Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2024, 2024:27
https://doi.org/10.28919/cmbn/8426
ISSN: 2052-2541

THE ROLE OF LINEAR TYPE OF HARVESTING ON TWO COMPETITIVE SPECIES INTERACTION

DINA SULTAN AL-JAF*
Department of Mathematics, College of Since, University of Baghdad, Iraq

Abstract: Harvesting's effect on living organisms is one of the most important factors affecting the way they reproduce. In this paper, we proposed a mathematical model to study the effect of harvest on two species with competition between them. The Holling type –II functional response has been considered, four equilibrium points are biologically accepted, The locall stability of these points was analyzed , and The stability of the positive point was analyzed locally and globally. the local bifurcation of the system at these points was also studied, and they were all of the type saddle-node bifurcation. In addition, the persist conditions were found. Finally, the system was analyzed numerically.

Keywords: harvest; competition interaction; local stability; global stability; local bifurcation; persistant.
2020 AMS Subject Classification: 92D40.

1. INTRODUCTION

In the ecosystem, there is a group of living organisms that interact with each other and compete. Mathematics has a major role in explaining these interactions between these organisms.This is done by formulating mathematical models that describe these interactions; among these

*Corresponding author
E-mail address: dina.s@sc.uobaghdad.edu.iq
Received January 03, 2024

interactions or effects are the effects of competition, harvest, coexistence, and others [1], [6], [13], [20], [37]. It is not limited to formulating mathematical models in the ecosystem but mathematics has a major role in formulating mathematical models in broad fields such as medicine, biology, chemistry, and others. [2], [5], [6], [8], [9], [12],[18], [24]- [29], [34], [35], [38], [39]. The dynamic relationship between living organisms is considered one of the most important relationships that researchers have been interested in studying on a large scale since ancient times. This relationship still plays a major role in researchers' interest in studying it [3], [10], [11], [18], [19], [37], [41], [44], [46]. The study of the effect of harvesting is not limited to organisms within the same variety, Rather, it studies the effect of harvesting on living organisms within different species, as is the case with compete, its study is not limited to organisms within one species However, there are many studies on the effect of competition on organisms within different species [7], [14], [15], [22], [30], [32]- [34], [36], [38]-[40], [42], [43],[45],[47], [49]. The effect of harvesting on the ecosystem is an important part of researchers, interest in studying environmental models. There are also different types of harvesting functions, including linear harvesting, harvesting that depends on the presence of a constant, non-linear harvesting, proportional harvesting that depends on density, and others [21], [23], [33]. In this paper, a two-species model telling both harvesting of type linear and competition is proposed and analyzed. The effects of the harvest and competition on the dynamic behavior of the two species model fit throughout analytically as fine as numerically. Also studied the bifurcation and got a clear result.

2. ASSUMPTIONS OF THE MODEL

Suppose we have a model consisting of two species x and $y,$ with competition between them according to Lotka Volterra type (2) of functional response, both of which are affected by the harvested.

Can be offered the model by the following two differential equations:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right) - \frac{\beta_1 xy}{\alpha_1 + x} - e_1q_1x = xf_1(x,y)$$

$$\frac{dy}{dt} = sy\left(1-\frac{y}{l}\right) - \frac{\beta_2 xy}{\alpha_2 + y} - e_2q_2y = yf_2\,(x,y) \qquad (1)$$

Here, model (1) has been analyzed with the initial conditions $x(0) \geq 0$ and $y(0) \geq 0$, where each parameter of system (1) is let to be positive, it can be described as follows: k, l are the carrying capacities of both x and y, respectively. In addition to their growth rate r and s; each of β_1 and β_2 are competition coefficients; α_1 , α_2 are half-saturation constants; finally e_1q_1 and e_2q_2 represents the harvesting rate of each of the above species.

3. BOUNDEDNESS

Theorem 1. All solutions $x(t)$ and $y(t)$ of the system (1), which start in R_+^2 are uniformly bounded.

Proof: let $(x(t)\,, y(t))$ be any solution of the system (1) with a nonnegative initial condition.

Now, for $w(t) = x(t) + y(t)$, we contain $\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$

$$\frac{dw}{dt} = rx - \frac{rx^2}{k} - \frac{\beta_1 xy}{\alpha_1 + x} - e_1q_1x + sy - \frac{sy^2}{l} - \frac{\beta_2 xy}{\alpha_2 + y} - e_2q_2y$$

Hence, $\frac{dw}{dt} + mw(t) \leq rk + sl = \mu$,

Where, $m = min\{e_1q_1x, e_2q_2y\}$,

Then , $w(t) \leq \mu - \mu e^{-mt} + w_0e^{-mt}$

Where $w_0 = w\big(x(0), y(0)\big)$

Hence, $\forall\, T \geq 0$ we have that $0 \leq w(T) \leq \mu)$

So, all solutions of system (1) which are initiated in R_+^2 are enter to the region

$\varphi = \{\big(x(t), y(t)\big) \in R_+^2 : w = x + y \leq \mu + \varepsilon,\ for\ any\ \varepsilon > 0\}$.

4. EXISTENCE OF EQUILIBRIA POINT AND THEIR STABILITY

In this section, the existence and the local stability analysis of all the equilibrium points of system (1) are studied, and four equilibrium points are found, namely:

1. The vanishing equilibrium point $\mathrm{E}_0 = (0,0)$.
2. The point on the $x-$axis is $E_1 = (\breve{x}, 0)$, we call it the axial equilibrium point where $\breve{x} = \frac{k(r-e_1q_1)}{r}$ exists if the following condition holds:
$$r > e_1q_1 \qquad (2)$$
3. The axial equilibrium point on the $y-$axis is given by $E_2 = (0, \bar{y})$, where $\bar{y} = \frac{l(s-e_2q_2)}{s}$ exists if the following condition holds:
$$s > e_2q_2 \qquad (3)$$
4. The positive equilibrium point $E_3 = (\dot{x}, \dot{y}\,)$, where $\dot{x} = \frac{l\alpha_2 a_1 + \dot{y}(a_1 l - s\alpha_2\beta_2 + s\dot{y})}{l\beta_2}$ exisit if the following condition holds:
$$a_1 l > s\alpha_2\beta_2 + s\dot{y} \qquad (4)$$
and y is the positive solution of the following polynomial
$$Ay^4 + By^3 + Cy^2 + Dy + E = 0$$
Where $A = \frac{-\,rs^2}{l^2\beta_2^2}$
$$B = \frac{2rs}{l\beta_2^2}\left(a_1 - \frac{s\alpha_2}{l}\right),$$
$$C = \frac{2rs\alpha_2 a_1}{l\beta_2^2} - \frac{rsa_3}{l\beta_2} - \frac{r}{\beta_2^2}\left(a_1 - \frac{s\alpha_2}{l}\right)^2,$$
$$D = \left(\frac{ra_3}{\beta_2} - \frac{2r\alpha_2 a_1}{\beta_2^2}\right)\left(a_1 - \frac{s\alpha_2}{l}\right) - k\beta_1,$$
$$E = a_2 + \frac{ra_1a_3\alpha_2}{\beta_2} - \frac{ra_1^2\alpha_2^2}{\beta_2^2}$$

Where $a_1 = s - e_2q_2 > 0, a_2 = rk\alpha_1 - e_1q_1 > 0$ and $a_3 = k - \alpha_1 > 0$

Therefore, by the discard rule of sign, the above equation has a positive root, say $\dot{y}$ either if the following condition

$$max\left\{\frac{s\alpha_2}{l},\frac{ls\beta_2 a_3 + l\left(a_1-\frac{s\alpha_2}{l}\right)^2}{2s\alpha_2},\frac{\alpha_2 a_1^2}{a_3}-\frac{a_2\beta_2}{ra_3\alpha_2}\right\} < a_1 < \frac{a_3\beta_2}{2\alpha_2}-\frac{k\beta_1\beta_2^2}{2r\alpha_2\left(a_1-\frac{s\alpha_2}{l}\right)}$$

hold. Or when B , C, D and E are all positive.

