frac{\Lambda_1}{\mu_a}\right\} \end{cases}$$

Positively invariant under system (1) with initial conditions

$S(0) \geq 0$, $E(0) \geq 0$, $A(0) \geq 0$, $I(0) \geq 0$,

$V(0) \geq 0$, $C(0) \geq 0$, $C_W(0) \geq 0$, $H(0) \geq 0$, $I_a(0) \geq 0$, $S_a(0) \geq 0$ *and* $R(0) \geq 0$.

Proof. Also, one assumes that:

$\frac{dN_h}{dt} = \Lambda - \mu_h N_h - \theta C$

$\frac{dN_h}{dt} = \Lambda - \mu_h N_h - \theta C \leq \Lambda - \mu_h N_h$

$\frac{dN_h}{dt} \leq \Lambda - \mu_h N_h$

$\Rightarrow N_h(t) \leq \frac{\Lambda}{\mu_h} + N_h(0)e^{-\mu_h t}$

If we wake limit $t \to \infty$ then $N_h(t) \leq \frac{\Lambda}{\mu}$.

It implies that the region Ω_h is a positively invariant set for the system (1).

$\Rightarrow N_h(t) \leq \frac{\Lambda}{\mu_h}$

subsequently, it can be proven that $N_a(t) \leq \frac{\Lambda_1}{\mu_a}$.

EXISTENCE OF SOLUTIONS

So the system (1) can be rewritten in the following form: $\psi(X) = AX + B(X)$

then

$$X = \begin{pmatrix} S(t) \\ E(t) \\ A(t) \\ I(t) \\ V(t) \\ C(t) \\ C_W(t) \\ H(t) \\ R(t) \\ S_a(t) \\ I_a(t) \end{pmatrix} \qquad B(X) = \begin{pmatrix} \frac{dS(t)}{dt} \\ \frac{dE(t)}{dt} \\ \frac{dA(t)}{dt} \\ \frac{dI(t)}{dt} \\ \frac{dV(t)}{dt} \\ \frac{dC(t)}{dt} \\ \frac{dC_W(t)}{dt} \\ \frac{dH(t)}{dt} \\ \frac{dR(t)}{dt} \\ \frac{dS_a(t)}{dt} \\ \frac{dI_a(t)}{dt} \end{pmatrix}$$

where

$$A = \begin{pmatrix} -\mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & -(\mu_3+\gamma) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \chi_2 & 0 & -(\mu_5+\sigma_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma(1-\rho) & 0 & A_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma\rho & 0 & 0 & -(\mu_7+n_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_2 & n_1 & -(\mu_8+\sigma_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & \sigma_2 & -\mu_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{11} \end{pmatrix}$$

$$A_1 = -(\alpha_1 + \alpha_2 + \mu_2)\ , A_2 = -(\mu_4 + \chi_1 + \chi_2)\ \ and\ A_3 = -(\mu_6 + n_2 + \theta)$$

and

$$B(X) = \begin{pmatrix} \Lambda - \beta_1 \frac{S(t)A(t)}{N_h} - \beta_2 \frac{S(t)I(t)}{N_h} - \beta_3 \frac{S(t)I_a(t)}{N_h} \\ \beta_1 \frac{S(t)A(t)}{N_h} + \beta_2 \frac{S(t)I(t)}{N_h} \\ 0 \\ \beta_3 \frac{S(t)I_a(t)}{N_h} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Lambda_1 - \beta_4 \frac{S_a(t)I_a(t)}{N_a} \\ -\beta_3 \frac{S(t)I_a(t)}{N_h} \end{pmatrix}$$

3. THE CONTROLLED MATHEMATICAL MODEL

The primary goal of this study is to curtail the incidence of bird flu infections among the populace. This objective is pursued through a collection of preventative measures, encompassing strategies like avoiding contact with infected birds and poultry, refraining from interactions with infected individuals, and administering vaccinations to those already infected. To accomplish this, we implement three control mechanisms denoted as u(t), v(t), and w(t). These controls correspond to the implementation of awareness programs involving information dissemination and education. They aim to enhance public understanding of the severity of the infection, its implications for human health, and methods to mitigate its spread.

