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A STUDY FOR FRACTIONAL ORDER EPIDEMIC MODEL OF COVID-19 SPREAD WITH VACCINATION

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Abstract. In this paper, we present a fractional bi-modal $SITR$ mathematical model to study the Covid-19 spread in a human population under vaccination influence. The study depends on the stability of the disease-free and endemic equilibrium. To demonstrate the validity of the results, we give a numerical example. The results show that the infected and treatment subpopulations decrease if the susceptible subpopulations are vaccinated. Moreover, the recovered subpopulation increased.

Keywords: Fractional-order derivative; bi-modal $SITR$ model; basic reproduction number; equilibrium.

2020 AMS Subject Classification: 26A33, 34C60, 34D23.

1. Introduction

Mathematical models are powerful tools for understanding and controlling infectious disease transmission. Mathematical models play an important role in quantifying and assessing the efficient control and preventive measures of infectious ailments [1]. It has been proved in multiple

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ways that mathematical modeling is a very flexible and efficient way of researching the dynamics of transmission of infectious ailments. Mathematical analysis and numerical simulations can be used to create and evaluate convincing control measures

The bi-modal $SITR$ compartment model is one of among the models of the spread of Covid 19 in the form of non-linear differential equations that are widely discussed [2, 3, 4]. In this model, the observed human population (N) is divided into five epidemiological sub-compartments denoted by individuals who are not yet infected with the virus (S_1), individuals who have some serious diseases or they are of an older age but they are not yet been infected with the virus (S_2), individuals who are infected by the virus and they can transmit it to others (I), individuals who are under treatment (T), and individuals who are recovered with medical treatment (R), as described by the compartment diagram in Figure 1 in literature [5] with the involve various parameters are described in Table 1 below.

TABLE 1. Parameter with description occuring in the bi-modal $SITR$ model

Parameter	**Description**
Λ_1	Influx rate of class S_1
Λ_2	Influx rate of class S_2
α	Rate of death rate of human population
β_1	Rate of transmission of subpopulation S_1 that infected
β_2	Rate of transmission of subpopulation S_2 that infected
μ	Rate of treatment
ρ	Rate of recovery from Covid-19

Based on that Figure and the assumptions in [5], the dynamics model for transmission of Covid-19 is given by the following nonlinear differential equations system [5]:

$$\begin{aligned}
\dot{S_1} &= \Lambda_1 - \beta_1 I S_1 - \alpha S_1, \\
\dot{S_2} &= \Lambda_2 - \beta_2 I S_2 - \alpha S_2 \\
\dot{I} &= (\beta_1 S_1 + \beta_2 S_2) I - (\alpha + \mu) I, \qquad (1) \\
\dot{T} &= \mu I - (\alpha + \rho) T, \\
\dot{R} &= \rho T - \alpha R.
\end{aligned}$$

with $N = S_1 + S_2 + I + T + R$ and the initial states $S_1(0) = S_{01}, S_2(0) = S_{02}, I(0) = I_0, T(0) = T_0, R(0) = R_0$.

Along with the development of the fractional-order differential equation, recently the issue of the development of mathematical models in the form of the fractional-order nonlinear differential equation has been widely discussed by many researchers, see [6, 7, 8, 9]. In this paper, we modify the model (1) by including the vaccination parameter with rate δ_1 for S_1 and rate δ_2 for S_2, and replacing the usual derivative into the fractional-order derivative of Caputo type such that the compartment diagram can be modified as follows:

