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EFFECT OF HUNTING COOPERATION AND FEAR IN A FOOD CHAIN MODEL WITH INTRASPECIFIC COMPETITION

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Abstract: Taking into account the significance of food chains in the environment, it demonstrates the interdependence of all living things and has economic implications for people. Hunting cooperation, fear, and intraspecific competition are all included in a food chain model that has been developed and researched. The study tries to comprehend how these elements affect the behavior of species along the food chain. We first examined the suggested model's solution properties before calculating every potential equilibrium point and examining the stability and bifurcation nearby. We have identified the factors that guarantee the global stability of the positive equilibrium point using the geometric approach. Additionally, the circumstances that would guarantee the continued existence of all living beings were computed. The theoretical findings were supported by numerical simulations, which also showed how altering parameter values affected the food chain's dynamic behavior.

Keywords: predator-prey; local stability; global stability; second additive compound matrix; hunting cooperation; fear effect; food chain; bifurcation analysis.

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1. INTRODUCTION

Prey-predator models have been used to explain a wide range of animal behavior, including the hunting and predation behaviors exhibited by predators and prey [1], [2], [3]. Several papers have

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suggested that predator-prey interactions follow a simple set of rules that govern the relationships between predators and their prey [3], [4]. Predation can be defined as a behavior in which one species attempts to kill another species. A predator is defined as a species that feeds on other species, while the prey is an animal that is targeted by a predator for consumption [5], [6]. Predators and prey are thought to have evolved distinct adaptations to survive in their different habitats [7]. The interaction between predators and prey can be influenced by a variety of factors; hence a lot of mathematical models had been developed to help us understand their behavior [3], [5]. In these models, the researchers considered various environmental factors that may affect the existence and stability of the model such as refuge of prey [8], [9], [10] harvesting [11], [12], [13], [14], disease [11], [15], seasonal variation [16], [17], delay [18], [19], sex structure and sexual favoritism [20], [21], Allee effects [22], [23], fear in prey populations [8], [12], [13], and many other factors affecting the dynamics of the system.

The basic model of prey-predator can be generalized to use with three or more species. Many researchers have extended the model into more realistic and complex forms by incorporating different types of habitats and considering multiple types of predator-prey relationships [11], [15], [24]. The relationship between these species can be put as a food chain [25], [26]. And food chain can be defined simply as a sequence of organisms feeding on each other [5], [27]. Food chains can be quite complex, involving a large number of species all competing for the same resources [28]. In addition, the behavior of individual animals within a food chain can have significant effects on the entire ecosystem.

The effects of fear induced by the predation process have been extensively examined in several theoretical models [29], [30]. Several studies have investigated the impact of predation on prey population dynamics by considering alternative strategies for escaping predation risk. In particular, the level of fear experienced by individual prey is generally associated with escape risk Two possible strategies employed by prey are referred to as "hiding" and "running away". The former entails a reduction in activity levels and a decrease in the likelihood of detection by predators [31].

The latter involves rapid movement and evasion of predators to prevent being killed. Both of these strategies have the potential to negatively influence prey population dynamics and survival. Conversely, reducing activity levels could result in a decreased food supply for reproduction, ultimately leading to a reduction in population size over the long term. Kumar and Kumari [32] showed that an increase in fear factors can make the system stable from chaotic dynamics, and large levels of fear can cause extinction of a population and they concluded that fear parameters can control chaotic dynamics in a food chain model. Pal [33] observed that high levels of fear can stabilize the eco-epidemiological model by excluding the periodic solution, and can also reduce the prey population as the level of fear increases. Maghool and Naji [34] showed that the increasing of the fear coefficients causes the chaotic regions to decrease and the solution to approach a stable point or periodic dynamics.

In ecosystems, some predators show cooperative behavior during hunting. This hunting cooperation usually creates fear in the prey community, in this case, prey tends to hide in a specific area or run away [35]. Hence, hunting cooperation has an indirect effect to increase the level of fear. For example, wolves are known to hunt in packs to hunt prey in large groups. Wolves in a pack can hunt faster than a single wolf to get at the same prey. Thus, prey (such as deer) that see a wolf pack, feel fear of wolves and will hide or run a safe distance away. Thus, the presence of hunting cooperation can increase the risk of prey that escaping from the predators by decreasing the probability of predator attack. Many researchers studied the impacts of hunting cooperation and fear effect in their suggested models separately or together. In [36], Belew and Melese observed that increasing the hunting cooperation of the predator population can decrease both the prey and predator populations, while the fear factor has a stabilizing effect on the dynamics of the system. The researchers in [37], showed that an increase in hunting cooperation by predators can create fear in prey populations, which can lead to a decrease in their birth rate. This can destabilize the ecosystem and lead to periodic oscillations. The strength of cooperation and fear can determine the stability of the ecosystem.

Another important factor that may affect the dynamics of the system is intraspecific competition, which is occurs between the individuals of the same species [30]. It occurs when the available resource is limited and the individuals in a species compete for this resource. In this situation, individuals with a larger amount of resources win, and individuals with less amount of resources lose out.

In a model of a three-species food chain, the impacts of hunting cooperation and fear are examined in this article. The model also takes into account the impact of intraspecific competition. The remainder of the article is structured as follows. The mathematical model is built in section 2. Section 3 provides evidence for the solution's positivity and boundness. Section 4 examines the stability of equilibria and analyzes the presence of equilibria. Section 5 talks about the system's persistence. Section 6 examines the global stability of the equilibrium points. The bifurcation analysis is provided in section 7. Section 8 provides the model's numerical simulation. The results discussion and final observations are offered at the end.

2. MATHEMATICAL MODEL FORMULATION

This section proposes and investigates an ecological food chain system that takes into account fear cost, hunting cooperation potential, and intraspecific competition. The cost of fear is thought to have an impact on both the growth and death rates, as well as the consumption rate of middle predators. As a result, the following assumptions are used to mathematically formulate the described system.

1. Without predation, it is thought that the population of prey increases logistically. It pays a cost for its fear of the growth and death of those it fears due to the presence of the middle predator.
2. It is believed that the middle predator eats the prey in accordance with the Lotka-Volterra functional response, which is impacted by the fear cost brought on by the presence of the top predator. Additionally, the middle predator's rate of death is impacted by the fear cost brought

on by the presence of top predators. The middle predator's individuals compete with one another within their own species.

3. According to the Lotka-Volterra functional response, which is influenced by the top predator's capability for hunting cooperatively, it is assumed that the top predator eats the middle predator. The attack rate, let's say $a_2 > 0$, can therefore be enhanced by the cooperative term to become $(a_2 + hZ)$, where $h \geq 0$ denotes the top predator cooperation hunting rate [38]. The top predator individuals compete with one another within their own species.

Accordingly, the following set of nonlinear differential equations of the first order can be used to characterize the movements of the stated ecological food chain system.

$$\begin{aligned} \frac{dX}{dT} &= \frac{rX}{1+k_1Y} - (1+b_1Y)d_1X - c_1X^2 - \frac{a_1XY}{1+k_2Z} \\ \frac{dY}{dT} &= \frac{e_1a_1XY}{1+k_2Z} - (1+b_2Z)d_2Y - c_2Y^2 - (a_2+hZ)YZ, \\ \frac{dZ}{dT} &= e_2(a_2+hZ)YZ - d_3Z - c_3Z^2 \end{aligned} \tag{1}$$

where $X(T)$, $Y(T)$, and $Z(T)$ stand for the density of the prey's community, the density of the middle predator's community, and the density of the top predator's community at time T respectively. Obviously, $\mathbb{R}^3_+ = \{(X,Y,Z) \in \mathbb{R}^3 : X(T) \geq 0, Y(T) \geq 0, Z(T) \geq 0\}$ describes the domain of the system (1). Furthermore, all the parameters are set to be nonnegative and defined in Table (1).

Table 1: Parameters description

Parameters	description
$r > 0$	The rate of birth of X
$k_1 \geq 0$	The intensity of fear in X
$k_2 \geq 0$	The intensity of fear in Y
$b_1 \geq 0$	The intensity of fear that affects the death of X
$b_2 \geq 0$	The intensity of fear that affects the death of Y
$d_1 \in (0,1)$	The rate of mortality for X

$d_2 \in (0,1)$	The rate of mortality for Y
$d_3 \in (0,1)$	The rate of mortality for Y
$c_1 > 0$	The intraspecific competition rate in X
$c_2 > 0$	The intraspecific competition rate in Y
$c_3 > 0$	The intraspecific competition rate in Z
$a_1 > 0$	The per-capita consumption rate of Y
$a_2 > 0$	The attack rate of Z
$e_1 \in (0,1)$	The conversion rate from X biomass into Y biomass
$e_2 \in (0,1)$	The conversion rate from Y biomass into Z biomass
$h > 0$	The cooperation of hunting rate of Z

Be aware that when utilizing variables scaling $rT = t$, $x = \frac{c_1}{r}X$, $y = \frac{a_1}{r}Y$, and $z = \frac{a_2}{r}Z$ the total quantity of parameters in system (1) is reduced from 16 to 12, system (2) adopts the following dimensionless form:

$$\begin{aligned}\frac{dx}{dt} &= x\left(\frac{1}{1+w_1y} - w_2(1+w_3y) - x - \frac{y}{1+w_4z}\right) := xg_1(x,y,z),\\ \frac{dy}{dt} &= y\left(\frac{w_5x}{1+w_4z} - w_6(1+w_7z) - w_8y - (1+w_9z)z\right) := yg_2(x,y,z),\\ \frac{dz}{dt} &= z(w_{10}(1+w_9z)y - w_{11} - w_{12}z) := zg_3(x,y,z),\end{aligned} \tag{2}$$

with the starting conditions:

$$x(0) = x_0 \geq 0,\ y(0) = y_0 \geq 0, z(0) = z_0 \geq 0. \tag{3}$$

However, the dimensionless parameters are given by:

$$w_1 = \frac{k_1 r}{a_1},\ w_2 = \frac{d_1}{r},\ w_3 = \frac{b_1 r}{a_1},\ w_4 = \frac{k_2 r}{a_2},\ w_5 = \frac{e_1 a_1}{c_1}, w_6 = \frac{d_2}{r},$$

$$w_7 = \frac{b_2 r}{a_2},\ w_8 = \frac{c_2}{a_1},\ w_9 = \frac{h r}{a_2^2},\ w_{10} = \frac{e_2 a_2}{a_1}, w_{11} = \frac{d_3}{r},\ w_{12} = \frac{c_3}{a_2}.$$

Keep in mind that system (2) has a single solution that belongs to $\mathbb{R}_+^3$ because the functions on the right-hand side are continuous and have continuous partial derivatives.

3. POSITIVITY AND BOUNDEDNESS

In this section, the following theorems studied the positivity and uniformly bounded properties of system (2) solutions with the conditions (3).

Theorem 1: *With initial conditions (3), all system (2) solutions are permanently positive.*

Proof: The proof is direct and hence it is omitted.

Theorem 2: *Uniform bounds exist for all system (2) solutions that start in the positive octant* $\mathbb{R}_+^3$.

Proof: The first equation of system (2) shows that, $\frac{dx}{dt} \leq x(1-x)$, and the solution of this inequality is given by $x(t) \leq \frac{x_0}{e^{-t}(1-x_0)-x_0}$, where x_0 is the initial value of x at $t = 0$. Therefore, the solution $x(t)$ as $t \to \infty$ satisfies that $x \leq 1$.

