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ANALYSIS OF SIR MODEL WITH OPTIMAL CONTROL STRATEGY FOR A SIMPLE TRAFFIC CONGESTION PROCESS

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Abstract. Traffic analysis on highways at the macroscopic level is very similar to the analysis of the spread of infectious diseases, namely the susceptible-infected-recover (SIR) model. We propose the SIR model with a control variable. The dynamics with fixed control and stability of the model are analyzed. Sensitivity analysis was also carried out. Variable control is applied as an effort to regulate or change the duration of the green light at an intersection. We obtain an optimal control strategy when the control is time-dependent. Numerical results show the positive impacts of implementing the control to susceptible vehicles and treatment for congested vehicles. We have also done an efficiency analysis, which shows that control is more effective than without control.

Keywords: SIR; traffic congestion; optimal control.

2020 AMS Subject Classification: 37N25, 37C75, 93B35.

1. Introduction

Various studies and research have been conducted to improve the quality of traffic, one of which is smooth travel without congestion. The microscopic approach in modelling urban traffic is computationally consuming and requires large computations to calibrate the parameters [1].

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Meanwhile, traffic analysis on highways at the macroscopic level is very similar to the analysis of the spread of infectious diseases, namely the susceptible-infected-recover (SIR) model. In recent years, the classic model of SIR began to be used by some researchers to model the dynamics of traffic congestion propagation. Some study results can be found in [1]-[4]. Wu J. et al. in [2] have modelled traffic congestion that occurs at one node in a traffic system network that can transmit/cause congestion at neighbouring points, and if not handled quickly, gradually, the congestion can propagate to further nodes. The SIR model was used in [2] to simulate the spread of bottlenecks quite well. Pu C et al. [3] also examined the spread of SIR epidemics in traffic networks, observing the effects of several factors, namely load distribution and homogeneity of load distribution. Network density or homogeneity will increase the spread of epidemic jams. Meanwhile, in [1] the SIR model was proposed to represent the dynamics of congestion propagation in a traffic network, and simulations were carried out for six different metropolitan cities to obtain congestion distribution patterns at the macroscopic level. The existence of a basic reproduction number (R_0) for congestion spread in urban networks was determined and is similar to that for the infectious disease model. Chen Y. et al.[4] examined the spread pattern of traffic network congestion, which is important in network performance analysis, namely by proposing an SIS-CP model based on virus transmission theory to model density propagation patterns within large-scale networks. SIS-CP can best predict the ratio of dense links during peak hours. The propagation of traffic jams in foggy weather situations was modelled by Jiao Yao et al.[5] by combining the SIR model with cellular automata. The analysis and simulations were able to give the best speed limit in foggy road conditions. Also, Chen K. [6] established the SIR model with different scenarios on a road. Another way to reduce traffic congestion is to provide real-time information on traffic conditions to vehicles on the road, with a vehicle-to-vehicle communication process between vehicles. Indrakanti T. [7] proposed a model for the propagation process of information received by vehicles based on the SIR model. A research study by Ashfaq in [8] investigated congestion propagation in urban cities using the SIR model with two macroscopic parameters: propagation rate and recovery rate. There are two levels of traffic assignment model: link level flow and network level congestion pattern. It also investigates the dynamical congestion alternative approach model using the Reaction Diffusion model, which involves a higher computational time.

The number of vehicles in major cities is increasing every year. This is not comparable to the availability of highway capacity, which results in heavy traffic and congestion in various places [9]. Looking at these conditions, effective solutions are needed to solve the problem of traffic congestion, one of which is by improving traffic light management at intersections. Traffic lights are generally used to regulate traffic in each lane to move in turns so as not to cause congestion. However, problems often arise where improper traffic time settings can cause very long vehicle queues. We observe that the number of vehicle arrivals and the duration of traffic lights in one cycle often cause long queues of vehicles at the location of traffic lights. Therefore, it is necessary to develop a model to regulate the duration of the red light so that there is no increase in the number of queues at traffic intersections [9]. Based on previous research mentioned, SIR is suitable for analyzing urban traffic congestion and can effectively predict and infer the changing trend of congestion. To the best of the authors' knowledge, previous studies have not included variable controls in the model. In this study, we propose adding variable control to the SIR model and then analyzing the stability of the model and the existence of optimal control qualitatively. The simplified model concerning conditions on a one-way (free-way) road section where there is an intersection with a traffic light at the end of the section. During peak hours, a buildup of vehicles (congestion) occurs when the light is red since the traffic-light cycle remains normal (constant) while the volume of vehicles increases sharply. Variable control is applied as an effort to regulate or change the duration of the green light by the traffic system operator. This paper will discuss the stability analysis of the optimal control model of SIR and its controllability. Subsequent sections of this paper are organized as follows. Section 2, the mathematical SIR model with control is introduced together with basic considerations. The Boundedness and positivity of the solutions of the proposed model are established followed by determining the equilibrium points and reproduction number are discussed in Section 3. The analysis of model stability at equilibrium points is presented in Section 4. In Section 5, we will discuss the optimum control characteristics. We performed numerical simulations in Section 6 to understand the positive impact of control and determine the best strategy for managing traffic at the intersection. The conclusion is given in Section 7.

2. Formulation of SIR Model With Control

Consider the population of vehicles at time $t \geq 0$ on a busy main road where traffic congestion often occurs at the traffic light intersection. The number of vehicles (N) on the main road is assumed constant and is divided into three sub-populations, namely, the population of vehicles vulnerable to being affected by traffic jams denoted as $S(t)$ or known as susceptible compartment; then compartment $I(t)$ is the number of vehicles that have been 'infected', i.e. vehicles that are caught in the traffic congestion; and lastly vehicles that have recovered, $R(t)$, are vehicles free from the congestion.

