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REACTION-DIFFUSION IN TRYPANOSOMIASIS-MALARIA CO-INFECTION TRANSMISSION DYNAMICS

A. O. SANGOTOLA[1], O.S. OBABIYI[2,*]

[1]Department of Physical Sciences, Bells University of Technology, Ota, Nigeria

[2]Department of Mathematics, University of Ibadan, Nigeria

Abstract. An epidemic model for trypanosomiasis-malaria co-infection dynamics is formulated to study the effect of diffusion on the disease dynamics. The basic reproduction number is calculated using the next generation matrix approach. The local stability of the spatially homogeneous disease-free equilibrium of both the trypanosomiasis and malaria aspect of the model are obtained through Routh-Hurwitz criteria. The local asymptotically stability of the spatially homogeneous disease-free equilibrium of the co-infection is also examined. The global stability of the spatially homogeneous disease free-equilibrium is established by using a suitable Lyapunov functional.

Keywords: stability; equilibrium; basic reproduction number; Lyapunov functional.

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1. INTRODUCTION

Reaction diffusion equations are used to describe many physical phenomena in science. A reaction-diffusion equation takes the form:

$$\frac{\partial u}{\partial t} = D\triangle u + F(u) \tag{1}$$

*Corresponding author

E-mail address: os.obabiyi@ui.edu.ng

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where $u \in \mathbf{R^m}$ denotes biological or physical phenomenon, $D \in \mathbf{R^{m\times m}}$ denotes the diffusion coefficient matrix, $\triangle$ is associated Laplacian with respect to the diffusion of the organism and $F(u)$ denotes the biological or chemical reactions.

Reaction-diffusion models have gained the attention of many researchers in epidemiology by looking at the effect of diffusion on disease spread. Lofti et al. [1] studied the dynamics of a specific nonlinear incidence rate on a reaction diffusion SIR model with homogeneous Neumann boundary condition. The global stability of both the disease free and endemic equilibria are established through suitable Lyapunov functionals.

Hattaf and Yousfi [2] constructed Lyapunov functionals to study the global stability of some diffusion equations in biology. The idea of constructing the Lyapunov functions in the ordinary differential case are employed to the obtain the Lyapunov functionals for the diffusion equations. This technique is also amplified in [3-5].

Wang et al. [6] studied a reaction-diffusion SIR via environmental driven infection in heterogeneous space. The global dynamics of the disease free equilibrium point for homogeneous and heterogeneous case were examined. Elawi and Al Agha [7] examined reaction-diffusion within host malaria dynamics with cell-mediated immune response and antibody. The global stability of all the possible equilibrium points were determined by selecting suitable Lyapunov functionals and using LaSalle invariance principle. Liu et al. [8] established an SIVR model with diffusion, spatially heterogeneous, latent infection, and incomplete immunity in the Neumann boundary condition. The stability of the equilibrium point is established in relation to the basic reproduction number. The operator semi group method was also used to prove the dynamic behaviour of the model.

The goal of this work is to investigate the local and global stability dynamics of the disease free equilibrium point of the reaction-diffusion of trypanosomiasis-malaria model in a spatially homogeneous space.

2. Model Formulation

The formulated model for the co-dynamics of trypanosomiasis and malaria divides human populations N_h into susceptible humans N_{ss},humans exposed to trypanosomiasis only N_{es}, humans infected with trypanosomiasis only N_{is},humans recovered from trypanosomiasis only

N_{rs},humans exposed to malaria only N_{se}, humans infected with malaria only N_{si},humans recovered from malaria only N_{sr}, N_{rr} represents those who are not susceptible to either infection but may includes those who are still exposed or infected with either infection or both. The model is designed in such a way that a susceptible individual person becomes exposed to trypanosomiasis or malaria after an effective contact with an infectious tsetse fly or mosquito with transmission rate ϕ_h and β_h respectively. The susceptible human compartment is increased as a result of new recruitment at rate Λ_h and as result of loss of immunity to both trypanosomiasis and malaria respectively at rate ω_t and ω_m respectively. The model takes into account both the natural death, disease-induced mortality and co-infection parameter ρ_m and ρ_t for both malaria and trypanosomiasis respectively.

The mosquito compartment is divided into three classes namely susceptible mosquitoes $S_m(t)$, exposed mosquitoes $E_m(t)$ and infectious mosquitoes $I_m(t)$. The total mosquitoes population is given by $N_m(t)$. We divide the tsetse fly population into three sub cases namely susceptible tsetse flies $S_t(t)$, exposed tsetse flies $E_t(t)$ and infectious tsetse flies $I_t(t)$ while the total size of the tsetse fly population at any given time t is denoted by $N_t(t)$.

Let $D_1, D_2, \cdots, D_{14}$ represents the diffusion constants of

$N_{ss}, N_{es}, N_{is}, N_{rs}, N_{se}, N_{si}, N_{sr}, N_{rr}, S_t, E_t, I_t, S_m, E_m, I_m$ respectively.

Thus, we have the following system of equations.

$$
\begin{aligned}
\frac{\partial N_{ss}(t,x)}{\partial t} &= D_1 \triangle N_{ss} + \Lambda_h - \frac{c_t \phi_h N_{ss} I_t}{N_h} - \frac{c_m \beta_h N_{ss} I_m}{N_h} - \mu_h N_{ss} + \omega_t N_{rs} + \omega_m N_{sr} \\
\frac{\partial N_{es}(t,x)}{\partial t} &= D_2 \triangle N_{es} + \frac{c_t \phi_h N_{ss} I_t}{N_h} - (\mu_h + \sigma_t) N_{es} - \frac{\rho_m \lambda_m N_{es} I_m}{N_h} \\
\frac{\partial N_{is}(t,x)}{\partial t} &= D_3 \triangle N_{is} + \sigma_t N_{es} - (\mu_h + \delta_a + \theta_t) N_{is} - \frac{\rho_m \lambda_m N_{is} I_m}{N_h} \\
\frac{\partial N_{rs}(t,x)}{\partial t} &= D_4 \triangle N_{rs} + \theta_t N_{is} - (\mu_h + \omega_t) N_{rs} + \varepsilon_m N_{rr} - \frac{\lambda_m N_{rs} I_m}{N_h} \\
\frac{\partial N_{se}(t,x)}{\partial t} &= D_5 \triangle N_{se} + \frac{c_m \beta_h N_{ss} I_m}{N_h} - (\mu_h + \sigma_m) N_{se} - \frac{\rho_t \lambda_t N_{se} I_t}{N_h} \\
\frac{\partial N_{si}(t,x)}{\partial t} &= D_6 \triangle N_{si} + \sigma_m N_{se} - (\mu_h + \delta_b + \theta_m) N_{si} - \frac{\rho_t \lambda_t N_{si} I_t}{N_h} \\
\frac{\partial N_{sr}(t,x)}{\partial t} &= D_7 \triangle N_{sr} + \theta_m N_{si} - (\mu_h + \omega_m) N_{sr} + \varepsilon_t N_{rr} - \frac{\lambda_t N_{sr} I_t}{N_h} \\
\frac{\partial N_{rr}(t,x)}{\partial t} &= D_8 \triangle N_{rr} + \frac{\rho_m \lambda_m (N_{es} + N_{is}) I_m}{N_h} + \frac{\rho_t \lambda_t (N_{se} + N_{si}) I_t}{N_h} - (\mu_h + \varepsilon_t + \varepsilon_m) N_{rr} \\
&\quad + \frac{\lambda_m N_{rs} I_m}{N_h} + \frac{\lambda_t N_{sr} I_t}{N_h} \\
\frac{\partial S_t(t,x)}{\partial t} &= D_9 \triangle S_t + \Lambda_t - \frac{c_t \phi_t N_{is} S_t}{N_t} - \mu_t S_t \\
\frac{\partial E_t(t,x)}{\partial t} &= D_{10} \triangle E_t + \frac{c_t \phi_t N_{is} S_t}{N_t} - (\alpha_t + \mu_t) E_t \\
\frac{\partial I_t(t,x)}{\partial t} &= D_{11} \triangle I_t + \alpha_t E_t - (\delta_t + \mu_t) I_t \\
\frac{\partial S_m(t,x)}{\partial t} &= D_{12} \triangle S_m + \Lambda_m - \frac{c_m \beta_m N_{si} S_m}{N_m} - \mu_m S_m \\
\frac{\partial E_m(t,x)}{\partial t} &= D_{13} \triangle E_m + \frac{c_m \beta_m N_{si} S_m}{N_m} - (\alpha_m + \mu_m) E_m \\
\frac{\partial I_m(t,x)}{\partial t} &= D_{14} \triangle I_m + \alpha_m E_m - (\delta_m + \mu_m) I_m
\end{aligned} \tag{2}
$$