Otherwise, system (1) could not have a positive fixed point depending on the sign of B , C, D and E

Now, the local behavior of the above stable points has been calculated by finding the Jacobian matrix of the system (1) around each point and extracting the eigenvalues of the matrix. Where Jacobian matrix of the system (1) at (x, y) be able to be written as:

$$J=\begin{bmatrix} r-2\frac{r}{k}x-\frac{\alpha_1\beta_1 y}{(\alpha_1+x)^2}-e_1q_1 & -\frac{\beta_1 x}{\alpha_1+x} \\ -\frac{\beta_2 y}{\alpha_2+y} & s-2\frac{s}{l}y-\frac{\alpha_2\beta_2 x}{(\alpha_2+y)^2}-e_2q_2 \end{bmatrix} \quad (5)$$

The Jacobian matrix of the system (1) at the vanishing critical point $\mathrm{E}_0=(0{,}0)$ can be written in the following form:

$$J(E_0)=\begin{bmatrix} r-e_1q_1 & 0 \\ 0 & s-e_2q_2 \end{bmatrix} \quad (6)$$

Hence, the eigenvalues of $J(E_0)$ will be in the following form

$\lambda_{0x}=r-e_1q_1$ and $\lambda_{0y}=s-e_2q_2$

This means that the point E_0 is locally asymptotically stable if and only if the following conditions are met

$$r < e_1q_1 \text{ and } s < e_2q_2 \quad (7)$$

The Jacobian matrix of the system (1) at $\mathrm{E}_1=(\check{x},0)$.can be written in the following form:

$$J(E_1)=\begin{bmatrix} r-2\frac{r}{k}\check{x}-e_1q_1 & -\frac{\beta_1\check{x}}{\alpha_1+\check{x}} \\ 0 & s-\frac{\beta_2\check{x}}{{\alpha_2}^2}-e_2q_2 \end{bmatrix} \quad (8)$$

Hence, the eigenvalues of $J(E_1)$ will be in the following form

$\lambda_{1x}=-r+e_1q_1<0$ under existence condition

and $\lambda_{1y}=s-\frac{\beta_2\check{x}}{\alpha_2}-e_2q_2$

This means that the point E_1 is locally asymptotically stable if and only if the following condition is met

$$s < \frac{\beta_2 \check{x}}{\alpha_2} + e_2 q_2 \tag{9}$$

The Jacobian matrix of the system (1) at $\mathrm{E}_2 = (0, \bar{y})$ can be written in the following form:

$$J(E_2) = \begin{bmatrix} r - \frac{\beta_1 \bar{y}}{\alpha_1} - e_1 q_1 & 0 \\ -\frac{\beta_2 \bar{y}}{\alpha_2 + \bar{y}} & s - 2\frac{s}{l}\bar{y} - e_2 q_2 \end{bmatrix} \tag{10}$$

Hence, the eigenvalues of $J(E_1)$ will be in the following

$$\lambda_{2x} = r - \frac{\beta_1 \bar{y}}{\alpha_1} - e_1 q_1$$

and $\lambda_{2y} = -s + e_2 q_2 < 0$ under existence condition

This means that the point E_2 is locally asymptotically stable if and only if the following condition is met

$$r < \frac{\beta_1 \bar{y}}{\alpha_1} + e_1 q_1 \tag{11}$$

Finally, The Jacobian matrix of the system (1) at $\mathrm{E}_3 = (\dot{x}, \dot{y})$ can be written in the following form:

$$J(E_3) = \begin{bmatrix} r - 2\frac{r}{k}\dot{x} - \frac{\alpha_1 \beta_1 \dot{y}}{(\alpha_1 + \dot{x})^2} - e_1 q_1 & -\frac{\beta_1 \dot{x}}{\alpha_1 + \dot{x}} \\ -\frac{\beta_2 \dot{y}}{\alpha_2 + \dot{y}} & s - 2\frac{s}{l}\dot{y} - \frac{\alpha_2 \beta_2 \dot{x}}{(\alpha_2 + \dot{y})^2} - e_2 q_2 \end{bmatrix} \tag{12}$$

Then, computing $|J(E_3) - I\lambda| = 0$ gives:

$\lambda^2 - Tr\big(J(E)\big)\lambda + Det\big(J(E_3)\big) = 0,$

where, $Tr\big(J(E_3)\big) = r - \frac{2r\dot{x}}{k} - \frac{\alpha_1 \beta_1 \dot{y}}{(\alpha_1 + \dot{x})^2} - e_1 q_1 + s - 2\frac{s}{l}\dot{y} - \frac{\alpha_2 \beta_2 \dot{x}}{(\alpha_2 + \dot{y})^2} - e_2 q_2,$

$Det\big(J(E_3)\big) = \left[r - 2\frac{r}{k}\dot{x} - \frac{\alpha_1 \beta_1 \dot{y}}{(\alpha_1 + \dot{x})^2} - e_1 q_1\right]\left[s - 2\frac{s}{l}\dot{y} - \frac{\alpha_2 \beta_2 \dot{x}}{(\alpha_2 + \dot{y})^2} - e_2 q_2\right] - \left[\frac{\beta_1 \beta_2 \dot{x}\dot{y}}{(\alpha_1 + \dot{x})(\alpha_2 + \dot{y})}\right].$

By the Routh–Hurwitz criterion, E_3 is a locally asymptotical stable point if and only if $Tr\big(J(E_3)\big) < 0$ and $Det\big(J(E_3)\big) > 0$.

Theorem 2. Suppose that E_3 of system (1) is locally asymptotically stable in R_+^2, and let

The following terms verified:

$$\frac{r}{k} > \frac{\beta_1 \dot{y}}{R_1} \text{ and } \frac{s}{l} > \frac{\beta_2 \dot{x}}{R_2} \tag{13}$$

Then E_3 is globally asymptotically stable in the R_+^2.

Proof:

Consider the following positive definite function

$$U(x,y) = \left[x - \dot{x} - \dot{x} ln \frac{x}{\dot{x}}\right] + \left[y - \dot{y} - \dot{y} ln \frac{y}{\dot{y}}\right]$$

Clearly $U: R_+^2 \to R$ is C^1 . Now

$$\frac{dU}{dt} = \frac{x-\dot{x}}{x}\frac{dx}{dt} + \frac{y-\dot{y}}{y}\frac{dy}{dt}$$

$$= -(x-\dot{x})^2\left[\frac{r}{k} - \frac{\beta_1 \dot{y}}{R_1}\right] - (x-\dot{x})(y-\dot{y})\left[\frac{\beta_1}{\alpha_1+x} + \frac{\beta_2}{\alpha_2+y}\right] - (y-\dot{y})^2\left[\frac{s}{l} - \frac{\beta_2 \dot{x}}{R_2}\right]$$

Clearly, $\frac{dU}{dt} < 0$ under the local stability condition and condition (13).

Hence, U is strictly a Lyapunov function. Thus, we obtained global asymptotically stable in the R_+^2 at E_3.

5. LOCAL BIFURCATION

Sotomayor's theory was used in this part of the research to find out whether stable states have local bifurcation conditions close to them. Most researchers used Sotomayors' theory to study some different types of bifurcation, such as saddle nodes transcortical and pitchfork bifurcation [4], [12], [16], [22], [26], [31], [32], [48]. Further, model (1) can be reformulated as follows:

$\frac{dN}{dt} = F(N)$ with $N = \begin{bmatrix} x \\ y \end{bmatrix}$, and $F = \begin{bmatrix} xf_1(x,y) \\ yf_2(x,y) \end{bmatrix}$.

Now, the Jacobian matrix at any point is given by (10). Then, for non-zero vector $A = (a_1, a_2,)^T$:

$$DF = \begin{bmatrix} \left(r - e_1 q_1 - \frac{2rx}{k} - \frac{\alpha_1 \beta_1 y}{(\alpha_1 + x)^2}\right) a_1 - \frac{\beta_1 x}{(\alpha_1 + x)} a_2 \\ -\frac{\beta_2 y}{(\alpha_2 + y)} a_1 + \left(s - e_2 q_2 - \frac{2sy}{l} - \frac{\alpha_2 \beta_2 x}{(\alpha_2 + y)^2}\right) a_2 \end{bmatrix}$$

and,

$$D^2F(X,X) = \begin{bmatrix} \left(\frac{\alpha_1\beta_1 y}{(\alpha_1+x)^3} - \frac{r}{k}\right)2a_1^2 - \frac{2\alpha_1\beta_1}{(\alpha_1+x)^2}a_1a_2 \\ \frac{-2\alpha_2\beta_2 a_1 a_2}{(\alpha_2+y)^2} - \left(\frac{s}{l} - \frac{\alpha_2\beta_2 x}{(\alpha_2+y)^3}\right)2a_2^2 \end{bmatrix}$$

$$D^3F(X,X) = \begin{bmatrix} \frac{-6\alpha_1\beta_1 y a_1^3}{(\alpha_1+x)^4} + \frac{6\alpha_1\beta_1 a_1^2 a_2}{(\alpha_1+x)^3} \\ \frac{2\alpha_2\beta_2 a_1 a_2^2}{(\alpha_2+y)^3} + \frac{6\alpha_2\beta_2 x a_2^3}{(\alpha_2+y)^4} \end{bmatrix}$$

Theorem 3. If $r = r^*$, then system (1) takes a saddle-node bifurcation at E_0.

Proof: Based on the $J(E_0)$, given in (4) system (1) has a zero eigenvalue at E_0, it's called $\lambda_{0x} = 0$, at $r^* = e_1q_1$ and $J^*(E_0) = J(E_0, r^*)$, becomes:

$$J^*(E_0, r^*) = \begin{bmatrix} 0 & 0 \\ 0 & s - e_2q_2 \end{bmatrix}$$

Now, assume $A^{[0]} = \left(a_1^{[0]}, a_2^{[0]}\right)^T$ is an eigenvector related with $\lambda_{0x} = 0$, then

$(J^*(E_0) - \lambda_{0x}I)A^{[0]} = 0$ gives $A^{[0]} = \left(a_1^{[0]}, 0\right)^T$. where $a_1^{[0]}$ be any nonzero real number.