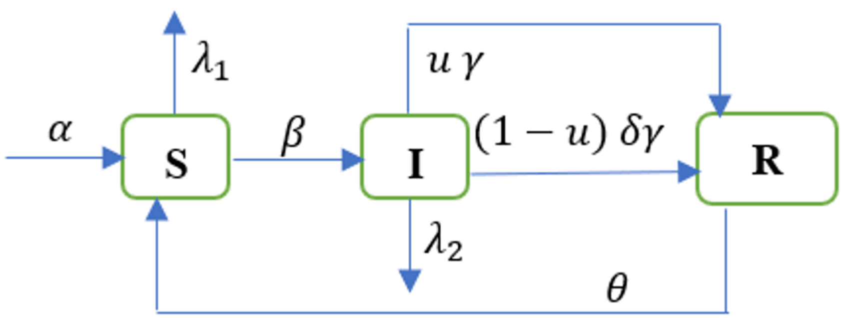


FIGURE 1. Proposed SIR Model with Control of Simple Traffic Congestion Process

Vehicles arriving or entering the main road at peak rush hour become susceptible population ($S(t)$) and are assumed to flow at an average rate α. The average transmission rate to the congested vehicles, $I(t)$, from the susceptible population, is denoted by the parameter β. A small portion of susceptible vehicles on the main road exits the road at a rate λ_1. At the intersection, vehicles can turn left directly without stopping, thus exiting the congestion at a proportion rate λ_2. The proportion of green light in a single traffic-light cycle is denoted by δ. Once the green light is on, vehicles leave (free) from the congestion state and transfer to $R(t)$ at an average rate γ. A small portion, θ, of the recovered vehicles may enter the main road again. The variable control $0 \leq u(t) \leq 1$ is an intervention applied to adjust the duration of the green light to eliminate traffic congestion. The compartment diagram can be traced in Figure 1. Incorporating all the assumptions above, the differential equation system forms the SIR model as follows

$$\begin{aligned}
\frac{dS}{dt} &= \alpha - \lambda_1 S - \beta SI + \theta R \\
\frac{dI}{dt} &= \beta SI - \lambda_2 I - [u + (1-u)\,\delta]\,\gamma I \\
\frac{dR}{dt} &= [u + (1-u)\,\delta]\,\gamma I - \theta R
\end{aligned} \tag{1}$$

with initial conditions $S(0) \geq 0, I(0) \geq 0, R(0) \geq 0$.

3. Boundedness of Solutions, Equilibrium Points and Basic Reproduction Number

We discuss the positivity and boundedness of the solutions, followed by the existence of the equilibrium points of the system (1) for a fixed value of control parameter u. We also derive the basic reproduction number when the control parameter is assumed fixed. From system (1), we have

$$\frac{dS}{dt} = -S(\lambda_1 + \beta I) + \alpha + \theta R \geq -S(\lambda_1 + \beta I).$$

Integrating we get

$$S(t) \geq S(0)\, e^{-\int_0^t (\lambda_1 + \beta I) dt} \geq 0.$$

In a similar way we can obtain

$$I(t) \geq I(0)\, e^{-(\lambda_2 + k)t} \geq 0$$

and

$$R(t) \geq R(0)\, e^{-\theta t} \geq 0.$$

Thus, we have shown that all variable solutions of the system (1) are positive. Now we will prove the solutions are bounded [10]. Let $\lambda = \min(\lambda_1, \lambda_2)$ and $k = [u + (1-u)\,\delta]\,\gamma$. Summation of the three equations of the system (1) gives $\frac{dN}{dt} = \alpha - \lambda_1 S - \lambda_2 I < \alpha - \lambda\,(N - R)$, that is N not bounded. To overcome this, we shift the system (1) along the S axis (see [11]). Here, substitutions are made $(S, I, R) \to (P, I, R)$ where $P = S + \eta R$, thus the global properties of the new system also apply to the system (1) and vice versa. Thus, system (1) became

$$\begin{aligned} \frac{dP}{dt} &= \alpha - \lambda_1(P-\eta R) - \beta(P-\eta R)I + \theta R \\ \frac{dI}{dt} &= \beta(P-\eta R)I - \lambda_2 I - [u+(1-u)\delta]\gamma I \\ \frac{dR}{dt} &= kI - \theta R \end{aligned} \tag{2}$$

The phase space of system (2) is the quadrant $\mathbb{R}^3_+$. The following theorem shows that system (2) is well-posed therefore the same applies to system (1).

Theorem 1.

The region $D = \left\{(P,I,R) \in \mathbb{R}^3_+ | P+I+R \leq \frac{\alpha}{\lambda^*}\right\}$ is a positively invariant set for system (2).

Proof. From the system (2), we can write $\widehat{N'} = P' + I' + R'$, that is $\frac{d\widehat{N}}{dt} = \alpha - \lambda_1 P - (\lambda_2 - \eta k) I - (\theta - \lambda_1)\eta R$. Now let $\lambda^* = \min(\lambda_1, (\lambda_2 - \eta k), (\theta - \lambda_1)\eta R)$. Thus, $\frac{d\widehat{N}}{dt} < \alpha - \lambda^*(P+I+R)$, integrate and evaluate $\limsup \widehat{N(t)}$ as $t \to \infty$ we obtain $\lim\limits_{t\to\infty} \sup \widehat{N(t)} \leq \frac{\alpha}{\lambda^*}$. Therefore, solutions of (2) are bounded and D is a positively invariant set. □

3.1. Equilibrium Points and Basic Reproduction Number.

The existence of the equilibrium points can be determined from (1) by solving the relations $\frac{dS}{dt} = 0, \frac{dI}{dt} = 0$ and $\frac{dR}{dt} = 0$ or written as

$$\begin{aligned} &\alpha - \lambda_1 S - \beta SI + \theta R = 0 \\ &\beta SI - \lambda_2 I - [u+(1-u)\delta]\gamma I = 0 \\ &[u+(1-u)\delta]\gamma I - \theta R = 0 \end{aligned} \tag{3}$$

By solving (3) we obtained the congestion-free (non-endemic) equilibrium point (CFE) at $E^0 = \left(S^0, I^0, R^0\right) = \left(\frac{\alpha}{\lambda_1}, 0, 0\right)$. When the system reaches the point E^0, traffic congestion disappears.