with $(t,\mathbf{x}) \in Q = (0,T) \times \Omega$ with spatial domain Ω

The system is imposed with Neumann boundary condition and it is given by

$$\frac{\partial N_{ss}(t,\mathbf{x})}{\partial n} = \frac{\partial N_{es}(t,\mathbf{x})}{\partial n} = \frac{\partial N_{is}(t,\mathbf{x})}{\partial n} = \frac{\partial N_{rs}(t,\mathbf{x})}{\partial n} = \frac{\partial N_{se}(t,\mathbf{x})}{\partial n} = \frac{\partial N_{si}(t,\mathbf{x})}{\partial n} = \frac{\partial N_{sr}(t,\mathbf{x})}{\partial n} = 0$$

$$\frac{\partial N_{rr}(t,\mathbf{x})}{\partial n} = \frac{\partial S_t(t,\mathbf{x})}{\partial n} = \frac{\partial E_t(t,\mathbf{x})}{\partial n} = \frac{\partial I_t(t,\mathbf{x})}{\partial n} = \frac{\partial S_m(t,\mathbf{x})}{\partial n} = \frac{\partial E_m(t,\mathbf{x})}{\partial n} = \frac{\partial I_m(t,\mathbf{x})}{\partial n} = 0$$

with $(t,\mathbf{x}) \in (0,T) \times \partial\Omega$ and initial condition

$$N_{ss}(0,\mathbf{x}) = N_{ss}^0, N_{es}(0,\mathbf{x}) = N_{es}^0, N_{is}(0,\mathbf{x}) = N_{is}^0, N_{rs}(0,\mathbf{x}) = N_{rs}^0,$$

$$N_{se}(0,\mathbf{x}) = N_{se}^0, N_{si}(0,\mathbf{x}) = N_{si}^0, N_{rr}(0,\mathbf{x}) = N_{rr}^0, S_t(0,\mathbf{x}) = S_t^0,$$

$$E_t(0,\mathbf{x}) = E_t^0, I_t(0,\mathbf{x}) = I_t^0, S_m(0,\mathbf{x}) = S_m^0, E_m(0,\mathbf{x}) = E_m^0, I_m(0,\mathbf{x}) = I_m^0, \mathbf{x} \in \Omega$$

3. Main Results

3.1. Trypanosomiasis.

The Trypanosomiasis only aspect of the model (2) is given by

$$\left.\begin{aligned}
\frac{\partial N_{ss}(t,x)}{\partial t} &= D_1 \triangle N_{ss} + \Lambda_h - \frac{c_t \phi_h N_{ss} I_t}{N_h} - \mu_h N_{ss} + \omega_t N_{rs} \\
\frac{\partial N_{es}(t,x)}{\partial t} &= D_2 \triangle N_{es} + \frac{c_t \phi_h N_{ss} I_t}{N_h} - (\mu_h + \sigma_t) N_{es} \\
\frac{\partial N_{is}(t,x)}{\partial t} &= D_3 \triangle N_{is} + \sigma_t N_{es} - (\mu_h + \delta_a + \theta_t) N_{is} \\
\frac{\partial N_{rs}(t,x)}{\partial t} &= D_4 \triangle N_{rs} + \theta_t N_{is} - (\mu_h + \omega_t) N_{rs} \\
\frac{\partial S_t(t,x)}{\partial t} &= D_9 \triangle S_t + \Lambda_t - \frac{c_t \phi_t N_{is} S_t}{N_t} - \mu_t S_t \\
\frac{\partial E_t(t,x)}{\partial t} &= D_{10} \triangle E_t + \frac{c_t \phi_t N_{is} S_t}{N_t} - (\alpha_t + \mu_t) E_t \\
\frac{\partial I_t(t,x)}{\partial t} &= D_{11} \triangle I_t + \alpha_t E_t - (\delta_t + \mu_t) I_t
\end{aligned}\right\} \tag{3}$$

TABLE 1. The description of parameters of model (2)

Definition	Symbols
Recruitment term of the susceptible humans	Λ_h
Biting rate of Mosquito	c_m
Biting rate of Tsetse fly	c_t
Probability that a bite by an infectious mosquito result in transmission of disease to human	β_h
Probability that a bite by an infectious tsetse fly result in transmission of disease to human	ϕ_h
Probability that a bite results in transmission of parasite to a susceptible mosquito	β_m
Probability that a bite results in transmission of parasite to a susceptible tsetse fly	ϕ_t
Progression rate of humans exposed to trypanosomiasis to infectious class	σ_m
Progression rate of humans exposed to malaria to infectious class	δ_m
Per capita transition rate of recovered humans from malaria	ω_m
Per capita transition rate of recovered humans from trypanosomiasis	ω_t
Natural death rate of humans	μ_h
co-infection parameter for malaria	ρ_m
co-infection parameter for trypanosomiasis	ρ_t
loss of immunity to malaria in the co-infection	ε_m
loss of immunity to trypanosomiasis in the co-infection	ε_t
effective treatment rate of humans infected with trypanosomiasis only	θ_t
effective treatment rate of humans infected with malaria only	θ_m
Recruitment rate of susceptible mosquito only	Λ_m
Recruitment rate of susceptible tsetse fly	Λ_t
Progression rate of the exposed mosquito to infectious mosquito	α_m
Progression rate of the exposed tsetse fly to infectious tsetse fly	α_t
Natural death rate of mosquito	μ_m
Natural death rate of tsetse fly	μ_t
Disease induced death rate of humans infected with malaria	δ_b
Disease induced death rate of humans infected with trypanosomiasis	δ_a
Disease induced death rate of mosquito	δ_m
Disease induced death rate of tsetse fly	δ_t

The system has a disease free equilibrium $\pi_0^t = (N_{ss}^{0t}, N_{es}^{0t}, N_{is}^{0t}, N_{rs}^{0t}, S_t^{0t}, E_t^{0t}, I_t^{0t})$ given below as

$$\pi_0^t = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, 0, \frac{\Lambda_t}{\mu_t}, 0, 0\right) \tag{4}$$

The basic reproduction number of (3) can be obtained easily using the next generation matrix method[10,11] and is given by

$$R_{0t} = \sqrt{\frac{{c_t}^2 \phi_h \phi_t \alpha_t \sigma_t}{(\mu_h + \sigma_t)(\mu_h + \delta_a + \theta_t)(\alpha_t + \mu_t)(\delta_t + \mu_t)}} \tag{5}$$

3.1.1. *Local Stability I.*

Theorem 3.1. The spatially homogeneous disease-free equilibrium solution π_0^t of (3) is locally asymptotically stable if $R_{0t} < 1$ and unstable otherwise.

Proof. System (3) is linearized about arbitrary spatially homogeneous equilibrium point $(\overline{N_{ss}}, \overline{N_{es}}, \overline{N_{is}}, \overline{N_{rs}}, \overline{S_t}, \overline{E_t}, \overline{I_t})$. Let there exist small perturbations $N_{s1}, N_{e1}, N_{i1}, N_{r1}, S_1, E_1, I_1$ as discussed in [12]. The resulting differential equation is given by