Assume that $B^{[0]} = \left(b_1^{[0]}, b_2^{[0]}\right)^T$ is an eigenvector related with $\lambda_{0x} = 0$ of the $\left(J^*(E_0, r^*)\right)^T$.

$$J^{*T}(E_0, r^*) = \begin{bmatrix} 0 & 0 \\ 0 & s - e_2q_2 \end{bmatrix}$$

Then $((J^*(E_0))^T - \lambda_{0x}I)B^{[0]} = 0$ gives $B^{[0]} = \left(b_1^{[0]}, 0\right)^T$, where $b_1^{[0]}$ be any nonzero real number.

Now, to know whether the bifurcation of the saddle nodes meets the conditions, the following is calculated

$$\frac{\partial F}{\partial r} = F_r(N, r) = \left(\frac{\partial f_1}{\partial r}, \frac{\partial f_2}{\partial r}\right)^T = \left(1 - \frac{x}{k}, 0\right)^T$$

So $F_r(E_0, r^*) = (1,0)^T$ and hence $\left(B^{[0]}\right)^T F_r = \left(b_1^{[0]}, 0\right)^T (1,0)^T = b_1^{[0]} \neq 0$.

So, the first condition was fulfilled through the bifurcation of the saddle node, while the bifurcation through the transcortical and the pitchfork did not fulfill its conditions.

Accordingly

$$D^2F_r(E_0, r^*)A^{[0]} = \left(\frac{-2e_1q_1(a_1^{[0]})^2}{k}, 0\right)^T$$

Hence,

$$\left(B^{[0]}\right)^T\left[D^2F_r(E_0, r^*)A^{[0]}\right] = \left(b_1^{[0]}, 0\right)^T\left(\frac{-2\,e_1q_1(a_1^{[0]})^2}{k}, 0\right)^T = -2e_1q_1b_1^{[0]}(a_1^{[0]})^2 \neq 0.$$

This means that the bifurcation of the saddle nodes fulfills the second condition. So, the bifurcation of the saddle nodes of system (1) is verified at E_0 with $r = r^*$.

Theorem 4: If $s = s^*$, then system (1) takes a saddle-node bifurcation at E_1.

Proof: depending on $J(E_1)$ given in (7), then system (1) has a zero eigenvalue at E_1which's called $\lambda_{1y} = 0$, at $s^* = \frac{\beta_2\hat{x}}{\alpha_2} + e_2q_2$ and $J^*(E_1) = J(E_1, s^*)$, becomes:

$$J^*(E_1, s^*) = \begin{bmatrix} -r + e_1q_1 & \dfrac{-\beta_1\hat{x}}{\alpha_1 + \hat{x}} \\ 0 & 0 \end{bmatrix}$$

Now, Assume that $A^{[1]} = \left(a_1^{[1]}, a_2^{[1]}\right)^T$ is an eigenvector related with $\lambda_{1y} = 0$, thus

$\left(J^*(E_1) - \lambda_{1\mathrm{y}}I\right)A^{[1]} = 0$ gives $A^{[1]} = \left(\frac{-\beta_1k}{r\alpha_1 + k(r - e_1q_1)}a_2^{[1]}, a_2^{[1]}\right)^T$. where $a_2^{[1]}$ be any nonzero real number.

Assume that $B^{[1]} = \left(b_1^{[1]}, b_2^{[1]}\right)^T$ is an eigenvector related to $\lambda_{1\mathrm{y}} = 0$ of the matrix $\left(J^*(E_1, s^*)\right)^T$.

$$J^{*T}(E_1, s^*) = \begin{bmatrix} -r + e_1q_1 & 0 \\ \dfrac{-\beta_1\hat{x}}{\alpha_1 + \hat{x}} & 0 \end{bmatrix}$$

Then $((J^*(E_1))^T - \lambda_{1y}I)B^{[1]} = 0$ gives $B^{[1]} = \left(0, b_2^{[1]}\right)^T$, where $b_2^{[1]}$ be any nonzero real number.

Now, to know whether the bifurcation of the saddle nodes meets the conditions, the following is calculated

$$\frac{\partial F}{\partial s} = F_s(N,s) = \left(\frac{\partial f_1}{\partial s}, \frac{\partial f_2}{\partial s}\right)^T = \left(0, 1 - \frac{y}{l}\right)^T$$

So $F_s(E_1, s^*) = (0,1)^T$ and hence $\left(B^{[1]}\right)^T F_s = \left(0, b_2^{[1]}\right)^T (0,1)^T = b_2^{[1]} \neq 0$. So, the first condition was fulfilled through the bifurcation of the saddle node, while the bifurcation through the transcortical and the pitchfork did not fulfill its conditions.

Now,

$$D^2F_s(E_1, s^*)A^{[1]} = \begin{bmatrix} \frac{-2r(a_1^{[1]})^2}{k} - \frac{2\alpha_1\beta_1 a_1^{[1]} a_2^{[1]}}{(\alpha_1 + \hat{x})^2} \\ \frac{-2\beta_2 a_1^{[1]} a_2^{[1]}}{\alpha_2^2} - \frac{2s^*\left(a_2^{[1]}\right)^2}{l} - \frac{2\beta_2\hat{x}(a_2^{[1]})^2}{\alpha_2^2} \end{bmatrix}$$

Hence,

$$\left(B^{[1]}\right)^T\left[D^2F_s(E_1, s^*)A^{[1]}\right] = \left(0, b_2^{[1]}\right)^T \begin{bmatrix} \frac{-2r(a_1^{[1]})^2}{k} - \frac{2\alpha_1\beta_1 a_1^{[1]} a_2^{[1]}}{(\alpha_1 + \hat{x})^2} \\ \frac{-2\beta_2 a_1^{[1]} a_2^{[1]}}{\alpha_2^2} - \frac{2s^*\left(a_2^{[1]}\right)^2}{l} - \frac{2\beta_2\hat{x}(a_2^{[1]})^2}{\alpha_2^2} \end{bmatrix}$$

$$= -2b_2^{[1]}\left[\frac{\beta_2 a_1^{[1]} a_2^{[1]}}{\alpha_2} + \left(\frac{s^*}{l} + \frac{\beta_2\hat{x}}{\alpha_2^2}\right)(a_2^{[2]})^2\right] \neq 0$$

This means that the bifurcation of the saddle nodes fulfills the second condition. So, the bifurcation of the saddle nodes of system (1) is verified at E_1 with $s = s^*$.

Theorem 5. If $r = r^{**}$, then system (1) at E_2 takings a saddle-node bifurcation provided

$$k\beta_1^2\bar{y} + r^{**}(a_1^{[2]})^2\alpha_1^2 > \beta_1\alpha_1 a_1^{[2]} a_2^{[2]} \tag{14}$$

Proof: Depend on $J(E_2)$ given in (10), system (1) has a zero eigenvalue at E_2 which's called $\lambda_{2x} = 0$, at $r^{**} = \frac{\beta_1\bar{y}}{\alpha_1} + e_1q_1$ and $J^*(E_2) = J(E_2, r^{**})$, becomes:

$$J^{*}(E_2, r^{**}) = \begin{bmatrix} 0 & 0 \\ \dfrac{-\beta_2\bar{y}}{\alpha_2+\bar{y}} & -s+e_2q_2 \end{bmatrix}$$

Now, assume that $A^{[2]} = \left(a_1^{[2]}, a_2^{[2]}\right)^T$ is an eigenvector corresponding to $\lambda_{2x} = 0$, thus

$(J^{*}(E_2) - \lambda_{2x}I)A^{[2]} = 0$ gives $A^{[2]} = \left(a_1^{[2]}, \frac{\beta_2\bar{y}}{(s-e_2q_2)(\alpha_2+\bar{y})}a_1^{[2]}\right)^T$. where $a_1^{[2]}$ be any nonzero real number.

Let $B^{[2]} = \left(b_1^{[2]}, b_2^{[2]}\right)^T$ be an eigenvector related with $\lambda_{2x} = 0$ of the matrix $\left(J^{*}(E_2, r^{**})\right)^T$.

$$J^{*T}(E_2, r^{**}) = \begin{bmatrix} 0 & \dfrac{-\beta_2\bar{y}}{\alpha_2+\bar{y}} \\ 0 & -s+e_2q_2 \end{bmatrix}$$

Then $((J^{*}(E_2))^T - \lambda_{2x}I)B^{[2]} = 0$ gives $B^{[2]} = \left(b_1^{[2]}, 0\right)^T$, where $b_1^{[2]}$ be any nonzero real number.

Now, to know whether the bifurcation of the saddle nodes meets the conditions, the following is calculated

$$\frac{\partial F}{\partial r} = F_r(N, r) = \left(\frac{\partial f_1}{\partial r}, \frac{\partial f_2}{\partial r}\right)^T = \left(1 - \frac{x}{k}, 0\right)^T$$

So $F_r(E_2, r^{**}) = (1, 0)^T$and hence $\left(B^{[2]}\right)^T F_r = \left(b_1^{[2]}, 0\right)^T (1,0)^T = b_1^{[2]} \neq 0$. So, the first condition was fulfilled through the bifurcation of the saddle node, while the bifurcation through the transcortical and the pitchfork did not fulfill its conditions.