Next, the basic reproduction number, R_0, is defined as the number of secondary infections or congested vehicles produced by a single congested vehicle in a completely susceptible population. We use the next-generation-matrix (NGM) [12], [13], FV^{-1}, to find R_0, where R_0 is the spectral radius of matrix FV^{-1} at the equilibrium point, E^0. In epidemic models, F is a new infection transmission vector and the transition matrix V is the displacement between compartments [14]. Here, $F = \left[\beta S^0\right] = \left[\frac{\alpha\beta}{\lambda_1}\right]$ and

$$V^{-1} = \left[\frac{1}{\lambda_2 + [u + (1-u)\,\delta]\,\gamma}\right]$$

. Therefore, we can define the basic reproduction number for the system (1) as follows

$$R_0 = \frac{\alpha\beta}{\lambda_1\,(\lambda_2 + [u + (1-u)\,\delta]\,\gamma)} \tag{4}$$

It is clear that $R_0 < 1$ if and only if $\alpha\beta < \lambda_1\,(\lambda_2 + [u + (1-u)\,\delta]\,\gamma)$. From equations in (3) we also determine the traffic congestion (endemic) equilibrium point: $E^* = (S^*, I^*, R^*)$ as follows.

Let $k = [u + (1-u)\,\delta]\,\gamma$, we have from (4), $R_0 = \frac{\alpha\beta}{\lambda_1(\lambda_2 + k)}$. Thus, solving (3) we obtain

$$\begin{aligned}
P^* &= \frac{\lambda_2 + k}{\beta} + \eta\frac{\alpha k}{\theta\lambda_2}(1 - R_0) > 0,\ \text{if } R_0 > 1 \text{ (endemic)}\\
S^* &= \frac{\lambda_2 + k}{\beta} > 0\\
R^* &= \frac{\alpha k}{\theta\lambda_2}(1 - R_0) > 0,\ \text{if } R_0 > 1 \text{ (endemic)}\\
I^* &= \frac{k}{\lambda_2}(1 - R_0) > 0,\ \text{if } R_0 > 1 \text{ (endemic)}
\end{aligned} \tag{5}$$

Clearly, $R_0 > 1$ which means an endemic or traffic jam occur.

4. Stability Analysis

Next, we shall investigate the stability at the congestion-free equilibrium point $E^0 = \left(S^0, I^0, R^0\right) = \left(\frac{\alpha}{\lambda_1}, 0, 0\right)$ for fixed control. The system (1) is linearized at E^0, the Jacobian matrix or variational matrix corresponding to system (1) is

$$J\left(E^0\right) = \begin{pmatrix} -\lambda_1 & -\beta S^0 & \theta \\ 0 & \beta S^0 - \lambda_2 - k & 0 \\ 0 & k & -\theta \end{pmatrix} = \begin{pmatrix} -\lambda_1 & -\beta\frac{\alpha}{\lambda_1} & \theta \\ 0 & \beta\frac{\alpha}{\lambda_1} - \lambda_2 - k & 0 \\ 0 & k & -\theta \end{pmatrix} \tag{6}$$

The following result ensures the local stability of the model.

Theorem 2. If $R_0 < 1$, then congestion free equilibrium point E^0 is asymptotically stable and if $R_0 > 1$ then it is unstable.

Proof. The characteristic roots or eigenvalues of the variational matrix J at E^0 are $-\lambda_1, (\lambda_2+k)(R_0-1)$, and $-\theta$. The matrix $J\left(E^0\right)$ have negative eigenvalues when $R_0<1$ or when or $\alpha\beta<\lambda_1(\lambda_2+k)$. Therefore, according to the Routh-Hurwitz stability criterium, the equilibrium point E^0 is locally asymptotically stable. Otherwise, if $R_0>1$, then E^0 is unstable. □

Furthermore, we will prove the global stability ([15] and [16]) of the congestion free equilibrium point.

Theorm 3. If $R_0\leq 1$ than the congestion free equilibrium point E^0 is globally asymptotically stable.

Proof. Consider the following Lyapunov function defined and continuous for all $S,I,R\geq 0$,

$$L(S,I,R)=S^0\left(\frac{S}{S^0}-\ln\frac{S}{S^0}\right)+I^0\left(\frac{I}{I^0}-\ln\frac{I}{I^0}\right)+R^0\left(\frac{R}{R^0}-\ln\frac{R}{R^0}\right).$$

The function L satisfies $\frac{\partial L}{\partial X}=1-\frac{X^0}{X}$ for $X=S,I,R$. Then the derivative for the Lyapunov function along the system (1) is as follows:

$$\frac{dL}{dt}=\left(1-\frac{S^0}{S}\right)\frac{dS}{dt}+\left(1-\frac{I^0}{I}\right)\frac{dI}{dt}+\left(1-\frac{R^0}{R}\right)\frac{dR}{dt}.$$

After simplification and since $\alpha=\lambda_1 S^0$, we have

$$\begin{aligned}\frac{dL}{dt}&=2\alpha-\lambda_1 S-\frac{\alpha^2}{\lambda_1 S}-\frac{\alpha\theta}{\lambda_1 S}R+\frac{\alpha\beta}{\lambda_1}I-\lambda_2 I\\&=2\lambda_1 S^0-\lambda_1 S-\lambda_1\frac{S^{0^2}}{S}-\frac{\alpha\theta}{\lambda_1 S}R-\left(\lambda_2-\frac{\alpha\beta}{\lambda_1}\right)I\\&=\lambda_1 S^0\left(2-\frac{S}{S^0}-\frac{S^0}{S}\right)-\frac{\alpha\theta}{\lambda_1 S}R-\left(\lambda_2-\frac{\alpha\beta}{\lambda_1}\right)I\\&=\lambda_1 S^0\left(\frac{S}{S^0}-1\right)\left(\frac{S^0}{S}-1\right)-\frac{\alpha\theta}{\lambda_1 S}R-\left(\lambda_2-\frac{\alpha\beta}{\lambda_1}\right)I\end{aligned}$$