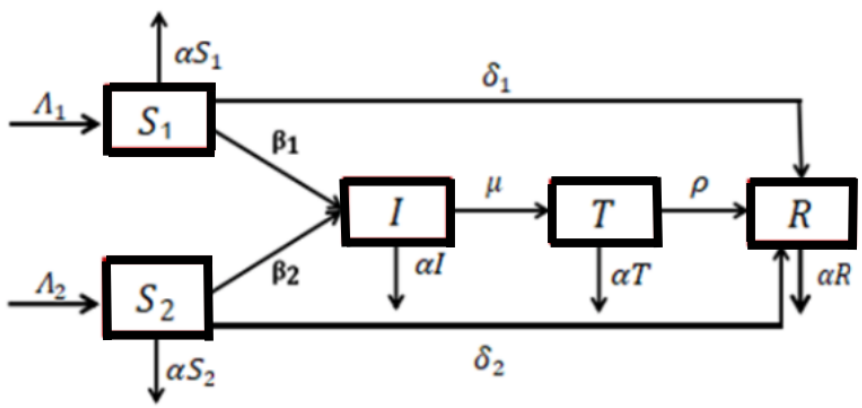

FIGURE 1. Compartment diagram for bi-modal $SITR$ model

The transmission model of Covid-19 spread which corresponds to compartment diagram Figure 1 in the form of the fractional order nonlinear differential equation is given by the following system:

$$
\begin{aligned}
\Delta^{(\gamma)} S_1 &= \Lambda_1 - \beta_1 I S_1 - (\alpha + \delta_1) S_1, \\
\Delta^{(\gamma)} S_2 &= \Lambda_2 - \beta_2 I S_2 - (\alpha + \delta_2) S_2 \\
\Delta^{(\gamma)} I &= (\beta_1 S_1 + \beta_2 S_2) I - (\alpha + \mu) I, \\
\Delta^{(\gamma} T &= \mu I - (\alpha + \rho) T, \\
\Delta^{(\gamma)} R &= \delta_1 S_1 + \delta_2 S_2 + \rho T - \alpha R,
\end{aligned} \tag{2}
$$

where $\Delta^{(\gamma)}$ is the fractional order Caputo derivative operator of order γ with $0 < \gamma < 1$. As a new bi-modal $SITR$ model, we study the stability of the disease-free equilibrium and endemic equilibrium of the model (2). Moreover, we will study the effect of vaccination to decrease the total of infected and treatment populations. To the best of the author's knowledge, this issue has not been solved yet to date. Therefore the results of this work constitute a novelties at once a new contribution in the field of fractional-order epidemic dynamic.

The paper is organized as follows: Section 2 considers basic preliminaries about Caputo fractional derivative and stability of the fractional-order nonlinear system. The main result of this article is presented in the section 3. Section 4 concludes the paper.

2. Basic Preliminaries

Here, we provide some primary preliminaries and results regarding the fractional operators. The Caputo fractional derivative of order γ with $n-1<\gamma<n,\ n\in\mathbb{N}$ for the integrable vector function $\mathbf{h}:[0,\infty)\to\mathbb{R}^n$, is defined by

$$\Delta^{(\gamma)}\mathbf{h}(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t(t-\tau)^{n-\gamma-1}\Delta^{(m)}\mathbf{h}(\tau)d\tau \tag{3}$$

where $\Gamma(.)$ is the Euler Gamma function [10], and $\Delta^{(m)}\mathbf{h}(.)$ is the usual m-th derivative of function $\mathbf{h}(.)$ with $m\in\mathbb{N}$.

Let us consider the fractional-order nonlinear system involving Caputo derivative

$$\Delta^{(\gamma)}\mathbf{h}(t)=\mathbf{g}(t,\mathbf{h}(t)),\ \mathbf{h}(0)=\mathbf{h}_0, \tag{4}$$

where $\mathbf{h}(t)\in\mathbb{R}^n$ is the state vector of the system (4), $\mathbf{g}:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$. The linear version of the system (4) can be written as

$$\Delta^{(\gamma)}\mathbf{h}(t)=\mathscr{A}\mathbf{h}(t)), \tag{5}$$

where $\mathscr{A}$ is a n by n matrix.

One important thing of the system (4) is stability of equilibrium point. When talking about stability, one is interested in the behavior of the solutions of (4) for $t\to\infty$. The point $\mathbf{h}^*$ is said the equilibrium point of the system (4) if $\mathbf{g}(t,\mathbf{h}^*)=\mathbf{0}$. Note that the equilibrium point is a constant solution to the dynamic system (4) [11].