Now,

$$D^2F_s(E_2, s^{*})A^{[2]} = \begin{bmatrix} \left(\frac{\beta_1\bar{y}}{\alpha_1^2} - \frac{r^{**}}{k}\right) 2(a_1^{[2]})^2 - \frac{2\beta_1 a_1^{[2]} a_2^{[2]}}{\alpha_1} \\ \frac{-2\alpha_2\beta_2 a_1^{[2]} a_2^{[2]}}{(\alpha_2+\bar{y})^2} - \frac{2s\left(a_2^{[2]}\right)^2}{l} \end{bmatrix}$$

Hence,

$$\left(B^{[2]}\right)^T\left[D^2F_s(E_2,s^*)A^{[2]}\right]=\left(b_1^{[2]},0\right)^T\begin{bmatrix}\frac{-2r^{**}(a_1^{[2]})^2}{l}-\frac{2\beta_1 a_1^{[2]}a_2^{[2]}}{\alpha_1}+\frac{2\beta_1^2\bar{y}}{\alpha_1^2}\\ \frac{-2\alpha_2\beta_2 a_1^{[2]}a_2^{[2]}}{(\alpha_2+\bar{y})^2}-\frac{2s\left(a_2^{[2]}\right)^2}{l}\end{bmatrix}$$

$$=2b_1^{[2]}\left[\frac{r^{**}(a_1^{[2]})^2\alpha_1^2-\beta_1\alpha_1 a_1^{[2]}a_2^{[2]}+k\beta_1^2\bar{y}}{\alpha_1^2 k}\right]\neq 0 \text{ Under condition (14).}$$

This means that the bifurcation of the saddle nodes fulfills the second condition. So, the bifurcation of the saddle nodes of system (1) is verified at E_2 with $r=r^{**}$

Theorem 6. If s permissions through

$$s^{**}=-\frac{\beta_1\beta_2\dot{y}\dot{x}}{(\alpha_1+\dot{x})(\alpha_2+\dot{y})a_{11}}+2\frac{s^{**}\dot{y}}{l}+\frac{\alpha_2\beta_2\dot{x}}{(\alpha_2+\dot{y})^2}+e_2q_2$$

then system (1) takes a saddle-node bifurcation at E_3 only if the following conditions hold

$$r>\alpha_1\beta_1 k\dot{y} \qquad 15(a)$$

$$\dot{y}\neq l \qquad 15(b)$$

$$r+s-e_2q_2<2\frac{r}{k}\dot{x}+\frac{\alpha_1\beta_1\dot{y}}{(\alpha_1+\dot{x})^2}+e_1q_1+2\frac{s}{l}\dot{y}+\frac{\alpha_2\beta_2\dot{x}}{(\alpha_2+\dot{y})^2} \qquad 15(c)$$

Proof: $J(E_3)$, given by (12) at $s=s^{**}$ can be inscribed as:

$$J^*(E_3,s^{**})=\begin{bmatrix} r-2\frac{r}{k}\dot{x}-\frac{\alpha_1\beta_1\dot{y}}{(\alpha_1+\dot{x})^2}-e_1q_1 & -\frac{\beta_1\dot{x}}{\alpha_1+\dot{x}}\\ -\frac{\beta_2\dot{y}}{\alpha_2+\dot{y}} & s^{**}-2\frac{s^{**}}{l}\dot{y}-\frac{\alpha_2\beta_2\dot{x}}{(\alpha_2+\dot{y})^2}-e_2q_2\end{bmatrix}$$

The calculation tells that $Det(J^*(E_3))=0$. Then $J^*(M_3)$ has a zero eigenvalue, say $\lambda_{3y}=0$ with the second eigenvalue $\lambda_{32}=r+s-e_2q_2-2\frac{r}{k}\dot{x}-\frac{\alpha_1\beta_1\dot{y}}{(\alpha_1+\dot{x})^2}-e_1q_1-2\frac{s}{l}\dot{y}-\frac{\alpha_2\beta_2\dot{x}}{(\alpha_2+\dot{y})^2}<0$

under condition 15(c). Now, let $A^{[3]}=\left(a_1^{[3]},a_2^{[3]}\right)^T$ be the eigenvector corresponding to $\lambda_{3y}=0$, thus $\left(J^*(E_3)-\lambda_{3y}I\right)A^{[3]}=0$ gives $A^{[3]}=\left(\frac{(s^{**}L-2s^{**}\dot{y}-e_2q_2L)(\alpha_2+\dot{y})^2-\alpha_2\beta_2\dot{x}L}{\beta_2\dot{y}L(\alpha_2+\dot{y})}a_2^{[3]},a_2^{[3]}\right)^T$.

where $a_2^{[3]}$ be any nonzero real number.

Let $B^{[3]}=\left(b_1^{[3]},b_2^{[3]}\right)^T$ be an eigenvector related to $\lambda_{3y}=0$ of the $\left(J^*(E_3,s^{**})\right)^T$.

$$J^{*T}(E_3, s^{**}) = \begin{bmatrix} r - 2\frac{r}{k}\dot{x} - \frac{\alpha_1\beta_1\dot{y}}{(\alpha_1+\dot{x})^2} - e_1q_1 & -\frac{\beta_2\dot{y}}{\alpha_2+\dot{y}} \\ -\frac{\beta_1\dot{x}}{\alpha_1+\dot{x}} & s^{**} - 2\frac{s^{**}}{l}\dot{y} - \frac{\alpha_2\beta_2\dot{x}}{(\alpha_2+\dot{y})^2} - e_2q_2 \end{bmatrix}$$

Then $((J^*(E_3))^T - \lambda_{3y}I)B^{[3]} = 0$ gives :

$B^{[3]} = \left(\frac{[(s^{**}l - 2s^{**}\dot{y} - e_2q_{2l})(\alpha_2+\dot{y})^2 - \alpha_2\beta_2\dot{x}l](\alpha_1+\dot{x})}{\beta_1\dot{x}l(\alpha_2+\dot{y})^2} b_2^{[3]}, b_2^{[3]}\right)^T$, where $b_2^{[3]}$ be any nonzero real number.

Now, to know whether the bifurcation of the saddle nodes meets the conditions, the following is calculated

$$\frac{\partial F}{\partial s} = F_s(N,s) = \left(\frac{\partial f_1}{\partial s}, \frac{\partial f_2}{\partial s}\right)^T = \left(0, 1 - \frac{y}{l}\right)^T$$

So $F_s(E_3, s^{**}) = \left(0,1 - \frac{\dot{y}}{l}\right)^T$ and $\left(B^{[3]}\right)^T F_{s^{**}} = \left(b_1^{[3]}, b_2^{[3]}\right)^T \left(0, 1 - \frac{\dot{y}}{l}\right)^T = \left(1 - \frac{\dot{y}}{l}\right) b_2^{[3]} \neq 0$ under condition 15(b).

Therefore, the first condition was fulfilled through the bifurcation of the saddle node, while the bifurcation through the transcortical and the pitchfork did not fulfill its conditions.

Now,

$$D^2F_s(E_3, s^{**})A^{[3]} = \begin{bmatrix} \left(\frac{\alpha_1\beta_1\dot{y}}{(\alpha_1+\dot{x})^3} - \frac{r}{k}\right) 2(a_1^{[3]})^2 - \frac{2\alpha_1\beta_1}{(\alpha_1+\dot{x})^2} a_1^{[3]} a_2^{[3]} \\ \frac{-2\alpha_2\beta_2 a_1^{[3]} a_2^{[3]}}{(\alpha_2+\dot{y})^2} - \left(\frac{s^{**}}{l} - \frac{\alpha_2\beta_2\dot{x}}{(\alpha_2+\dot{y})^3}\right) 2(a_2^{[3]})^2 \end{bmatrix}$$

Hence,

$$\left(B^{[3]}\right)^T\left[D^2F_s(E_1, s^{**})A^{[3]}\right] = \left(b_1^{[3]}, b_2^{[3]}\right)^T \begin{bmatrix} \left(\frac{\alpha_1\beta_1\dot{y}}{(\alpha_1+\dot{x})^3} - \frac{r}{k}\right) 2(a_1^{[3]})^2 - \frac{2\alpha_1\beta_1}{(\alpha_1+\dot{x})^2} a_1^{[3]} a_2^{[3]} \\ \frac{-2\alpha_2\beta_2 a_1^{[3]} a_2^{[3]}}{(\alpha_2+\dot{y})^2} - \left(\frac{s^{**}}{l} - \frac{\alpha_2\beta_2\dot{x}}{(\alpha_2+\dot{y})^3}\right) 2(a_2^{[3]})^2 \end{bmatrix}$$

$= b_1^{[3]} \left[\frac{2\left(a_1^{[3]}\right)^2(-r + \alpha_1\beta_1 k\dot{y}) - 2\alpha_1\beta_1 a_1^{[3]} a_2^{[3]}}{k(\alpha_1+\dot{x})^2}\right] - 2b_2^{[3]} \left[\frac{\alpha_2\beta_2 l a_1^{[3]} a_2^{[3]} + (s^{**} + \alpha_2\beta_2\dot{x})(a_2^{[3]})^2}{l(\alpha_2+\dot{y})^2}\right] \neq 0$ under condition 15(a).