$$\left.\begin{aligned}
\frac{\partial N_{s1}}{\partial t} &= D_1 \triangle N_{s1} + c_{11} N_{s1} + c_{12} N_{e1} + c_{13} N_{i1} + c_{14} N_{r1} + c_{15} S_1 + c_{16} E_1 + c_{17} I_1 \\
\frac{\partial N_{e1}}{\partial t} &= D_2 \triangle N_{e1} + c_{21} N_{s1} + c_{22} N_{e1} + c_{23} N_{i1} + c_{24} N_{r1} + c_{25} S_1 + c_{26} E_1 + c_{27} I_1 \\
\frac{\partial N_{i1}}{\partial t} &= D_3 \triangle N_{i1} + c_{31} N_{s1} + c_{32} N_{e1} + c_{33} N_{i1} + c_{34} N_{r1} + c_{35} S_1 + c_{36} E_1 + c_{37} I_1 \\
\frac{\partial N_{r1}}{\partial t} &= D_4 \triangle N_{r1} + c_{41} N_{s1} + c_{42} N_{e1} + c_{43} N_{i1} + c_{44} N_{r1} + c_{45} S_1 + c_{46} E_1 + c_{47} I_1 \\
\frac{\partial S_1}{\partial t} &= D_9 \triangle S_1 + c_{51} N_{s1} + c_{52} N_{e1} + c_{53} N_{i1} + c_{54} N_{r1} + c_{55} S_1 + c_{56} E_1 + c_{57} I_1 \\
\frac{\partial E_1}{\partial t} &= D_{10} \triangle E_1 + c_{61} N_{s1} + c_{62} N_{e1} + c_{63} N_{i1} + c_{64} N_{r1} + c_{65} S_1 + c_{66} E_1 + c_{67} I_1 \\
\frac{\partial I_1}{\partial t} &= D_{11} \triangle I_1 + + c_{71} N_{s1} + c_{72} N_{e1} + c_{73} N_{i1} + c_{74} N_{r1} + c_{75} S_1 + c_{76} E_1 + c_{77} I_1
\end{aligned}\right\} \tag{6}$$

where $c_{ij} (i, j = 1, 2, \cdots, 7)$ are the elements of the Jacobian matrix at the disease free equilibrium, π_0^t.

Let there exist a series solution of (6) of the form:

$$\left.\begin{aligned} N_{s1} &= \sum_k N_{sk} e^{\lambda t} coskx \\ N_{e1} &= \sum_k N_{ek} e^{\lambda t} coskx \\ N_{i1} &= \sum_k N_{ik} e^{\lambda t} coskx \\ N_{r1} &= \sum_k N_{rk} e^{\lambda t} coskx \\ S_1 &= \sum_k S_k e^{\lambda t} coskx \\ E_1 &= \sum_k E_k e^{\lambda t} coskx \\ I_1 &= \sum_k I_k e^{\lambda t} coskx \end{aligned}\right\} \tag{7}$$

Equation (6) can thus be converted into:

$$\left.\begin{aligned} &(c_{11} - D_1k^2 - \lambda)N_{s1} + c_{12}N_{e1} + c_{13}N_{i1} + c_{14}N_{r1} + c_{15}S_1 + c_{16}E_1 + c_{17}I_1 = 0 \\ &c_{21}N_{s1} + (c_{22} - D_2k^2 - \lambda)N_{e1} + c_{23}N_{i1} + c_{24}N_{r1} + c_{25}S_1 + c_{26}E_1 + c_{27}I_1 = 0 \\ &c_{31}N_{s1} + c_{32}N_{e1} + (c_{33} - D_3k^2 - \lambda)N_{i1} + c_{34}N_{r1} + c_{35}S_1 + c_{36}E_1 + c_{37}I_1 = 0 \\ &c_{41}N_{s1} + c_{42}N_{e1} + c_{43}N_{i1} + (c_{44} - D_4k^2 - \lambda)N_{r1} + c_{45}S_1 + c_{46}E_1 + c_{47}I_1 = 0 \\ &c_{51}N_{s1} + c_{52}N_{e1} + c_{53}N_{i1} + c_{54}N_{r1} + (c_{55} - D_9k^2 - \lambda)S_1 + c_{56}E_1 + c_{57}I_1 = 0 \\ &c_{61}N_{s1} + c_{62}N_{e1} + c_{63}N_{i1} + c_{64}N_{r1} + c_{65}S_1 + (c_{66} - D_{10}k^2 - \lambda)E_1 + c_{67}I_1 = 0 \\ &c_{71}N_{s1} + c_{72}N_{e1} + c_{73}N_{i1} + c_{74}N_{r1} + c_{75}S_1 + c_{76}E_1 + (c_{77} - D_{11}k^2 - \lambda)I_1 = 0 \end{aligned}\right\} \tag{8}$$

The Jacobian matrix with diffusion for the system (8) is given by

$$
(9)\quad \begin{pmatrix}
c_{11}-D_1k^2 & 0 & 0 & \omega_t & 0 & 0 & -c_t\phi_h \\
0 & c_{22}-D_2k^2 & 0 & 0 & 0 & 0 & c_t\phi_h \\
0 & \sigma_t & c_{33}-D_3k^2 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma_t & c_{44}-D_4k^2 & 0 & 0 & 0 \\
0 & 0 & -c_t\phi_t & 0 & c_{55}-D_9k^2 & 0 & 0 \\
0 & 0 & c_t\phi_t & 0 & 0 & c_{66}-D_{10}k^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_t & c_{77}-D_{11}k^2
\end{pmatrix}
$$

where $c_{11}=-\mu_h, c_{22}=-(\mu_h+\sigma_t), c_{33}=-(\mu_h+\delta_a+\theta_t), c_{44}=-(\mu_h+\omega_t), c_{55}=-\mu_t, c_{66}=-(\alpha_t+\mu_t), c_{77}=-(\delta_t+\mu_t)$.

$-\mu_h-D_1k^2$, $-\mu_t-D_9k^2$ and$-(\mu_h+\omega_t)-D_4k^2$ are part of the roots of the characteristics equation of (9). The other eigenvalues are evaluated from the resulting variational sub matrix below.

(10)

$$
\begin{pmatrix}
-(\mu_h+\sigma_t)-D_2k^2 & 0 & 0 & c_t\phi_h \\
\sigma_t & -(\mu_h+\delta_a+\theta_t)-D_3k^2 & 0 & 0 \\
0 & c_t\phi_t & -(\alpha_t+\mu_t)-D_{10}k^2 & 0 \\
0 & 0 & \alpha_t & -(\delta_t+\mu_t)-D_{11}k^2
\end{pmatrix}
$$

The sub matrix (10) can be represented below.

$$
(11)\quad \begin{pmatrix}
a_{11} & 0 & 0 & a_{14} \\
a_{21} & a_{22} & 0 & 0 \\
0 & a_{32} & a_{33} & 0 \\
0 & 0 & a_{43} & a_{44}
\end{pmatrix}
$$

The resulting characteristic equation is of the form $p(\lambda)=\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4=0$ where

$a_1=-(a_{11}+a_{22}+a_{33}+a_{44})$

$a_2=a_{11}a_{22}+a_{11}a_{33}+a_{11}a_{44}+a_{22}a_{33}+a_{22}a_{44}+a_{33}a_{44}$

$a_3=-(a_{11}a_{22}a_{33}+a_{11}a_{22}a_{44}+a_{11}a_{33}a_{44}+a_{22}a_{33}a_{44})$

$a_4=a_{11}a_{22}a_{33}a_{44}-a_{43}a_{32}a_{21}a_{14}$

The condition for stability by Routh-Hurwitz gives $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$ and $a_1a_2a_3 > a_3^2 + a_1^2 + a_1^2a_4$.

Clearly, $a_1 > 0, a_2 > 0, a_3 > 0$.

a_4 can be expanded below as:

$a_4 = c_t^2\phi_t\phi_h\sigma_t\alpha_t(1 - R_{0t}^2) + (\mu_h + \sigma_t)(\mu_h + \delta_a + \theta_t)(\alpha_t + \mu_t)D_{11}k^2 + (\mu_h + \sigma_t)(\mu_h + \delta_a + \theta_t)(\delta_t + \mu_t)D_{10}k^2 + (\mu_h + \sigma_t)(\mu_h + \delta_a + \theta_t)D_{10}D_{11}k^4 + (\mu_h + \sigma_t)(\alpha_t + \mu_t)(\delta_t + \mu_t)D_3k^2 + (\mu_h + \sigma_t)(\alpha_t + \mu_t)D_3D_{11}k^4 + (\mu_h + \sigma_t)(\delta_t + \mu_t)D_3D_{10}k^4 + (\mu_h + \sigma_t)D_3D_{10}D_{11}k^6 + (\mu_h + \delta_a + \theta_t)(\alpha_t + \mu_t)(\delta_t + \mu_t)D_2k^2 + (\mu_h + \delta_a + \theta_t)(\alpha_t + \mu_t)D_2D_{11}k^4 + (\mu_h + \delta_a + \theta_t)(\delta_t + \mu_t)D_2D_{10}k^4(\mu_h + \delta_a + \theta_t)D_2D_{10}D_{11}k^6 + (\alpha_t + \mu_t)(\delta_t + \mu_t)D_2D_3k^4 + (\alpha_t + \mu_t)D_2D_3D_{11}k^6 + (\delta_t + \mu_t)D_2D_3D_{10}k^6 + D_2D_3D_{10}D_{11}k^8$

$a_4 > 0$ provided that $R_{0t} < 1$.