Since $S\neq S^0$, then $\left(\frac{S}{S^0}-1\right)\left(\frac{S^0}{S}-1\right)<0$; and $\frac{\alpha\beta}{\lambda_1}<\lambda_2<\lambda_2+k$ as $R_0=\frac{\alpha\beta}{\lambda_1(\lambda_2+k)}<1$. Therefore $\frac{dL}{dt}\leq 0$ if $R_0\leq 1$. Further for $S=S^0, R=0$ and $I=0$, the Lyapunov derivative function $\frac{dL}{dt}=0$. Based on the LaSalle invariant principle [17] and [18], this proofs that the free equilibrium point E^0 is globally asymptotically stable □

5. Sensitivity Analysis

The basic reproduction number R_0 also acts as initial congestion transmission. A normalized sensitivity index is then determined to investigate the relative change of parameters (appeared in R_0) on the value of R_0. This is used to measure which parameter that has the most impact on R_0. The normalized sensitivity index is defined as follows [14].

$$I_p^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0}, \tag{7}$$

where p is the set of parameters, $p = \{\alpha, \lambda_1, \beta, \lambda_2, \delta, \gamma\}$ From (7), we have $I_\alpha^{R_0} = I_\beta^{R_0} = 1$,

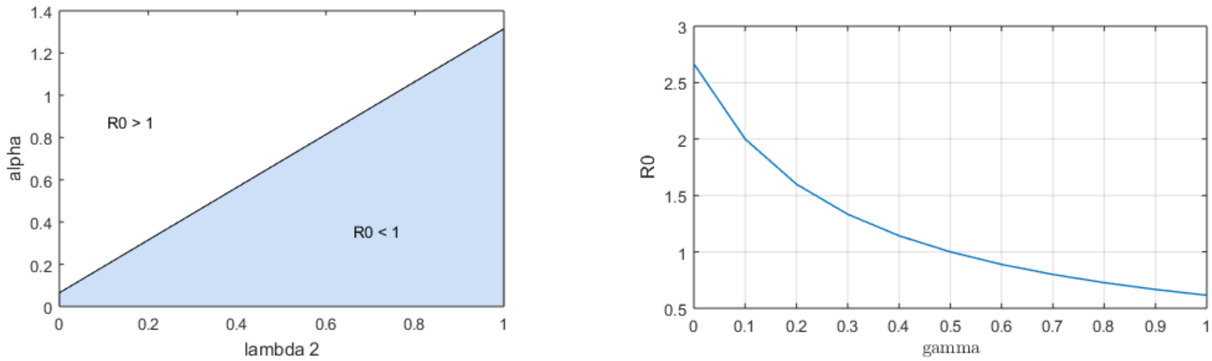
$$I_{\lambda_1}^{R_0} = -\frac{\alpha\beta}{\lambda_1^2[\lambda_2 + k]} \frac{\lambda_1^2[\lambda_2 + k]}{\alpha\beta} = -1 < 0$$

$$I_{\lambda_2}^{R_0} = -R_0 \frac{\lambda_1 \lambda_2}{\alpha\beta} < 0$$

$$I_{\gamma}^{R_0} = -R_0 \frac{k\lambda_2}{\alpha\beta} < 0$$

$$I_{\delta}^{R_0} = -R_0 \frac{\lambda_1 (1-u)\gamma\delta}{\alpha\beta} < 0$$

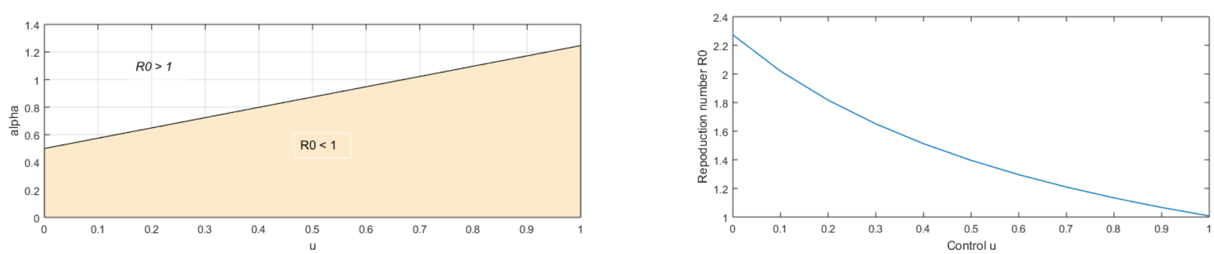
Thus, parameters $\lambda_1, \lambda_2, \delta$ and γ have negative impact on R_0, meaning as $\lambda_1, \lambda_2, \delta$ or γ increases, then R_0 decreases. Meanwhile, the parameters α and β have a positive impact. We assume the parameter values are $\delta = 0.104, \alpha = 0.4, \lambda_1 = 0.1, \beta = 0.08, \lambda_2 = 0.3 and \theta = 0.01$. By substituting fixed parameter values except for γ, we can analyze the normalized sensitivity index of R_0 towards the proportion rate of vehicles free from congestion, γ. It can be seen from the graphic plotted in Figure 2b that $\gamma = 0.5$ is the threshold, when $\gamma > 0.5$, then R_0 is < 1 meaning the road is always free from traffic congestion.



(A) (B)

FIGURE 2. (a). Area $R_0 < 1$ or $R_0 > 1$ using combination α and λ_2. (b) Graph of R_0 vs γ

In equation (4), as arrival rate α increases, R_0 also increases, in other words, the curve of α increases monotonically with respect to R_0. Meanwhile in Figure 2a, the diagram shows how combination of parameters α and λ_2 that will produce $R_0 < 1$ or $R_0 > 1$. Next, we substitute fix parameter values except α and u in the equation $R_0 = 1$, and obtain $\alpha = 0.5 + 0.7476u$. Figure 3 depicts how R_0 can be determined by relying on α and u qualitatively. When $\alpha < 0.5$ we can see that R_0 is always less than 1, meaning the free-way is always congestion-free.



(A) (B)

FIGURE 3. (a) R_0 sensitivity diagram of parameter α(alpha) and control u with threshold line $R_0 = 1$. The colored area defines conditions with no traffic congestion (free-flow). (b) The reproduction number reduces to 1 as the control increases towards the maximum value of 1.