Definition 2.1. [10, 12] *Let* $\mathbf{h}^*$ *is an equilibrium point of the fractional-order system (4).*

(1). $\mathbf{h}^*$ *is said to be stable if for* $\varepsilon > 0$, *there exists a* $\eta_\varepsilon > 0$ *such that*

$$\|\mathbf{h}(t_0) - \mathbf{h}^*\| < \eta_\varepsilon \to \|\mathbf{h}(t) - \mathbf{h}^*\| < \varepsilon \ \textit{for} \quad t \geq t_0.$$

(2). $\mathbf{h}^*$ *is said to be asymtotically stable if it is stable and* $\lim_{t\to\infty} \mathbf{h}(t) = \mathbf{h}^*$.

Theorem 2.2. [10, 12] *The equilibrium point* $\mathbf{h}^*$ *of the fractional-order linear system (5) with* $0 < \gamma < 1$ *is asymptotically stable if*

$$|\arg(r_i)| > \frac{1}{2}\gamma\pi, \tag{6}$$

where $r_i,\ i = 1, 2, \cdots, n$ *are eigenvalues of the matrix* $\mathscr{A}$.

Theorem 2.3. [10, 12] *For* $0 < \gamma < 1$, *the equilibrium point* $\mathbf{h}^*$ *of the system (4) is asymptotically stable if*

$$|\arg(r)| > \frac{1}{2}\gamma\pi, \tag{7}$$

for all roots r of the equation

$$|J_{\mathbf{h}^*} - rI| = 0 \tag{8}$$

where $J_{\mathbf{h}^*}$ *is the Jacobian matrix of system (4) at the equilibrium* $\mathbf{h}^*$.

3. Asymptotic Stability of the Equilibria

By following the procedure in [12, 14], it is easy to show that the solution of the model under consideration is restricted to the feasible region given by

$$\mathscr{U} = \left\{(S_1, S_2, I, T, R) \in \mathbb{R}^5_+ \ : \ 0 \leq N(t) \leq \frac{\Lambda_1 + \Lambda_2}{\alpha}\right\}$$

if the initial conditions $S_1(0) = S_{01} \geq 0,\ S_2(0) = S_{02} \geq 0,\ I(0) = I_0 \geq 0,\ T(0) \geq 0,\ R(0) = R_0 \geq 0$.

It is well-known in epidemiology that the dynamical behavior of the model (2) depends on the basic reproductive number. By using the next generation method, the basic reproduction number for the model (2) is given by

$$\mathscr{R}_0 = \frac{1}{\alpha + \mu}\Big(\frac{\beta_1\Lambda_1}{\alpha + \delta_1} + \frac{\beta_2\Lambda_2}{\alpha + \delta_2}\Big). \tag{9}$$

In order to find the equilibrium point of the model (2), we must solve the following equations:

$$\Delta^{(\gamma)}S_1 = \Delta^{(\gamma)}S_2 = \Delta^{(\gamma)}I = \Delta^{(\delta)}T = \Delta^{(\delta)}R = 0.$$

By assuming $I = 0$, one finds the disease-free equilibrium of the fractional order Covid-19 model (2), denoted by $\mathscr{E}^0 = (S_1^0, S_2^0, I^0, T^0, R^0)$, is as follows:

$$S_1^0 = \frac{\Lambda_1}{\alpha+\delta_1},\ S_2^0 = \frac{\Lambda_2}{\alpha+\delta_2},\ I^0 = 0,\ T^0 = 0,\ R^0 = \frac{\delta_1\Lambda_1}{\mu(\alpha+\delta_1)} + \frac{\delta_2\Lambda_2}{\mu(\alpha+\delta_2)}. \tag{10}$$

Moreover, the endemic equilibrium of the fractional order Covid-19 model, denoted by $\mathscr{E}^* = (S_1^*, S_2^*, I^*, T^*, R^*)$, is as follows:

$$\begin{aligned} S_1{}^* &= \frac{\Lambda_1}{\beta_1 I^* + \alpha + \delta_1}, \\ S_2{}^* &= \frac{\Lambda_2}{\beta_2 I^* + \alpha + \delta_2}, \\ I^* &= \frac{b + \sqrt{b^2 + 4bc}}{2}, \\ T^* &= \frac{\mu I^*}{\alpha + \rho}, \\ R^* &= \frac{\delta_1 S_1^* + \delta_2 S_2^* + \rho I^*}{\alpha}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} b &= \frac{\Lambda_1 + \Lambda_2}{\alpha + \mu} - \frac{(\alpha+\delta_2)\beta_1}{\beta_2} - \frac{(\alpha+\delta_1)\beta_2}{\beta_1}, \\ c &= \frac{\Lambda_1(\alpha+\delta_2)}{(\alpha+\mu)\beta_2} + \frac{\Lambda_2(\alpha+\delta_1)}{(\alpha+\mu)\beta_1} - \frac{(\alpha+\delta_1)(\alpha+\delta_2)}{\beta_1\beta_2}. \end{aligned}$$