$a_1a_2a_3$ and $a_3^2 + a_1^2a_4$ can be expanded below as:

$a_1a_2a_3 = a_{11}^3a_{22}^2a_{33} + a_{11}^3a_{22}a_{33}^2 + 3a_{11}^2a_{22}^2a_{33}^2 + 8a_{11}^2a_{22}^2a_{33}a_{44} + 8a_{11}^2a_{22}a_{33}^2a_{44} + a_{11}^2a_{22}^3a_{33} + a_{11}a_{22}^3a_{33}^2 + a_{11}a_{22}^3a_{33}a_{44} + 8a_{11}a_{22}^2a_{33}^2a_{44} + a_{11}^2a_{22}a_{33}^3 + a_{11}a_{22}^2a_{33}^3 + a_{11}a_{22}a_{33}^3a_{44} + 8a_{11}^2a_{22}a_{33}a_{44}^2 + 8a_{11}a_{22}^2a_{33}a_{44}^2 + 8a_{11}a_{22}a_{33}^2a_{44}^2 + a_{11}^3a_{22}^2a_{44} + a_{11}^3a_{22}a_{33}a_{44} + +a_{11}^3a_{22}a_{44}^2 + 3a_{11}^2a_{22}^2a_{44}^2 + a_{11}^2a_{22}^3a_{44} + a_{11}a_{22}^3a_{44}^2 + a_{11}^2a_{22}a_{44}^3 + a_{11}a_{22}^2a_{44}^3 + a_{11}a_{22}a_{33}a_{44}^3 + a_{11}^3a_{22}a_{33}a_{44} + a_{11}^3a_{33}^2a_{44} + a_{11}^3a_{33}a_{44}^2 + 3a_{11}^2a_{33}^2a_{44}^2 + a_{11}^2a_{33}^3a_{44} + a_{11}a_{22}a_{33}^3a_{44} + a_{11}a_{33}^3a_{44}^2 + a_{11}^2a_{33}a_{44}^3 + a_{11}a_{22}a_{33}a_{44}^3 + a_{11}a_{33}^2a_{44}^3 + a_{11}a_{22}^3a_{33}a_{44} + a_{22}^3a_{33}^2a_{44} + a_{22}^3a_{33}a_{44}^2 + 3a_{22}^2a_{33}^2a_{44}^2 + a_{11}a_{22}a_{33}^2a_{44} + a_{22}^2a_{33}^3a_{44} + a_{22}a_{33}^3a_{44}^2 + a_{11}a_{22}a_{33}a_{44}^3 + a_{22}^2a_{33}a_{44}^3 + a_{22}a_{33}^2a_{44}^3$

$a_3^2 + a_1^2a_4 = a_{11}^2a_{22}^2a_{33}^2 + 4a_{11}^2a_{22}^2a_{33}a_{44} + 4a_{11}^2a_{22}a_{33}^2a_{44} + 4a_{11}a_{22}^2a_{33}^2a_{44} + a_{11}^2a_{22}^2a_{44}^2 + 4a_{11}^2a_{22}a_{33}a_{44}^2 + 4a_{11}a_{22}^2a_{33}a_{44}^2 + a_{11}^2a_{33}^2a_{44}^2 + 4a_{11}a_{22}a_{33}^2a_{44}^2 + a_{22}^2a_{33}^2a_{44}^2 + a_{11}^3a_{22}a_{33}a_{44} + a_{11}a_{22}^3a_{33}a_{44} + a_{11}a_{22}a_{33}^3a_{44} - a_{11}^2a_{43}a_{32}a_{21}a_{14} - 2a_{11}a_{22}a_{43}a_{32}a_{21}a_{14} - 2a_{11}a_{33}a_{43}a_{32}a_{21}a_{14} - 2a_{11}a_{44}a_{43}a_{32}a_{21}a_{14} - a_{22}^2a_{43}a_{32}a_{21}a_{14} - 2a_{22}a_{33}a_{43}a_{32}a_{21}a_{14} - 2a_{22}a_{44}a_{43}a_{32}a_{21}a_{14} - a_{33}^2a_{43}a_{32}a_{21}a_{14} - 2a_{33}a_{44}a_{43}a_{32}a_{21}a_{14} - a_{44}^2a_{43}a_{32}a_{21}a_{14}$

Clearly, $a_1a_2a_3 > a_3^2 + a_1^2a_4$.

3.1.2. *Global Stability I.*

Theorem 3.2. The spatially homogeneous disease-free equilibrium solution π_0^t of (3) is globally asymptotically stable if $R_{0t} < 1$.

Proof. Consider the Lyapunov function:

$$\begin{aligned} V_1 &= \frac{c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\left[N_{ss}-N_{ss}^{0t}-N_{ss}^{0t}ln\left(\frac{N_{ss}}{N_{ss}^{0t}}\right)\right]+\frac{c_t\phi_t\alpha_t\sigma_t N_{es}}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)} \\ &+\frac{c_t\phi_t\alpha_t N_{is}}{(\mu_h+\delta_a+\theta_t)}+\alpha_t E_t+(\alpha_t+\mu_t)I_t \end{aligned} \tag{12}$$

The time derivative of V_1 is given by

$$\begin{aligned} \dot{V}_1 &= \frac{c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\left(1-\frac{N_{ss}^{0t}}{N_{ss}}\right)\dot{N}_{ss}+\frac{c_t\phi_t\alpha_t\sigma_t\dot{N}_{es}}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)} \\ &+\frac{c_t\phi_t\alpha_t\dot{N}_{is}}{(\mu_h+\delta_a+\theta_t)}+\alpha_t\dot{E}_t+(\alpha_t+\mu_t)\dot{I}_t \end{aligned} \tag{13}$$

Substituting the reaction part of (3) gives

$$\begin{aligned} \dot{V}_1 &= \frac{c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\left(1-\frac{N_{ss}^{0t}}{N_{ss}}\right)\left(\Lambda_h-\frac{c_t\phi_h N_{ss}I_t}{N_h}-\mu_h N_{ss}\right) \\ &+\frac{c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\left(\frac{c_t\phi_h N_{ss}I_t}{N_h}-(\mu_h+\sigma_t)N_{es}\right) \\ &+\frac{c_t\phi_t\alpha_t}{(\mu_h+\delta_a+\theta_t)}(\sigma_t N_{es}-(\mu_h+\delta_a+\theta_t)N_{is}) \\ &+\alpha_t\left(\frac{c_t\phi_t N_{is}S_t}{N_t}-(\alpha_t+\mu_t)E_t\right)+(\alpha_t+\mu_t)(\alpha_t E_t-(\delta_t+\mu_t)I_t) \end{aligned} \tag{14}$$

Hence,

$$\begin{aligned} \dot{V}_1 &\leq -\frac{\mu_h c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\frac{(N_{ss}-N_{ss}^{0t})^2}{N_{ss}} \\ &+\frac{c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\left(\frac{c_t\phi_h N_{ss}^{0t}I_t}{N_h}\right)-(\alpha_t+\mu_t)(\delta_t+\mu_t)I_t \end{aligned} \tag{15}$$

Thus,

$$\dot{V}_1 \leq -\frac{\mu_h c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\frac{(N_{ss}-N_{ss}^{0t})^2}{N_{ss}}+(\alpha_t+\mu_t)(\delta_t+\mu_t)I_t(R_{0t}^2-1) \tag{16}$$