Figure 3(b) shows that when control u is applied then the reproduction number, R_0, will decrease until it reaches $R_0 = 1$.

6. CHARACTERIZATION OF THE OPTIMAL CONTROL

The strategy of the optimal control is to minimize the number of population $I(t)$, which is the number of vehicles in the traffic congestion as well as the cost of implementing the control. We assume the cost due to congestion is a nonlinear function of population I because the traffic jam causes a sharp nonlinear cost from burning fuel and time lost during the bottleneck from each vehicle. The cost to implement controlling the traffic light as a function of the effectiveness is also assumed nonlinear. Thus, the aim of the optimal control problem is to minimize the objective functional or cost function $J(U)$ set as,

$$J(u) = \int_0^t \left(mI^2 + nu^2\right) dt$$

where m and n are positive weight costs that relate to vehicles in the traffic congestion and the implementation of control, respectively. Therefore, the optimal control problem is written as

$$\min J(u) = \int_0^t \left(mI^2 + nu^2\right) dt$$

subject to system (1) with initial conditions $S(0) \geq 0, I(0) \geq 0, R(0) \geq 0$, and $u(t) \in U = \{0 \leq u(t) \leq 1, t \in [0,T]\}$, where terminal time $T > 0$.

Theorem 5. There exists an optimal solution $u^*(t)$ such that $J(u^*) = \min J(u)$ subject to constraints of state variables system (1) with $S(0) \geq 0, I(0) \geq 0, R(0) \geq 0$, and $u(t) \in U$.

Proof. The integrand of the objective functional $J(u)$ is a convex function of u since n is positive. Furthermore, the control space U is a closed and convex region. We have $J(u) \geq nu^2$, hence there exists $k_1 = n \geq 0$ and $k_2 \geq 1$ such that$J(u) \geq k_1 \|u\|^{k_2}$ The optimal control is bounded and therefore there exists an optimal control u^* that minimizes $J(u)$ for $t \in (0,T)$. □

Apply the Pontryagin's maximum principle To solve the optimal control problem we define the Hamiltonian function [19] which is the sum of the integrand of $J(u)$ and the inner product of the state function with the adjoint variables. The corresponding Hamiltonian is defined as follows:

$$H(t) = mI^2 + nu^2 + z_s \frac{dS}{dt} + z_I \frac{dI}{dt} + z_R \frac{dR}{dt}. \tag{8}$$

To obtain the optimal solution we use the following theorem.

Theorem 6. Consider optimal control u^* and optimal solutions $\hat{S},\hat{I},\hat{R}$ of system (1) that minimizes $J(u)$ over U. Then there exists adjoint variables $z_S(t), z_I(t), z_R(t)$ satisfying the co-state system

$$\begin{aligned}\frac{dz_S}{dt} &= -\frac{\partial H}{\partial S} = z_S\lambda_1 + (z_S - z_I)\,\beta I \\ \frac{dz_I}{dt} &= -\frac{\partial H}{\partial I} = -2mI + (z_S - z_I)\,\beta S + \lambda_2 z_I + (z_I - z_R)\,[u + (1-u)\,\delta]\,\gamma \\ \frac{dz_R}{dt} &= -\frac{\partial H}{\partial R} = (z_R - z_S)\,\theta\end{aligned} \tag{9}$$

with transversality condition given by:

$$z_S(t) = z_I(t) = z_R(t) = 0. \tag{10}$$

Thus, the optimal control is obtained

$$u^* = \frac{(z_I^* - z_R^*)\widehat{I}(1-\delta)\,\gamma}{2n}.$$

Proof. Use Pontryagin's Maximum Principle.

The Hamiltonian:

$$\begin{aligned}H(S(t),I(t),R(t),u(t),z_S(t),z_I(t),z_R(t)) &= mI^2 + nu^2 + z_S(a - \lambda_1 S - \beta SI + \lambda R) \\ &+ z_I(\beta SI - \lambda_2 I - [u + (1-u)d]\lambda I) + z_R([u + (1-u)d]\lambda I - \lambda R)\end{aligned}$$

Differentiate the Hamiltonian with respect to each state variable, gives the co-state system (9):

$$\frac{dz_S}{dt} = -\frac{\partial H}{\partial S}, \frac{dz_I}{dt} = -\frac{\partial H}{\partial I}, \frac{dz_R}{dt} = -\frac{\partial H}{\partial R}.$$

There exists u^* such that $H\left(S^*,I^*,R^*,u^*,z_S^*,z_I^*,z_R^*\right) \leq H\left(S^*,I^*,R^*,u,z_S^*,z_I^*,z_R^*\right)$ for all $u(0,T) \to [0,1]$. In this case, u^* satisfies

$$\left.\frac{\partial H}{\partial u}\right|_{u^*} = 2nu^* - (z_R^* - z_I^*)\,(1-\delta)\,\gamma I^* = 0.$$

Therefore, we obtain

$$u^* = \frac{(1-\delta)\,\gamma I^*\,(z_R^* - z_I^*)}{2n}$$

By using the property of the control space U, we have the following conditions. If $\frac{\partial H}{\partial u} < 0$,then $u^* = 0$, conversely, if $\frac{\partial H}{\partial u} > 0$, at t, we take $u^* = 1$. Therefore, we can rewrite in compact form the optimal control variable by

$$u^* = \max\left\{0, \min\left(1, \frac{(1-\delta)\,\gamma I^*\,(z_R^* - z_I^*)}{2n}\right)\right\}$$

Here, (z_I^*, z_R^*) is the solution of system (9), (10). □

7. Numerical Simulation

In this section, we first simulate numerical results when the system (1) is without control, there is forward bifurcation when $R_0 = 1$, and plot results from sensitivity analysis. Further, we simulate the dynamic system (1) with the control.