We will analyze the stability of these two equilibrium points. First of all, the Jacobian matrix of the vector field corresponding to model (2) is

$$J = \begin{bmatrix} -(\beta_1 I + \alpha + \delta_1) & 0 & -\beta_1 S_1 & 0 & 0 \\ 0 & -(\beta_2 I + \alpha + \delta_2) & -\beta_2 S_2 & 0 & 0 \\ -\beta_1 I & \beta_1 I & \beta_1 S_1 + \beta_2 S_2 - (\alpha+\mu) & 0 & 0 \\ 0 & 0 & \mu & -(\alpha+\rho) & 0 \\ \delta_1 & \delta_1 & 0 & \rho & -\alpha \end{bmatrix}. \tag{12}$$

The stability of the disease-free equilibrium $\mathscr{E}^0$ is given in the following theorem.

Theorem 3.1. *If $\mathscr{R}_0 < 1$, then the disease-free equilibrium $\mathscr{E}^0$ is asymptotically stable and becomes unstable when $\mathscr{R}_0 \geq 1$.*

Proof. Note that, the Jacobian matrix (12) at $\mathscr{E}^0 = \left(\frac{\Lambda_1}{\alpha+\delta_1}, \frac{\Lambda_2}{\alpha+\delta_2}, 0, 0, \frac{\delta_1\Lambda_1}{\mu(\alpha+\delta_1)} + \frac{\delta_2\Lambda_2}{\mu(\alpha+\delta_2)}\right)$ is given by:

$$J_{\mathscr{E}^0} = \begin{bmatrix} -(\alpha+\delta_1) & 0 & -\frac{\beta_1\Lambda_1}{\alpha+\delta_1} & 0 & 0 \\ 0 & -(\alpha+\delta_2) & -\frac{\beta_2\Lambda_2}{\alpha+\delta_1} & 0 & 0 \\ 0 & 0 & \frac{\beta_1\Lambda_1}{\alpha+\delta_1} + \frac{\beta_2\Lambda_2}{\alpha+\delta_2} - (\alpha+\mu) & 0 & 0 \\ 0 & 0 & \mu & -(\alpha+\rho) & 0 \\ \delta_1 & \delta_1 & 0 & \rho & -\alpha \end{bmatrix}. \tag{13}$$

This implies that the characteristic equation of (13) is

$$\begin{aligned} |J_{\mathscr{E}^0} - rI| &= [-\alpha - r]\Big[-(\alpha+\rho) - r\Big]\Big[-(\alpha+\lambda_1) - r\Big]\Big[-(\alpha+\lambda_2) - r\Big] \times \\ &\quad \Big[(\beta_1 S_1^0 + \beta_2 S_2^0 - (\alpha+\mu)) - r\Big] \\ &= 0. \end{aligned} \tag{14}$$

Based on the equation (14) one find that the eigenvalues of $J_{\mathscr{E}^0}$ are

$$r_1 = -\alpha, r_2 = -(\alpha+\rho), r_3 = -(\alpha+\delta_1), r_4 = -(\alpha+\delta_2)$$

and

$$r_5 = \beta_1 S_1^0 + \beta_2 S_2^0 - (\alpha+\mu) = \Big(\frac{\beta_1\Lambda_1}{\alpha+\delta_1} + \frac{\beta_2\Lambda_2}{\alpha+\delta_2}\Big) - (\alpha+\mu).$$

One can see that all eigenvalues of (14) satisfy $|\arg(r_j)| > \frac{\gamma\pi}{2}$ if

$$\Big(\frac{\beta_1\Lambda_1}{\alpha+\delta_1} + \frac{\beta_2\Lambda_2}{\alpha+\delta_2}\Big) - (\alpha+\mu) < 0. \tag{15}$$

that is

$$\mathscr{R}_0 = \frac{1}{\alpha+\mu}\Big(\frac{\beta_1\Lambda_1}{\alpha+\delta_1} + \frac{\beta_2\Lambda_2}{\alpha+\delta_2}\Big) < 1. \tag{16}$$