Next, the Lyapunov functional for the reaction diffusion system (3) is given by

$$W_1=\int_\Omega V_1(N_{ss}(x,t),N_{es}(x,t),N_{is}(x,t),E_t(x,t),I_t(x,t))dx$$

The time derivative of W_1 is given by

$$\begin{aligned}\frac{dW1}{dt} &= \int_\Omega \left(\frac{D_1 c_t \phi_t \alpha_t \sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\left(1-\frac{N_{ss}^{0t}}{N_{ss}}\right)\Delta N_{ss}\right)dx \\ &+ \int_\Omega \left(\frac{D_2 c_t \phi_t \alpha_t \sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\Delta N_{es}\right)dx + \int_\Omega \left(D_3 \frac{c_t \phi_t \alpha_t}{(\mu_h+\delta_a+\theta_t)}\Delta N_{is}\right)dx \\ &+ \int_\Omega (D_{10}\alpha_t \Delta E_t)\,dx + \int_\Omega (D_{11}(\alpha_t+\mu_t)\Delta I_t)\,dx + \int_\Omega \frac{dV_1}{dt}dx \end{aligned} \tag{17}$$

Applying Green's first identity and simplifying, we have

$$\begin{aligned}\frac{dW_1}{dt} &\leq \int_\Omega \left[-\frac{\mu_h c_t \phi_t \alpha_t \sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\frac{(N_{ss}-N_{ss}^{0t})^2}{N_{ss}} + (\alpha_t+\mu_t)(\delta_t+\mu_t)I_t(R_{0t}^2-1)\right]dx \\ &- \frac{D_1 N_{ss}^{0t} c_t \phi_t \alpha_t \sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\int_\Omega \frac{|\nabla_x N_{ss}|^2}{N_{ss}^2}dx \end{aligned} \tag{18}$$

Whenever $R_{0t} \leq 1$, we have $\frac{dW_1}{dt} \leq 0$. Hence, the disease free equilibrium point π_0^t is stable and $\frac{dW_1}{dt} = 0$ if $N_{ss} = N_{ss}^{0t}$ and $I_t(R_{0t}^2-1) = 0$. The maximum compact invariant set in $\{(N_{ss}, N_{es}, N_{is}, E_t, I_t | \frac{dW_1}{dt} = 0)\}$ is the singleton π_0^t. Thus, the disease free equilibrium is globally asymptotically stable by LaSalle invariance principle [13].

3.2. Malaria. The Malaria aspect of the model (2) with diffusion is given by

$$\left.\begin{aligned}\frac{\partial N_{ss}(t,x)}{\partial t} &= D_1 \triangle N_{ss} + \Lambda_h - \frac{c_m \beta_h N_{ss} I_m}{N_h} - \mu_h N_{ss} + \omega_m N_{sr} \\ \frac{\partial N_{se}(t,x)}{\partial t} &= D_5 \triangle N_{se} + \frac{c_m \beta_h N_{ss} I_m}{N_h} - (\mu_h+\sigma_m)N_{se} \\ \frac{\partial N_{si}(t,x)}{\partial t} &= D_6 \triangle N_{si} + \sigma_m N_{se} - (\mu_h+\delta_b+\theta_m)N_{si} \\ \frac{\partial N_{sr}(t,x)}{\partial t} &= D_7 \triangle N_{sr} + \theta_m N_{si} - (\mu_h+\omega_m)N_{sr} \\ \frac{\partial S_m(t,x)}{\partial t} &= D_{12} \triangle S_m + \Lambda_t - \frac{c_m \beta_m N_{si} S_m}{N_m} - \mu_m S_m \\ \frac{\partial E_m(t,x)}{\partial t} &= D_{13} \triangle E_m + \frac{c_m \beta_m N_{si} S_m}{N_m} - (\alpha_m+\mu_m)E_m \\ \frac{\partial I_m(t,x)}{\partial t} &= D_{14} \triangle I_m + \alpha_m E_m - (\delta_m+\mu_m)I_m \end{aligned}\right\} \tag{19}$$

The system has a disease free equilibrium $\pi_0^m = (N_{ss}^{0m}, N_{se}^{0m}, N_{si}^{0m}, N_{sr}^{0m}, S_m^{0m}, E_m^{0m}, I_m^{0m})$ given below as

$$\pi_0^m = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, 0, \frac{\Lambda_m}{\mu_m}, 0, 0\right) \tag{20}$$

The basic reproduction number of (13) can be obtained easily using the next generation matrix method[10,11] and is given by

$$R_{0m} = \sqrt{\frac{{c_m}^2 \beta_h \beta_m \alpha_m \sigma_m}{(\mu_h + \sigma_m)(\mu_h + \delta_b + \theta_m)(\alpha_m + \mu_m)(\delta_m + \mu_m)}} \tag{21}$$

3.2.1. *Local Stability II.*

Theorem 3.3. The spatially homogeneous disease-free equilibrium solution π_0^m of (19) is locally asymptotically stable if $R_{0m} < 1$ and unstable otherwise.

Proof. System (19) is linearized about arbitrary spatially homogeneous equilibrium point $(\overline{N_{ss}}, \overline{N_{se}}, \overline{N_{si}}, \overline{N_{sr}}, \overline{S_m}, \overline{E_m}, \overline{I_m})$. Let there exist small perturbations $N_{1s}, N_{1e}, N_{1i}, N_{1r}, S_1, E_1, I_1$. The resulting differential equation is given by

$$\left.\begin{aligned}
\frac{\partial N_{1s}}{\partial t} &= D_1 \triangle N_{1s} + d_{11}N_{1s} + d_{12}N_{1e} + d_{13}N_{1e} + d_{14}N_{1r} + d_{15}S_1 + d_{16}E_1 + d_{17}I_1 \\
\frac{\partial N_{1e}}{\partial t} &= D_5 \triangle N_{1e} + d_{21}N_{1s} + d_{22}N_{1e} + d_{23}N_{1i} + d_{24}N_{1r} + d_{25}S_1 + d_{26}E_1 + d_{27}I_1 \\
\frac{\partial N_{1i}}{\partial t} &= D_6 \triangle N_{1i} + d_{31}N_{1s} + d_{32}N_{1e} + d_{33}N_{1i} + d_{34}N_{1r} + d_{35}S_1 + d_{36}E_1 + d_{37}I_1 \\
\frac{\partial N_{1r}}{\partial t} &= D_7 \triangle N_{1r} + d_{41}N_{1s} + d_{42}N_{1e} + d_{43}N_{1i} + d_{44}N_{1r} + d_{45}S_1 + d_{46}E_1 + d_{47}I_1 \\
\frac{\partial S_1}{\partial t} &= D_{12} \triangle S_1 + d_{51}N_{1s} + d_{52}N_{1e} + d_{53}N_{1i} + d_{54}N_{1r} + d_{55}S_1 + d_{56}E_1 + d_{57}I_1 \\
\frac{\partial E_1}{\partial t} &= D_{13} \triangle E_1 + + d_{61}N_{1s} + d_{62}N_{1e} + d_{63}N_{1i} + d_{64}N_{1r} + d_{65}S_1 + d_{66}E_1 + d_{67}I_1 \\
\frac{\partial I_1}{\partial t} &= D_{14} \triangle I_1 + d_{71}N_{1s} + d_{72}N_{1e} + d_{73}N_{1i} + d_{74}N_{1r} + d_{75}S_1 + d_{76}E_1 + d_{77}I_1
\end{aligned}\right\} \tag{22}$$

where $d_{ij}(i, j = 1, 2, \cdots, 7)$ are the elements of the Jacobian matrix at the disease free equilibrium, π_0^m.