7.1. Simulation without Control.

First, consider the basic reproduction number R_0 in (4) without control u, that is $u = 0$. Thus, we have $I^* = \frac{\lambda_1(\lambda_2+\delta\gamma)}{\lambda_2\beta}(R_0 - 1)$. It is found that I^* exists if $R_0 \geq 1$. We can confirm this by plotting I^* as a function of R_0, where R_0 varies. This is given in Figure 4. It shows that there exists a forward bifurcation with $R_0 = 1$ as the bifurcation point. When $R_0 \leq 1$, we have $I^* = 0$ i.e. there is no congestion, and when $R_0 \geq 1$, we get $I^* \geq 0$, i.e. there is congestion with the number of vehicles in the congestion increases linearly as R_0 increases.

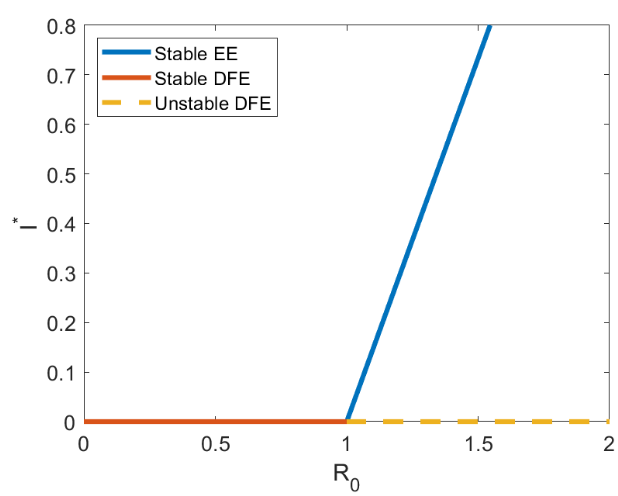


FIGURE 4. A forward bifurcation diagram appeared in the system

We can directly see from Equation (4) that decreasing the value of α or β, or increasing the value of $\lambda_1, \lambda_2, \delta$, or γ will reduce the number R_0. This is confirmed in Figure 5. We can observe that decreasing the value of α and β will reduce linearly the value of R_0. Meanwhile, increasing the value of $\lambda_1, \lambda_2, \delta$, and γ will reduce the value of R_0 exponentially

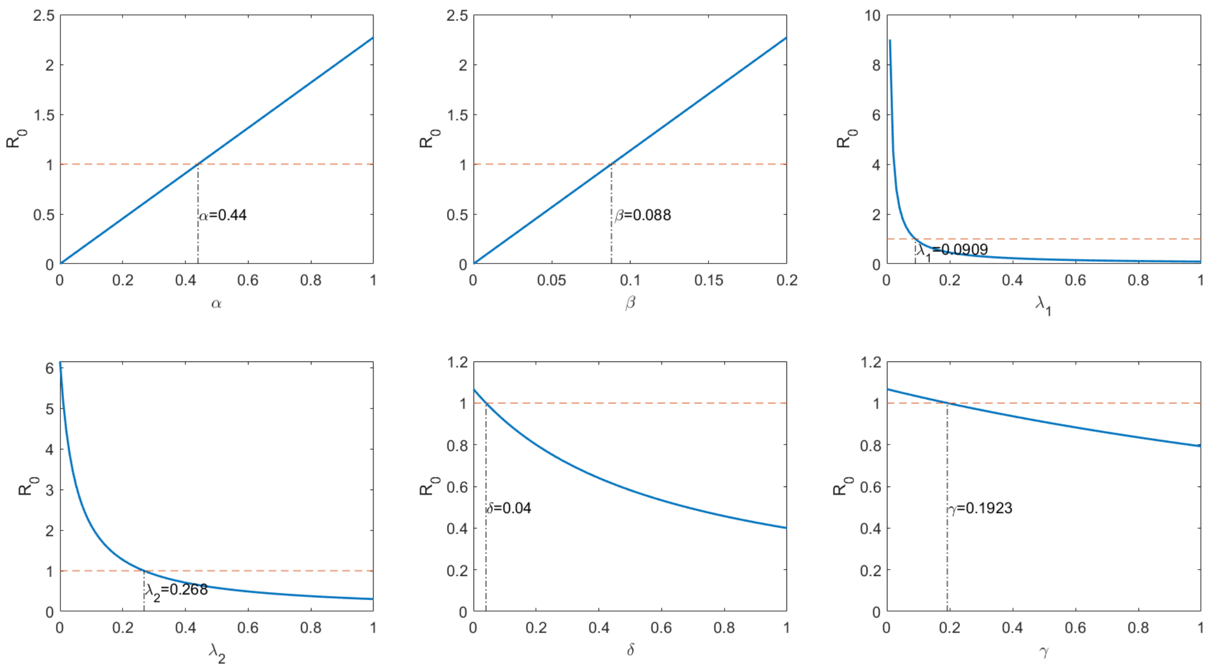


FIGURE 5. Plot of parameters versus the basic reproduction number.

To observe which parameter that has the most impact on R_0, we calculate the normalized sensitivity index, or elasticity index, that is defined as in (7). We present the bar plot in Figure 6 to compare the elasticity index for all parameters. It is found that parameters α, β, λ_1 have most impact proportionally on R_0, and followed by λ_2. Thus, in order to control the congestion maximally, we can reduce the number of susceptible vehicles entering the main road (that is an interpretation of α), or reduce the transmission rate of the congested vehicles by keeping a certain distance between the susceptible and infected vehicles (this is an interpretation of β), or increase the number of susceptible vehicles that exit the main road (this is an interpretation of λ_1). To do that in practice, for the case of α, we can add a traffic light on the main road far before; for the case of β, we can add a road sign on the main road that says, for example, be aware of the congestion ahead; for the case of λ_1, some of the susceptible vehicles need to move to the side road to exit the main road.