This means that the bifurcation of the saddle nodes fulfills the second condition. So the bifurcation of the saddle nodes of system (1) is verified at E_3 with $s = s^{**}$.

6. PERSISTENCE ANALYSIS

In this section, we will study the persistence of model (1). We say that the model persists if and only if each species is alive. Mathematically, it means that system (1) persists if the solution of a system with a positive initial condition does not have an omega limit set on the boundary of its domain. For more detail, see [50]. Now, we can establish the persistence condition of system (1) in the following theorem.

Theorem 7. Suppose that the following sets of conditions

$$r > e_1 q_1 \text{ and } s > e_2 q_2 \tag{16}$$

hold. Then, system (1) persists.

Proof: Let $w(x,y) = x^a y^b$, such that a and b are positive constants. Clear that $w(x,y) > 0$, for each $(x,y) \in R_+^2$ and when x or $y \to 0$ then $w(x,y) \to 0$ therefore ,

$$\varphi(x,y) = \frac{\dot{w}}{w} = b\left[s\left(1 - \frac{y}{l}\right) - \frac{\beta_2 x}{\alpha_2 + y} - e_2 q_2\right] + a\left[r\left(1 - \frac{x}{k}\right) - \frac{\beta_1 y}{\alpha_1 + x} - e_1 q_1\right]$$

Now, since exclusive probable omega limit sets to the system (1) on the border of xy- plane are the critical points E_0, E_1 and E_2. So, system (1) is uniformly persistent. conditional that $\varphi(x,y) > 0$ at the E_0, E_1 and E_2.

Now, meanwhile

$$\varphi(E_0) = b[s - e_2 q_2] + a[r - e_1 q_1] \, ;$$

$$\varphi(E_1) = b\left[s - \frac{\beta_2 \check{x}}{\alpha_2} - e_2 q_2\right] ;$$

$$\varphi(E_2) = a\left[r - \frac{\beta_1 \bar{y}}{\alpha_1} - e_1 q_1\right] .$$

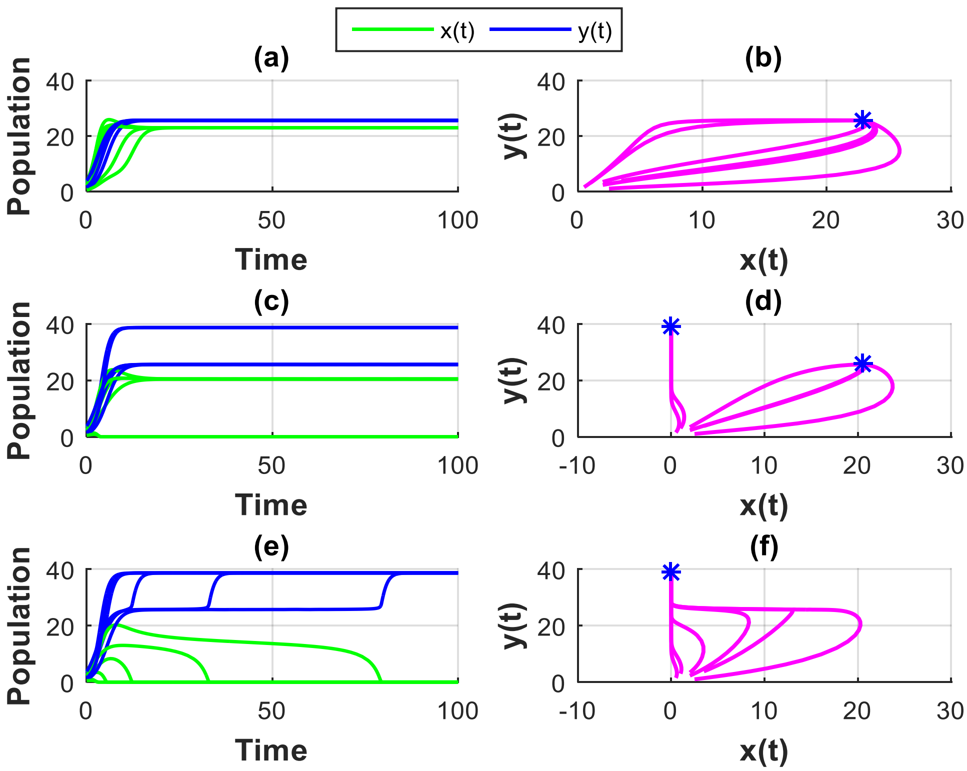
It follows that, $\varphi(E_0) > 0$, $\varphi(E_1) > 0$ and $\varphi(E_2) > 0$ under conditions 15. Then, system (1) is uniformly persistent. ▪

7. NUMERICAL ANALYSIS WITH DISCUSSION

In this section, system (1) is studied numerically using MATLAB, and a phase plane with a time series are obtained with the following set of parameters:

$r = 1.2;\ \beta_1 = 0.2;\ \alpha_1 = 0.3;\ e_1 = 0.1;\ q_1 = 0.4; l = 40;\ s = 0.9;\ \beta_2 = 0.3;\ \alpha_2 = 0.4;\ e_2 = 0.1; q_2 = 0.3; k = 30.$ (17)

Now, for different values of the growth rate of the first species x (r) and keeping all parameters as shown in Eq. (17), We noticed a clear effect on the system dynamics (1) as shown in Fig. (1) (a-f).



Fig(1):(a) time series of system(1)at data given in (16) with $r = 1.1$ **(b)** phase plane of(a).

(c) time series of system(1)at data given in (16) with $r = 0.9$ **(d)** phase plane of (c).

(e) time series of system(1)at data given in (16) with $r = 0.74$ **(f)** phase plane of (e).

According to the above figure, it is observed that the phase plane of system (1) approaches a global asymptotic equilibrium point $E_3 = (22.9,25.56)$ for $r \geq 1.1$ As shown in the Fig. (1)(a-b), while it approaches to two different points, $E_2 = (0,38.67)$ and $E_3 = (20.44,25.59)$ for $0.75 \leq r \leq 1.1$ as show in the Fig (1)(c-d) ,finally for $r < 0.75$ system (1) approaches to $E_2 = (0,38.67)$ and losses the persist as show in Fig.(1)(e-f). In order to know the effect of the harvest of the species x (i.e., parameter q_1) with all parameters as shown in Eq. (16), We noticed a clear effect on system dynamics (1) as shown in the Fig.(2) (a-f)

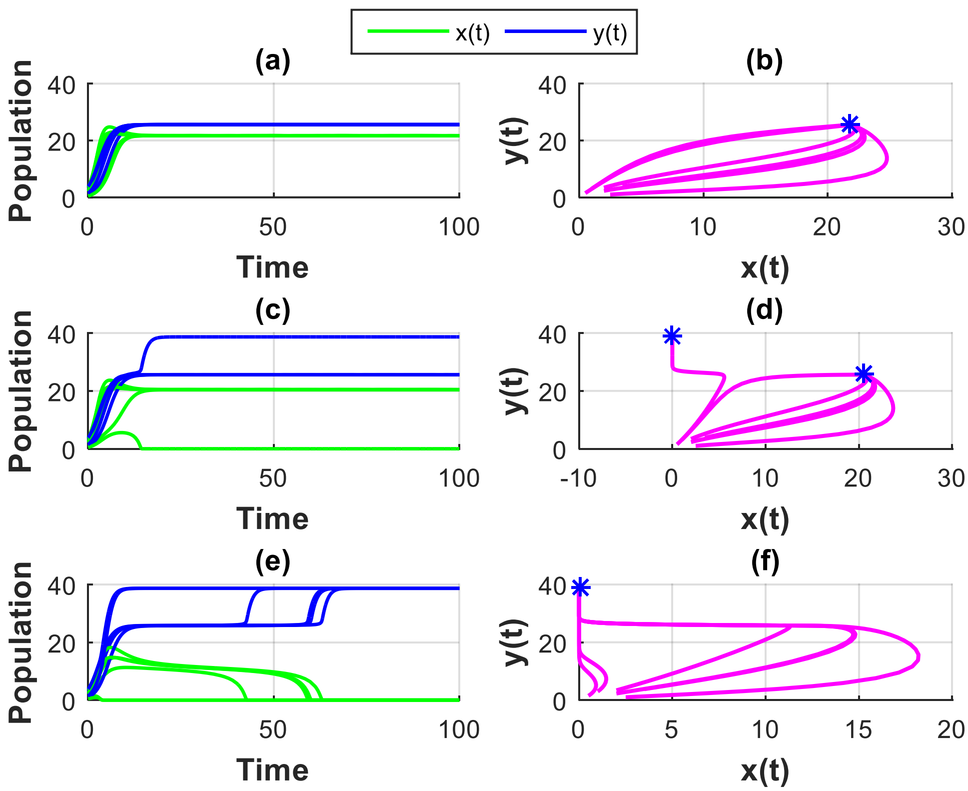


Fig. (2): (a) time series of system(1)at data given in (16) with $q_1 = 1$ **(b)** phase plane of(a).

(c) time series of system(1)at data given in (16) with $q_1 = 1.36$ **(d)** phase plane of (c).