Moreover, at least one eigenvalue satisfy $|\arg(r_j)| < \frac{\gamma\pi}{2}$ when $\mathscr{R}_0 > 1$. Hence, $\mathscr{E}^0$ is locally asymptotically stable if $\mathscr{R}_0 < 1$ and becomes unstable if $\mathscr{R}_0 > 1$. □

We now consider the stability of the endemic equilibrium $\mathscr{E}^*$. The Jacobian matrix $J_{\mathscr{E}^*}$ is found by subtituting the endemic equilibrium $\mathscr{E}^*$ into (12). This implies that the eigenvalue of the matrice $J_{\mathscr{E}^*}$ is obtained by solving the following characteristic equation

$$(-\alpha - r)(-(\alpha+\rho) - r)p(r) = 0. \tag{17}$$

with $p(r)$ is the following third order polynomial

$$\begin{aligned} p(r) = \Big[& r^3 + \Big(\alpha + \mu + \beta_1 S_1^* + \beta_2 S_2^* - \frac{\Lambda_1}{S_1^*} - \frac{\Lambda_2}{S_2^*}\Big) r^2 + \Big((\alpha+\mu)\Big(\frac{\Lambda_1}{S_1^*} + \frac{\Lambda_2}{S_2^*}\Big) - \beta_1\Lambda_1 - \beta_2\Lambda_2 \\ & - \frac{\Lambda_1 \beta_2 S_2^*}{S_1^*} - \frac{\Lambda_2 \beta_1 S_1^*}{S_2^*} + \frac{\Lambda_2\Lambda_2}{S_1^* S_2^*} - \beta_1^2 S_1^* I^* - \beta_2^2 S_2^* I^*\Big) r + \Big((\alpha + \mu + \beta_1 S_1^* + \beta_2 S_2^*)\frac{\Lambda_1\Lambda_2}{S_1^* S_2^*} \\ & + \frac{\Lambda_1 \beta_2^2 S_2^* I^*}{S_1^*} + \frac{\Lambda_2 \beta_1^2 S_1^* I^*}{S_2^*} \Big]. \end{aligned} \tag{18}$$

One can see that the roots of (17) are $r_1 = -\alpha,\ r_2 = -(\alpha+\rho)$ and r_3, r_4, r_5 are the roots of $p(r)$. It is obvious that both r_1 and r_2 are negative, and thus Theorem 2.3 is satisfied. Since the algebraic form of solution of the equation (18) is quite complicated, we solve it numerically to find r_3, r_4, and r_5.

In order to show the validity of the results, let us examine the numerical example in [5], where $\Lambda_1 = 0.2, \Lambda_2 = 0.05, \beta_1 = 0.2, \beta_2 = 0.4, \alpha = 0.5, \mu = 0.1, \rho = 0.3$. The initial conditions are $S_1(0) = 0.45, S_2(0) = 0.15, I(0) = 0.1, T(0) = 0.2$ and $R(0) = 0.1$. To execute the model (2), let $\delta_1 = 0.15, \delta_2 = 0.08$. Base on these parameter values, we find the basic reproduction number $\mathscr{R}_0 = 0.4589$, thus the disease free equiblirium point is $\mathscr{E}^0 = (0.5, 0.1515, 0, 0, 0.8712)$. Graph of subpopulation S_1, S_2, I, T, R for $\gamma = 0.8$ is given in the Figure 2.

Furthermore, for the above data let us replace $\Lambda_1 = 0.6$ and $\Lambda_2 = 0.08$, then we have the basic reproduction number $\mathscr{R}_0 = 1.1342$, thus the equilibrium is endemic. Graph of subpopulation S_1, S_2, I, T, R for $\gamma = 0.8$ is given in the Figure 3. One can see that the infected and treatment subpopulations decrease if the subpopulations S_1 and S_2 are vaccinated. Moreover, the recovered subpopulation increased.