Now, let $G(t) = w_5 w_{10} x(t) + w_{10} y(t) + z(t)$, then simple calculation gives

$$\frac{dG}{dt} \leq w_5 w_{10}(2 - w_2) - w_5 w_{10} x - w_6 w_{10} y - w_{11} z,$$

$$\frac{dG}{dt} \leq L - \delta G,$$

where $L = w_5 w_{10}(2 - w_2)$ and $\delta = \min\{1, w_6, w_{11}\}$, with the survival condition $1 - w_2 > 0$. Therefore, by the Gronwall lemma for differential inequality [39], $G(t) \leq \frac{L}{\delta}$ as $t \to \infty$. So, all the solutions of system (2) are uniformly bounded in the following region:

$$\left\{(x, y, z) \in \mathbb{R}_+^3 | 0 \leq x(t) \leq 1, 0 \leq w_5 w_{10} x(t) + w_{10} y(t) + z(t) \leq \frac{L}{\delta}\right\}. \quad \square$$

4. EXISTENCE AND THE STABILITY OF EQUILIBRIA

The existence of non-negative equilibria is examined, and the stability of these critical points is established. The non-negative equilibrium points are determined as follows:

The entire extinction of prey and predator species corresponds to the trivial equilibrium point $E_0(0, 0, 0)$ which always exists.

The absences of the predator species correspond to the axial equilibrium point $E_1(1 - w_2, 0,0)$ which exists when $w_2 < 1$. This condition is known as a survival condition of the prey species in the absence of their predators (top and middle).

The boundary equilibrium point $E_2(\bar{x},\bar{y},0)$, which corresponds to the absence of the top predator species exists when

$$\bar{x} = \frac{w_6+w_8 y}{w_5}, \tag{4}$$

and $\bar{y}$ is the positive root of the quadratic equation

$$\alpha_0 + \alpha_1 y + \alpha_2 y^2 = 0,$$

where

$$\alpha_0 = w_5(1-w_2) - w_6.$$

$$\alpha_1 = -(w_2 w_3 w_5 + w_8 + w_5 + w_1 w_2 w_5 + w_1 w_6) < 0.$$

$$\alpha_2 = -w_1(w_2 w_3 w_5 + w_8 + w_5) < 0.$$

It is clear that a unique positive root exists if:

$$w_6 < w_5(1-w_2), \tag{5}$$

Hence, the value of $\bar{y}$ can be written as:

$$\bar{y} = \frac{-\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2\alpha_0}}{2\alpha_2}. \tag{6}$$

Obviously, when $1 - w_2 < 0$, the boundary equilibrium point does not exist.

The positive equilibrium point $E_3(x^*,y^*,z^*)$ is found by solving system (2) for $x > 0$, $y > 0$, and $z > 0$. Direct computation gives that

$$\begin{aligned} x^* &= \frac{(1+w_4 z^*)\left[w_6 w_{10}(1+w_7 z^*)(1+w_9 z^*) + w_{10} z(1+w_9 z^*)^2 + w_8(w_{11}+w_{12} z^*)\right]}{w_5 w_{10}(1+w_9 z^*)}, \\ y^* &= \frac{w_{11}+w_{12} z^*}{(1+w_9 z^*) w_{10}}. \end{aligned} \tag{7}$$

While, z^* denotes the positive root of the following sixth-order equation.

$$N_1 z^6 + N_2 z^5 + N_3 z^4 + N_4 z^3 + N_5 z^2 + N_6 z + N_7 = 0, \tag{8}$$

where

$$N_1 = w_4^2 w_9^2 w_{10}(w_9 w_{10} + w_1 w_{12})$$

$$\begin{aligned} N_2 = w_4 w_9 w_{10}[&3 w_4 w_9 w_{10} + w_4 w_9 w_{10} w_6 w_7 + 2 w_9^2 w_{10} + w_1 w_4 w_9 w_{11} \\ &+ 2 w_1 w_4 w_{12} + w_1 w_4 w_6 w_7 w_{12} + 2 w_1 w_9 w_{12}] \end{aligned}$$

$$N_3 = 3w_4^2 w_9 w_{10}^2 + 2w_4^2 w_6 w_7 w_9 w_{10}^2 + 6w_4 w_9^2 w_{10}^2 + w_4^2 w_6 w_9^2 w_{10}^2 + 2w_4 w_6 w_7 w_9^2 w_{10}^2 + w_9^3 w_{10}^2 + 2w_1 w_4^2 w_9 w_{10} w_{11} + w_1 w_4^2 w_6 w_7 w_9 w_{10} w_{11} + 2w_1 w_4 w_9^2 w_{10} w_{11} + w_1 w_4^2 w_{10} w_{12} + w_1 w_4^2 w_6 w_7 w_{10} w_{12} + 4w_1 w_4 w_9 w_{10} w_{12} + w_1 w_4^2 w_6 w_9 w_{10} w_{12} + 2w_1 w_4 w_6 w_7 w_9 w_{10} w_{12} + w_4^2 w_8 w_9 w_{10} w_{12} + w_1 w_9^2 w_{10} w_{12} + w_1 w_4^2 w_8 w_{12}^2$$

$$N_4 = w_4^2 w_{10}^2 + w_4^2 w_6 w_7 w_{10}^2 + 6w_4 w_9 w_{10}^2 + 2w_4^2 w_6 w_9 w_{10}^2 + 4w_4 w_6 w_7 w_9 w_{10}^2 + 3w_9^2 w_{10}^2 - w_4 w_5 w_9^2 w_{10}^2 + w_2 w_4 w_5 w_9^2 w_{10}^2 + 2w_4 w_6 w_9^2 w_{10}^2 + w_6 w_7 w_9^2 w_{10}^2 + w_1 w_4^2 w_{10} w_{11} + w_1 w_4^2 w_6 w_7 w_{10} w_{11} + 4w_1 w_4 w_9 w_{10} w_{11} + w_1 w_4^2 w_6 w_9 w_{10} w_{11} + 2w_1 w_4 w_6 w_7 w_9 w_{10} w_{11} + w_4^2 w_8 w_9 w_{10} w_{11} + w_1 w_9^2 w_{10} w_{11} + 2w_1 w_4 w_{10} w_{12} + w_1 w_4^2 w_6 w_{10} w_{12} + 2w_1 w_4 w_6 w_7 w_{10} w_{12} + w_4^2 w_8 w_{10} w_{12} + 2w_1 w_9 w_{10} w_{12} + w_1 w_2 w_4 w_5 w_9 w_{10} w_{12} + w_2 w_3 w_4 w_5 w_9 w_{10} w_{12} + 2w_1 w_4 w_6 w_9 w_{10} w_{12} + w_1 w_6 w_7 w_9 w_{10} w_{12} + 2w_4 w_8 w_9 w_{10} w_{12} + 2w_1 w_4^2 w_8 w_{11} w_{12} + w_1 w_2 w_3 w_4 w_5 w_{12}^2 + 2w_1 w_4 w_8 w_{12}^2$$

$$N_5 = 2w_4 w_{10}^2 + w_4^2 w_6 w_{10}^2 + 2w_4 w_6 w_7 w_{10}^2 + 3w_9 w_{10}^2 - 2(1 - w_2) w_4 w_5 w_9 w_{10}^2 + 4w_4 w_6 w_9 w_{10}^2 + 2w_6 w_7 w_9 w_{10}^2 - w_5 w_9^2 w_{10}^2 + w_2 w_5 w_9^2 w_{10}^2 + w_6 w_9^2 w_{10}^2 + 2w_1 w_4 w_{10} w_{11} + w_1 w_4^2 w_6 w_{10} w_{11} + 2w_1 w_4 w_6 w_7 w_{10} w_{11} + w_4^2 w_8 w_{10} w_{11} + 2w_1 w_9 w_{10} w_{11} + w_1 w_2 w_4 w_5 w_9 w_{10} w_{11} + w_2 w_3 w_4 w_5 w_9 w_{10} w_{11} + 2w_1 w_4 w_6 w_9 w_{10} w_{11} + w_1 w_6 w_7 w_9 w_{10} w_{11} + 2w_4 w_8 w_9 w_{10} w_{11} + w_1 w_4^2 w_8 w_{11}^2 + w_1 w_{10} w_{12} + w_1 w_2 w_4 w_5 w_{10} w_{12} + w_2 w_3 w_4 w_5 w_{10} w_{12} + 2w_1 w_4 w_6 w_{10} w_{12} + w_1 w_6 w_7 w_{10} w_{12} + 2w_4 w_8 w_{10} w_{12} + w_5 w_9 w_{10} w_{12} + w_1 w_2 w_5 w_9 w_{10} w_{12} + w_2 w_3 w_5 w_9 w_{10} w_{12} + w_1 w_6 w_9 w_{10} w_{12} + w_8 w_9 w_{10} w_{12} + 2w_1 w_2 w_3 w_4 w_5 w_{11} w_{12} + 4w_1 w_4 w_8 w_{11} w_{12} + w_1 w_5 w_{12}^2 + w_1 w_2 w_3 w_5 w_{12}^2 + w_1 w_8 w_{12}^2$$

$$N_6 = -(1 - w_2) w_5 w_{10}^2 [w_4 + 2w_9] + w_{10}^2 [1 + 2w_4 w_6 + w_6 w_7 + 2w_6 w_9] + w_{10} w_{11} [w_1 + w_1 w_2 w_4 w_5 + w_2 w_3 w_4 w_5 + 2w_1 w_4 w_6 + w_1 w_6 w_7 + 2w_4 w_8 + w_5 w_9 + w_1 w_2 w_5 w_9 + w_2 w_3 w_5 w_9 + w_1 w_6 w_9 + w_8 w_9] + w_1 w_4 w_{11}^2 [w_2 w_3 w_5 + 2w_8] + w_{10} w_{12} [w_5 + w_1 w_2 w_5 + w_2 w_3 w_5 + w_1 w_6 + w_8] + 2w_1 w_{11} w_{12} [w_5 + w_2 w_3 w_5 + w_8]$$

$$N_7 = -(1 - w_2) w_5 w_{10}^2 + w_6 w_{10}^2 + w_{10} w_{11} [w_5 + w_1 w_2 w_5 + w_2 w_3 w_5 + w_1 w_6 + w_8] + w_1 w_{11}^2 [w_5 + w_2 w_3 w_5 + w_8]$$

If one set of the following requirements is met, direct calculation reveals that equation (8) has a single positive root.

$$\left.\begin{array}{c} N_4 > 0, N_6 < 0, N_7 < 0 \\ N_4 > 0, N_5 > 0, N_6 > 0, N_7 < 0 \\ N_4 < 0, N_5 < 0, N_6 < 0, N_7 < 0 \end{array}\right\} \tag{9}$$

The Jacobian matrix of system (2) is generated at any position (x, y, z), as shown below, to examine the behavior near it.

$$J(x, y, z) = \begin{bmatrix} x\frac{\partial g_1}{\partial x} + g_1 & x\frac{\partial g_1}{\partial y} & x\frac{\partial g_1}{\partial z} \\ y\frac{\partial g_2}{\partial x} & y\frac{\partial g_2}{\partial y} + g_2 & y\frac{\partial g_2}{\partial z} \\ z\frac{\partial g_3}{\partial x} & z\frac{\partial g_3}{\partial y} & z\frac{\partial g_3}{\partial z} + g_3 \end{bmatrix}, \tag{10}$$

where

$$\frac{\partial g_1}{\partial x} = -1, \frac{\partial g_1}{\partial y} = -w_2 w_3 - \frac{w_1}{(1+w_1 y)^2} - \frac{1}{1+w_4 z}, \frac{\partial g_1}{\partial z} = \frac{w_4 y}{(1+w_4 z)^2},$$

$$\frac{\partial g_2}{\partial x} = \frac{w_5}{1+w_4 z}, \frac{\partial g_2}{\partial y} = -w_8,\ \frac{\partial g_2}{\partial z} = -\left(1 + w_6 w_7 + 2w_9 z + \frac{w_4 w_5 x}{(1+w_4 z)^2}\right),$$

$$\frac{\partial g_3}{\partial x} = 0, \frac{\partial g_3}{\partial y} = w_{10}(1 + w_9 z),\ \frac{\partial g_2}{\partial z} = -w_{12} + w_9 w_{10} y.$$

Theorem 3: *For system (2)*

i. *The point E_0 is asymptotically stable locally if $w_2 > 1$.*

ii. *The point E_1 is asymptotically stable locally if $w_6 > w_5(1 - w_2)$.*

iii. *The point E_2 is asymptotically stable locally if $\bar{y} < \frac{w_{11}}{w_{10}}$.*

Proof: (i) Depending on the general Jacobian matrix given by (10), the Jacobian matrix at $E_0(0,0,0)$ is given by:

$$J_0(0,0,0) = \begin{bmatrix} 1 - w_2 & 0 & 0 \\ 0 & -w_6 & 0 \\ 0 & 0 & -w_{11} \end{bmatrix}. \tag{11}$$

The eigenvalues of J_0 are $\lambda_1 = 1 - w_2$, $\lambda_2 = -w_6$, and $\lambda_3 = -w_{11}$. So, if $w_2 > 1$, then the three eigenvalues are negative, and E_0 is an asymptotically stable locally or stable node. That means if the birth of the prey is less than its death, both prey and predators (top and middle) populations will be extinct.