(e) time series of system(1)at data given in (16) with $q_1 = 3.1$ **(f)** phase plane of (e).

Noticeably, as q_1 when it decreases The numerical analysis we analyzed shows that the phase plane of system (1) approaches to global asymptotic equilibrium point $E_3 = (21.67,25.57)$ for $q_1 \leq 1.35$ As shown in the Fig. (2)(a-b), While it approaches to two different points, $E_2 =$

(0,38.67) and $E_3 = (20.42,25.59)$ for $1.35 < q_1 \leq 3$ as show in the Fig (2)(c-d) ,finally for $q_1 > 3$ system (1) losses persist and approach to an axial point $E_2 = (0,38.67)$ as show in Fig.(2)(e-f).

In specific proportions to influence competition of the species x (i.e. parameter β_1) with all parameters as shown in Eq. (16), We noticed a clear effect on the system dynamics (1) as shown in the Fig. (3) (a-f)

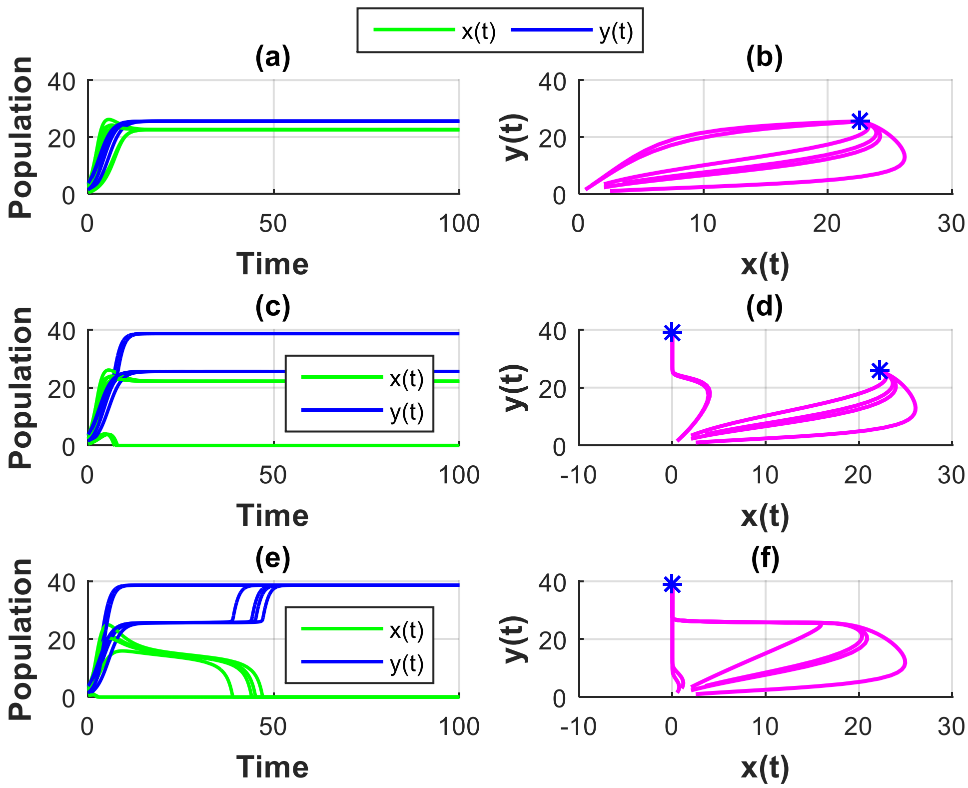


Fig.(3): (a) time series of system(1)at data given in (16) with $\beta_1 = 0.23$ **(b)** phase plane of(a).

(c) time series of system(1)at data given in (16) with $\beta_1 = 0.24$ **(d)** phase plane of (c).

(e) time series of system(1)at data given in (16) with $\beta_1 = 0.34$ **(f)** phase plane of (e).

Remarkably, for a small value of $\beta_1, \beta_1 \leq 0.23$ the numerical analysis we analyzed shows that the phase plane of system (1) approaches the global asymptotic equilibrium point $E_3 = (22.57,25.56)$ as shown in the Fig. (3)(a-b), While with a slight change of β_1, $0.23 < \beta_1 \leq 0.33$ system (1) approaches to two different points, $E_2 = (0,38.67)$ and $E_3 = (22.17,25.57)$ show in the Fig (3)(c-d), finally for $\beta_1 > 0.33$ system (1) losses persist and approach an axial point $E_2 = (0,38.67)$ as show in Fig.(3)(e-f).

In order to know the effect of the growth rate of species y (i.e., parameter s) with all parameters as shown in Eq. (16), We noticed a pure effect on the system dynamics (1) as shown in the Fig. (4) (a-h)

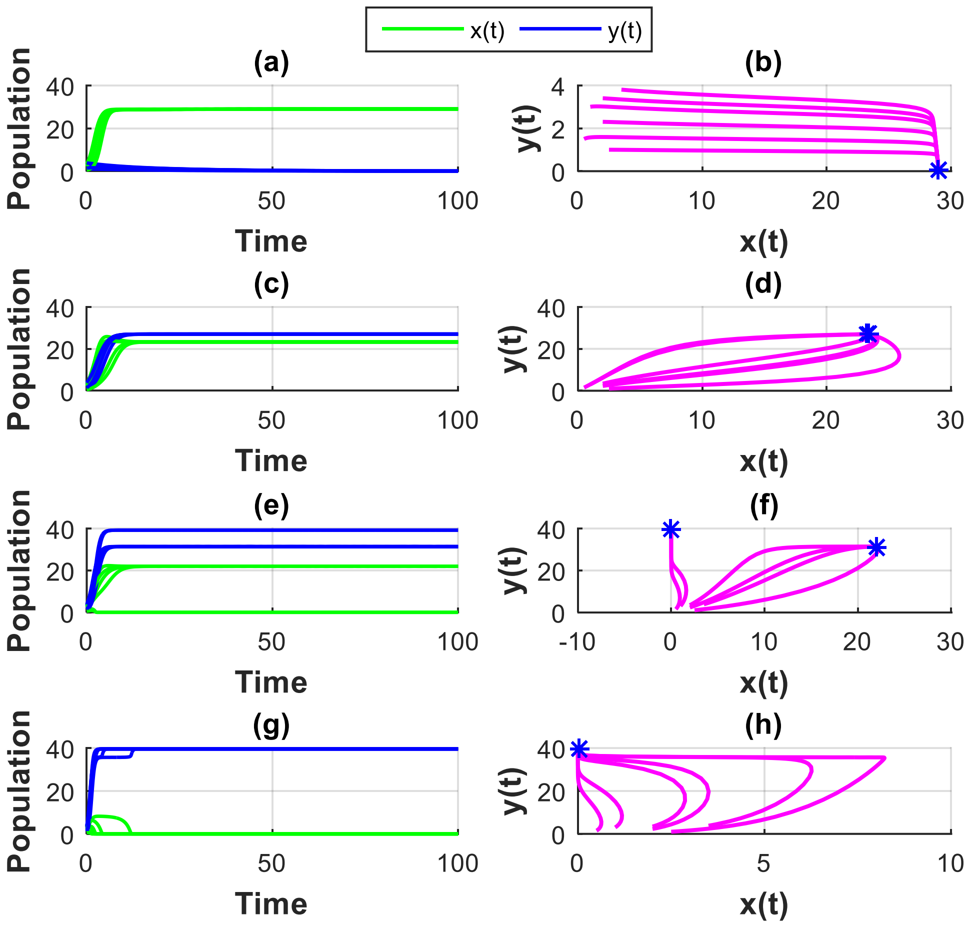


Fig. (4): (a) time series of system(1) with data given in (16) at $s = 0.29$ **(b)** phase plane of(a).

(c) time series of system(1) with data given in (16) at $s = 1$ **(d)** phase plane of (c).

(e) time series of system(1) with data given in (16) at $s = 1.5$ **(f)** phase plane of (e).

(g) time series of system(1) with data given in (16) at $s = 2.91$ **(h)** phase plane of (e).

For a small value of s, ($s \leq 0.32$) the numerical analysis we analyzed shows that the phase plane of system (1) approaches to $E_1 = (28.99,0)$ and losses persist as shown in the Fig. (4)(a-b), While with a slight change of s, $0.32 < s \leq 1$ system (1) approaches to the global asymptotic equilibrium point $E_3 = (23.27,27)$ as shown in Fig (4)(c-d). However, the solution of system (1) approaches to two different points, $E_2 = (0,39.2)$ and $E_3 = (21.96,31.34)$ as $1 < s < 2.91$ shown in the Fig (4)(e-f), finally for $s \geq 2.91$ system (1) losses persist once again and approach to axial point $E_2 = (0,39.59)$ as show in Fig (4)(g-h).

