Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2023, 2023:135
https://doi.org/10.28919/cmbn/8285
ISSN: 2052-2541

MATHEMATICAL MODELING AND OPTIMAL CONTROL STRATEGY FOR A DISCRETE-TIME CHOLERA MODEL

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Abstract. This paper aims to develop and investigate the optimal combination of control interventions for a discrete mathematical cholera model. The population is divided into four compartments: susceptible individuals, symptomatic infected individuals, individuals undergoing treatment, and recovered individuals. The objective is to identify the most effective strategy for minimizing the incidence of cholera cases, susceptible individuals, and symptomatic infected individuals. Three specific control strategies are being considered: the implementation of awareness programs through media and educational channels, the prevention of contact through security campaigns, and the implementation of specific interventions such as sanitation and water treatment. The environmental control strategy aims to reduce the environmental burden of cholera bacteria and minimize the risk of infection through specific interventions. Pontryagin's maximum principle in discrete time characterizes the optimal control strategy. Numerical simulations using MATLAB are conducted to demonstrate the effectiveness of the optimization strategy.

Keywords: cholera; optimal control; treatment; Pontryagin's maximum principle.
2020 AMS Subject Classification: 92C60.

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Received October 15, 2023

1. INTRODUCTION

Cholera presents a significant global public health challenge, resulting in the tragic loss of tens of thousands of lives [1]. The transmission rate of cholera within a community is influenced by a combination of social and environmental factors. Seasonal fluctuations in contact rates contribute to a recurring pattern of cholera outbreaks, mirroring observations in specific cholera-endemic communities [2, 3].

Accurate estimates of Vibrio cholera infection prevalence in endemic populations are crucial, as is a detailed comprehension of the relationship between the bacteria's dosage and its virulence[2]. Cholera outbreaks exhibit a cyclical pattern due to seasonal variations in contact rates, a phenomenon observed in certain cholera-endemic communities[4].

In 2001, researchers expanded Capasso's model by incorporating the concentration of Vibrio cholerae in the water supply, representing the environmental aspect, into a foundational SIR model. They employed a logistic function to model the incidence, capturing the saturation effect. Additionally, in 2006, Hartley and colleagues built upon Codeco's model by introducing a hyperinfectious state of the pathogen, reflecting the highly contagious nature of freshly shed Vibrio cholerae observed in laboratory settings [4].

To effectively combat the cholera epidemic, interventions should concentrate on reducing the transmission risk of the highly contagious, short-lived form of toxigenic Vibrio cholerae. It is also essential to assess the presence of similar highly infectious states in other prevalent diseases. If identified, these states should be integrated into disease prevention models for comprehensive and targeted interventions. Codeco's model was further modified by Joh et al. in 2009 [5]. By explicitly considering a minimum infectious dose of the pathogen necessary to cause infection, interventions should prioritize minimizing the risk associated with transmitting the highly contagious, short-lived form of toxigenic Vibrio cholerae. Furthermore, evaluating the presence of comparable highly infectious states in other prevalent diseases, accounting for the minimum infectious dose, is crucial. Incorporating this factor into disease prevention models will enable more precise and effective targeted interventions [6].

Awareness initiatives play a crucial role in mitigating the transmission of infectious diseases. By informing susceptible populations, these programs reduce the likelihood of contact transmission, contributing to disease control. Given the rapid dissemination of outbreak information through social media and increased global travel, awareness becomes even more vital. It not only diminishes the probability of transmission but also positively influences the overall dynamics of the epidemic, exerting a significant impact on disease spread [7, 8, 9].

In this paper, we investigate the optimal control of the $SICR-B$ model that considers the concentration of bacteria in the dynamic cholera model. At any given time $k \geq 0$, the overall population $N_k$is clustered into 4 clusters, namely individuals susceptible to cholera infection at time k, S_k, individuals infected by cholera displaying symptoms at time k, I_k, individuals under treatment via center at time k,C_k, (C, for Centers), and individuals declared recovered from cholera at time k, R_k. On the other hand, for a cluster of bacteria at time k, B_k, we use a control simulating an awareness program. Pontryagin maximum principle, in discrete time, is used to characterize the optimal control. The numerical simulation is carried out using MATLAB. The obtained results confirm the performance of the optimization [10] Figure(1).