(ii) The Jacobian matrix at $E_1(1 - w_2, 0,0)$ is given by:

$$J_1(E_1) = \begin{bmatrix} w_2 - 1 & -(1 - w_2)(1 + w_1 + w_2 w_3) & 0 \\ 0 & w_5(1 - w_2) - w_6 & 0 \\ 0 & 0 & -w_{11} \end{bmatrix}. \tag{12}$$

The eigenvalues of J_1 are $\lambda_1 = w_2 - 1$, $\lambda_2 = w_5(1-w_2) - w_6$, and $\lambda_3 = -w_{11}$. It is clear that $\lambda_1 < 0$ due to the condition of existence. Also, if $w_6 > w_5(1-w_2)$, then all three eigenvalues are negative, and E_1 is an asymptotically stable locally or stable node.

(iii) The Jacobian matrix at $E_2(\bar{x}, \bar{y}, 0)$ is given by:

$$J_2(E_2) = \begin{bmatrix} -\bar{x} & -\bar{x}\left(1 + w_2 w_3 + \frac{w_1}{(1+w_1\bar{y})^2}\right) & w_4\bar{x}\bar{y} \\ w_5\bar{y} & -w_8\bar{y} & -\bar{y}(1 + w_6 w_7 + w_4 w_5 \bar{x}) \\ 0 & 0 & w_{10}\bar{y} - w_{11} \end{bmatrix} = [j_{ik}]. \quad (13)$$

The characteristics equation of J_2 is

$$(\lambda^2 + A_1\lambda + A_2)[\lambda - (w_{10}\bar{y} - w_{11})] = 0, \quad (14)$$

where $A_1 = -(j_{11} + j_{22})$ and $A_2 = (j_{11}j_{22} - j_{12}j_{21})$. Obviously, $A_1 > 0$ and $A_2 > 0$, hence λ_1 and λ_2 have negative real parts. Moreover, $\lambda_3 = w_{10}\bar{y} - w_{11}$ will be negative if $\bar{y} < \frac{w_{11}}{w_{10}}$, and that makes E_2 an asymptotically stable locally. □

Theorem 4: *The point $E_3(x^*, y^*, z^*)$, is an asymptotically stable locally if*

$$y^* < \frac{w_{12}}{w_9 w_{10}}. \quad (15)$$

$$w_4 w_5 < \frac{(1+w_4z^*)^3(1+w_6w_7+2w_9z^*)}{y^* - (1+w_4z^*)x^*}. \quad (16)$$

Proof: At E_3, the Jacobian matrix can be written as

$$J_3(\mathrm{E}_3) = [a_{ij}]_{3\times3},$$

where the components of J_3 are as follows

$$a_{11} = -x^* < 0\ ,\ a_{12} = -x^*\left(w_2w_3 + \frac{w_1}{(1+w_1y^*)^2} + \frac{1}{1+w_4z^*}\right) < 0,\ a_{13} = \frac{w_4x^*y^*}{(1+w_4z^*)^2} > 0,$$

$$a_{21} = \frac{w_5y^*}{1+w_4z^*} > 0,\ a_{22} = -w_8y^* < 0,\ a_{23} = -y^*\left(1 + w_6w_7 + 2w_9z^* + \frac{w_4w_5x^*}{(1+w_4z^*)^2}\right) < 0,$$

$$a_{31} = 0,\ a_{32} = w_{10}(1 + w_9z^*)z^* > 0,\ a_{33} = (w_{10}w_9y^* - w_{12})z^*.$$

The corresponding characteristic equation is

$$\lambda^3 + \omega_1\lambda^2 + \omega_2\lambda + \omega_3 = 0,$$

where

$$\omega_1 = -(a_{11} + a_{22} + a_{33}),$$

$$\omega_2 = a_{22}a_{33} + a_{11}(a_{22} + a_{33}) - (a_{12}a_{21} + a_{23}a_{32}),$$

$$\omega_3 = a_{32}(a_{11}a_{23} - a_{13}a_{21}) + a_{33}(a_{12}a_{21} - a_{11}a_{22}),$$

Moreover

$$\omega_1\omega_2 - \omega_3 = -2a_{11}a_{22}a_{33} - (a_{11} + a_{22})[a_{11}a_{22} - a_{12}a_{21}] - a_{11}a_{33}(a_{11} + a_{33}) - (a_{22} + a_{33})[a_{22}a_{33} - a_{23}a_{32}] + a_{13}a_{21}a_{32}$$

Condition (15) ensure that $a_{33} < 0$, hence $\omega_1 > 0$ and condition (16) ensure that $\omega_3 > 0$ and $\omega_1\omega_2 - \omega_3 > 0$. Accordingly, E_3 is an asymptotically stable locally due to the Routh-Hurwitz criterion [40]. □

Table (2) summarizes the conditions for the existence of equilibrium points and their local stability criteria.

Table 2: Summary of existence and the stability of equilibria

Equilibrium Point	**Existence Conditions**	**Stability Criteria**
$E_0(0,0,0)$	always exists	$w_2 > 1$
$E_1(1 - w_1, 0,0)$	$w_2 < 1$	$w_2 > \frac{w_5 - w_6}{w_5}$
$E_2(\bar{x}, \bar{y}, 0)$	$w_2 < \frac{w_5 - w_6}{w_5}$	$\bar{y} < \frac{w_{11}}{w_{10}}$
$E_3(x^*, y^*, z^*)$	$0 < 1 - A < y^*(1 + w_1 y^*)$	$\frac{w_{12}}{w_9 w_{10}} < y^*$ $w_4 w_5 < \frac{(1 + w_4 z^*)^3(1 + w_6 w_7 + 2w_9 z^*)}{y^* - (1 + w_4 z^*)x^*}$

Table (2) makes it evident that due to the E_0's stability, no additional equilibrium points exist in the system (2). But when E_0 is unstable, either E_1 is stable and E_2 doesn't exist, or E_1 is unstable and E_2 exists.

5. PERSISTENCE

This section examines system (2)'s persistence. It is entirely understood that the system is regarded as persistent if and only if none of its species have gone extinct. Accordingly, system (2) persists if its domain's border planes do not have an omega limit set on its trajectory, which starts at a positive initial point.

Theorem 5: *System (2) has no periodic dynamics on the border planes.*

Proof: It is obvious that, system (2) contains only one subsystem, which is located in the xy-plane.

$$\begin{aligned} \frac{dx}{dt} &= x\left(\frac{1}{1+w_1 y} - w_2(1+w_3 y) - x - y\right) := f_1(x,y) \\ \frac{dy}{dt} &= y(w_5 x - w_6 - w_8 y) := f_2(x,y) \end{aligned} \quad . \qquad (17)$$

It is easily confirmed that subsystem (17) has a positive equilibrium point $E(\bar{x},\bar{y})$ in the interior of border plane xy-plane, which coincides with $E_2(\bar{x},\bar{y},0)$.

Therefore, Bendixson–Dulac theorem [41], [42], can be utilized to check the possibility of existence of the periodic dynamics on the positive quadrant of the border plane xy-plane.

Let $B_1(x,y) = \frac{1}{xy}$. Clearly, this function is positive and C^1 function in the interior of $\mathbb{R}^2_+$ of xy-plane. Now

$$\Delta_1(x,y) = \frac{\partial}{\partial x}(B_1 \cdot f_1) + \frac{\partial}{\partial y}(B_1 \cdot f_2) = -\left(\frac{1}{y} + \frac{w_8}{x}\right)$$

Obviously, $\Delta_1(x,y)$ does not change their sign and does not vanish, hence according to Bendixson–Dulac theorem there are no periodic dynamics in the interior of the positive quadrant of the xy −plane. □

Theorem 6: *Provided the following requirements are met, System (2) is uniformly persistent:*

i. $w_2 < 1$

ii. $w_6 < (1 - w_2)w_5$

iii. $\bar{y} > \frac{w_{11}}{w_{10}}$

Proof: Let $\varphi(x,y,z) = x^{p_1} y^{p_2} z^{p_3}$ be the average Lyapunov function [43], where p_1, p_2, and p_3 are positive constants. Obviously, φ is positive for all (x,y,z) in the first octant $\mathbb{R}^3_+$. Moreover, $\varphi \to 0$ when any one of the three variables goes to zero.

$$\psi(x,y,z) = \frac{\varphi'}{\varphi} = \sum_{i=1}^{3} p_i g_i(x,y,z),$$

where g_i for all $i = 1,2,3$ are given in system (2).

$$\psi(E_0) = (1 - w_2)p_1 - w_6 p_2 - w_{11} p_3.$$

$$\psi(E_1) = \big((1 - w_2)w_5 - w_6\big)p_2 - w_{11} p_3.$$

$$\psi(E_2) = p_3(w_{10}\bar{y} - w_{11}).$$

It is clear that $\psi(E_0) > 0$ if condition (i) is satisfied and p_1 is chosen to be sufficiently large in relation to p_2 and p_3. The expression $\psi(E_1) > 0$ holds under condition (ii) when p_2 is selected sufficiently larger than p_3. Finally, $\psi(E_3) > 0$ if condition (iii) holds with any choice of p_3. Hence, the average Lyapunov function is positive and the proof is complete [43]. □

6. GLOBAL STABILITY

This section examines the global stability of the positive equilibrium point E_3. It is widely known that an equilibrium point is asymptotically stable globally with regard to an open set D if it is asymptotically stable locally and its basin of attraction contains D, as shown in [28]. Consequently, a second additive compound matrix was used to discuss its stability.

Theorem 6: *The point E_3 is asymptotically stable globally in the $D \subset \mathbb{R}^3$ provided that*

$$\Theta < \mathrm{w}_8 y_{min}, \tag{18}$$

where all the new symbols are given in the proof.

Proof: System (2) can be reformulated in vector form as follows:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} xg_1(x,y,z) \\ yg_2(x,y,z) \\ zg_3(x,y,z) \end{bmatrix}, \tag{19}$$

where $\mathbf{F}: D \longrightarrow \mathbb{R}^3$ is a $C^1(D)$ and D is a simply connected open set.

The equilibrium point E_3 is said to be globally stable in D provided that it's locally stable and all the trajectories approach to E_3.