In order to know the impact of competition of the species y (i.e., parameter β_2) with all parameters as shown in Eq. (17), We noticed a clear effect on the system dynamics (1) as shown in the figure (5) (a-d):

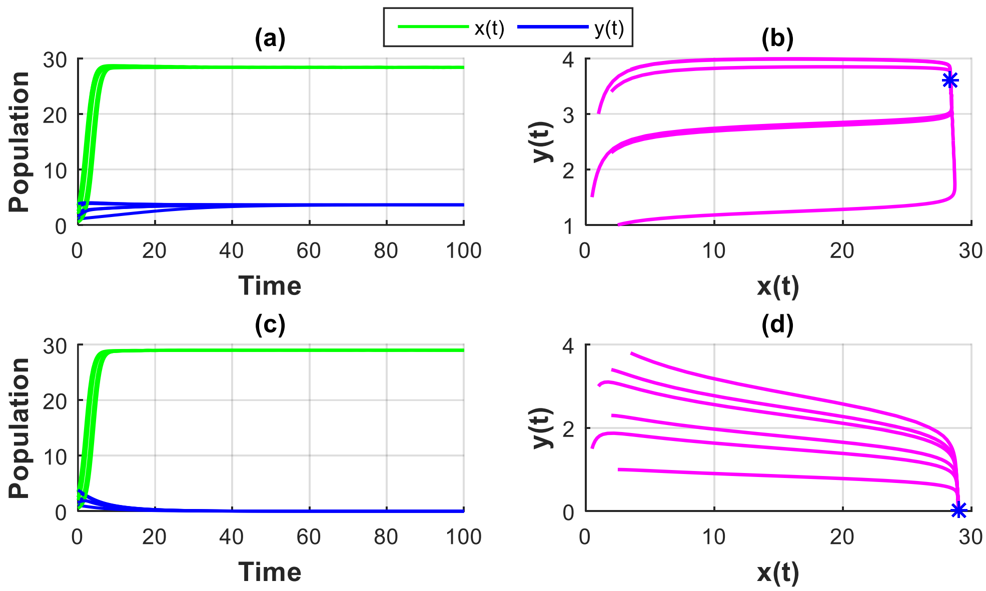


Fig. (5) (a) time series of system(1) with data given in (16) at $\beta_2 = 0.8$ **(b)** phase plane of(a).
(c) time series of system(1) with data given in (16) at $\beta_2 = 0.99$ **(d)** phase plane of (c).

With a small change in the values of β_2, $\beta_2 < 0.89$ notice that the solution of system (1) approaches to a global asymptotically stable point $E_3 = (28.45,3.15)$, as shown in Fig. (5) (a-b), while system (1) losses it persists and approaches to axial equilibrium point $E_1 = (29,0)$ for $\beta_2 \geq 0.89$ as shown in Fig. (5) (c-d).

Finally, the dynamic behavior of the system (1) is studied under the influence of harvesting of species y (i.e. parameter q_2) with all parameters as shown in Eq. (17) as shown in the figure (6) (a-d):

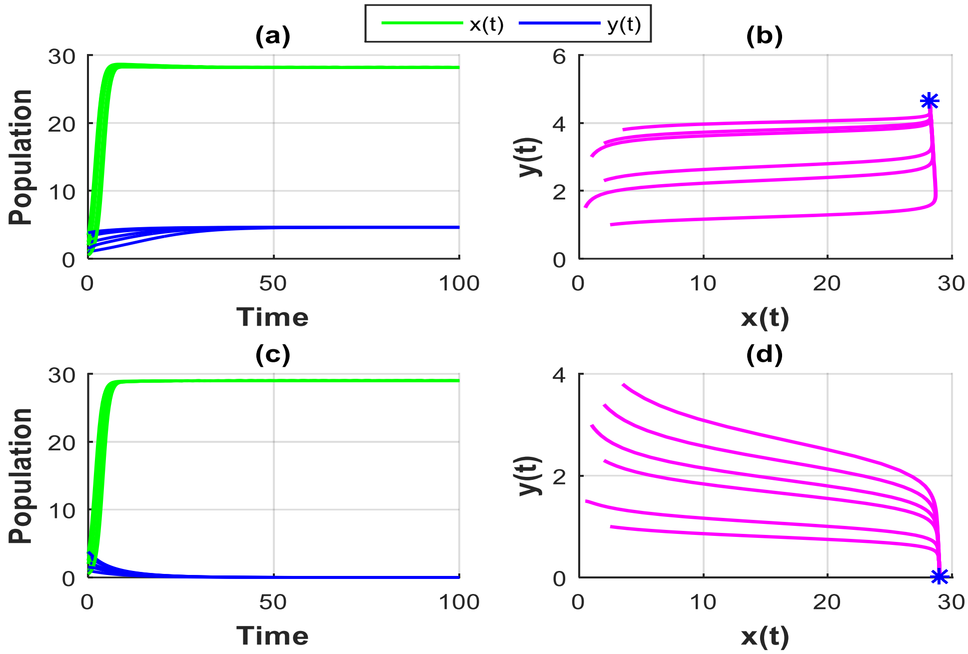


Fig.(6): **(a)** time series of system(1) with data given in (16) at $q_2 = 5$ **(b)** phase plane of (a).

(c) time series of system(1) with data given in (16) at $q_2 = 7$ **(d)** phase plane of (c).

As the effects of the harvest $q_2$increase until $(q_2 \leq 5.99)$ We noticed that the dynamic behavior of system (1) still persist and approaches to global asymptotically stable point $E_3 = (28.19,4.63)$ as show in Fig.(6)(a-b), while system (1) losses persist and approaches to $E_1 = (28.99,0)$ for $(q_2 > 5.99)$ as show in Fig.(6)(c-d).

8. CONCLUSION

In order to study the effect of harvest and competition on an ecological system consisting of two species, with studying the effect of changing the growth rate for both species. The dynamic behavior of the system (1) is studied theoretically by finding conditions for local stability of points and global stability of positive points with finding the conditions for the bifurcation. And the persist conditions of the system (1) as well derived.

Now, we conclude that the effects of changes in parameters on the dynamic behavior of system (1) based on the numerical study in section (7) are as follows:

Regarding the effect of a change in the growth rate of the first species while keeping all parameters as they are in the Eq. (16) we observed that the phase plane of system (1) approaches to globally asymptotically equilibrium point $E_3$for ($r \geq 1.1$), while the solution of system (1) approaches to E_2 and E_3 when ($0.75 \leq r \leq 1.1$) According to the initial conditions, if the initial condition falls within the basin of attraction for the point E_2 , the solution approaches to E_2 while if the initial condition falls within the basin of attraction for the point E_3 , the solution approaches to E_3. Finally for ($r < 0.75$) system (1) losses persistence.

The effect of varying harvest rate and v competition rate of the first species on the dynamical behavior of system (1) (i.e. parameter q_1 and β_1) respectively, it has the same effect as the growth rate on the dynamic behavior of the system(1) .

Regarding the effect of varying of growth rate of second species y (i.e. parameter s) on the dynamical behavior of system (1) while keeping all parameters as they are in the Eq. (17) we observed that for small value of growth rate ($s \leq 0.32$) the phase plane of system (1) loss persist and approaches to E_1 ,while approaches to globally asymptotically equilibrium point E_3 when($0.32 < s \leq 1$) , while the solution of system (1) approaches to E_2 and E_3 when ($1 < s < 2.91$) ,according to the initial conditions, if the initial condition falls within the basin of attraction for the point E_2 , the solution approaches to E_2 while if the initial condition falls within the basin of attraction for the point E_3 ,finally for ($s \geq 2.91$) system (1) also losses persistence and approaches to E_2.

Regarding the effect of harvest rate of the second species on the dynamical behavior of system (1) while keeping all parameters as they are in the Eq. (16) we observed that for ($q_2 \leq 5.99$) the phase plane of system (1) approaches to globally asymptotically equilibrium point in the $Int. R_+^2$,while for ($q_2 > 5.99$) system (1) losses persistence. Regarding the effect of varying of competition rate of the second species on the dynamical behavior of system (1) while keeping all parameters as they are in the Eq. (17) we observed that for small value of competition rate

($\beta_2 < 0.89$) the phase plane of system (1) approaches to globally asymptotically equilibrium point in the $Int. R_+^2$,while for ($\beta_2 \geq 89$) system (1) losses persistence.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

REFERENCES

[1] A. Manju, R. Pathak, Persistence and optimal harvesting of prey-predator model with Holling Type III functional response, Int. J. Eng. Sci. Technol. 4 (2012), 78-96.

[2] M. Ahmed, S. Jawad, The role of antibiotics and probiotics supplements on the stability of gut flora bacteria interactions, Commun. Math. Biol. Neurosci. 2023 (2023), 33. https://doi.org/10.28919/cmbn/7879.

[3] M. Al Nuaimi, S. Jawad, Modelling and stability analysis of the competitional ecological model with harvesting, Commun. Math. Biol. Neurosci. 2022 (2022), 47. https://doi.org/10.28919/cmbn/7450.

[4] Z.Kh. Alabacy, A.A. Majeed, The local bifurcation analysis of two preys stage-structured predator model with anti-predator behavior, J. Phys.: Conf. Ser. 2322 (2022), 012061. https://doi.org/10.1088/1742-6596/2322/1/012061.

[5] W.F. Alfwzan, K. Abuasbeh, A. Raza, et al. Computational analysis of the coronavirus epidemic model involving nonlinear stochastic differential equations, AIP Adv. 13 (2023), 085318. https://doi.org/10.1063/5.0161001.