Let there exist a series solution of (22) of the form:

$$\left.\begin{aligned} N_{1s} &= \sum_k N_{ks} e^{\lambda t} coskx \\ N_{1e} &= \sum_k N_{ke} e^{\lambda t} coskx \\ N_{1i} &= \sum_k N_{ki} e^{\lambda t} coskx \\ N_{1r} &= \sum_k N_{kr} e^{\lambda t} coskx \\ S_1 &= \sum_k S_k e^{\lambda t} coskx \\ E_1 &= \sum_k E_k e^{\lambda t} coskx \\ I_1 &= \sum_k I_k e^{\lambda t} coskx \end{aligned}\right\} \tag{23}$$

Equation (22) can thus be converted into:

$$\left.\begin{aligned} &(d_{11} - D_1 k^2 - \lambda) N_{1s} + d_{12} N_{1e} + d_{13} N_{1i} + d_{14} N_{1r} + d_{15} S_1 + d_{16} E_1 + d_{17} I_1 = 0 \\ &d_{21} N_{1s} + (d_{22} - D_5 k^2 - \lambda) N_{1e} + d_{23} N_{1i} + d_{24} N_{1r} + d_{25} S_1 + d_{26} E_1 + d_{27} I_1 = 0 \\ &d_{31} N_{1s} + d_{32} N_{1e} + (d_{33} - D_6 k^2 - \lambda) N_{1i} + d_{34} N_{1r} + d_{35} S_1 + d_{36} E_1 + d_{37} I_1 = 0 \\ &d_{41} N_{1s} + d_{42} N_{1e} + d_{43} N_{1i} + (d_{44} - D_7 k^2 - \lambda) N_{1r} + d_{45} S_1 + d_{46} E_1 + d_{47} I_1 = 0 \\ &d_{51} N_{1s} + d_{52} N_{1e} + d_{53} N_{1i} + d_{54} N_{1r} + (d_{55} - D_{12} k^2 - \lambda) S_1 + d_{56} E_1 + d_{57} I_1 = 0 \\ &d_{61} N_{1s} + d_{62} N_{1e} + d_{63} N_{1i} + d_{64} N_{1r} + d_{65} S_1 + (d_{66} - D_{13} k^2 - \lambda) E_1 + d_{67} I_1 = 0 \\ &d_{71} N_{1s} + d_{72} N_{1e} + d_{73} N_{1i} + d_{74} N_{1r} + d_{75} S_1 + d_{76} E_1 + (d_{77} - D_{14} k^2 - \lambda) I_1 = 0 \end{aligned}\right\} \tag{24}$$

The Jacobian matrix with diffusion for the system (24) is given by

(25)
$$\begin{pmatrix} d_{11}-D_1k^2 & 0 & 0 & \omega_m & 0 & 0 & -c_m\beta_h \\ 0 & d_{22}-D_5k^2 & 0 & 0 & 0 & 0 & c_m\beta_h \\ 0 & \sigma_m & d_{33}-D_6k^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_m & d_{44}-D_7k^2 & 0 & 0 & 0 \\ 0 & 0 & -c_m\beta_m & 0 & d_{55}-D_{12}k^2 & 0 & 0 \\ 0 & 0 & c_m\beta_m & 0 & 0 & d_{66}-D_{13}k^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_m & d_{77}-D_{14}k^2 \end{pmatrix}$$

where

$d_{11}=-\mu_h, d_{22}=-(\mu_h+\sigma_m), d_{33}=-(\mu_h+\delta_b+\theta_m), d_{44}=-(\mu_h+\omega_m), d_{55}=-\mu_m, d_{66}=-(\alpha_m+\mu_m), d_{77}=-(\delta_m+\mu_m)$.

$-\mu_h-D_1k^2$, $-\mu_m-D_{12}k^2$ and $-(\mu_h+\omega_m)-D_7k^2$ are part of the roots of the characteristics equation of (25). The other eigenvalues are evaluated from the resulting variational sub matrix below.

(26)
$$\begin{pmatrix} -(\mu_h+\sigma_m)-D_5k^2 & 0 & 0 & c_m\beta_h \\ \sigma_m & -(\mu_h+\delta_b+\theta_m)-D_6k^2 & 0 & 0 \\ 0 & c_m\beta_m & -(\alpha_m+\mu_m)-D_{13}k^2 & 0 \\ 0 & 0 & \alpha_m & -(\delta_m+\mu_m)-D_{14}k^2 \end{pmatrix}$$

The sub matrix (26) can be represented below as:

(27)
$$\begin{pmatrix} b_{11} & 0 & 0 & b_{14} \\ b_{21} & b_{22} & 0 & 0 \\ 0 & b_{32} & b_{33} & 0 \\ 0 & 0 & b_{43} & b_{44} \end{pmatrix}$$

The resulting characteristic equation is of the form $p(\lambda)=\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4=0$ where

$b_1=-(b_{11}+b_{22}+b_{33}+b_{44})$

$b_2=b_{11}b_{22}+a_{11}b_{33}+b_{11}b_{44}+b_{22}b_{33}+b_{22}b_{44}+b_{33}b_{44}$

$b_3=-(b_{11}b_{22}b_{33}+b_{11}b_{22}b_{44}+b_{11}b_{33}b_{44}+b_{22}b_{33}b_{44})$

$b_4 = b_{11}b_{22}b_{33}b_{44} - b_{43}b_{32}b_{21}b_{14}$

The condition for stability by Routh-Hurwitz gives $b_1 > 0, b_2 > 0, b_3 > 0, b_4 > 0$ and $b_1b_2b_3 > b_3^2 + b_1^2 + b_1^2b_4$.

Clearly, $b_1 > 0, b_2 > 0, b_3 > 0$.

b_4 can be expanded below as:

$b_4 = c_m^2\beta_m\beta_h\sigma_m\alpha_m(1 - R_{0m}^2) + (\mu_h + \sigma_m)(\mu_h + \delta_b + \theta_m)(\alpha_m + \mu_m)D_{14}k^2 + (\mu_h + \sigma_m)(\mu_h + \delta_b + \theta_m)(\delta_m + \mu_m)D_{13}k^2 + (\mu_h + \sigma_m)(\mu_h + \delta_b + \theta_m)D_{13}D_{14}k^4 + (\mu_h + \sigma_m)(\alpha_m + \mu_m)(\delta_m + \mu_m)D_6k^2 + (\mu_h + \sigma_m)(\alpha_m + \mu_m)D_6D_{14}k^4 + (\mu_h + \sigma_m)(\delta_m + \mu_m)D_6D_{13}k^4 + (\mu_h + \sigma_m)D_6D_{13}D_{14}k^6 + (\mu_h + \delta_b + \theta_m)(\alpha_m + \mu_m)(\delta_m + \mu_m)D_5k^2 + (\mu_h + \delta_b + \theta_m)(\alpha_m + \mu_m)D_5D_{14}k^4 + (\mu_h + \delta_b + \theta_m)(\delta_m + \mu_m)D_5D_{13}k^4(\mu_h + \delta_b + \theta_m)D_5D_{13}D_{14}k^6 + (\alpha_m + \mu_m)(\delta_m + \mu_m)D_5D_6k^4 + (\alpha_m + \mu_m)D_5D_5D_{14}k^6 + (\delta_m + \mu_m)D_5D_6D_{13}k^6 + D_5D_6D_{13}D_{14}k^8$

$b_4 > 0$ provided that $R_{0m} < 1$.

$b_1b_2b_3$ and $b_3^2 + b_1^2b_4$ can be expanded below as:

$b_1b_2b_3 = b_{11}^3b_{22}^2b_{33} + b_{11}^3b_{22}b_{33}^2 + 3b_{11}^2b_{22}^2b_{33}^2 + 8b_{11}^2b_{22}^2b_{33}b_{44} + 8b_{11}^2b_{22}b_{33}^2b_{44} + b_{11}^2b_{22}^3b_{33} + b_{11}b_{22}^3b_{33}^2 + b_{11}b_{22}^3b_{33}b_{44} + 8b_{11}b_{22}^2b_{33}^2b_{44} + b_{11}^2b_{22}b_{33}^3 + b_{11}b_{22}^2b_{33}^3 + b_{11}b_{22}b_{33}^3b_{44} + 8b_{11}^2b_{22}b_{33}b_{44}^2 + 8b_{11}b_{22}^2b_{33}b_{44}^2 + 8b_{11}b_{22}b_{33}^2b_{44}^2 + b_{11}^3b_{22}^2b_{44} + b_{11}^3b_{22}b_{33}b_{44} + b_{11}^3b_{22}b_{44}^2 + 3b_{11}^2b_{22}^2b_{44}^2 + b_{11}^2b_{22}^3b_{44} + b_{11}b_{22}^3b_{44}^2 + b_{11}^2b_{22}b_{44}^3 + b_{11}b_{22}^2b_{44}^3 + b_{11}b_{22}b_{33}b_{44}^3 + b_{11}^3b_{22}b_{33}b_{44} + b_{11}^3b_{33}^2b_{44} + b_{11}^3b_{33}b_{44}^2 + 3b_{11}^2b_{33}^2b_{44}^2 + b_{11}^2b_{33}^3b_{44} + b_{11}b_{22}b_{33}^3b_{44} + b_{11}b_{33}^3b_{44}^2 + b_{11}^2b_{33}b_{44}^3 + b_{11}b_{22}b_{33}b_{44}^3 + b_{11}b_{33}^2b_{44}^3 + b_{11}b_{22}^3b_{33}b_{44} + b_{22}^3b_{33}^2b_{44} + b_{22}^3b_{33}b_{44}^2 + 3b_{22}^2b_{33}^2b_{44}^2 + b_{11}b_{22}b_{33}^2b_{44} + b_{22}^2b_{33}^3b_{44} + b_{22}b_{33}^3b_{44}^2 + b_{11}b_{22}b_{33}b_{44}^3 + b_{22}^2b_{33}b_{44}^3 + b_{22}b_{33}^2b_{44}^3$

