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# DYNAMIC BEHAVIOR OF A HARVESTED LOGISTIC MODEL INCORPORATING FEEDBACK CONTROL

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**Abstract.** We study the dynamic behavior of a logistic model with feedback control and harvesting. The solutions of the model are shown to be non-negative and uniformly bounded. We prove that the extinction point always exists and it is locally and globally asymptotically stable if the harvesting constant ($b$) is greater than one. For smaller harvesting constant, i.e. when $b < 1$, the model has also a positive equilibrium point which is locally and globally stable. These theoretical results are confirmed by our numerical simulations.

**Keywords:** logistic model; feedback control; harvesting, local stability; global stability; Lyapunov function.
**2010 AMS Subject Classification:** 34C25, 92D25, 34D20, 34D40.

## 1. INTRODUCTION

The logistic equation

$$\frac{dN(\tau)}{d\tau} = rN(\tau)\left(1 - \frac{N(\tau)}{K}\right), \tag{1}$$

is one of single-species population growth models and has been implemented as the basis for the development of several population dynamics models. In equation (1), $N(\tau)$ represents the

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population size at time $\tau$, while the parameters $r$ and $K$ are the intrinsic growth rate and environmental carrying capacity, respectively. It can be shown that if $N(0) > 0$, then the solution to equation (1) has the property that

$$\lim_{\tau\to\infty} N(\tau) = N^* = K,$$

namely that the population grows until it reaches saturation level $(K)$. However, as explained by Gopalsamy and Weng [1], the equilibrium $N^*$ in some situation is not desirable (or unattainable), and thus the population is required to be stable at a smaller value. This requirement can be achived by introducing feedback control variables into the system (1). Feedback control is a process whereby if a certain component of a system changes, then other components will also undergo a series of changes, which in turn affect the component that initially changed [2]. This feedback control can be achieved through biological control. A species can be also influenced by negative feedbacks in the environment in which it lives, such as the accumulation of toxic residues or artificial control adjustments. In this regard, Fan and Wang [3] proposed a logistics model with feedback control

$$\begin{aligned} \frac{dN(\tau)}{d\tau} &= rN(\tau)\left(1 - \frac{N(\tau)}{K} - au(\tau)\right),\ N(0) \geq 0 \\ \frac{du(\tau)}{d\tau} &= -eu(\tau) + CN(\tau),\ u(0) \geq 0, \end{aligned} \tag{2}$$

where $a, e$ and $C$ are positive parameters. Models of population growth of a single species with feedback control have attracted the interest of many researchers, for example see [5, 6, 7, 4]. Recently, feedback control has also been implemented to epidemic models in [8]. It has been shown that feedback control is effective in controlling and treating the disease. Based on this result, feedback control models for epidemic models have been widely applied for disease control purposes, see for example see [9, 10, 11, 12].

Harvesting of species has been widely practiced in aquaculture, agriculture, forestry and wildlife conservation for various reasons. Of course, harvesting policies and their optimization must be adjusted to the objectives. In general, harvesting is optimized in such a way as to maintain species conservation. However, for pest control, harvesting is performed with the aim of eradicating them. Hence, species harvesting is of great interest to many scholars. For example, the effect of harvesting in the predator-prey interaction has been studied in [13, 14, 15, 16, 17, 18, 19].

The effects of harvesting have not only been studied on predator-prey interactions, but also on populations of a single species. In a real sense, the population of a species always interacts with populations of other species, but the growth of a single population can be observed on a laboratory scale or in artificial breeding. Artificial breeding is carried out by humans because it provides many necessary roles in production or for economic purposes. Society needs to develop the resources of a single population for a long time and continuously, and give full play to the best use value based on the lowest possible consumption costs, so the control and prediction of a single population is very important [20]. Harvesting of a single species and its optimization have attracted many researchers, see [21, 22, 23, 24, 25, 26]. However, as far as we know, the effect of harvesting on a single species population with feedback control has not been studied in any literature. Thus, in this paper we modify the logistic model with feedback control (2) by adding a linear harvesting of the population and study the dynamic behavior of the model. The proposed model is given by

$$\begin{aligned}\frac{dN(\tau)}{d\tau} &= rN(\tau)\left(1-\frac{N(\tau)}{K}-au(\tau)\right)-BN(\tau),\ N(0)\geq 0\\ \frac{du(\tau)}{d\tau} &= -eu(\tau)+CN(\tau),\ u(0)\geq 0,\end{aligned} \tag{3}$$

where $B$ is the harvesting constant. For simplicity, the model (3) are simplified by introducing variable transformation $(N,\tau)\to\left(Kx,\frac{t}{r}\right)$ to get the following non-dimensional system