Consequently, we present a controlled mathematical model depicting the propagation of bird flu within the context of Chile. This model is governed by the following system of differential equations:

$$
(2)\quad \begin{cases}
\frac{dS(t)}{dt} = \Lambda - \beta_1(1-u(t))\frac{S(t)A(t)}{N_h} - \beta_2(1-u(t))\frac{S(t)I(t)}{N_h} - \beta_3(1-v(t))\frac{S(t)I_a(t)}{N_h} - \mu_1 S(t) \\
\frac{dE(t)}{dt} = \beta_1(1-u(t))\frac{S(t)A(t)}{N_h} + \beta_2(1-u(t))\frac{S(t)I(t)}{N_h} - \mu_2 E(t) - \alpha_1 E(t) - \alpha_2 E(t) \\
\frac{dA(t)}{dt} = \alpha_1 E(t) - \mu_4 A(t) - \chi_1 A(t) - \chi_2 A(t) \\
\frac{dI(t)}{dt} = \alpha_2 E(t) - \mu_3 I(t) - \gamma\rho I(t) + \chi_1 A(t) - \gamma(1-\rho)I(t) + \beta_3(1-v(t))\frac{S(t)I_a(t)}{N_h} - w(t)I(t) \\
\frac{dV(t)}{dt} = \chi_2 A(t) - \mu_5 V(t) - \sigma_1 V(t) \\
\frac{dC(t)}{dt} = \gamma(1-\rho)I(t) - (\theta + \mu_7)C(t) - n_2 C(t) - w(t)C(t) \\
\frac{dC_\omega(t)}{dt} = \gamma\rho I(t) - \mu_6 C_\omega(t) - n_1 C_\omega(t) - w(t)C_\omega(t) \\
\frac{dH(t)}{dt} = n_1 C_\omega(t) + n_2 C(t) - \mu_8 H(t) - \sigma_2 H(t) + w(t)I(t) + w(t)C(t) + w(t)C_\omega(t) \\
\frac{dR(t)}{dt} = \sigma_1 V(t) + \sigma_2 H(t) - \mu_9 R(t) \\
\frac{dS_a(t)}{dt} = \Lambda_1 - \mu_{10} S_a(t) - \beta_4 \frac{S_a(t)I_a(t)}{N_a} \\
\frac{dI_a(t)}{dt} = \beta_4 \frac{S_a(t)I_a(t)}{N_a} - \mu_{11} I_a(t) - \beta_3(1-v(t))\frac{S(t)I_a(t)}{N_h}
\end{cases}
$$

3.1. THE OPTIMAL CONTROL PROBLEM

The problem is to minimize the objective functional

$$J(u,v) = I(T) + C(T) + C_\omega(T) + I_a(T) + \int_0^T \left[I(t) + C(T) + C_\omega(T) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) + \frac{A_3}{2}w^2(t) \right] dt$$

Where $A_1 \geq 0$, $A_2 \geq 0$ and $A_3 \geq 0$ are the cost coefficients. They are selected to weigh the relative importance of $u(t)$, $v(t)$ and $w(t)$ at time t; T is the final time. In other words, we seek the optimal controls u^*, v^* and w^* such that

$$J(u^*, v^*, w^*) = \min_{u,v,w \in U} (J(u,v,w))$$

where U is the set of admissible control defined by

$U = \{(u,v,w)/0 \leq u_{\min} \leq u(t) \leq u_{\max} \leq 1, 0 \leq v_{\min} \leq v(t) \leq v_{\max} \leq 1 and 0 \leq w_{\min} \leq w(t) \leq w_{\max} \leq 1/t \in [0, T_f]\}$

3.2 THE OPTIMAL CONTROL: EXISTENCE AND CHARACTERIZATION

Proof: We first show existence of solutions of the system (1) there after we will prove the existence of optimal control. Consider the control problem with system (...), there exists an optimal $J(u^*, v^*, w^*) \in U^3$, such that $J(u^*, v^*, w^*) = \min_{u,v,w \in U} (J(u,v,w))$.

Proof: The existence of the optimal control can be obtained using a result by Fleming and Rishel [13],checking the following steps.

- It follows that the set of controls and corresponding state variables is nonempty. We will use a simplified version of an existence results [14] theorem 7.1.1.

- $J(u,v,w)$ is convex in U.

- The control space $U = \{(u,v,w)/(u,v,w)$ is measurable.

$0 \leq u_{\min} \leq u(t) \leq u_{\max} \leq 1, 0 \leq v_{\min} \leq v(t) \leq v_{\max} \leq 1 and 0 \leq w_{\min} \leq w(t) \leq w_{\max} \leq 1/t \in \left[0, T_f\right]\}$ is convex and closed by definition.