$b_3^2 + b_1^2b_4 = b_{11}^2b_{22}^2b_{33}^2 + 4b_{11}^2b_{22}^2b_{33}b_{44} + 4b_{11}^2b_{22}b_{33}^2b_{44} + 4b_{11}b_{22}^2b_{33}^2b_{44} + b_{11}^2b_{22}^2b_{44}^2 + 4b_{11}^2b_{22}b_{33}b_{44}^2 + 4b_{11}b_{22}^2b_{33}b_{44}^2 + b_{11}^2b_{33}^2b_{44}^2 + 4b_{11}b_{22}b_{33}^2b_{44}^2 + b_{22}^2b_{33}^2b_{44}^2 + b_{11}^3b_{22}b_{33}b_{44} + b_{11}b_{22}^3b_{33}b_{44} + b_{11}b_{22}b_{33}^3b_{44} - b_{11}^2b_{43}b_{32}b_{21}b_{14} - 2b_{11}b_{22}b_{43}b_{32}b_{21}b_{14} - 2b_{11}b_{33}b_{43}b_{32}b_{21}b_{14} - 2b_{11}b_{44}b_{43}b_{32}b_{21}b_{14} - b_{22}^2b_{43}b_{32}b_{21}b_{14} - 2b_{22}b_{33}b_{43}b_{32}b_{21}b_{14} - 2b_{22}b_{44}b_{43}b_{32}b_{21}b_{14} - b_{33}^2b_{43}b_{32}b_{21}b_{14} - 2b_{33}b_{44}b_{43}b_{32}b_{21}b_{14} - b_{44}^2b_{43}b_{32}b_{21}b_{14}$

Clearly, $b_1b_2b_3 > b_3^2 + b_1^2b_4$.

3.2.2. *Global Stability II.*

Theorem 3.4. The spatially homogeneous disease-free equilibrium solution π_0^m of (19) is globally asymptotically stable if $R_{0m} < 1$.

Proof. Consider the Lyapunov function:

$$V_2 = \frac{c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\left[N_{ss}-N_{ss}^{0m}-N_{ss}^{0m}ln\left(\frac{N_{ss}}{N_{ss}^{0m}}\right)\right]+\frac{c_m\beta_m\alpha_m\sigma_m N_{se}}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}$$
$$+\frac{c_m\beta_m\alpha_m N_{si}}{(\mu_h+\delta_b+\theta_m)}+\alpha_m E_m+(\alpha_m+\mu_m)I_m \tag{28}$$

The time derivative of V_2 is given by

$$\dot{V_2} = \frac{c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\left(1-\frac{N_{ss}^{0m}}{N_{ss}}\right)\dot{N_{ss}}+\frac{c_t\beta_m\alpha_m\sigma_m\dot{N_{se}}}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}$$
$$+\frac{c_m\beta_m\alpha_m\dot{N_{si}}}{(\mu_h+\delta_b+\theta_m)}+\alpha_m\dot{E_m}+(\alpha_m+\mu_m)\dot{I_m} \tag{29}$$

Substituting the reaction part of (19) gives

$$\dot{V_2} = \frac{c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\left(1-\frac{N_{ss}^{0m}}{N_{ss}}\right)\left(\Lambda_h-\frac{c_m\beta_h N_{ss}I_m}{N_h}-\mu_h N_{ss}\right)$$
$$+\frac{c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\left(\frac{c_m\beta_h N_{ss}I_m}{N_h}-(\mu_h+\sigma_m)N_{se}\right)$$
$$+\frac{c_m\beta_m\alpha_m}{(\mu_h+\delta_b+\theta_m)}\left(\sigma_m N_{se}-(\mu_h+\delta_b+\theta_m)N_{si}\right)$$
$$+\alpha_m\left(\frac{c_m\beta_m N_{se}S_m}{N_m}-(\alpha_m+\mu_m)E_m\right)+(\alpha_m+\mu_m)\left(\alpha_m E_m-(\delta_m+\mu_m)I_m\right) \tag{30}$$

Hence,

$$\dot{V_2} \leq -\frac{\mu_h c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\frac{(N_{ss}-N_{ss}^{0m})^2}{N_{ss}}$$
$$+\frac{c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\left(\frac{c_m\beta_h N_{ss}^{0m}I_m}{N_h}\right)-(\alpha_m+\mu_m)(\delta_m+\mu_m)I_m \tag{31}$$

Thus,

$$\dot{V_2} \leq -\frac{\mu_h c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\frac{(N_{ss}-N_{ss}^{0m})^2}{N_{ss}}+(\alpha_m+\mu_m)(\delta_m+\mu_m)I_m(R_{0m}^2-1) \tag{32}$$

Next, the Lyapunov functional for the reaction diffusion system (19) is given by

$$W_2=\int_\Omega V_2(N_{ss}(x,t),N_{se}(x,t),N_{si}(x,t),E_m(x,t),I_m(x,t))dx$$

The time derivative of W_2 is given by

$$\begin{aligned}\frac{dW_2}{dt} &= \int_\Omega \left(\frac{D_1 c_m \beta_m \alpha_m \sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)} \left(1-\frac{N_{ss}^{0m}}{N_{ss}}\right) \Delta N_{ss} \right) dx \\ &+ \int_\Omega \left(\frac{D_5 c_m \beta_m \alpha_m \sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)} \Delta N_{se} \right) dx + \int_\Omega \left(D_6 \frac{c_m \beta_m \alpha_m}{(\mu_h+\delta_b+\theta_m)} \Delta N_{si} \right) dx \\ &+ \int_\Omega (D_{13} \alpha_m \Delta E_m)\, dx + \int_\Omega (D_{14}(\alpha_m+\mu_m)\Delta I_m)\, dx + \int_\Omega \frac{dV_2}{dt} dx \end{aligned} \tag{33}$$

Applying Green's first identity and simplifying, we have

$$\begin{aligned}\frac{dW_2}{dt} &< \int_\Omega \left[-\frac{\mu_h c_t \beta_m \alpha_m \sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)} \frac{(N_{ss}-N_{ss}^{0m})^2}{N_{ss}} + (\alpha_m+\mu_m)(\delta_m+\mu_m) I_t (R_{0m}^2-1) \right] dx \\ &- \frac{D_1 N_{ss}^{0m} c_m \beta_m \alpha_m \sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)} \int_\Omega \frac{|\nabla_x N_{ss}|^2}{N_{ss}^2} dx \end{aligned} \tag{34}$$

Whenever $R_{0m} \leq 1$, we have $\frac{dW_2}{dt} \leq 0$. Hence, the disease free equilibrium point π_0^m is stable and $\frac{dW_2}{dt} = 0$ if $N_{ss} = N_{ss}^{0m}$ and $I_m(R_{0m}^2 - 1) = 0$. The maximum compact invariant set in $\{(N_{ss}, N_{se}, N_{si}, E_m, I_m | \frac{dW_2}{dt} = 0)\}$ is the singleton π_0^m. Thus, the disease free equilibrium is globally asymptotically stable by LaSalle invariance principle [13].

3.3. Co-infection Model Stability Analysis.

The system (2) has a disease free equilibrium

$\pi^0 = (N_{ss}^{0t}, N_{es}^{0t}, N_{is}^{0t}, N_{rs}^{0t}, S_t^{0t}, E_t^{0t}, I_t^{0t}, N_{rr}^{tm}, N_{se}^{0m}, N_{si}^{0m}, N_{sr}^{0m}, S_m^{0m}, E_m^{0m}, I_m^{0m})$ given below as

$$\pi_0 = \left(\frac{\Lambda_h}{\mu_h}, 0,0,0, \frac{\Lambda_t}{\mu_t}, 0,0,0,0,0,0, \frac{\Lambda_m}{\mu_m}, 0,0 \right) \tag{35}$$

The basic reproduction number of (2) can be obtained easily using the next generation matrix method[10,11] and is given by

$$R_0 = max\{R_{0t}, R_{0m}\} \tag{36}$$

where R_{0t} and R_{0m} are defined by (5) and (21) respectively.