Let $A(\mathbf{X})$ be a 3×3 matrix-valued function that is C^1 for $\mathbf{X} \in D$, and assume that $A^{-1}(\mathbf{X})$ exists and continuous for $\mathbf{X} \in D$, which means D represents the compact absorbing set, so that they define by:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{y}{z} & 0 \\ 0 & 0 & \frac{y}{z} \end{pmatrix}, A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{z}{y} & 0 \\ 0 & 0 & \frac{z}{y} \end{pmatrix}.$$

Consider

$$B = A_{\mathbf{F}} A^{-1} + A J^{[2]} A^{-1},$$

where $A_{\mathbf{F}}$ is the matrix of the directional derivatives in the direction of **F** and $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix (10) [44], [45].

$$A_{\mathbf{F}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{y'}{z} - \frac{y\,z'}{z^2} & 0 \\ 0 & 0 & \frac{y'}{z} - \frac{y\,z'}{z^2} \end{pmatrix},$$

$$J^{[2]} = \begin{pmatrix} -x - \mathrm{w}_8 y & -y\left(1 + w_6 w_7 + 2 w_9 z + \frac{w_4 w_5 x}{(1+w_4 z)^2}\right) & -\frac{w_4 x y}{(1+w_4 z)^2} \\ w_{10}(1 + w_9 z) z & -x + (w_{10} w_9 y - w_{12}) z & -x\left(w_2 w_3 + \frac{w_1}{(1+w_1 y)^2} + \frac{1}{1+w_4 z}\right) \\ 0 & \frac{w_5 y}{1+w_4 z} & -w_8 y + (w_{10} w_9 y - w_{12}) z \end{pmatrix}.$$

Accordingly, the matrix B can be written as:

$$B = \left(b_{ij}\right)_{3\times 3},$$

where

$$b_{11} = -x - \mathrm{w}_8 y,$$

$$b_{12} = -z\left(1 + w_6 w_7 + 2 w_9 z + \frac{w_4 w_5 x}{(1+w_4 z)^2}\right),$$

$$b_{13} = -\frac{w_4 x z}{(1+w_4 z)^2},$$

$$b_{21} = w_{10}(1 + w_9 z) y,$$

$$b_{22} = \frac{y'}{y} - \frac{z'}{z} - x + (w_{10} w_9 y - w_{12}) z,$$

$$b_{23} = -x\left(w_2 w_3 + \frac{w_1}{(1+w_1 y)^2} + \frac{1}{1+w_4 z}\right),$$

$$b_{31} = 0,$$

$$b_{32} = \frac{w_5 y}{1+w_4 z},$$

$$b_{33} = \frac{y'}{y} - \frac{z'}{z} - w_8 y + (w_{10} w_9 y - w_{12}) z.$$

Therefore, the matrix B can be reformulated in block form at E_3 as:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

with

$$B_{11} = (b_{11}), B_{12} = [b_{12} \quad b_{13}], B_{21} = \begin{bmatrix} b_{21} \\ 0 \end{bmatrix}, B_{22} = \begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix}.$$

Consider the vector norm $\|\cdot\|$ in $\mathbb{R}^3$ as $\|(u_1, u_2, u_3)\| = \max\{|u_1|, |u_2| + |u_3|\}$, where (u_1, u_2, u_3) represents the vector in $\mathbb{R}^3$. Let $\mu(B)$ be the Lozinski measure with respect to the considered norm. Then by using a similar argument as in [46] or [47] it follows that

$$\mu(B) \leq \sup \{k_1, k_2\} \equiv \sup \{\mu_1(B_{11}) + |B_{12}|, \mu_1(B_{22}) + |B_{21}|\},$$

where $|B_{12}|$, and $|B_{21}|$ are matrix norms with respect to the $\mathcal{L}^1$ vector norm and μ_1 represents the Lozinski measure with respect to the $\mathcal{L}^1$ vector norm. Thus, it is obtained:

$$\mu_1(B_{11}) = B_{11} = -(x + \mathrm{w}_8 y).$$

$$\mu_1(B_{22}) = \max \{b_{22} + |b_{32}|, b_{33} + |b_{23}|\},$$

where

$$b_{22} + |b_{32}| = \frac{y'}{y} - \frac{z'}{z} - x + (w_{10} w_9 y - w_{12}) z + \frac{w_5 y}{1 + w_4 z}.$$

$$b_{33} + |b_{23}| = \frac{y'}{y} - \frac{z'}{z} - w_8 y + (w_{10} w_9 y - w_{12}) z + x \left(w_2 w_3 + \frac{w_1}{(1 + w_1 y)^2} + \frac{1}{1 + w_4 z} \right).$$

Accordingly, it easy to verify that

$$\mu_1(B_{22}) = \frac{y'}{y} - \frac{z'}{z} - w_8 y + (w_{10} w_9 y - w_{12}) z + x \left(w_2 w_3 + \frac{w_1}{(1 + w_1 y)^2} + \frac{1}{1 + w_4 z} \right).$$

Also, we have

$$\begin{aligned} |B_{12}| &= \max \left\{ \left| -z \left(1 + w_6 w_7 + 2 w_9 z + \frac{w_4 w_5 x}{(1 + w_4 z)^2} \right) \right|, \left| -\frac{w_4 x z}{(1 + w_4 z)^2} \right| \right\} \\ &= z \left(1 + w_6 w_7 + 2 w_9 z + \frac{w_4 w_5 x}{(1 + w_4 z)^2} \right) \end{aligned}.$$

$$|B_{21}| = |w_{10}(1 + w_9 z) y| + |0| = w_{10}(1 + w_9 z) y$$

Hence, it is resulted that

$$k_1 = \mu_1(B_{11}) + |B_{12}| = -(x + \mathrm{w}_8 y) + z\left(1 + w_6 w_7 + 2w_9 z + \frac{w_4 w_5 x}{(1+w_4 z)^2}\right).$$

$$\begin{aligned} k_2 &= \mu_1(B_{22}) + |B_{21}| = \frac{y'}{y} - \frac{z'}{z} - w_8 y + (w_{10} w_9 y - w_{12})z \\ &\quad + x\left(w_2 w_3 + \frac{w_1}{(1+w_1 y)^2} + \frac{1}{1+w_4 z}\right) + w_{10}(1 + w_9 z)y \\ &= \frac{(w_5+1)x}{1+w_4 z} - w_6(1 + w_7 z) - w_8 y - (1 + w_9 z)z + w_{11} \\ &\quad - w_8 y + w_{10} w_9 y z + x\left(w_2 w_3 + \frac{w_1}{(1+w_1 y)^2}\right) \end{aligned}.$$

Consequently, the following has resulted

$$k_1 \leq -\mathrm{w}_8 y_{min} + z_{max}\left(1 + w_6 w_7 + 2w_9 z_{max} + \frac{w_4 w_5}{(1+w_4 z_{min})^2}\right),$$

$$k_2 \leq \frac{(w_5+1)}{1+w_4 z_{min}} - w_8 y_{min} + w_{11} + w_{10} w_9 y_{max} z_{max} + w_2 w_3 + \frac{w_1}{(1+w_1 y_{min})^2},$$

where y_{min}, y_{max}, z_{min}, and z_{max} are positive constants represent the bounds of y and z in the interior of D. Now, define the positive constant Θ as

$$\Theta = \max\left\{z_{max}\left(1 + w_6 w_7 + 2w_9 z_{max} + \frac{w_4 w_5}{(1+w_4 z_{min})^2}\right), \frac{(w_5+1)}{1+w_4 z_{min}} + w_{11} + w_{10} w_9 y_{max} z_{max} + w_2 w_3 + \frac{w_1}{(1+w_1 y_{min})^2}\right\}.$$

Then, we obtain that

$$k_i \leq -\mathrm{w}_8 y_{min} + \Theta, \mathrm{i} = 1,2$$

Therefore, $\mu(B) \leq -(\mathrm{w}_8 y_{min} - \Theta)$, hence the proof is done provided that the condition (18) is met.

7. BIFURCATION ANALYSIS

This section attempts to determine the possibility of the occurrence of bifurcations in the system (2) and specify the conditions of obtaining saddle-node, transcritical, and pitchfork bifurcation near equilibria. Recall the equation (19), then it is simply to obtain that:

$$D^2\mathbf{F}(\mathbf{X})(\mathbf{V}, \mathbf{V}) = \left(c_{i1}^{[2]}\right)_{3\times 3}, \tag{20}$$

where $\mathbf{V} = (v_1, v_2, v_3)^T$ is a general vector with

$$c_{11}^{[2]} = \frac{2v_1v_3w_4y}{(1+w_4z)^2} - 2v_1^2 + 2x\left(\frac{v_2^2w_1^2}{(1+w_1y)^3} - \frac{v_3^2w_4^2y}{(1+w_4z)^3} + \frac{v_2v_3w_4}{(1+w_4z)^2}\right)$$
$$+v_1v_2\left(-2w_2w_3 - \frac{2w_1}{(1+w_1y)^2} - \frac{2}{1+w_4z}\right)$$

$$c_{21}^{[2]} = -2v_2(v_3 + v_2w_3 + v_3w_6w_7) - 2v_3^2w_9y - 4v_2v_3w_9z + \frac{2v_3^2w_4^2w_5xy}{(1+w_4z)^3}$$
$$-\frac{2v_3w_4w_5(v_2x+v_1y)}{(1+w_4z)^2} + \frac{2v_1v_2w_5}{1+w_4z}$$

$$c_{31}^{[2]} = 2v_3\big(-v_3w_{12} + w_{10}(v_2 + v_3w_9y + 2v_2w_9z)\big)$$

Furthermore, we have

$$D^3\mathbf{F}(\mathbf{X})(\mathbf{V},\mathbf{V},\mathbf{V}) = \left(c_{i1}^{[3]}\right)_{3\times 3}, \tag{21}$$

$$c_{11}^{[3]} = \frac{6v_1v_2^2w_1^2}{(1+w_1y)^3} - \frac{6v_2^3w_1^3x}{(1+w_1y)^4} + \frac{6v_2v_3w_4(v_1-v_3w_4x+v_1w_4z)}{(1+w_4z)^3} - \frac{6v_3^2w_4^2y(v_1-v_3w_4x+v_1w_4z)}{(1+w_4z)^4}.$$

$$c_{21}^{[3]} = -\frac{6v_3(v_1w_4w_5(1+w_4z)(v_2-v_3w_4y+v_2w_4z)+v_3(v_3w_4^3w_5xy+v_2(1+w_4z)(-w_4^2w_5x+w_9(1+w_4z)^3)))}{(1+w4z)^4}.$$

$$c_{31}^{[3]} = 6v_2v_3^2w_{10}w_9.$$

Theorem 7: *A transcritical bifurcation takes place at the trivial point E_0 of the system (2) when the parameters meet $w_2 = w_2^* = 1$, where w_2 is taken as the bifurcation parameter.*

Proof: In the following, we demonstrate the transversality requirement of transcritical bifurcation at $w_2 = w_2^*$ by using Sotomayor's theorem [48], [49]. At E_0 with $w_2 = w_2^*$ the matrix (11) becomes:

$$J_0^* = D\mathbf{F}(E_0, w_2^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -w_6 & 0 \\ 0 & 0 & -w_{11} \end{bmatrix}.$$