All the right hand sides of equation of system are continuous, bounded above by a sum of bounded control and state, and can be written as a linear function of u,v and w with coefficients depending on time and state.

The integrate in the objective functional is Creally convex on U.

$I(T)+C(T)+C_{\omega}(T)+I_a(T)+\frac{A_1}{2}u^2(t)+\frac{A_2}{2}v^2(t)+\frac{A_3}{2}w^2(t)$.

It rest to show that there exists constants and satisfies then from Fleming and Rishel [13], we conclude that there exists an optimal control.

We have according to the theorem of Pontryagine [15, 16, 19, 20, 21].

Proof: The Hamiltonian is defined as follows

$$H = I(T)+C(T)+C_{\omega}(T)+I_a(T)+\frac{A_1}{2}u^2(t)+\frac{A_2}{2}v^2(t)+\frac{A_3}{2}w^2(t) + \sum_{i=1}^{11}\lambda_i(t).f_i(S,E,A,I,V,C,C_{\omega},H,R,S_a,I_a)$$

where f_i is the right optimal controls (u^*,v^*,w^*) and the solutions

$S^*,E^*,I^*,V^*,C^*,{C_{\omega}}^*,H^*,V^*,R^*,{S_a}^*$ and ${I_a}^*$ of the corresponding state (1).

There exists adjoint variables ${\lambda_1}', ... \; and \;\; {\lambda_{11}}'$ satisfying

Then from Fleming and Rishel [13] we conclude that there exists an optimal control.

$$f_1(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \Lambda - \beta_1(1-u(t))\frac{S(t)A(t)}{N_h} - \beta_2(1-u(t))\frac{S(t)I(t)}{N_h} - \beta_3(1-v(t))\frac{S(t)I_a(t)}{N_h} - \mu_1 S(t)$$

$$f_2(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \beta_1(1-u(t))\frac{S(t)A(t)}{N_h} + \beta_2(1-u(t))\frac{S(t)I(t)}{N_h} - \mu_2 E(t) - \alpha_1 E(t) - \alpha_2 E(t)$$

$$f_3(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \alpha_1 E(t) - \mu_4 A(t) - \chi_1 A(t) - \chi_2 A(t)$$

$$f_4(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \alpha_2 E(t) - \mu_3 I(t) - \gamma\rho I(t) + \chi_1 A(t) - \gamma(1-\rho)I(t) + \beta_3(1-v(t))\frac{S(t)I_a(t)}{N_h} - w(t)I(t)$$

$$f_5(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \chi_2 A(t) - \mu_5 V(t) - \sigma_1 V(t)$$

$$f_6(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \gamma(1-\rho)I(t) - (\theta+\mu_7)C(t) - n_2 C(t) - w(t)C(t)$$

$$f_7(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \gamma\rho I(t) - \mu_6 C_\omega(t) - n_1 C_\omega(t) - w(t)C_\omega(t)$$

$$f_8(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = n_1 C_\omega(t) + n_2 C(t) - \mu_8 H(t) - \sigma_2 H(t) + w(t)I(t) + w(t)C(t) + w(t)C_\omega(t)$$

$$f_9(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \sigma_1 V(t) + \sigma_2 H(t) - \mu_9 R(t)$$

$$f_{10}(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \Lambda_1 - \mu_{10} S_a(t) - \beta_4 \frac{S_a(t)I_a(t)}{N_a}$$

$$f_{11}(S,E,A,I,V,C,C_\omega,H,R,S_a,I_a) = \beta_4 \frac{S_a(t)I_a(t)}{N_a} - \mu_{11} I_a(t) - \beta_3(1-v(t))\frac{S(t)I_a(t)}{N_h}$$

Where

$$\lambda_1{}' = -\frac{\partial H}{\partial S} = \left(\beta_1\frac{A}{N_h} + \beta_2\frac{I}{N_h}\right)(1-u(t))(\lambda_1-\lambda_2) + \beta_3\frac{I_a}{N_h}(1-v(t))(\lambda_{11}-\lambda_4+\lambda_1) + \lambda_1\mu_1$$

$$\lambda_2{}' = -\frac{\partial H}{\partial E} = \alpha_1(\lambda_2-\lambda_3) + \alpha_2(\lambda_2-\lambda_4) + \lambda_2\mu_2 + \lambda_3\mu_4$$