3.3.1. *Local Stability III.*

Theorem 3.5. The disease free equilibrium π_0 of (2) is locally asymptotically stable if $R_0 < 1$ and unstable otherwise.

Proof. It follows from theorem (3.1) and theorem (3.3).

3.3.2. *Global Stability III.*

Theorem 3.6. The spatially homogeneous disease-free equilibrium solution π_0 of (2) is globally asymptotically stable if $R_{0t}, R_{0m} < 1$.

Proof. Consider the Lyapunov function:

$$\begin{aligned} V_3 &= \frac{c_t\phi_t\alpha_t\sigma_t N_{es}}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)} + \frac{c_t\phi_t\alpha_t N_{is}}{(\mu_h+\delta_a+\theta_t)} + \alpha_t E_t + (\alpha_t+\mu_t)I_t \\ &+ \frac{c_m\beta_m\alpha_m\sigma_m N_{se}}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)} + \frac{c_m\beta_m\alpha_m N_{si}}{(\mu_h+\delta_b+\theta_m)} + \alpha_m E_m + (\alpha_m+\mu_m)I_m \end{aligned} \tag{37}$$

The time derivative of V_3 is given by

$$\begin{aligned} \dot{V}_3 &= \frac{c_t\phi_t\alpha_t\sigma_t \dot{N}_{es}}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)} + \frac{c_t\phi_t\alpha_t \dot{N}_{is}}{(\mu_h+\delta_a+\theta_t)} + \alpha_t \dot{E}_t + (\alpha_t+\mu_t)\dot{I}_t \\ &+ \frac{c_m\beta_m\alpha_m\sigma_m \dot{N}_{se}}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)} + \frac{c_m\beta_m\alpha_m \dot{N}_{si}}{(\mu_h+\delta_b+\theta_m)} + \alpha_m \dot{E_m} + (\alpha_m+\mu_m)\dot{I_m} \end{aligned} \tag{38}$$

Substituting the reaction part of (2) gives

$$\begin{aligned} \dot{V}_3 &= \frac{c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\left(\frac{c_t\phi_h N_{ss} I_t}{N_h} - (\mu_h+\sigma_t)N_{es} - \frac{\rho_m\lambda_m N_{es} I_m}{N_h}\right) \\ &+ \frac{c_t\phi_t\alpha_t}{(\mu_h+\delta_a+\theta_t)}\left(\sigma_t N_{es} - (\mu_h+\delta_a+\theta_t)N_{is} - \frac{\rho_m\lambda_m N_{es} I_m}{N_h}\right) \\ &+ \alpha_t\left(\frac{c_t\phi_t N_{is} S_t}{N_t} - (\alpha_t+\mu_t)E_t\right) + (\alpha_t+\mu_t)\left(\alpha_t E_t - (\delta_t+\mu_t)I_t\right) \\ &+ \frac{c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\left(\frac{c_m\beta_h N_{ss} I_m}{N_h} - (\mu_h+\sigma_m)N_{se} - \frac{\rho_t\lambda_t N_{se} I_t}{N_h}\right) \\ &+ \frac{c_m\beta_m\alpha_m}{(\mu_h+\delta_b+\theta_m)}\left(\sigma_m N_{se} - (\mu_h+\delta_b+\theta_m)N_{si} - \frac{\rho_t\lambda_t N_{se} I_t}{N_h}\right) \\ &+ \alpha_m\left(\frac{c_m\beta_m N_{si} S_m}{N_m} - (\alpha_m+\mu_m)E_m\right) + (\alpha_m+\mu_m)\left(\alpha_m E_m - (\delta_m+\mu_m)I_m\right) \end{aligned} \tag{39}$$

Simplifying and ignoring some terms gives

$$\begin{aligned} \dot{V}_3 &< \frac{c_t^2\phi_h\phi_t\alpha_t\sigma_t I_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)} - (\alpha_t+\mu_t)(\delta_t+\mu_t)I_t \\ &+ \frac{c_m^2\beta_h\beta_m\alpha_m\sigma_m I_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)} - (\alpha_m+\mu_m)(\delta_m+\mu_m)I_m \end{aligned} \tag{40}$$

Further simplification gives

$$\begin{aligned}\dot{V}_3 &< (\alpha_t+\mu_t)(\delta_t+\mu_t)I_t\left(\frac{{c_t}^2\phi_h\phi_t\alpha_t\sigma_t}{(\mu_h+\sigma_t)(\mu_h+\delta_a+\theta_t)(\alpha_t+\mu_t)(\delta_t+\mu_t)}-1\right)\\ &+(\alpha_m+\mu_m)(\delta_m+\mu_m)I_m\left(\frac{{c_m}^2\beta_h\beta_m\alpha_m\sigma_m}{(\mu_h+\sigma_m)(\mu_h+\delta_b+\theta_m)(\alpha_m+\mu_m)(\delta_m+\mu_m)}-1\right) \qquad (41)\end{aligned}$$

Hence,

$$\dot{V}_3 < (\alpha_t+\mu_t)(\delta_t+\mu_t)I_t(R_{0t}^2-1)+(\alpha_m+\mu_m)(\delta_m+\mu_m)I_m(R_{0m}^2-1) \qquad (42)$$

Next, the Lyapunov functional for the reaction diffusion system is given by

$$W_3=\int_\Omega V_3(N_{es}(x,t),N_{is}(x,t),E_t(x,t),I_t(x,t),N_{se}(x,t),N_{si}(x,t),E_m(x,t),I_m(x,t))dx$$

The time derivative of W_3 is given by

$$\begin{aligned}\frac{dW_3}{dt} &= \int_\Omega\left(\frac{D_2c_t\phi_t\alpha_t\sigma_t}{(\mu_h+\delta_a+\theta_t)(\mu_h+\sigma_t)}\Delta N_{es}\right)dx+\int_\Omega\left(D_3\frac{c_t\phi_t\alpha_t}{(\mu_h+\delta_a+\theta_t)}\Delta N_{is}\right)dx\\ &+\int_\Omega(D_{10}\alpha_t\Delta E_t)\,dx+\int_\Omega(D_{11}(\alpha_t+\mu_t)\Delta I_t)\,dx\\ &+\int_\Omega\left(\frac{D_5c_m\beta_m\alpha_m\sigma_m}{(\mu_h+\delta_b+\theta_m)(\mu_h+\sigma_m)}\Delta N_{se}\right)dx+\int_\Omega\left(D_6\frac{c_m\beta_m\alpha_m}{(\mu_h+\delta_b+\theta_m)}\Delta N_{si}\right)dx\\ &+\int_\Omega(D_{13}\alpha_m\Delta E_m)\,dx+\int_\Omega(D_{14}(\alpha_m+\mu_m)\Delta I_m)\,dx+\int_\Omega\frac{dV_3}{dt}dx \qquad (43)\end{aligned}$$

Applying Green's first identity and simplifying gives

$$\frac{dW_3}{dt} \leq \int_\Omega\left[(\alpha_t+\mu_t)(\delta_t+\mu_t)I_t(R_{0t}^2-1)+(\alpha_m+\mu_m)(\delta_m+\mu_m)I_m(R_{0m}^2-1)\right]dx \qquad (44)$$

Whenever $R_{0t}\leq 1$ and $R_{0m}\leq 1$, we have $\frac{dW_3}{dt}\leq 0$. Hence, the disease free equilibrium point π_0 is stable and $\frac{dW_3}{dt}=0$ if $I_t(R_{0t}^2-1)=0$ and $I_m(R_{0m}^2-1)=0$. The maximum compact invariant set in $\{N_{es},N_{is},E_t,I_t,N_{se},N_{si},E_m,I_m|\frac{dW_3}{dt}=0)\}$ is the singleton π_0. Thus, the disease free equilibrium is globally asymptotically stable by LaSalle invariance principle[13].

4. CONCLUSION

In this paper, the dynamics of a reaction-diffusion Trypanosomiasis-Malaria co-infection dynamics is investigated. The disease free equilibrium is locally and globally asymptotically stable for all diffusion coefficients if $R_{0t},R_{0m}<1$ and unstable otherwise.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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