Simple calculation shows that, there are zero eigenvalue with two negative eigenvalues, moreover $\mathbf{U}_1 = (1,0,0)^{\mathrm{T}}$ and $\mathbf{W}_1 = (1,0,0)^{\mathrm{T}}$ are the eigenvector corresponding to the zero eigenvalue of J_0^* and J_0^{*T} respectively. Also, by using equation (20), it's obtained:

$$\mathbf{F}_{w_2} = \begin{pmatrix} -(1+w_3y)x \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_{w_2}(E_0, w_2^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$D\mathbf{F}_{w_2}(E_0, w_2^*).\mathbf{U}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

$$D^2\mathbf{F}(E_0, w_2^*)(\mathbf{U}_1, \mathbf{U}_1) = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, straightforward computation shows that:

$$\mathbf{W}_1^{\mathrm{T}}\mathbf{F}_{w_2}(E_0, w_2^*) = 0.$$

$$\mathbf{W}_1^{\mathrm{T}}\left(D\mathbf{F}_{w_2}(E_0, w_2^*).\mathbf{U}_1\right) = -1 \neq 0.$$

$$\mathbf{W}_1^{\mathrm{T}}[D^2\mathbf{F}(E_0, w_2^*)(\mathbf{U}_1, \mathbf{U}_1)] = -2 \neq 0.$$

Therefore, system (2) has a transcritical bifurcation near E_0 due to Sotomayor's theorem [48], [49].□

Theorem 8: *A transcritical bifurcation takes place at the axial point E_1 of the system (2) when the parameters meet $w_6 = w_6^* = w_5(1 - w_2)$, where w_6 is taken as the bifurcation parameter.*

Proof: Similarly at E_1 with $w_6 = w_6^*$ equation (12) becomes:

$$J_1^* = D\mathbf{F}(E_1, w_6^*) = \begin{bmatrix} w_2 - 1 & (w_2 - 1)(1 + w_1 + w_2 w_3) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -w_{11} \end{bmatrix}.$$

It is simply to verify that, there are zero eigenvalue with two negative eigenvalues, and the corresponding eigenvectors for the zero eigenvalue of J_1^* and $J_1^{*\mathrm{T}}$ are $\mathbf{U}_2 = (-(1 + w_1 + w_2 w_3), 1, 0)^{\mathrm{T}}$ and $\mathbf{W}_2 = (0,1,0)$ respectively. Moreover, it's obtained:

$$\mathbf{F}_{w_6} = \begin{pmatrix} 0 \\ -(1 + w_7 z)y \\ 0 \end{pmatrix} \Longrightarrow \mathbf{F}_{w_6}(E_1, w_6^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$D\mathbf{F}_{w_6}(E_1, w_6^*).\mathbf{U}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

$$D^2\mathbf{F}(E_1, w_6^*)(\mathbf{U}_2, \mathbf{U}_2) = \begin{pmatrix} 2w_1^2(1 - w_1) \\ -2w_3 - 2(1 + w_1 + w_2 w_3)w_5 \\ 0 \end{pmatrix}.$$

Therefore, straightforward computation shows that:

$\mathbf{W}_2^{\mathrm{T}}\mathbf{F}_{w_6}(E_1, w_6^*) = 0.$

$\mathbf{W}_2^{\mathrm{T}}\left(D\mathbf{F}_{w_6}(E_1, w_6^*).\mathbf{U}_2\right) = -1 \neq 0.$

$\mathbf{W}_2^{\mathrm{T}}[D^2\mathbf{F}(E_1, w_6^*)(\mathbf{U}_2, \mathbf{U}_2)] = -2w_3 - 2(1 + w_1 + w_2 w_3)w_5 \neq 0.$

Consequently, a transcritical bifurcation near E_1. □

Theorem 9: *A transcritical bifurcation takes place at the boundary point E_2 of the system (2) when the parameters meet $w_{11} = w_{11}^* = w_{10}\bar{y}$, where w_{11} is taken as the bifurcation parameter provided that:*

$$w_{10}(\rho_2 + w_9\bar{y}) \neq w_{12}. \tag{22}$$

A pitchfork bifurcation takes place otherwise if the following condition holds

$$\rho_2 \neq 0. \tag{23}$$

Proof: From equation (13) the Jacobian matrix at E_2 with $w_{11} = w_{11}^*$, which is denoted by $J_2^* = D\mathbf{F}(E_2, w_{11}^*)$ becomes:

$$J_2^* = \begin{bmatrix} -\bar{x} & -\bar{x}\left(1 + w_2 w_3 + \frac{w_1}{(1+w_1\bar{y})^2}\right) & w_4\bar{x}\bar{y} \\ w_5\bar{y} & -w_8\bar{y} & -\bar{y}(1 + w_6 w_7 + w_4 w_5\bar{x}) \\ 0 & 0 & 0 \end{bmatrix} = \left(k_{ij}\right)_{3\times3}.$$

Direct computation shows that J_2^* has zero eigenvalue with the other two eigenvalues having negative real parts, also $\mathbf{U}_3 = (\rho_1, \rho_2, 1)^T$ and $\mathbf{W}_3 = (0,0,1)$, where $\rho_1 = \frac{k_{12}k_{23} - k_{13}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} > 0$ and $\rho_2 = \frac{k_{13}k_{21} - k_{11}k_{23}}{k_{11}k_{22} - k_{12}k_{21}}$, are the eigenvectors corresponding to the zero eigenvalue of J_2^* and J_2^{*T} respectively. Moreover, it is resulted that:

$$\mathbf{F}_{w_{11}} = \begin{pmatrix} 0 \\ 0 \\ -z \end{pmatrix} \Longrightarrow \mathbf{F}_{w_{11}}(E_2, w_{11}^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$D\mathbf{F}_{w_{11}}(E_2, w_{11}^*).\mathbf{U}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

$$D^2\mathbf{F}(E_2, w_{11}^*)(\mathbf{U}_3, \mathbf{U}_3) = \left(c_{i1}^{[2]}(E_2, w_{11}^*)\right)_{3\times1},$$

where

$$c_{11}^{[2]}(E_2, w_{11}^*) = 2\rho_1 w_4 \bar{y} - 2\rho_1^2 + 2\bar{x}\left(\frac{\rho_2^2 w_1^2}{(1+w_1\bar{y})^3} - w_4^2\bar{y} + \rho_2 w_4\right) + \rho_1\rho_2\left(-2w_2w_3 - \frac{2w_1}{(1+w_1\bar{y})^2} - 2\right).$$

$$c_{21}^{[2]}(E_2, w_{11}^*) = -2\rho_2(1+\rho_2 w_3 + w_6 w_7) - 2w_9\bar{y} + 2w_4^2 w_5 \bar{x}\bar{y} - 2w_4 w_5(\rho_2\bar{x} + \rho_1\bar{y}) + 2\rho_1\rho_2 w_5$$

$$c_{31}^{[2]}(E_2, w_{11}^*) = 2[-w_{12} + w_{10}(\rho_2 + w_9\bar{y})]$$

Therefore, straightforward computation using condition (22) shows that:

$$\mathbf{W}_3^{\mathrm{T}}\mathbf{F}_{w_{11}}(E_2, w_{11}^*) = 0.$$

$$\mathbf{W}_3^{\mathrm{T}}\left(D\mathbf{F}_{w_{11}}(E_2, w_{11}^*).\mathbf{U}_3\right) = -1 \neq 0.$$

$$\mathbf{W}_3^{\mathrm{T}}[D^2\mathbf{F}(E_2, w_{11}^*)(\mathbf{U}_3, \mathbf{U}_3)] = 2[-w_{12} + w_{10}(\rho_2 + w_9\bar{y})] \neq 0.$$

Hence a transcritical bifurcation takes place at E_2. On the other hand, if $w_{10}(\rho_2 + w_9\bar{y}) = w_{12}$, then $\mathbf{W}_3^{\mathrm{T}}[D^2\mathbf{F}(E_2, w_{11}^*)(\mathbf{U}_3, \mathbf{U}_3)] = 0$. While from equation (21), it is obtained that

$$D^3\mathbf{F}(E_2, w_{11}^*)(\mathbf{U}_3, \mathbf{U}_3, \mathbf{U}_3) = \left(c_{i1}^{[3]}(E_2, w_{11}^*)\right)_{3\times 1},$$

where

$$c_{11}^{[3]}(E_2, w_{11}^*) = \frac{6\rho_1\rho_2^2 w_1^2}{(1+w_1\bar{y})^3} - \frac{6\rho_2^3 w_1^3 \bar{x}}{(1+w_1\bar{y})^4} + 6\rho_2 w_4(\rho_1 - w_4\bar{x}) - 6w_4^2\bar{y}(\rho_1 - w_4\bar{x}).$$

$$c_{21}^{[3]}(E_2, w_{11}^*) = -6[\rho_1 w_4 w_5(\rho_2 - w_4\bar{y}) + (w_4^3 w_5 \bar{x}\bar{y} + \rho_2(w_9 - w_4^2 w_5 \bar{x}))].$$

$$c_{31}^{[3]}(E_2, w_{11}^*) = 6\rho_2 w_{10} w_9.$$

Therefore, by using condition (23), it is obtained that

$$\mathbf{W}_3^{\mathrm{T}}[D^3\mathbf{F}(E_2, w_{11}^*)(\mathbf{U}_3, \mathbf{U}_3, \mathbf{U}_3)] = 6\rho_2 w_{10} w_9 \neq 0.$$

Hence, pitchfork bifurcation takes place and the proof is done.

Theorem 10: *suppose that condition (15) is satisfied, then a saddle-node bifurcation takes place at positive equilibrium point $E_3(x^*, y^*, z^*)$ of the system (2) when the parameters meet $w_8 = w_8^*$, where w_8 is taken as the bifurcation parameter provided that the following conditions are met.*

$$a_{12}a_{21}a_{33} < [a_{13}a_{21} - a_{11}a_{23}]a_{32}. \tag{24}$$

$$\sigma_3 c_{11}^{[2]}(E_4, w_8^*) + \sigma_4 c_{21}^{[2]}(E_4, w_8^*) + c_{31}^{[2]}(E_4, w_8^*) \neq 0$$

with

$$w_8^* = \frac{a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}}{a_{11}a_{33}\, y^*}.$$

Proof: The Jacobian matrix of system (2) at E_3 with $w_8 = w_8^*$ can be written:

$$J_3^* = D\mathbf{F}(E_3, w_8^*) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22}^* & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix}.$$

where a_{ij} are given in the theorem (4), and $a_{22}^* = a_{22}(w_8^*)$. It is easy to verify that

$$\det(J_3^*) = \omega_3(w_8^*) = a_{32}(a_{11}a_{23} - a_{13}a_{21}) + a_{33}(a_{12}a_{21} - a_{11}a_{22}^*) = 0.$$

Therefore, the characteristic equation of the $J_3(E_3)$ at $w_8 = w_8^*$ that is given in theorem (4) can be written as

$$\lambda^3 + \omega_1\lambda^2 + \omega_2\lambda = \lambda[\lambda^2 + \omega_1\lambda + \omega_2] = 0.$$

Accordingly, J_3^* has zero eigenvalue with the other two eigenvalues given by $\lambda_{1,2} = -\frac{\omega_1 \pm \sqrt{\omega_1{}^2 - 4\omega_2}}{2}$, where $\omega_1 > 0$, $\omega_2 > 0$ due to condition (15). Hence the eigenvalues $\lambda_{1,2}$ have negative real parts. It is determined that $\mathbf{U}_4 = (\sigma_1, \sigma_2, 1)^T$, where $\sigma_1 = \frac{a_{12}a_{23} - a_{13}a_{22}}{a_{11}a_{22}^* - a_{12}a_{21}} > 0$, and $\sigma_2 = \frac{a_{13}a_{21} - a_{11}a_{23}}{a_{11}a_{22}^* - a_{12}a_{21}} > 0$ due to condition (24) and $\mathbf{W}_4 = (\sigma_3, \sigma_4, 1)$, where $\sigma_3 = \frac{a_{21}a_{32}}{a_{11}a_{22}^* - a_{12}a_{21}} > 0$, and $\sigma_4 = -\frac{a_{11}a_{32}}{a_{11}a_{22}^* - a_{12}a_{21}} > 0$ are the corresponding eigenvectors related with the zero eigenvalue of J_3^* and $J_3^{*\mathrm{T}}$ respectively.