2. Model Formulation

We propose a $SICR-B$ (Susceptible–Infectious–Centres-Recovered-Bacterial) type model and consider a class of bacterial concentration for the dynamics of cholera. The total population N_k is divided into four classes: susceptible S_k, infectious $I_k$with symptoms, in treatment through Centres C_k (C, for Centers), and recovered R_k at the time k, for $k > 0$. Furthermore, we consider a class B_k that reflects the bacterial concentration at the time k. We assume that there is a positive recruitment rate A into the susceptible class S_k and a positive natural death rate μ for all time k under study. Susceptible individuals can become infected with cholera at a rate $\beta_2 \frac{B_k}{\kappa+B_k}$ that is dependent on time k Note that $\beta_2 > 0$ the ingestion rate of the bacteria through contaminated sources κ is the half-saturation constant of the bacteria population, and $\frac{B_k}{\kappa+B_k}$ is the possibility of an infected individual having the disease with symptoms, given contact with contaminated sources[11] The infected individuals can accept to be in treatment through Centres for a period of time. During this time k they are isolated and subject to proper medication, at a rate α_1 The treatment through Centres can recover at a rate σ The disease-related death rates associated

with the individuals that are infected and in treatment through Centres are δ_1 and μ respectively. Each infected individual contributes to the increase of the bacterial concentration at rate $\wp$ On the other hand, the bacterial concentration can decrease at mortality rate δ_2 (See Table1).

2.1. Description of the Model.

- **The compartment S_k:** Susceptible individuals at time k. These are individuals who are not infected but are at risk of contracting cholera
- **The compartment I_k:** Infected individuals at time k. These are individuals who are currently infected with cholera.
- **The compartment C_k:** Treatment centers at time k are staffed with trained medical professionals who can monitor patients' conditions, administer necessary treatments, and provide medical care.
- **The compartment R_k:** Recovered or Removed individuals at time k. These are individuals who have recovered from cholera or have been effectively removed from the transmission process due to awareness programs.
- **The compartment B_k:** Represents the bacteria at time k that caused the infection, in the case of cholera.

Symbol	Parameter
A	New population is recruited into the model with a constant rate
μ	The natural death rate
β_1	transmission rate from human to human
β_2	transmission rate from environment to human
α_1 ; α_2	Recovery rate from cholera
κ	Environmental carrying capacity
δ_1	The death rate induced by the cholera
δ_2	Bacteria death rate
$\wp$	shedding rate of bacteria by infectious population
σ	Recovery rate from cholera sequentially

TABLE 1. Cholera model parameters

2.2. Model Equations. By the addition of the rates at which individuals enter the compartment and also by subtracting the rates at which people leave the compartment, we obtain a difference equation for the rate at which the individuals of each compartment change over discrete time.

Hence, we present the cholera disease infection model by the following system of difference equations(See[12]):

$$
\begin{aligned}
S_{k+1} &= S_k + \mathrm{A} - \mu S_k - \beta_1 \frac{S_k I_k}{N} - \beta_2 \frac{S_k B_k}{\kappa + B_k};\\
I_{k+1} &= I_k + \beta_1 \frac{S_k I_k}{N} - I_k(\mu + \delta_1) - (\alpha_1 + \alpha_2) I_k + \beta_2 \frac{S_k B_k}{\kappa + B_k};\\
C_{k+1} &= C_k + \alpha_1 I_k - \sigma C_k - \mu C_k; \qquad (1)\\
R_{k+1} &= R_k + \alpha_2 I_k + \sigma C_k - \mu R_k;\\
B_{k+1} &= B_k + \wp I_k - \delta_2 B_k;
\end{aligned}
$$

where $S_0 \succeq 0, I_0 \succeq 0, C_0 \succeq 0, R_0 \succeq 0, B_0 \succeq 0$ are the given initial states.

3. THE OPTIMAL CONTROL PROBLEM

Awareness Program through Media and Education (Control: u_k): Implementing educational campaigns through various media channels to raise awareness about cholera prevention and transmission.

Contact Prevention through Security Campaigns (Control:v_k): Conducting security campaigns to prevent direct contact between infected and susceptible individuals, Providing medical treatment to infected individuals, and offering psychological support. Continuous follow-up ensures proper recovery and reduces the chances of reinfection. thereby reducing the spread of the disease.

Pesticides for Cholera Bacteria (Control: w_k): Implementing pesticide measures to target cholera bacteria in affected areas, reducing environmental contamination.

So, the controlled mathematical system is given by the following system of difference equations:(See [13, 14, 15]).

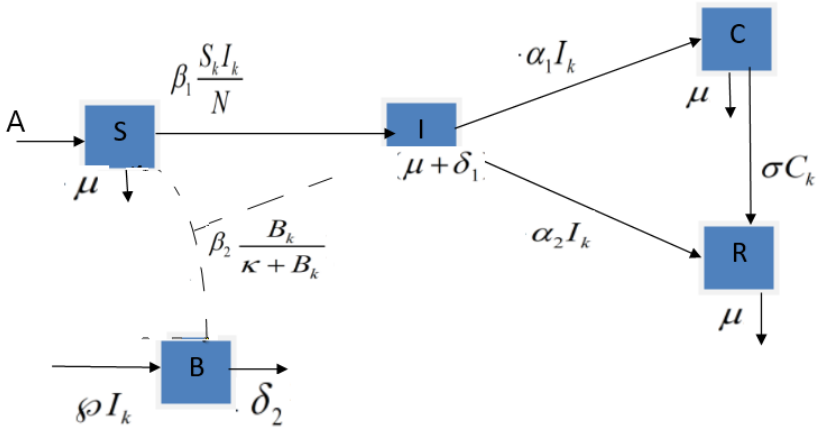

FIGURE 1. A compartmental diagram of the SIRC-B model(1)