4. Conclusion

We have find the fractional order bi-modal $SITR$ model for dynamic of Covid-19 spread. A numerical test that illustrating the result has been presented. The numerical test shows that the infected and treatment subpopulations decrease if the subpopulations S_1 and S_2 are vaccinated. Moreover, the recovered subpopulation increased. The analysis shows that the bi-modal $SITR$ model give the adequate information about spread of Covid-19.

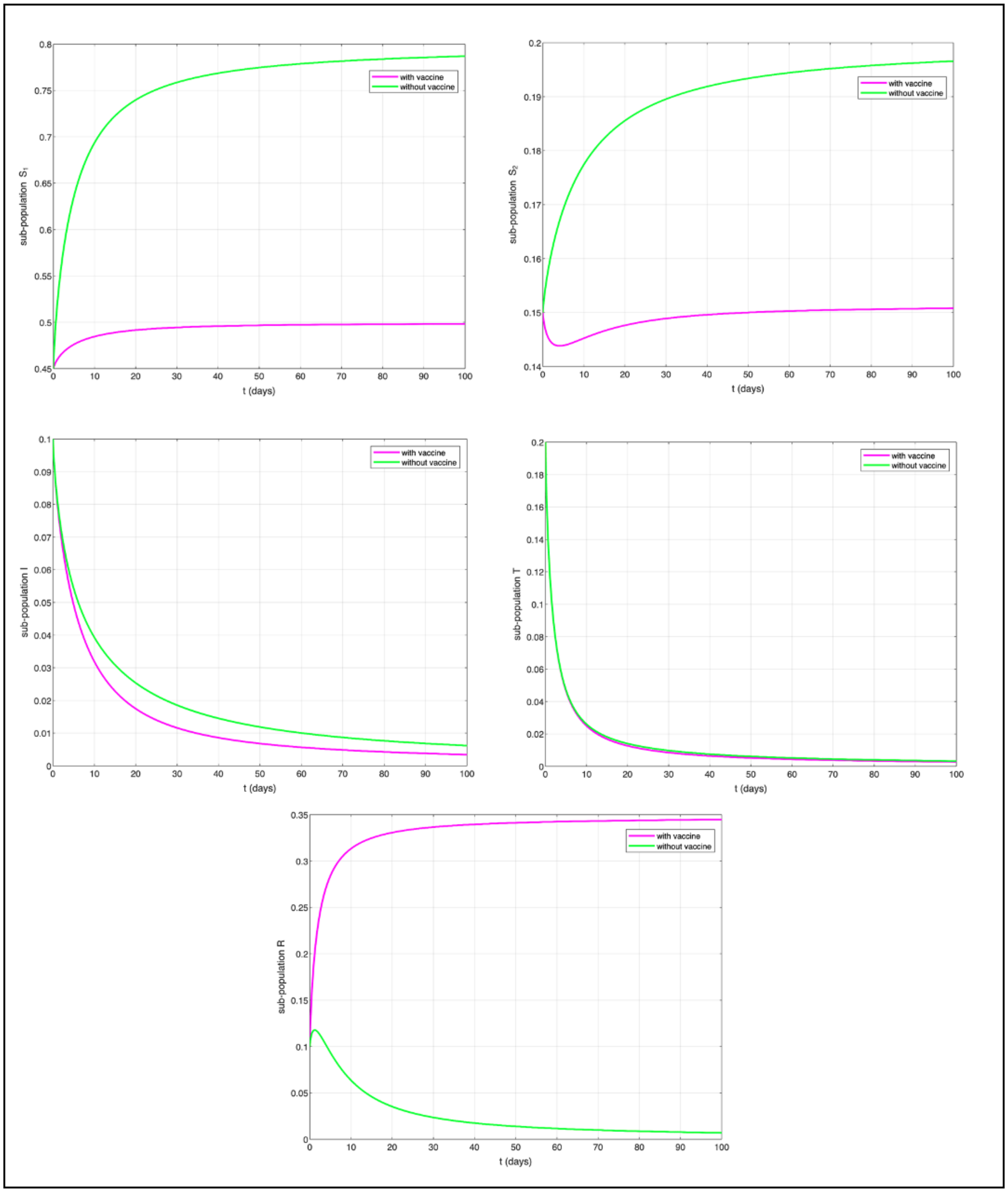

FIGURE 2. Curves of S_1, S_2, I, T, R for free disease equilibrium with $\gamma = 0.8$