Moreover, it's obtained:

$$\mathbf{F}_{w_8} = \begin{pmatrix} 0 \\ -y^2 \\ 0 \end{pmatrix} \Longrightarrow \mathbf{F}_{w_8}(E_3, w_8^*) = \begin{pmatrix} 0 \\ -(y^*)^2 \\ 0 \end{pmatrix}.$$

$$D^2\mathbf{F}(E_3, w_8^*)(\mathbf{U}_4, \mathbf{U}_4) = \left(c_{i1}^{[2]}(E_4, w_8^*)\right)_{3\times 1},$$

where

$$c_{11}^{[2]}(E_4, w_8^*) = \frac{2\sigma_1 w_4 y^*}{(1+w_4 z^*)^2} - 2\sigma_1^2 + 2x^*\left(\frac{\sigma_2^2 w_1^2}{(1+w_1 y^*)^3} - \frac{w_4^2 y^*}{(1+w_4 z^*)^3} + \frac{\sigma_2 w_4}{(1+w_4 z^*)^2}\right)$$
$$+\sigma_1\sigma_2\left(-2w_2 w_3 - \frac{2w_1}{(1+w_1 y^*)^2} - \frac{2}{1+w_4 z^*}\right)$$

$$c_{21}^{[2]}(E_4,w_8^*)=-2\sigma_2(1+\sigma_2w_3+w_6w_7)-2w_9y^*-4\sigma_2w_9z^*+\frac{2w_4^2w_5x^*y^*}{(1+w_4z^*)^3}-\frac{2w_4w_5(\sigma_2x^*+\sigma_1y^*)}{(1+w_4z^*)^2}+\frac{2\sigma_1\sigma_2w_5}{1+w_4z^*}$$

$$c_{31}^{[2]}(E_4,w_8^*)=2[-w_{12}+w_{10}(\sigma_2+w_9y^*+2\sigma_2w_9z^*)]$$

Therefore, straightforward computation using condition (22) shows that:

$$\mathbf{W}_4^{\mathrm{T}}\mathbf{F}_{w_8}(E_3,w_8^*)=-\sigma_4(y^*)^2\neq 0.$$

$$\mathbf{W}_4^{\mathrm{T}}[D^2\mathbf{F}(E_3,w_8^*)(\mathbf{U}_4,\mathbf{U}_4)]=\sigma_3c_{11}^{[2]}(E_4,w_8^*)+\sigma_4c_{21}^{[2]}(E_4,w_8^*)+c_{31}^{[2]}(E_4,w_8^*)\neq 0.$$

Hence a saddle-node bifurcation takes place at the positive equilibrium point E_3.

8. Numerical Analysis

In this section, the aspects of system (2) dynamics are explored. The primary objective is to develop an understanding of how the system behaves as its parameters are altered. To accomplish this, a set of hypothetical parameter values (25) is employed. The solutions are analyzed through phase portraits and time series analyses using Matlab and AutoPortrait [50].

Parameter:	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}
Value:	0.3	0.05	0.15	0.3	0.75	0.1	0.25	0.3	0.5	0.6	0.15	0.5

(25)

Based on the data set provided (25) it can be observed that system (2) has a positive equilibrium point E_3 that remains stable globally over time. This stability is evident when analyzing the phase portrait shown in Figure (1a) and the corresponding time series depicted in Figures (1b-1e). Additionally, Figures (1f-1h) provide projections of the portrait onto the $xy-$, $xz-$ and $yz-$ planes. These figures demonstrate the existence of a globally asymptotically stable point of system (2), in agreement with the analytical results.

Figure 2 demonstrates that when the value of the parameter w_2 is increased to fall within the range of $[0.4720, 0.8830]$, system (2) loss its persistence since point E_3 is approached to E_2, and extinction of the top predator is observed, while it continued approach to E_1 when the value of w_2 surpasses the threshold of 0.8830, resulting in the extinction of the middle predator as well.

Furthermore, if the value of w_2 exceeds 1, the point E_3 will be approached to E_0, signifying the extinction of all species. It is important to note the features represented in this figure and its subsequent ones, including stable equilibrium point (●), saddle equilibrium point (●), streamlines (--), trajectory (--), and nullcline (--).

In Figure 3, it can be observed that as the value of the parameter w_5 is decreased to fall within the range of $[0.0930, 0.2600]$, system (2) again loss its persistence and point E_3 is approached to E_2, and the top predator extinct. Furthermore, when the value of w_5 is reduced below the threshold of 0.0930, E_3 continues its approach to E_1, resulting in the extinction of the middle predator as well.

On the other hand, Figure 4, demonstrates that when the value of the parameter w_6 is increased to fall within the range of $[0.4260, 0.7230]$, point E_3 is approached, leading to its transformation into E_2, and the observation of top predator extinction with a loss of the system persistence. Furthermore, when the value of w_6 exceeds the threshold of 0.7230, an approach to E_1 is continued, resulting in the extinction of the middle predator as well.

Figure 5 displays the behavior of the system (2) as the parameter w_{10} is varied. It is observed that point E_3 transforms into E_2 when the value of w_{10} falls below 0.2760, resulting in the occurrence of extinction in the top predator. In contrast, when w_{11} is less than 0.3090, the transformation of E_2 into E_3 is depicted, as shown in Figure 6.

Finally, a very large value of the parameter w_1 causes E_3 to be moved toward E_2, resulting in the decrease of the top predator population till extinction, as Figure 7 shows.

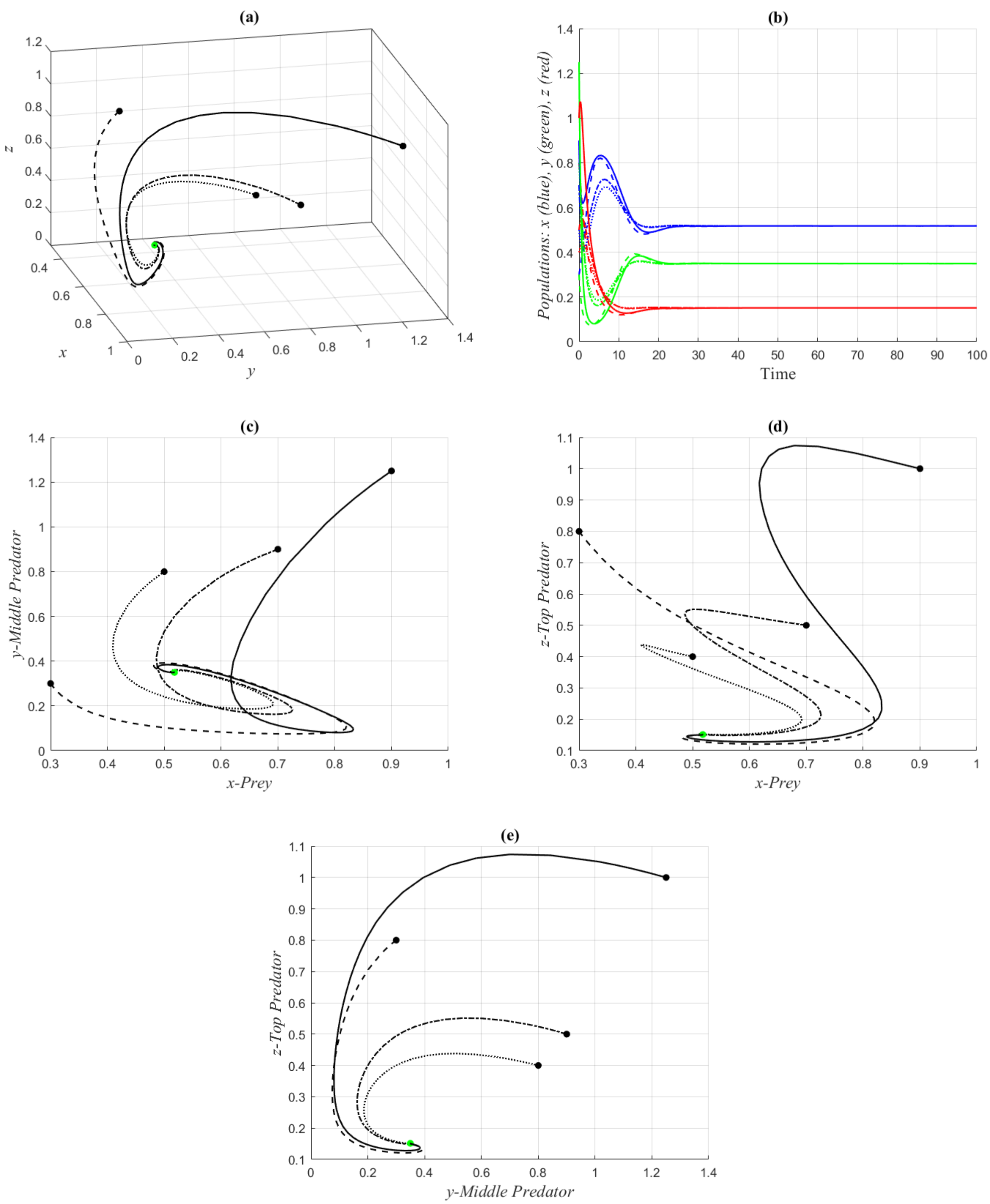
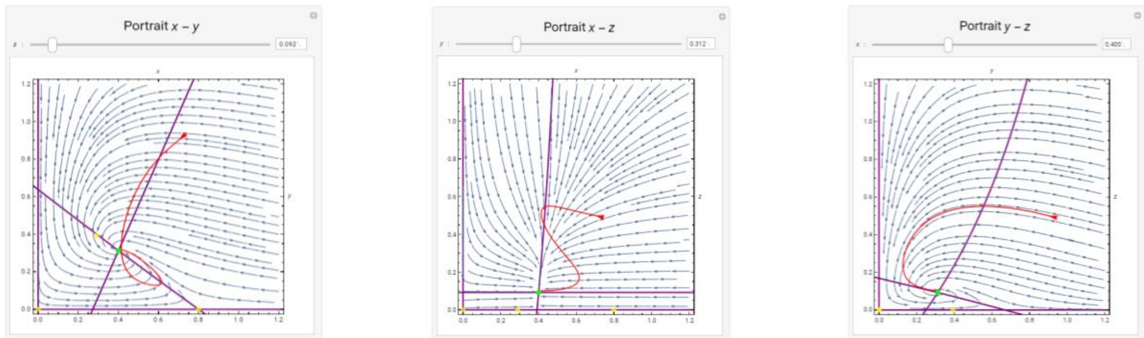
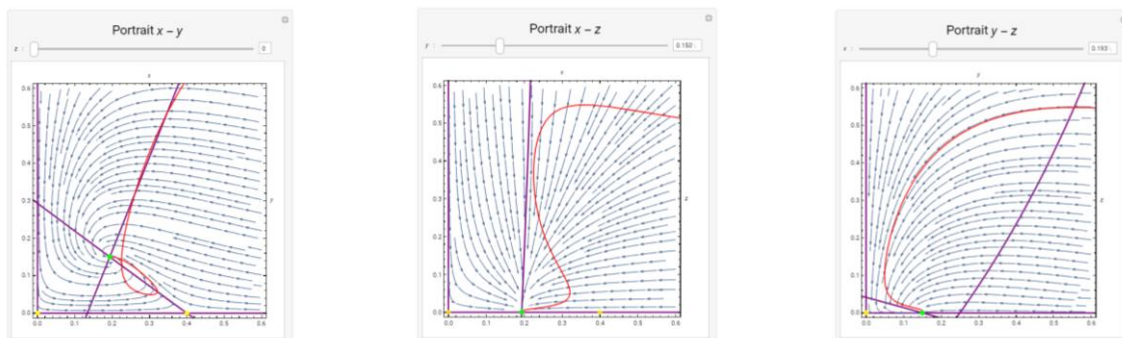