$$
\begin{aligned}
S_{k+1} &= S_k + \mathrm{A} - \mu S_k - \beta_1 \frac{S_k I_k}{N} - \beta_2 \frac{S_k B_k}{\kappa + B_k} - u_k S_k; \\
I_{k+1} &= I_k + \beta_1 \frac{S_k I_k}{N} - I_k(\mu + \delta_1) - (\alpha_1 + \alpha_2) I_k + \beta_2 \frac{S_k B_k}{\kappa + B_k} - v_k I_k; \\
C_{k+1} &= C_k + \alpha_1 I_k - \sigma C_k - \mu C_k + v_k I_k; \\
R_{k+1} &= R_k + \alpha_2 I_k + \sigma C_k - \mu R_k + u_k S_k; \\
B_{k+1} &= B_k + \wp I_k - \delta_2 B_k - w_k B_k;
\end{aligned} \tag{2}
$$

where $S_0 \succeq 0, I_0 \succeq 0, C_0 \succeq 0, R_0 \succeq 0, B_0 \succeq 0$ are the given initial states.

there are three controls u_k,v_k, and w_k the first control can be interpreted as the proportion to be adopted to awareness programs through media and education. the second control can be interpreted as the proportion of contact prevention through security campaigns. the third control can be interpreted as Pesticides for cholera bacteria at time k. The challenge that we face here is how to minimize the objective functional: (See [15],[20],[21])

$$
J(u_k; v_k; w_k) = A_{1;T} I_T + A_{2;T} B_T + \sum_{k=0}^{T-1} (A_{1;k} I_k + A_{2;k} B_k + \frac{1}{2} A_{3;k} {u_k}^2 + \frac{1}{2} A_{4;k} {v_k}^2 + \frac{1}{2} A_{5;k} w_k^2); \tag{3}
$$

Where the parameters $A_{3;k} \geq 0$,$A_{4;k} \geq 0$ and $A_{5;k} \geq 0$ are the cost coefficients. They are selected to weigh the relative importance of It and ut at time k. T is the final time. We are minimizing the number of infected individuals during the time steps $k = 0$ to $T - 1$, and at the final time also

minimizing the cost of administering the control. In other words, we seek the optimal control u^*,v^* and w^* such that

$$J(u^*;v^*;w^*) = \min_{(u,v,w)\in U_{ab}} J(u;v;w); \tag{4}$$

Where U is the set of admissible controls defined by

$$U_{ab} = \{(u_k;v_k;w_k) : a \le u_k \le b; c \le v_k \le d; e \le w_k \le f; k = 0,1,...,T-1\}; \tag{5}$$

In our approach, we utilize the discrete version of Pontryagin's Maximum Principle, a fundamental concept in optimal control theory(See [16, 17]). The key concept behind this principle is the introduction of an adjoint function, linking the system of difference equations with the objective function. This linkage results in the formulation of a function known as the Hamiltonian.

This principle transforms the problem of finding the optimal control to maximize the objective function while adhering to the state difference equation with initial conditions. It seeks to determine the control strategy that optimizes the Hamiltonian pointwise concerning the control variable. By doing so, it provides a systematic method to optimize the control measures applied in the context of our problem. We have the Hamiltonian, H_k at time step k defined by

$$H_k = A_{1,k}I_k + A_{2,k}B_k + \frac{1}{2}A_{3,k}u_k^2 + \frac{1}{2}A_{4,k}v_k^2 + \frac{1}{2}A_{5,k}w_k^2 + \sum_{i=1}^{5}\lambda_{i,k+1}f_{i,k+1}. \tag{6}$$

where $f_{j,k+1}$ is the right side of the difference equation of the j^{th} state variable at time step $k+1$(See [16],[17]).

Theorem 1.