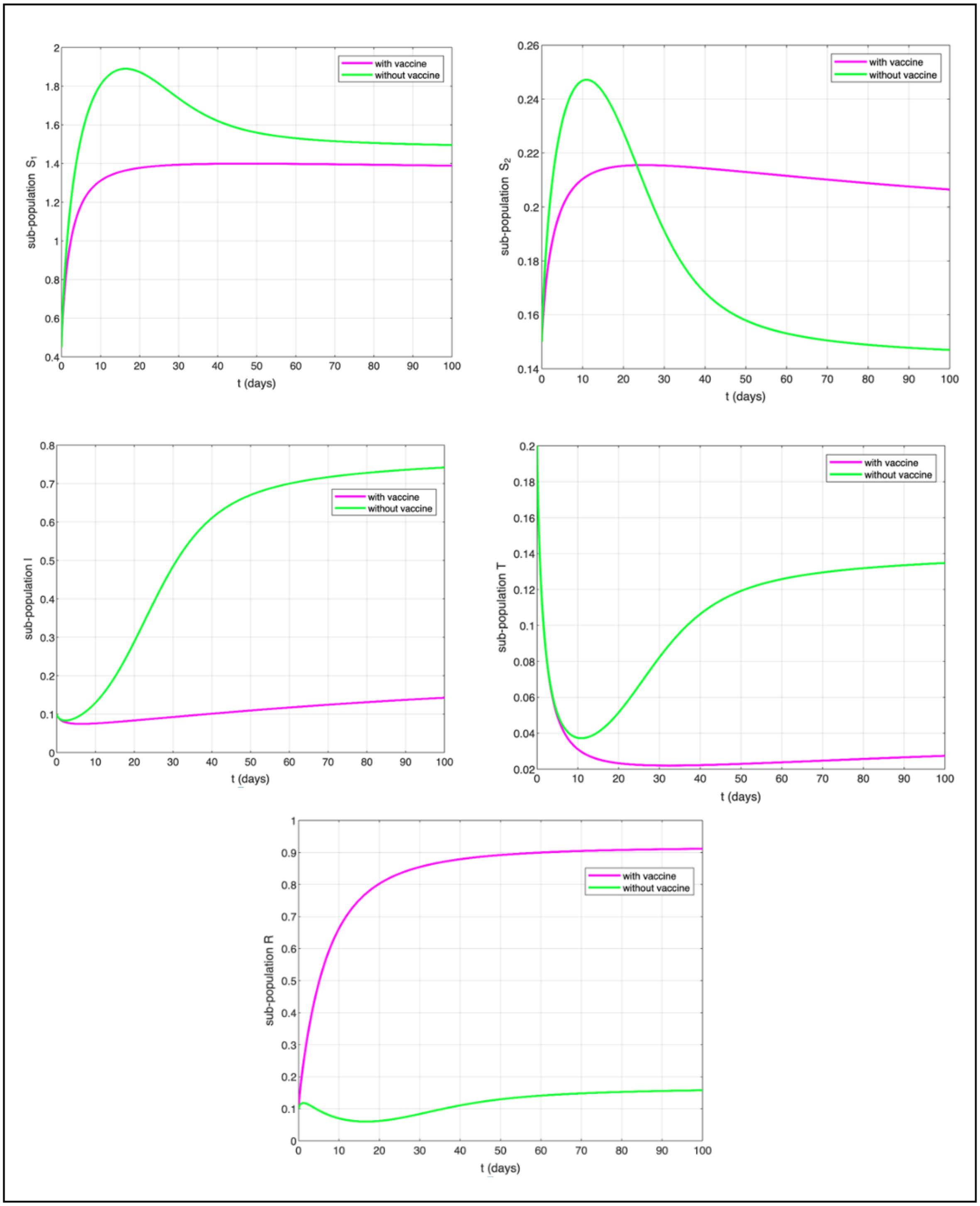

FIGURE 3. Curves of S_1, S_2, I, T, R for endemic equilibrium with $\gamma = 0.8$

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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