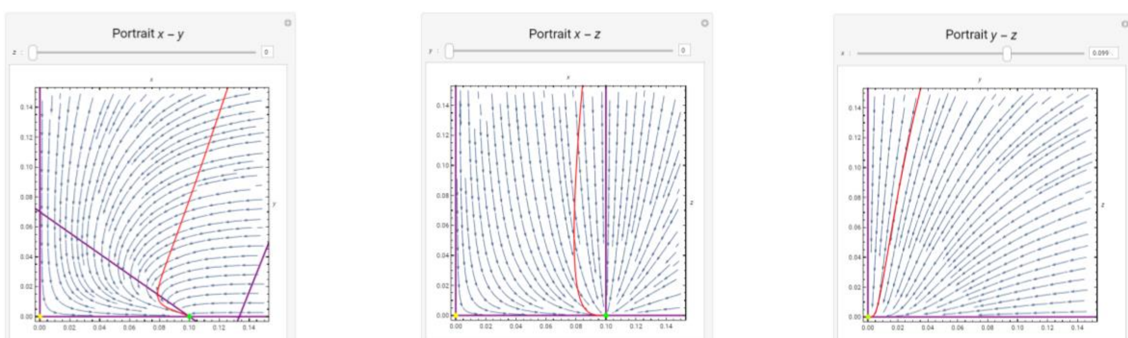
Figure 1: (a) The trajectories of system (2) approach asymptotically to the positive equilibrium point $E_3(0.5181,0.3496,0.1515)$ from different initial points with the data given by (25), **(b)** their corresponding time series, while **(c)**, **(d)**, **(e)** show the trajectories projection on the $xy-$, $xz-$ and $yz-$ plane respectively.



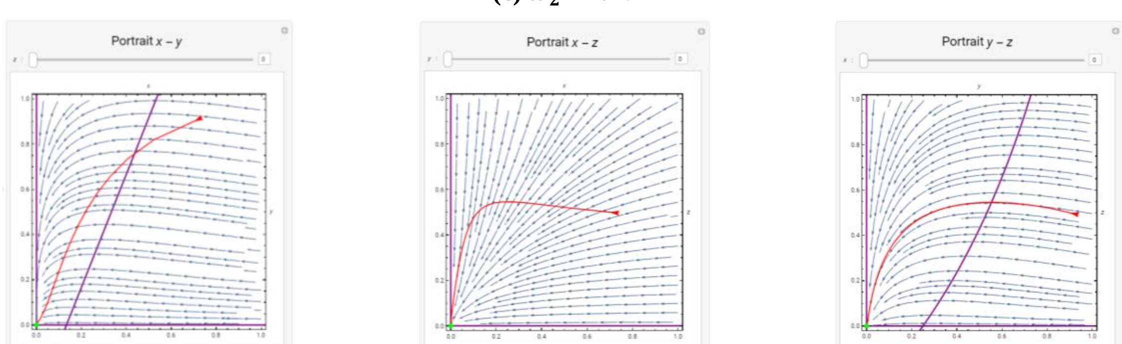
(a) $w_2 = 0.2$



(b) $w_2 = 0.6$

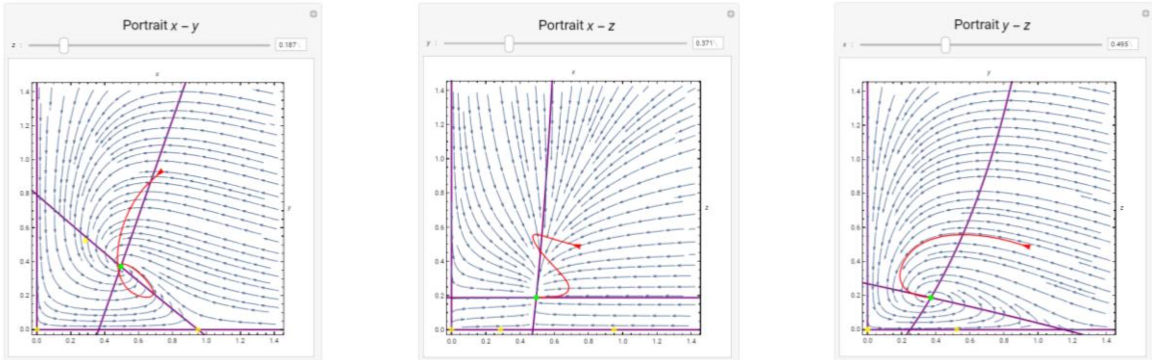


(c) $w_2 = 0.9$

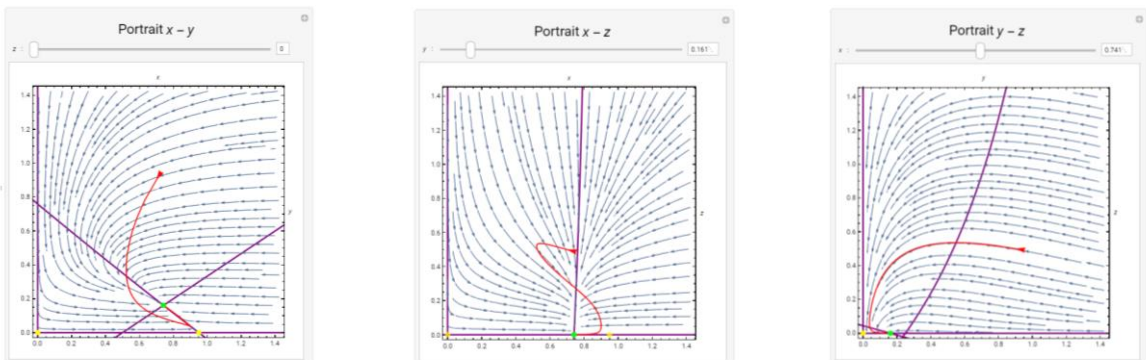


(d) $w_2 = 1.2$

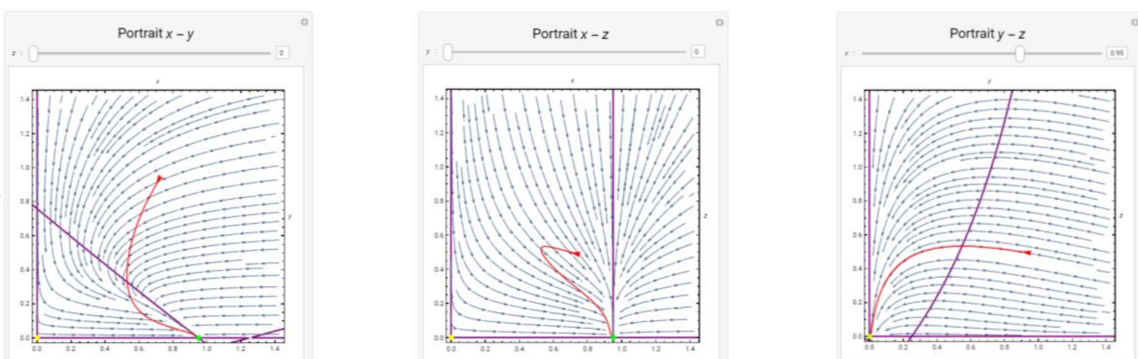
Figure 2: The phase portrait of the system (2) in the $xy-$, $xz-$, and $yz-$ plane, respectively, for different values of w_2.



(a) $\boldsymbol{w_5 = 0.9}$

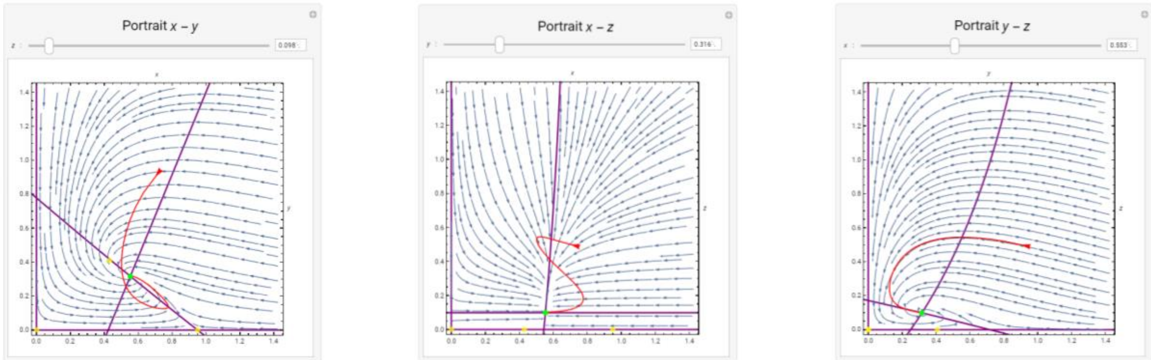


(b) $\boldsymbol{w_5 = 0.2}$

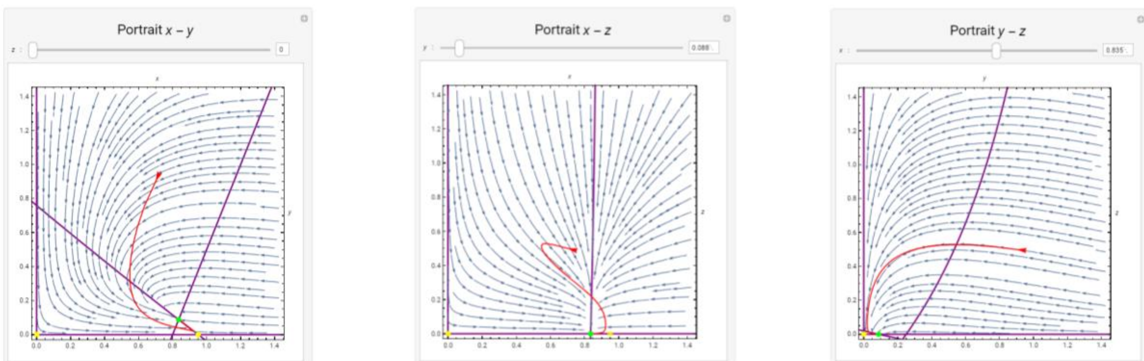


(c) $\boldsymbol{w_5 = 0.08}$

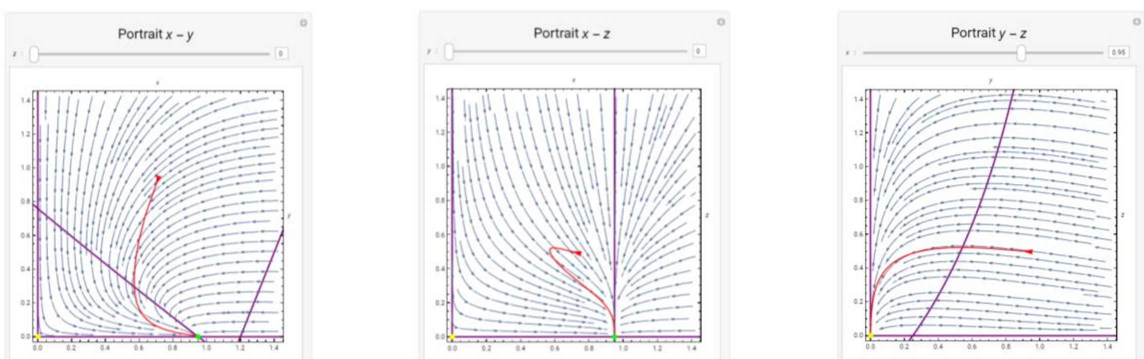
Figure 3: The phase portrait of the system (2) in the $xy-$, $xz-$, and $yz-$ plane, respectively, for different values of w_5.