Given an optimal control $u_k^* \in U_{ab}$, $v_k^* \in U_{ab}$ and $v_k^* \in U_{ab}$ and the solutions S_k^* ,I_k^*, C_k^* , R_k^* and B_k^* of the corresponding state system (2), there exists adjoint functions $\lambda_{1,k}\lambda_{2,k}\lambda_{3,k}$,$\lambda_{4,k}$ and $\lambda_{5,k}$satisfying:

$$
\begin{aligned}
(7)\quad \lambda_{1,k} &= \frac{\partial H_k}{\partial S_k} = \lambda_{1,k+1}(1-\mu-\beta_1\frac{I_k}{N}-\beta_2\frac{B_k}{\kappa+B_k}-u_k)+\lambda_{2,k+1}(\beta_1\frac{I_k}{N}+\beta_2\frac{B_k}{\kappa+B_k})+\lambda_{4,k+1}(u_k);\\
\lambda_{1,k} &= \frac{\partial H_k}{\partial S_k} = \beta_1\frac{I_k}{N}(\lambda_{2,k+1}-\lambda_{1,k+1})+u_k(\lambda_{4,k+1}-\lambda_{1,k+1})+\lambda_{1,k+1}(1-\mu-\beta_2\frac{B_k}{\kappa+B_k})\\
&\quad +\lambda_{2,k+1}(\beta_2\frac{B_k}{\kappa+B_k});\\
\lambda_{2,k} &= \frac{\partial H_k}{\partial I_k} = A_{1,k}+\beta_1\frac{S_k}{N}(\lambda_{2,k+1}-\lambda_{1,k+1})-v_k(\lambda_{2,k+1}+\lambda_{3,k+1})+\alpha_1(\lambda_{3,k+1}-\lambda_{2,k+1})\\
&\quad +\alpha_2(\lambda_{4,k+1}-\lambda_{2,k+1})+\lambda_{2,k+1}(1-(\mu+\delta_1))+\lambda_{5,k+1}\wp;\\
\lambda_{3,k} &= \frac{\partial H_k}{\partial C_k} = \sigma(\lambda_{3,k+1}+\lambda_{4,k+1})+\lambda_{3,k+1}(1-\mu);\\
\lambda_{4,k} &= \frac{\partial H_k}{\partial R_k} = \lambda_{4,k+1}(1-\mu);\\
\lambda_{5,k} &= \frac{\partial H_k}{\partial B_k} = \mathrm{A}_{2,k}+\beta_2(\frac{S_k(\kappa+B_k)-(S_kB_k)}{(\kappa+B_k)^2}(\lambda_{2,k+1}-\lambda_{1,k+1})+\lambda_{5,k+1}(1-\delta_2-w_k);
\end{aligned}
$$

With the transversality conditions at time T.

$$\lambda_{1;T}=\lambda_{3;T}=\lambda_{4;T}0;\lambda_{2;T}=A_{1;T}\ and\ \lambda_{5;k}=A_{2;T}$$

Furthermore, for $k=0,1,2\ldots T-1$, the optimal control u_k^*,$v_k^*$and w_k^*, is given by

$$u_k^* = min(b,max(a;\frac{\lambda_{3;k+1}-\lambda_{4;k+1}}{A_{3,k}}S_k));$$

$$
\begin{aligned}
(8)\quad v_k^* &= \min[d,\max(c;\frac{1}{A_{4,k}}(\lambda_{2,k+1}-\lambda_{3,k+1})I_k)];\\
w_k^* &= \min[f;\max(e,\frac{1}{A_{5,k}}\lambda_{5,k+1}B_k)];
\end{aligned}
$$

Proof. The Hamiltonian at time step k is given by

$$
\begin{aligned}
(9)\quad H_k &= A_{1,k}I_k+A_{2,k}B_k+\frac{1}{2}A_{3,k}u_k^2+\frac{1}{2}A_{4,k}v_k^2+\frac{1}{2}A_{5,t}w_k^2+\lambda_{1,k+1}(S_k+\mathrm{A}-\mu S_k-\beta_1\frac{S_kI_k}{N}\\
&\quad -\beta_2\frac{S_kB_k}{\kappa+B_k}-u_kS_k)+\lambda_{2,k+1}(I_k+\beta_1\frac{S_kI_k}{N}-I_k(\mu+\delta_1)-(\alpha_1+\alpha_2)I_k+\beta_2\frac{S_kB_k}{\kappa+B_k}-v_kI_k)\\
&\quad +\lambda_{3,k+1}(C_k+\alpha_1I_k-\sigma C_k-\mu C_k+v_kI_k)+\lambda_{4,k+1}(R_k+\alpha_2I_k+\sigma C_k-\mu R_k+u_kS_k)\\
&\quad +\lambda_{5,k+1}(B_k+\wp I_k-\delta_2B_k-w_kB_k);
\end{aligned}
$$

$$\lambda_{1;k}=\frac{\partial H_k}{\partial S_k};\lambda_{1;T}=0;$$

$$\lambda_{2;k} = \frac{\partial H_k}{\partial I_k}; \lambda_{2;T} = A_{1,T};$$

$$\lambda_{3;k} = \frac{\partial H_k}{\partial C_k}; \lambda_{3;T} = 0;$$

$$\lambda_{4;k} = \frac{\partial H_k}{\partial R_k}; \lambda_{4;T} = 0;$$

$$\lambda_{5;k} = \frac{\partial H_k}{\partial B_k}; \lambda_{5;T} = \mathrm{A}_{2,T};$$