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\Big(1-x(t)-au(t)\Big)-bx(t),\ x(0)\geq 0\\ \frac{du(t)}{dt} &= -eu(t)+cx(t),\ u(0)\geq 0,\end{aligned} \tag{4}$$

where $b=\dfrac{B}{r}$ and $c=CK$. This paper is organized as follows. In Section 2 we show that the non-negativity and boundedness of solutions of system (4). Next we investigate the existence and local stability of equilibrium points of (4) in Section 3. The global stability properties of all possible equilibria are discussed in Section 4. Numerical simulations of the model (4) are presented in Section 5 to illustrate the analytical results. Finally, we present some conclusions in Section 6.

## 2. NON-NEGATIVITY AND BOUNDEDNESS OF SOLUTION

System (4) describes the population of a single species, and thus we require that the solution of this model must be non-negative and bounded. The non-negativity of solution is stated in the following theorem.

**Theorem 1.** *All solutions of system* (4) *with positive initial values* $x(0) \geq 0$ *and* $u(0) \geq 0$ *are always non-negative.*

*Proof.* From the first equation of system (4), we can show that

$$x(t) = x(0)\exp\left(\int_0^t (1 - x(s) - au(s) - b)\, ds\right).$$

Then, it is clear that $x(t) \geq 0$ if $x(0) \geq 0$. Moreover, based on the second equation of system (4), we have that

$$\frac{du(t)}{dt}|_{u=0} = cx(t).$$

Since $x(t) \geq 0$, $\frac{du(t)}{dt}|_{u=0} \geq 0$ and therefore, by applying Proposition B.7 in [27] we can show that $u(t) \geq 0$ if $u(0) \geq 0$ and $x(0) \geq 0$. □

To show the boundedness of $x(t)$, we apply the comparison lemma as in [28]. The first equation of system (4) leads to

$$\frac{x(t)}{dt} = x(t)\Big(1 - x(t) - au(t) - b\Big) \leq x(t)\Big(1 - x(t)\Big).$$

Then, using the comparison theorem we get

$$\lim_{t\to\infty} \sup x(t) \leq 1.$$

Hence, $x(t)$ is uniformly bounded. Using the same method as in [1], we rewrite the second equation of system (4) as follows

$$\frac{d}{dt}\Big(u(t)\exp(et)\Big) = cx(t)\exp(et),$$

$$u(t)\exp(et) = u(0) + c\int_0^t x(s)\exp(es)\, ds$$

$$\leq u(0) + \frac{c}{e}\bar{x}\Big(\exp(et) - 1\Big);\ \bar{x} = \sup_{t\geq 0} x(t)$$

$$u(t) \leq u(0)\exp(-et) + \frac{c}{e}\bar{x}\left(1 - \exp(-et)\right), \; t > 0.$$

A summary of the results of the above analysis can be stated in the following theorem.

**Theorem 2.** *All solutions of system* (4) *with positive initial values* $x(0) \geq 0$ *and* $u(0) \geq 0$ *are uniformly bounded.*

## 3. Existence and Local Stability of Equilibria

**3.1. Existence of Equilibria.** The equilibrium points of (4) are determined by the following equation

$$x(t)\left(1 - x(t) - au(t)\right) - bx(t) = 0$$
$$-eu(t) + cx(t) = 0.$$

It is trivial to show that system (4) always admits the extinction equilibrium point $E^0 = (x_0, u_0) = (0,0)$. Furthermore, the system also has a unique positive equilibrium point $E^* = (x^*, u^*)$ where

$$x^* = \frac{e(1-b)}{e+ac}, \; u^* = \frac{c(1-b)}{e+ac}.$$

We find that the system (4) without feedback control and without harvesting has a positive equilibrium point $x^* = 1$. When the feedback control presents in the system but without harvesting, the positive equilibrium point is given by $x^* = \frac{e}{e+ac} < 1, u^* = \frac{c}{e+ac}$. The presence of both feedback control and harvesting in the system causes the positive equilibrium point to become smaller. Furthermore, it is clear that the positive equilibrium point $E^*$ exists only if $b < 1$. The existence of equilibrium points of system (4) is summarized in the following theorem.