(a) $\boldsymbol{w_6 = 0.2}$

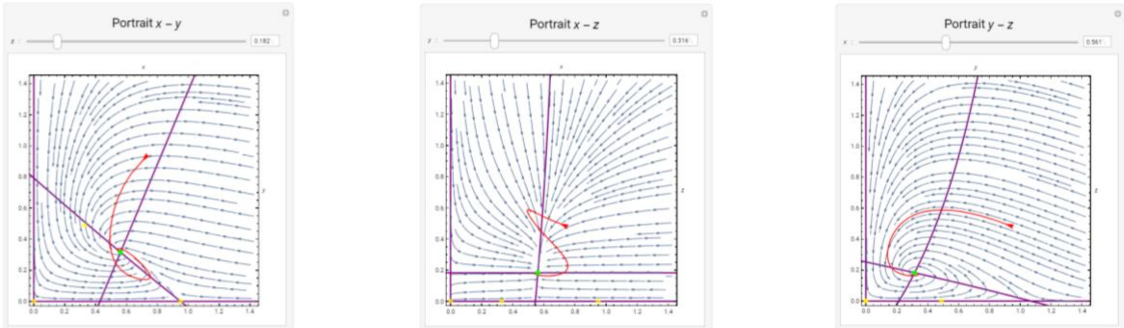


(b) $\boldsymbol{w_6 = 0.6}$

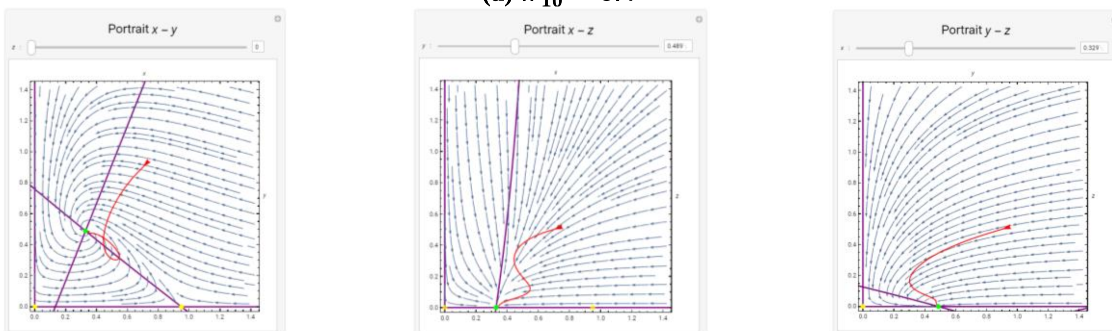


(c) $\boldsymbol{w_6 = 0.9}$

Figure 4: The phase portrait of the system (2) in the $xy-$, $xz-$, and $yz-$ plane, respectively, for different values of w_6.

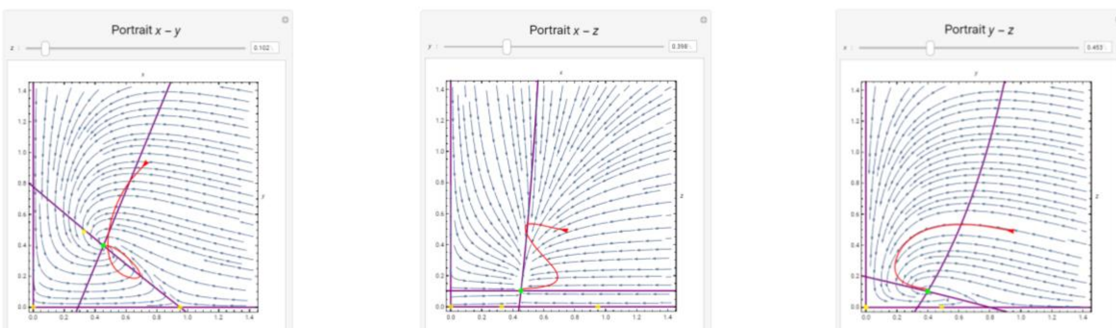


(a) $\mathbf{w_{10} = 0.7}$

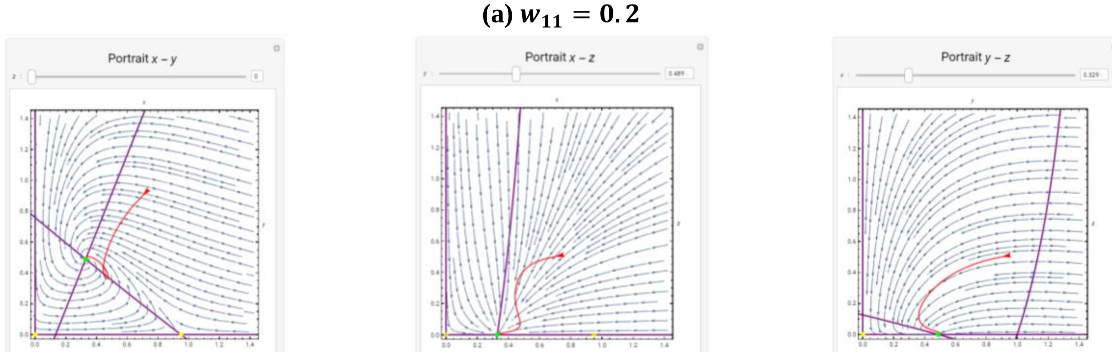


(b) $\mathbf{w_{10} = 0.1}$

Figure 5: The phase portrait of the system (2) in the $xy-$, $xz-$, and $yz-$ plane, respectively, for different values of w_{10}.

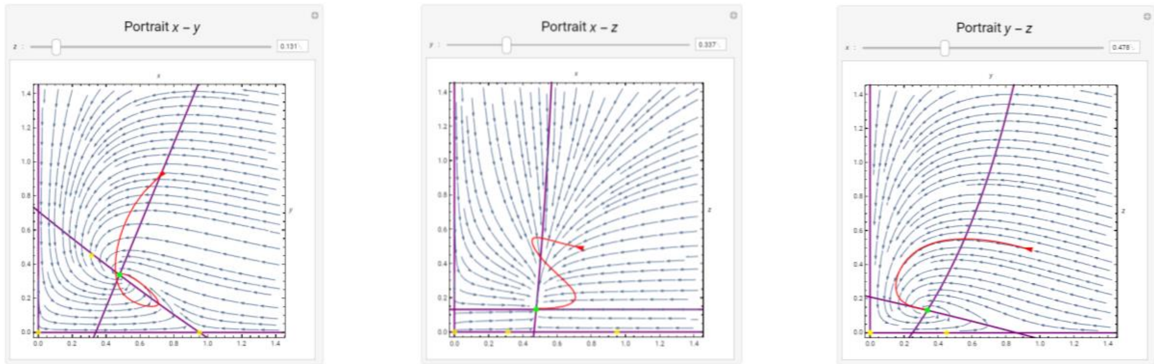


(a) $\mathbf{w_{11} = 0.2}$

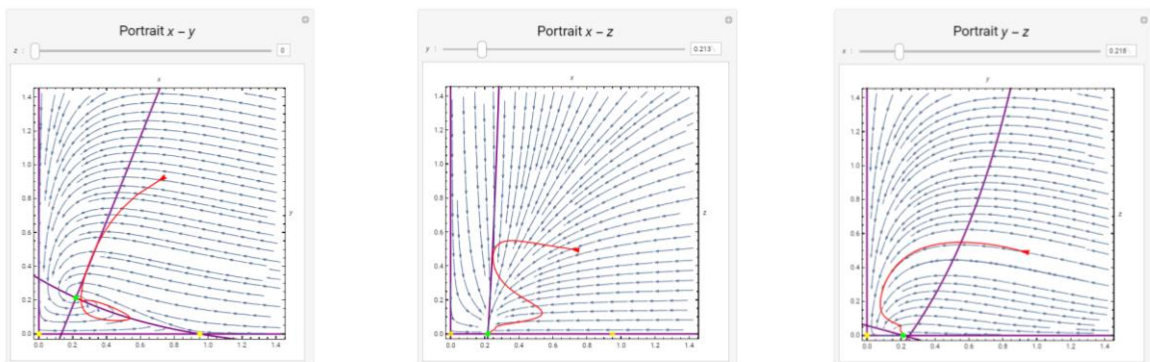


(b) $\mathbf{w_{11} = 0.7}$

Figure 6: The phase portrait of the system (2) in the $xy-$, $xz-$, and $yz-$ plane, respectively, for different values of w_{11}.



(a) $\boldsymbol{w_1 = 0.5}$



(b) $\boldsymbol{w_1 = 5}$

Figure 7: The phase portrait of the system (2) in the $xy-$, $xz-$, and $yz-$ plane, respectively, for different values of w_1.

Finally, varying the other parameters are carried out too and the obtained results can be given in Table 3 below.

Table 3: The dynamical behavior of the system (2) as a function of some parameters:

Parameter	The Dynamical behavior using the numerical simulation with data (25)
w_3	The system loses persistence as the parameter value increases, and the solution goes ultimately to the boundary equilibrium point.
w_4	The system loses persistence as the parameter value increases, and the solution goes ultimately to the boundary equilibrium point.
w_7	The system loses persistence as the parameter value increases, and the solution goes ultimately to the boundary equilibrium point.
w_8	The system loses persistence as the parameter value increases, and the solution goes ultimately to the boundary equilibrium point.
w_9	As the parameter value increases, the system still persists and goes ultimately to the 3D periodic attractor.
w_{12}	The system loses persistence as the parameter value increases, and the solution goes ultimately to the boundary equilibrium point.

9. Discussion and Conclusions

The effects of hunting cooperation, fear, and intraspecific competition are developed upon and explored from a dynamics standpoint in this research using a three-species food chain model. The qualitative properties of the solution are explained. Analyzing the existence of equilibria and looking at their stability are both done. The persistence requirements of the system are established. The study of the global stability of the positive equilibrium point is employed utilizing the geometric technique. We propose the bifurcation conditions that ensure local bifurcation around the equilibrium points. According to the results, all other points do not exist when the trivial equilibrium point is stable, but when this point is unstable, the axial points do exist, and one of them will be asymptotically stable. Therefore, a crucial bifurcation parameter in the food chain system is the death rate for prey species, and this is supported by theorem (7) and Figure (2).

Extensive numerical simulations are performed using the fictitious set of data provided in equation (25), and the observed results can be summed up as follows. This is done in order to better understand the effects of changing parameters and to confirm our acquired conclusions.

The rate of mortality for the prey, which has three bifurcation points, the rate of conversion of biomass from the prey into the middle predator, and the rate of mortality for the middle predator, both of which have two bifurcation points, are the factors that have the most impact on the dynamics of the system (2). One point serves as the bifurcation for all other factors. The biomass conversion rates from the prey to the middle and top predators are positively correlated with the stability of the positive equilibrium point. By approaching periodic dynamics, the top predator's hunting cooperative rate destabilizes the system and maintains its persistence. All other biological factors, however, have caused the system to become unstable as a result of losing the system's (2) permanence. Based on the above, fear in general and intraspecific competition lead to destabilization of the system and loss of its persistence, while cooperation in hunting leads to destabilization of the system and maintaining its persistence.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interests.

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