For, $k = 0, 1 \ldots T-1$ the optimal control u_k^* ,v_k^* and w_k^* can be solved from the optimality condition,

$$\frac{\partial H_k}{\partial u_k} = 0; \frac{\partial H_k}{\partial v_k} = 0; \frac{\partial H_k}{\partial w_k} = 0;$$

That is:

$$\begin{aligned}
&\tfrac{\partial H_k}{\partial u_k} = A_{3,k}u_k + S_k(\lambda_{4,k+1} - \lambda_{4,k+1}) = 0;\\
&u_k = \tfrac{1}{A_{3,k}}(\lambda_{1,k+1} - \lambda_{4,k+1})S_k;\\
&\tfrac{\partial H_k}{\partial v_k} = 0 = A_{4,k}v_k - \lambda_{2,k+1}I_k + \lambda_{3,k+1}I_k = A_{4,k}v_k + I_k(\lambda_{3,k+1} - \lambda_{2,k+1}) = 0;\\
&v_k = \tfrac{1}{A_{4,k}}(\lambda_{2,k+1} - \lambda_{3,k+1})I_k;\\
&\tfrac{\partial H_k}{\partial w_k} = A_{5,k}w_k - \lambda_{5,k+1}B_k = 0;\\
&w_k = \tfrac{1}{A_{5,k}}\lambda_{5,k+1}B_k;
\end{aligned}$$

By the bounds in U_{ab} of the controls, it is easy to obtain u_k^* ,v_k^* and w_k^* in the form of (8)

4. NUMERICAL SIMULATION

In this section, we present the numerical results obtained from solving a two-point boundary value problem within the formulated optimality system. The problem is characterized by distinct conditions at the initial and final time steps and involves state equations, adjoint equations, and control specifications. The state variables are initialized with initial conditions, and the adjoint variables are finalized with terminal conditions. Our approach to solving this optimality system is iterative. During each iteration, we progress by solving the state equations forward in time and then addressing the adjoint equations backward in time. The process begins with an initial guess for the controls, which are refined using control characterization before proceeding to the next iteration. This iterative cycle continues until consecutive iterations converge, guaranteeing, ensuring a gradual enhancement and optimization of control strategies.

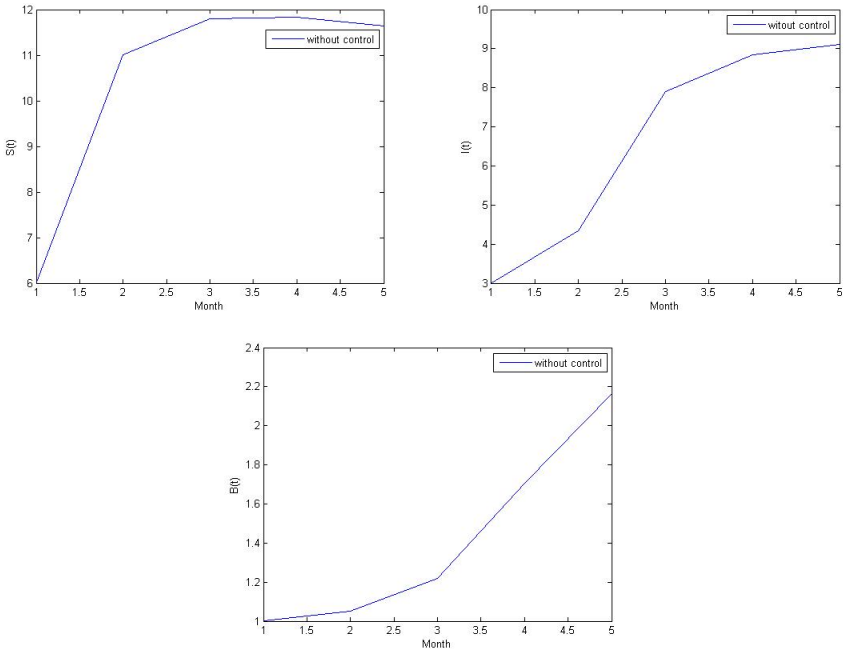

FIGURE 2. Here are the simulation results depicting the number of susceptible and infectious individuals and the concentration of bacteria without control.