**Theorem 3.** *System* (4) *has always an extinction equilibrium* $E^0 = (0,0)$. *The existence of positive equilibrium point is dependent on the value of b as follows.*

*(i). If* $b < 1$ *then system* (4) *also has a unique positive equilibrium point* $E^* = (x^*, u^*)$.

*(ii). Otherwise, if* $b \geq 1$ *then the positive equilibrium* $E^* = (x^*, u^*)$ *does not exist.*

**3.2. Local Stability of Equilibria.** To study the local stability of an equilibrium point $\hat{E} = (\hat{x}, \hat{u})$, we calculate the Jacobian matrix of system (4) evaluated at an equilibrium point $\hat{E}$ as follows

$$J(\hat{E}) = \begin{bmatrix} 1-2\hat{x}-a\hat{u}-b & -a\hat{x} \\ c & -e \end{bmatrix}. \tag{5}$$

Hence, the Jacobian matrix at the extinction point $E^0$ is

$$J(E^0) = \begin{bmatrix} 1-b & 0 \\ c & -e \end{bmatrix}.$$

Clearly that $J(E^0)$ has two eigenvalues, namely $\lambda_1 = 1-b$ and $\lambda_2 = -e < 0$. Thus, if $b < 1$ then we get $\lambda_1 > 0$ and $E^0$ is a hyperbolic saddle. On the other hand, if $b > 1$, we have $\lambda_1 < 0$, and then $E^0$ is asymptotically stable.

Similarly, the Jacobian matrix at the positive equilibrium point $E^*$ is

$$J(E^*) = \begin{bmatrix} 1-2x^*-au^*-b & -ax^* \\ c & -e \end{bmatrix}.$$

The determinant and the trace of $J(E^*)$ are respectively given by

$$Det(J(E^*)) = e(b+2x^*+au^*)+acx^* = e(1-b),$$
$$Tr(J(E^*)) = 1-2x^*-au^*-b-e = -e(1+\frac{e(1-b)}{ac+e}).$$

It is seen that if $b < 1$ then $Det(J(E^*)) > 0$ and $Tr(J(E^*)) < 0$. If $b > 1$ then $Det(J(E^*)) < 0$. Therefore, if $b < 1$ then $E^*$ is locally asymptotically stable. On the contrary, if $b > 1$ then $E^*$ is unstable. We summarize the local stability of equilibrium points in the following theorem.

**Theorem 4.** *The local stability properties of equilibrium points of system* (4) *is as follows.*

*(i). If $b > 1$ then the extinction point $E^0$ is asymptotically stable, while the positive equilibrium point $E^*$ is unstable.*

*(ii). If $b < 1$ then the extinction point $E^0$ is unstable, while the positive equilibrium $E^*$ is asymptotically stable.*

## 4. GLOBAL STABILITY

**Theorem 5.** *If $b > 1$ then the extinction point $E^0$ is globally asymptotically stable.*

*Proof.* When $b > 1$, Theorem 3 and 4 state that the system (4) only has the extinction equilibrium point $E^0$, which is locally asymptotically stable. Furthermore, Theorem 1 and 2 show that the solutions are always non-negative and uniformly bounded. The system (4) does not have a closed orbit in the first quadrant. This is caused by the fact that if a closed orbit exists then there must be a positive equilibrium in the interior of the closed orbit. Since $E^0$ is the only possible equilibrium point, hence the system does not have a closed orbit. Consequently, the extinction point $E^0$ is globally asymptotically stable. □

**Theorem 6.** *If $b < 1$ then the positive equilibrium point $E^*$ is globally asymptotically stable.*

*Proof.* We first consider a Lyapunov function

$$V(x,u) = \frac{c}{a}\left(x - x^* - x^* \ln\frac{x}{x^*}\right) + \frac{1}{2}(u-u^*)^2.$$