$$\lambda_3{}' = -\frac{\partial H}{\partial A} = \beta_1\frac{S}{N}(1-u(t))(\lambda_1-\lambda_2) + \chi_1(\lambda_3-\lambda_4) + \chi_2(\lambda_3-\lambda_5)$$

$$\lambda_4{}' = -\frac{\partial H}{\partial I} = -1 + \beta_2 \frac{S}{N_h}(1-u(t))(\lambda_1 - \lambda_2) + \gamma\rho(\lambda_4 - \lambda_7) + \gamma(1-\rho)(\lambda_4 - \lambda_6)$$
$$+ w(\lambda_4 - \lambda_8)$$
$$\lambda_5{}' = -\frac{\partial H}{\partial V} = \sigma_1(\lambda_5 - \lambda_9) + \lambda_5\mu_5$$
$$\lambda_6{}' = -\frac{\partial H}{\partial C} = -1 + (\lambda_6 - \lambda_8)(n_2 + w) + \lambda_6(\theta + \mu_6)$$
$$\lambda_7{}' = -\frac{\partial H}{\partial C_\omega} = -1 + (\lambda_7 - \lambda_8)(n_1 + w) + \lambda_7\mu_7$$
$$\lambda_8{}' = -\frac{\partial H}{\partial H} = \sigma_2(\lambda_8 - \lambda_9) + \lambda_8\mu_8$$
$$\lambda_9{}' = -\frac{\partial H}{\partial R} = \lambda_9\mu_9$$
$$\lambda_{10}{}' = -\frac{\partial H}{\partial S_a} = \beta_4 \frac{I_a}{N_a}(\lambda_{10} - \lambda_{11}) + \lambda_{10}\mu_{10}$$
$$\lambda_{11}{}' = -\frac{\partial H}{\partial I_a} = -1 + \beta_3 \frac{S}{N_h}(1-v(t))(\lambda_1 - \lambda_4 + \lambda_{11}) + \beta_4 \frac{S_a}{N_a}(\lambda_{10} - \lambda_{11}) + \lambda_{11}\mu_{11}$$

for $t \in [0, T_f]$ the optimal control $u^* v^* and\ w*$ can be solved from the optimality condition . that are

$$-\frac{\partial H}{\partial u} = -A_1 u - \lambda_1\left(\beta_1 \frac{S(t)A(t)}{N_h} + \beta_2 \frac{S(t)I(t)}{N}\right) - \lambda_2\left(-\beta_1 \frac{S(t)A(t)}{N_h} - \beta_2 \frac{S(t)I(t)}{N_h}\right) = 0$$
$$-\frac{\partial H}{\partial v} = -A_2 v - \lambda_1\left(\beta_3 \frac{S(t)I_a(t)}{N_h}\right) - \lambda_4\left(-\beta_3 \frac{S(t)I_a(t)}{N_h}\right) - \lambda_{11}\left(\beta_3 \frac{S(t)I_a(t)}{N_h}\right) = 0$$
$$-\frac{\partial H}{\partial w} = -A_3 w + \lambda_4(-I(t)) - \lambda_6(-C(t)) - \lambda_7(-C_\omega(t)) = 0$$

we have

$$u = \frac{(\lambda_2 - \lambda_1)}{A_1}\left(\beta_1 \frac{S(t)A(t)}{N_h} + \beta_2 \frac{S(t)I(t)}{N_h}\right)$$
$$v = \frac{(\lambda_4 - \lambda_1 - \lambda_{11})}{A_2}\left(\beta_3 \frac{S(t)I_a(t)}{N_h}\right)$$
$$w = \frac{1}{A_3}(I(\lambda_4 - \lambda_8) + C(\lambda_6 - \lambda_8) + C_\omega(\lambda_7 - \lambda_8))$$

4. SIMULATION

In this section, we present the outcomes derived from the optimized system composed of eleven equations. The problem entails both initial and final conditions at specified points. Initially, we utilize an estimated value for the initial transformation variables, solving for the adjacent variables in subsequent time steps. This iterative process continues, incorporating adjustments based on descriptions, until the convergence of successive iterations is attained. A MATLAB code is developed and compiled, employing the data provided in Table 1.