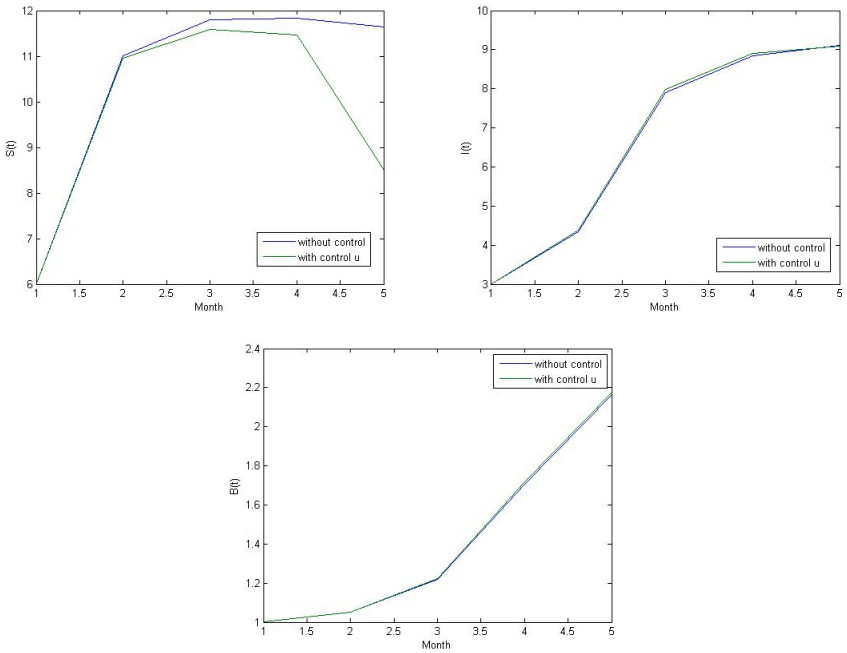

FIGURE 3. Shows the simulation results of the Susceptible individuals the infectious individuals and the concentration of bacteria with control u_k only.

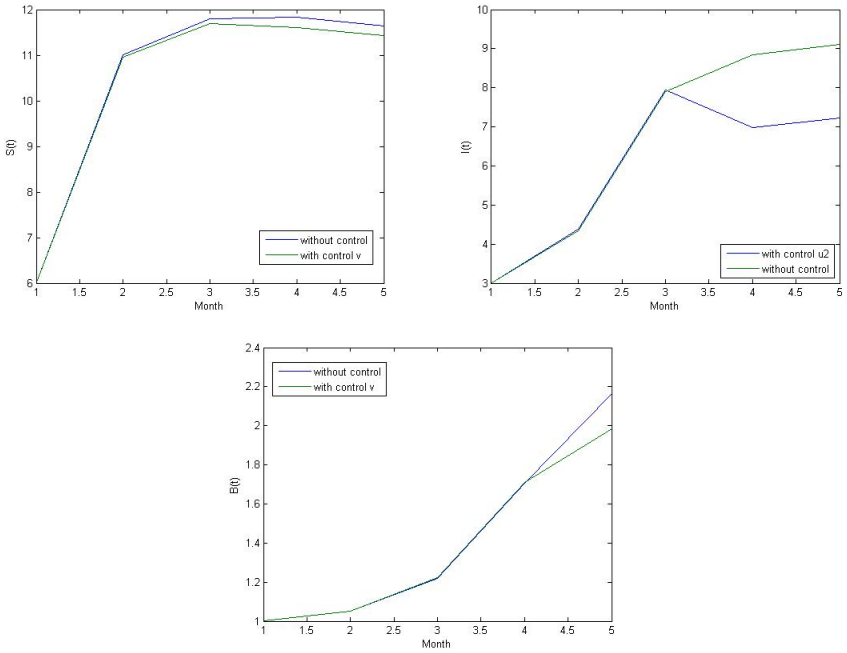

FIGURE 4. Shows the simulation results of the Susceptible individuals the infectious individuals and the concentration of bacteria with control v_k only.

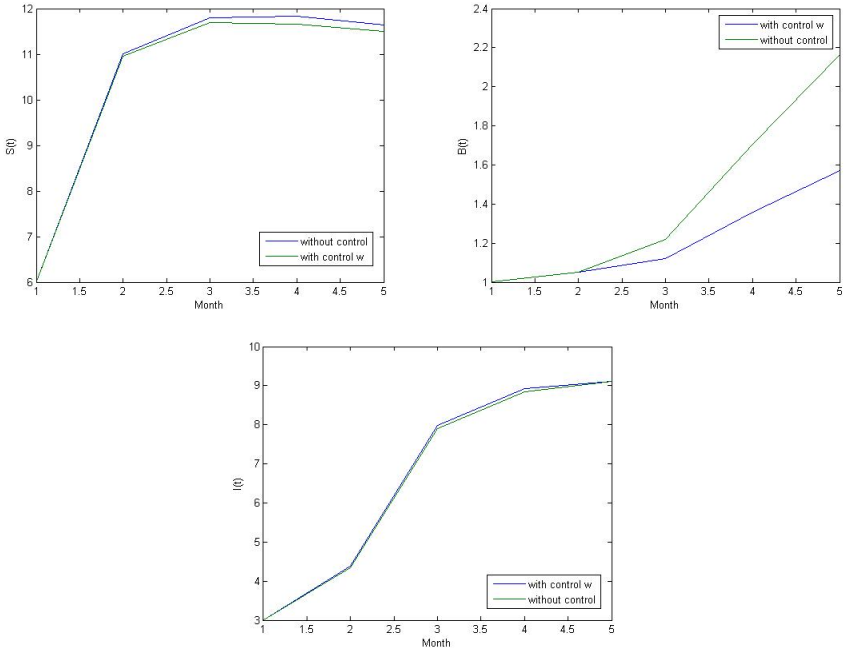

FIGURE 5. Shows the simulation results of the Susceptible individuals the infectious individuals and the concentration of bacteria with control w_k only.