The first derivative of $V(x,u)$ with respect to $t$ is given by

$$\begin{aligned}
\frac{dV}{dt} &= \frac{c}{a}\left(1-\frac{x^*}{x}\right)\frac{dx}{dt} + (u-u^*)\frac{du}{dt}\\
&= \frac{c}{a}\left(1-\frac{x^*}{x}\right)(1-x-au-b)x + (u-u^*)(-eu+cx)\\
&= \frac{c}{a}(x-x^*)(x-x^*+a(u-u^*)) + (u-u^*)(-eu+cx+eu^*-cx^*)\\
&= -\frac{c}{a}(x-x^*)^2 - (u-u^*)^2 \leq 0.
\end{aligned}$$

It is observed that the only invariant set on which $\frac{dV}{dt} = 0$ is the singleton $\{E^* = (x^*,u^*)\}$. Then, according to the LaSalle's invariance principle, $E^*$ is globally asymptotically stable. □

## 5. NUMERICAL SIMULATIONS

To confirm our previous theoretical results, in this Section we present some numerical simulation results of the model (4), which are obtained by the fourth-order Runge-Kutta method.

Because field data are not available, we apply hypothetical parameter value. For the first numerical simulation, we use the following parameter values

$$a = 0.8,\ b = 0.25,\ c = 0.5,\ e = 1. \tag{6}$$

System (4) with parameter values as in (6) has two equilibrium points: the extinction point $E^0 = (0,0)$ and the positive equilibrium point $E^* = (0,5357, 0.2679)$. Since $b < 1$, $E^0$ is unstable while $E^*$ is globally asymptotically stable. This behavior is clearly seen in Figure 1 where all solutions with various initial values are convergent to the positive equilibrium point $E^*$.

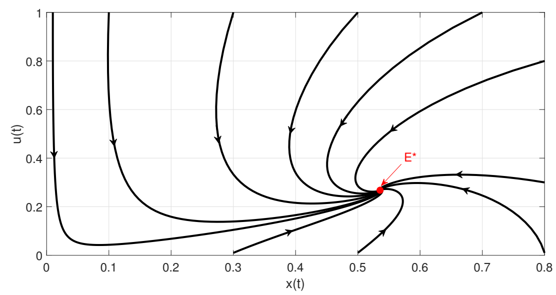


FIGURE 1. Phase portrait of system (4) with parameter values $a = 0.8, b = 0.25, c = 0.5$, and $e = 1$.

Next, we perform numerical simulations of system (4) with parameter values as in (6), except $b = 1.25$. Figure 2 shows that all numerical solutions with various initial values are convergent to the extincion point $E^0 = (0,0)$. This figure confirms our theoretical results that for $b > 1$, the system (4) does not have a positive equilibrium point and the extinction point $E^0$ is globally asymptotically stable.

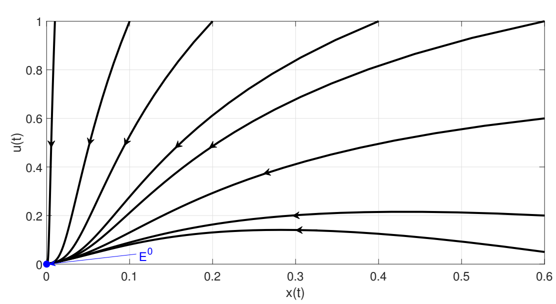


FIGURE 2. Phase portrait of system (4) with parameter values $a = 0.8, b = 1.25, c = 0.5$, and $e = 1$.

## 6. CONCLUSIONS

In this article, we investigate the effect of linear harvesting on a logistic model with feedback control. It is shown that all solutions of the logistic model with feedback control and harvesting are non-negative and uniformly bounded. The proposed model has always an extinction equilibrium point $E^0 = (0,0)$ which is globally asymptotically stable if the harvesting constant is greater than one $(b > 1)$. When the harvesting constant is less than one $(b < 1)$, then the system also has a positive equilibrium point $E^*$ which is globally asymtotically stable. The theoretical finding has been confirmed by our numerical simulations.

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### CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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