Figure 2 illustrates the progression of individuals exposed to viral infection. The graph demonstrates a gradual increase in the number of individuals exposed to the virus, with a noticeable acceleration after approximately 30 days. This swift rise signifies that symptoms of infection begin to manifest more prominently as time progresses, leading to the emergence of two categories of infected individuals. These categories include those carrying the virus without exhibiting symptoms (Figure 3) and those displaying symptoms (Figure 4). The count of asymptomatic virus carriers remains elevated over time, contributing significantly to the pool of individuals who eventually develop symptoms. This transition occurs due to interactions between susceptible individuals and virus-carrying birds or poultry, as well as contact with individuals already exposed to the virus.

The situation further complicates as it leads to the emergence of two distinct categories of infected individuals: those with severe symptoms (Figure 5) and those with non-severe symptoms (Figure 6). This distinction arises from the varying levels of symptom severity experienced by infected individuals.

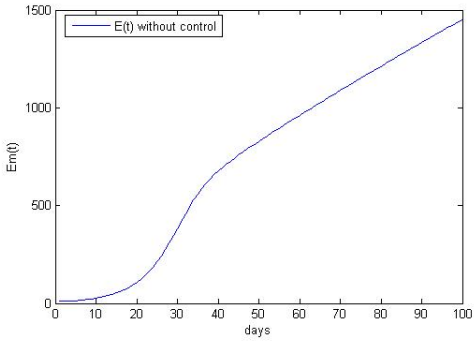

fig.2

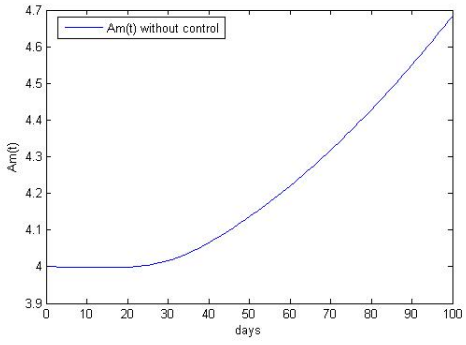

fig.3

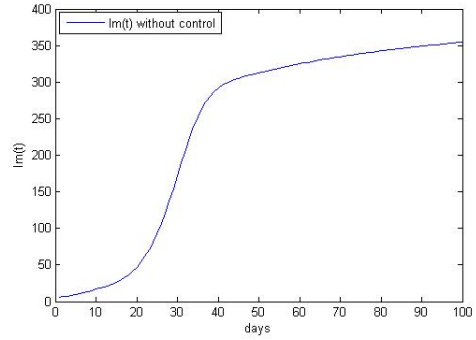

fig.4

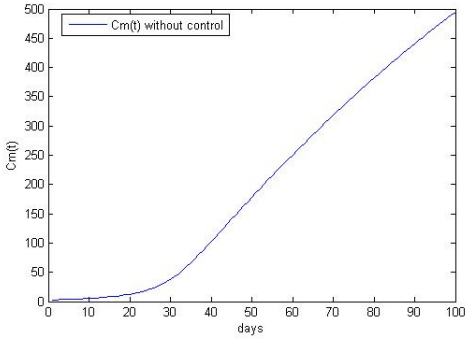

fig.5

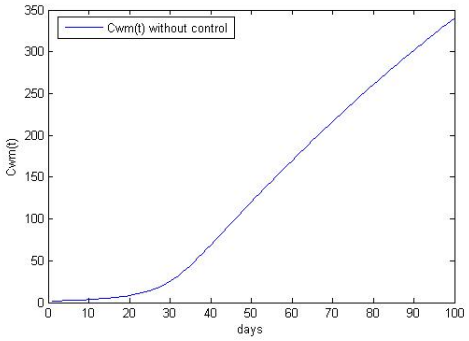

fig.6

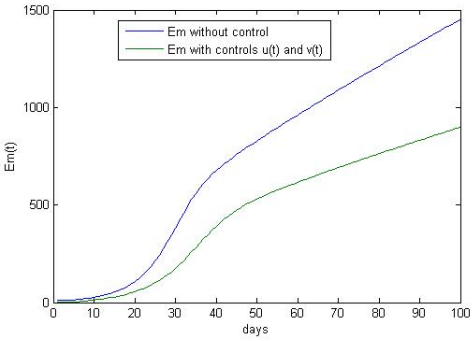

fig.7

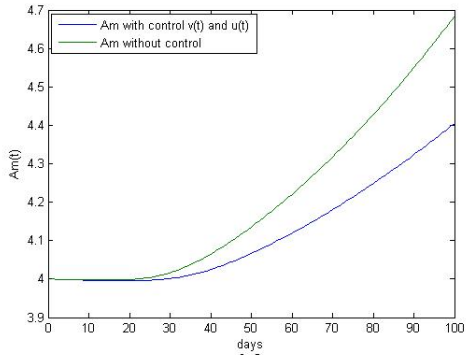

fig.8

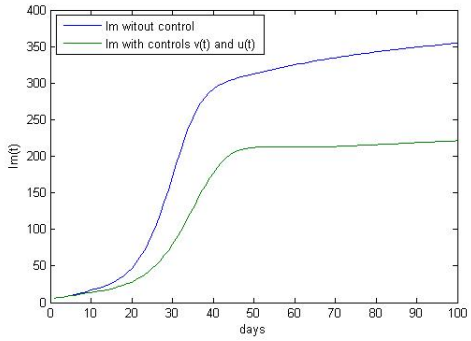

fig.9

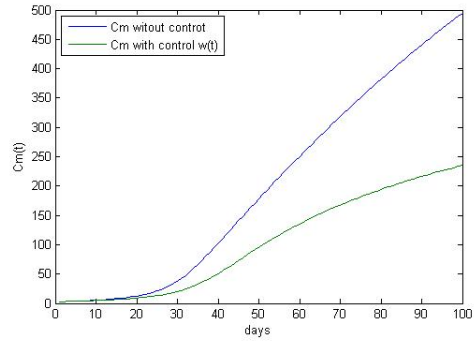

fig.10

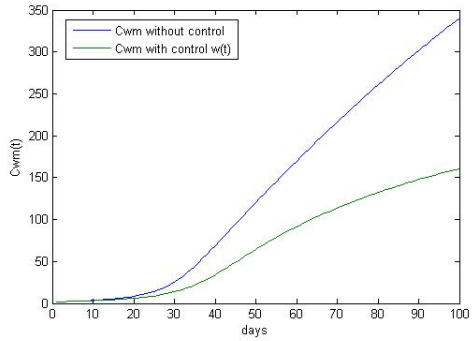

fig.11

Parameter	Value	Source
Λ	50	Assumed
β_1	0.25	Assumed
β_2	0.25	Assumed
β_3	0.23	assumed
β_4	0.23	Assumed
σ_1	0.28	Assumed
σ_2	0.28	Assumed
μ	0.06	17
n_1	0.08	Assumed
n_2	0.08	Assumed
γ	0.05	Assumed
Λ_1	30	Assumed
α_1	0.30	Assumed
α_2	0.21	Assumed
χ_1	0.01	Assumed
θ	0.08	18
χ_2	0.01	Assumed

TABLE 1. Liste of all parameters of system (1)

4.1. CHARACTERISATION OF THE OPTIMAL CONTROL

The control strategy proposed in this article serves multiple objectives, which are explored in the following subsections:

4.1.1 STRATEGY A: SENSITIZATION AND PREVENTION

Incorporating the optimal controls u(t) and v(t), this strategy concentrates on two key goals. The first is to raise awareness among the public about the gravity of avian flu and to equip them with protective measures against its spread. Employing methods such as awareness campaigns and protective initiatives (illustrated in Figures 7, 8, and 9), this strategy involves educating

citizens about the seriousness of avian flu through media channels. Preventive measures include avoiding contact with infected individuals, practicing regular hand hygiene (especially after sneezing), and seeking medical consultation if symptoms arise. The outcome of this strategy demonstrates a decrease in the numbers of infected individuals displaying symptoms, as well as those who remain asymptomatic.

4.1.2. FOLLOW UP ON MEDICAL TREATMENT: CONTROL $w(t)$

This strategy, implemented through control w(t), emphasizes the importance of prompt medical treatment and consultation for individuals exhibiting severe symptoms. The approach yields favorable outcomes, as depicted in Figures 10 and 11, showing an increased rate of treatment among individuals suffering from severe symptoms.

5. CONCLUSION

This research delved into a mathematical epidemiological model to analyze avian influenza infection dynamics. Following the presentation of the paper and a comprehensive review of related literature, a mathematical model was formulated to describe the dynamics of avian influenza within various age groups of Chile's population. The model aims to mitigate the infection rates among individuals with varying symptom presentations. Controls were introduced, including patient isolation and quarantine, mask usage, regular hand washing, and medical consultation, to combat the spread of the disease.

By applying control theory techniques, optimal control strategies were obtained and assessed. The results underscored the efficacy of the proposed control measures. In the future, further investigation will be conducted into the temporal and spatial aspects of the disease's propagation.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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