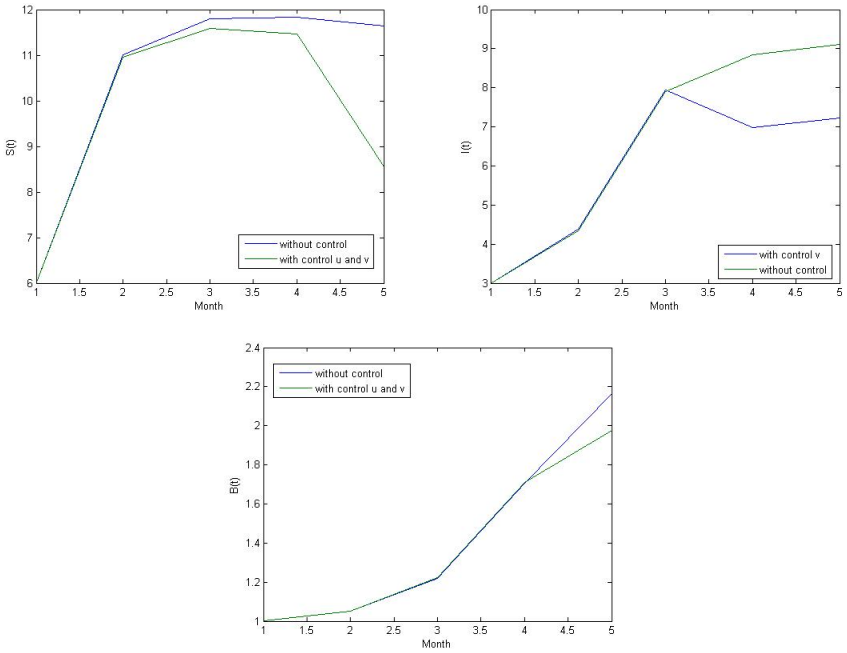

FIGURE 6. Shows the simulation results of the Susceptible individuals the infectious individuals and the concentration of bacteria with control u_k and v_k only.

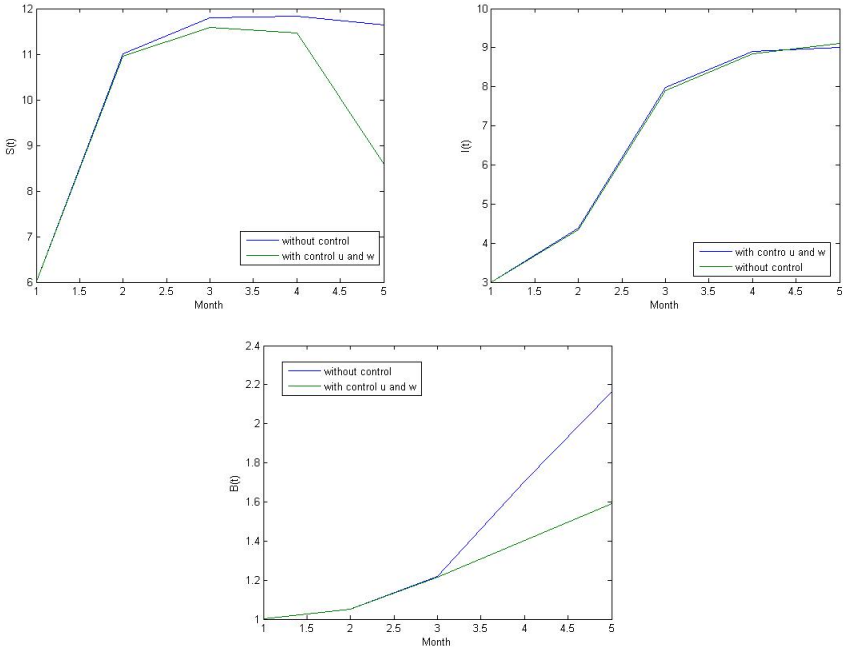

FIGURE 7. Shows the simulation results of the Susceptible individuals the infectious individuals and the concentration of bacteria with control u_k and w_k only.