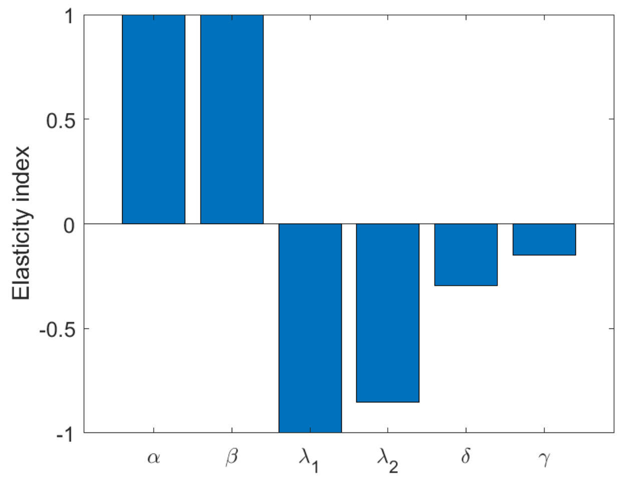


FIGURE 6. Elasticity index of parameters.

Another way to observe the sensitivity of R_0 is by plotting its contour plot where R_0 acts as a function of two parameters. The plots are presented in Figure 7. The figures show how the behavior of R_0 will change if two parameters are changed. Thus, by these figures, we can observe how two treatments will affect simultaneously the congestion.

7.2. Simulation with Control. We implement the forward-backward sweep numerical approach [19] to solve the optimal control system. The fourth-order Runge-Kutta method is applied for solving the state system and the adjoint system. The parameter values $\delta = 0.104, \gamma = 0.5$ are from the traffic condition at the intersection in [9]. The numerical simulation (Figure 8), when $R_0 < 1$, confirms the analytical result of the non-endemic equilibrium point, $E^0 = \left(\frac{\alpha}{\lambda_1}, 0, 0\right)$, in the case without the control ($u = 0$) where the remaining parameters are assumed as follows: $\alpha = 0.4, \lambda_1 = 0.1, \beta = 0.08, \lambda_2 = 0.3$ and $\theta = 0.01$. The reproduction number is obtained $R_0 = 0.9091 \leq 1$. The initial conditions used are given as follows: $S(0) = 50, I(0) = 4, R(0) = 0$. Here, the non-endemic (congestion-free) equilibrium point $E^0 = (4, 0, 0)$ is reached at $T = 300$. Traffic congestion or endemic state can occur when α increases, now assume $\alpha = 0.6$ (see Figure 3(a)), thus, the reproduction number becomes $R_0 = 1.3636 \geq 1$, meaning a traffic jam (endemic) occurs. By applying the control $u(t)$ the congestion is resolved faster compared to without the control. Figure 9 depicts the dynamics of the number of congested vehicles,I, the recovered vehicles, R, and the variable optimal control, $u(t)$, for both cases with and without the control when $T = 30$ and $T = 50$, respectively. From

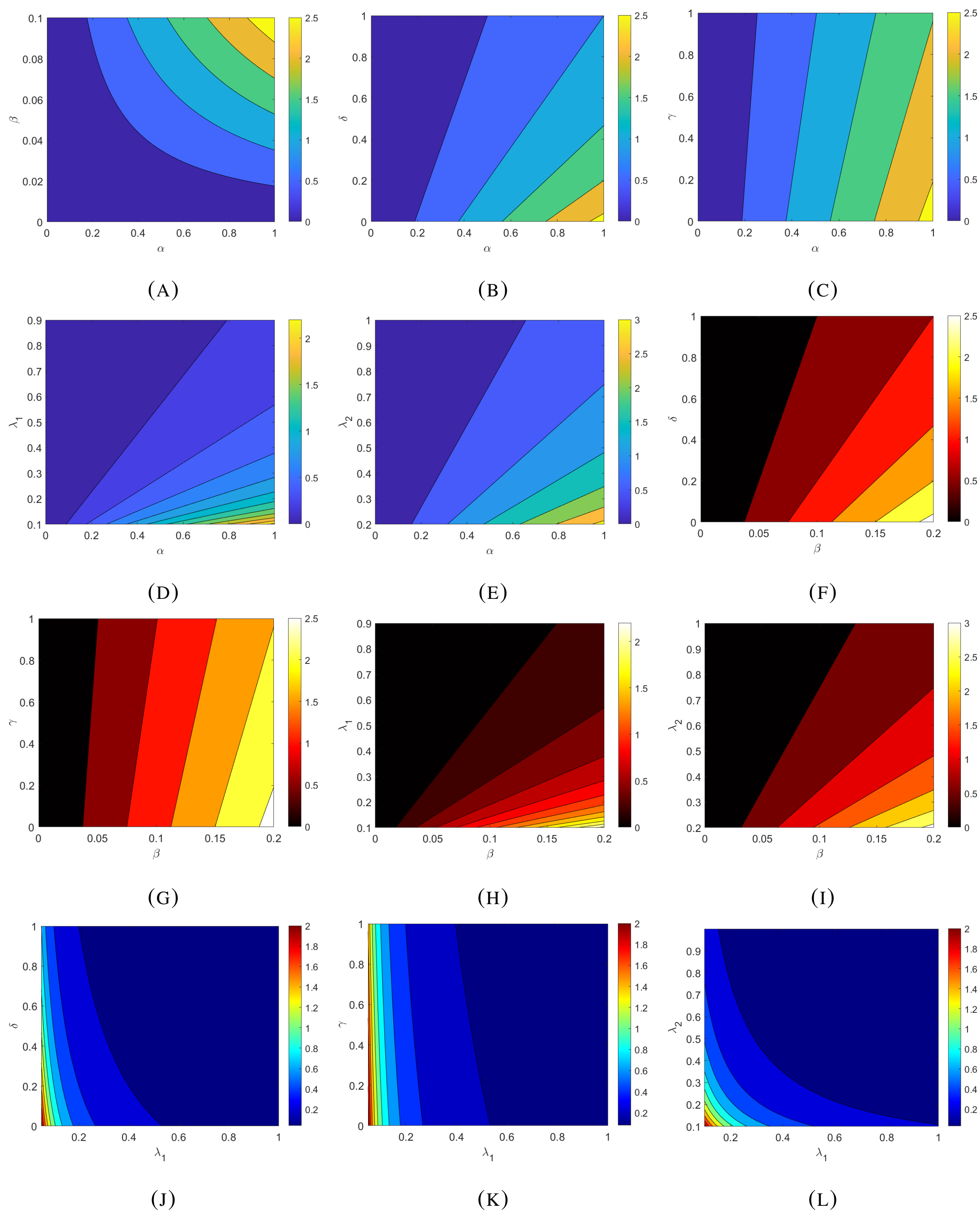
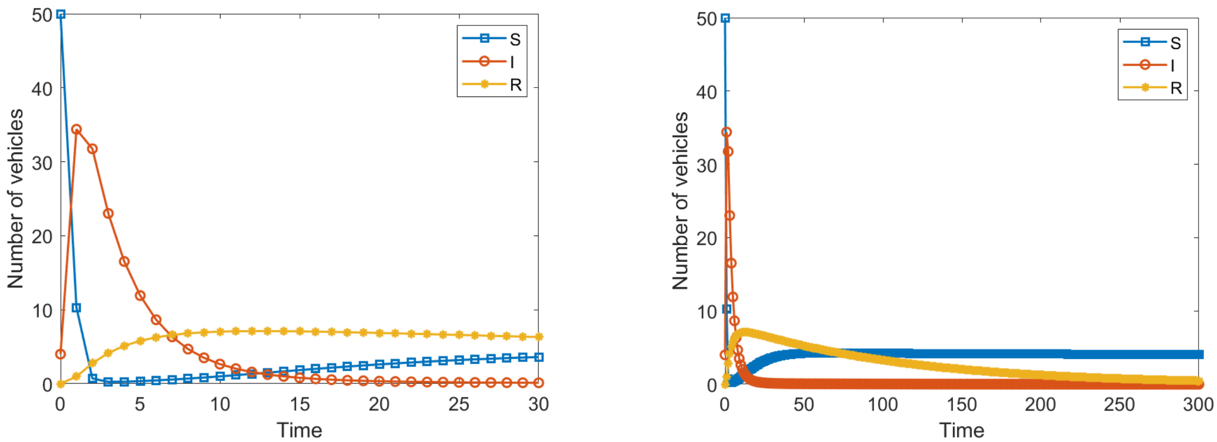
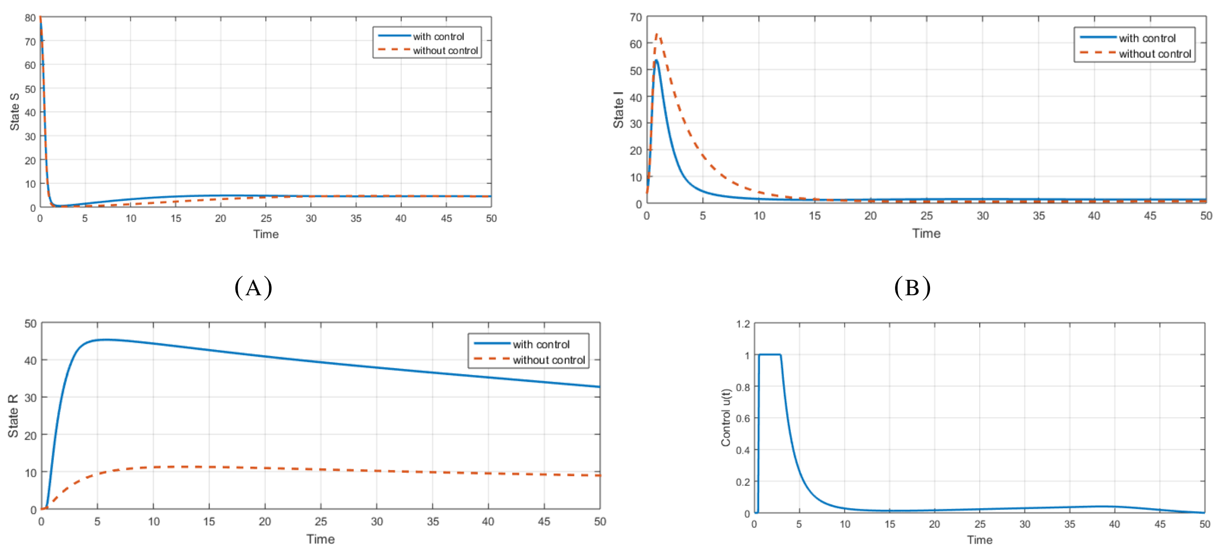


FIGURE 7. Contour plot of the basic reproduction number as a function of two parameters.



(A) (B)

FIGURE 8. Non-endemic equilibrium



(C) (D)

FIGURE 9. Dynamics of each state variables states in traffic congestion with and without control, and the optimal control variable $u(t)$.

the figures, we see that the optimal control u^* that is extending the duration of green light is effective for reducing traffic congestion.

From Figure 9 it can be seen that the optimal control effectively reduces congestion faster, when $t = 5$, the number of vehicles that are in a traffic jam is only below 5 compared to without control the number is still above 17, thus, the intervention successfully reduces traffic congestion with efficiency of around 75%. Conversely, the number of vehicles free from traffic jams increased significantly at $t = 5$, which is $R \approx 45$ compared to without control $R \approx 9$. The value of the control function $u(t)$, at the beginning, reaches the maximum and then decreases once vehicles in a congested state are recovered from the traffic congestion as shown in the plot for R which reached the peak 45. Thus, the existence of optimal control is effective in reducing congestion or vehicle buildup.

8. CONCLUSION

This paper deals with an SIR model with control for simple traffic congestion. A control on the length of green light at a busy intersection has been used. The congestion-free equilibrium point is locally asymptotically stable if the basic reproduction number R_0 is less than 1 and is unstable if $R_0 > 1$. The optimal control problem is to minimize the vehicles in traffic congestion and costs for implementation of the control. For the time-varying control problem, we discuss its optimal control problem, including the existence and uniqueness of optimal control and its solution. These problems are resolved by applying the optimal control theory. Numerical results show the positive impacts of implementing the control to susceptible vehicles and treatment for congested vehicles. We have also done an efficiency analysis, which shows that control is more effective than without control. This work is theoretical modelling and it can be further justified by using experimental results.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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