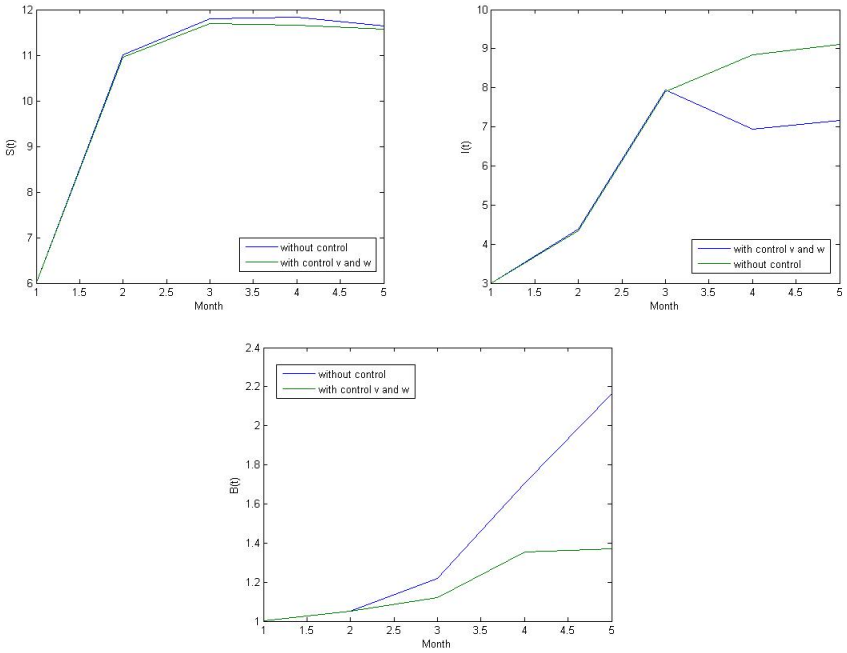

FIGURE 8. Shows the simulation results of the Susceptible individuals the infectious individuals and the concentration of bacteria with control v_k and w_k only.

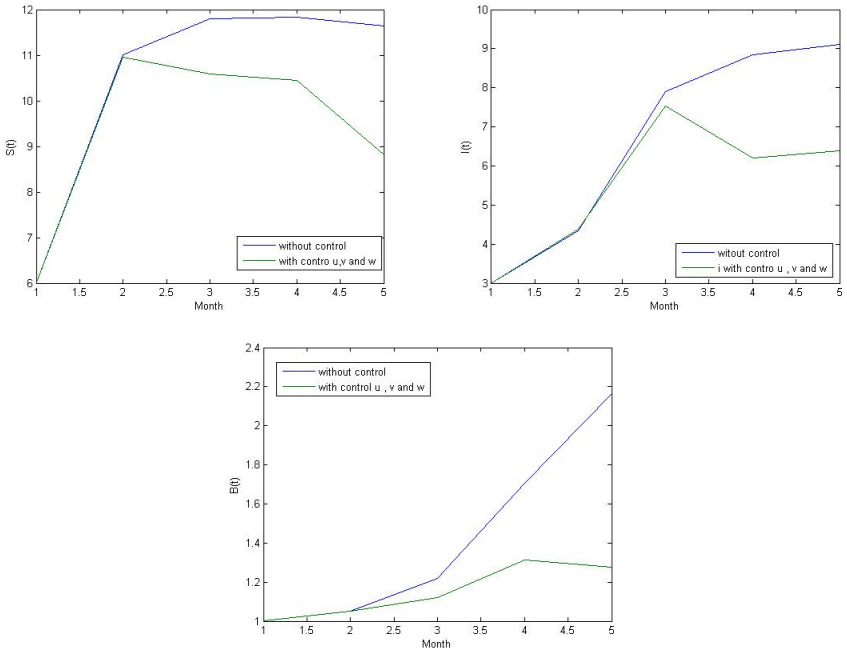

FIGURE 9. Shows the simulation results of the Susceptible individuals the infectious individuals and the concentration of bacteria with control u_k, v_k and w_k.

The initial conditions and parameters of the $SICR-B$ Model		
Parameter	Initial conditions	Reference
N	60	assumed
$S(0)$	6	assumed
$I(0)$	3	assumed
$B(0)$	1	assumed
$R(0)$	6	assumed
Values of model parameters of the $SICR-B$ Model		
Parameter	Value	Reference
A	10 per day	[18]
μ	0.15	assumed
β_2	0.75	assumed
α_1	0.21	assumed
α_2	0.31	assumed
κ	0.1	assumed
δ_1	0.25	assumed
δ_2	0.25/week	[19]
$\wp$	0.1	assumed
σ	0.2	assumed
κ	0.1	assumed

5. CONCLUSION

In this study, we have applied optimal control techniques within the field, concentrating particularly on the SICR-B model for cholera outbreaks. Our control parameter measures the number of susceptible and infectious individuals, as well as the concentration of bacteria. Through numerical simulations using MATLAB, our findings demonstrate that the proposed control strategy notably diminishes the number of individuals infected with cholera, concurrently minimizing the associated costs.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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