[6] R.U.D. Amalia, Fatmawati, Windarto, et al. Optimal control of predator-prey mathematical model with infection and harvesting on prey, J. Phys.: Conf. Ser. 974 (2018) 012050. https://doi.org/10.1088/1742-6596/974/1/012050.

[7] J. Bhattacharyya, S. Pal, Coexistence of competing predators in a coral reef ecosystem, Nonlinear Anal.: Real World Appl. 12 (2011), 965-978. https://doi.org/10.1016/j.nonrwa.2010.08.020.

[8] R. Bhattacharyya, B. Mukhopadhyay, On an eco-epidemiological model with prey harvesting and predator switching: Local and global perspectives, Nonlinear Anal.: Real World Appl. 11 (2010), 3824-3833. https://doi.org/10.1016/j.nonrwa.2010.02.012.

[9] G.J. Butler, S.B. Hsu, P. Waltman, Coexistence of competing predators in a chemostat, J. Math. Biol. 17 (1983), 133-151. https://doi.org/10.1007/bf00305755.

[10] J. Chattopadhyay, G. Ghosal, K.S. Chaudhuri, Nonselective harvesting of a prey-predator community with infected prey, Korean J. Comput. Appl. Math. 6 (1999), 601-616. https://doi.org/10.1007/bf03009952.

[11] G. Dai, M. Tang, Coexistence region and global dynamics of a harvested predator-prey system, SIAM J. Appl. Math. 58 (1998), 193-210. https://doi.org/10.1137/s0036139994275799.

[12] J. Datta, D. Jana, R.K. Upadhyay, Bifurcation and bio-economic analysis of a prey-generalist predator model with Holling type IV functional response and nonlinear age-selective prey harvesting, Chaos Solitons Fractals. 122 (2019), 229-235. https://doi.org/10.1016/j.chaos.2019.02.010.

[13] S. Dawud, S. Jawad, Stability analysis of a competitive ecological system in a polluted environment, Commun. Math. Biol. Neurosci. 2022 (2022), 70. https://doi.org/10.28919/cmbn/7520.

[14] Y. El Foutayeni, M. Khaladi, A bio-economic model of fishery where prices depend on harvest, Adv. Model. Optim. 14 (2012), 543-555.

[15] A.G. Fredrickson, G. Stephanopoulos, Microbial competition, Science. 213 (1981), 972-979. https://doi.org/10.1126/science.7268409.

[16] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, Springer, New York, (2013). https://doi.org/10.1007/978-1-4612-1140-2.

[17] S.K. Hassan, S.R. Jawad, The effect of mutual interaction and harvesting on food chain model, Iraqi J. Sci. 63 (2022), 2641-2649. https://doi.org/10.24996/ijs.2022.63.6.29.

[18] S.B. Hsu, Limiting behavior for competing species, SIAM J. Appl. Math. 34 (1978), 760-763. https://doi.org/10.1137/0134064.

[19] S.B. Hsu, S. Hubbell, P. Waltman, A mathematical theory for single-nutrient competition in continuous cultures of micro-organisms, SIAM J. Appl. Math. 32 (1977), 366-383. https://doi.org/10.1137/0132030.

[20] M.H. Ismaeel, A.A. Majeed, A proposed bio-economic model for the prey-predator fishery model with harvesting: toward modeling dynamics in applied mathematics, J. Phys.: Conf. Ser. 1362 (2019), 012142. https://doi.org/10.1088/1742-6596/1362/1/012142.

[21] S. Jawad, Study the dynamics of commensalism interaction with Michaels-Menten type prey harvesting, Al-Nahrain J. Sci. 25 (2022), 45-50.

[22] S.R. Jawad, M. Al Nuaimi, Persistence and bifurcation analysis among four species interactions with the influence of competition, predation and harvesting, Iraqi J. Sci. 64 (2023), 1369-1390. https://doi.org/10.24996/ijs.2023.64.3.30.

[23] S.S. Jawad, S.K. Hassan, Bifurcation analysis of commensalism intraction and harvisting on food chain model, Brazil. J. Biometrics, 41 (2023), 218-233. https://doi.org/10.28951/bjb.v41i3.609.

[24] S. Jawad, M. Winter, Z.-A.S.A. Rahman, et al. Dynamical behavior of a cancer growth model with chemotherapy and boosting of the immune system, Mathematics. 11 (2023), 406. https://doi.org/10.3390/math11020406.

[25] A.J. Kadhim, A.A. Majeed, Epidemiological model involving two diseases in predator population with holling type-II functional response, Int. J. Nonlinear Anal. Appl. 12 (2021), 2085-2107. https://doi.org/10.22075/ijnaa.2021.5349.

[26] E.M. Kafi, A.A. Majeed, The local bifurcation of an eco-epidemiological model in the presence of stage-structured with refuge, Iraqi J. Sci. 61 (2020), 2087-2105. https://doi.org/10.24996/ijs.2020.61.8.24.

[27] A.M. Kareem, S.N. Al-Azzawi, A stochastic differential equations model for the spread of coronavirus COVID-19: The case of Iraq, Iraqi J. Sci. 62 (2011), 1025-1035. https://doi.org/10.24996/ijs.2021.62.3.31.

[28] A.M. Kareem, S.N. Al-Azzawi, Comparison between deterministic and stochastic model for interaction (COVID-19) with host cells in humans, Baghdad Sci. J. 19 (2022), 1140-1147.

[29] H.S. Kareem, A.A. Majeed, A qualitative study of an eco-toxicant model with anti-predator behavior, Int. J. Nonlinear Anal. Appl. 12 (2021), 1861-1882. https://doi.org/10.22075/ijnaa.2021.5322.

[30] A.A. Majeed, Local bifurcation and persistence of an ecological system consisting of a predator and stage structured prey, Iraqi J. Sci. 54 (2013), 696-705.

[31] A.A. Majeed, Local bifurcation of four species syn-ecosymbiosis model, Iraqi J. Sci. 56 (2015), 474-491.

[32] A.A. Majeed, The bifurcation analysis of an ecological model involving stage structures in both populations with toxin, Italian J. Pure Appl. Math. 46 (2021), 171-183.

[33] A.A. Majeed, Z.K. Alabacy, The persistence and bifurcation analysis of an ecological model with fear effect involving prey refuge and harvesting, AIP Conf. Proc. 2394 (2022), 070001. https://doi.org/10.1063/5.0121877.

[34] A.A. Majeed, O.S. Ali, The dynamics of prey-predator model with harvesting involving disease in both populations, Int. J. Current Adv. Res. 6 (2017), 2222-2241.

[35] A.A. Majeed, A.J. Kadhim, The bifurcation analysis and persistence of the food chain ecological model with toxicant, J. Phys.: Conf. Ser. 1818 (2021), 012191. https://doi.org/10.1088/1742-6596/1818/1/012191.

[36] A.A. Majeed, R.R. Saadi, The persistence of prey-predator model with competition hosts, Int. J. Appl. Math. Res. 4 (2015), 351-361. https://doi.org/10.14419/ijamr.v4i2.4514.

[37] R. McGehee, R.A. Armstrong, Some mathematical problems concerning the ecological principle of competitive exclusion, J. Differ. Equ. 23 (1977), 30-52.

[38] D. Melese, O. Muhye, S.K. Sahu, Dynamical behavior of an eco-epidemiological model incorporating prey refuge and prey harvesting, Appl. Appl. Math.: Int. J. 15 (2020), 28.

[39] D. Pal, G.S. Mahaptra, G.P. Samanta, Optimal harvesting of prey-predator system with interval biological parameters: A bioeconomic model, Math. Biosci. 241 (2013), 181-187. https://doi.org/10.1016/j.mbs.2012.11.007.

[40] K. Ramesh, K.S. Reddy, C.P. Kumar, et al. Dynamics of bio economic and optimal harvesting in multi species fish population dynamical system, Ann. Roman. Soc. Cell Biol. 23 (2019), 24-39.

[41] C. Raymond, A. Hugo, M. Kung'aro, Modeling dynamics of prey-predator fishery model with harvesting: A bioeconomic model, J. Appl. Math. 2019 (2019), 2601648. https://doi.org/10.1155/2019/2601648.

[42] P. Roy, K.P. Das, P. Karmakar, et al. Role of harvesting in controlling chaos and disease propagation in predator-prey system with disease in prey, Int. J. Dyn. Syst. Differ. Equ. 9 (2019), 234-261.

[43] O.J. Schmitz, Predators affect competitors' coexistence through fear effects, Nature 570 (2019), 43-44. https://doi.org/10.1038/d41586-019-01712-7.

[44] H.L. Smith, The interaction of steady state and hopf bifurcations in a two-predator-one-prey competition model, SIAM J. Appl. Math. 42 (1982), 27-43. https://doi.org/10.1137/0142003.

[45] H.L. Smith, P.E. Waltman, The theory of the chemostat: dynamics of microbial competition, Cambridge University Press, Cambridge, 1995.

[46] F.M. Stewart, B.R. Levin, Partitioning of resources and the outcome of interspecific competition: A model and some general considerations, Amer. Naturalist. 107 (1973), 171-198.

[47] J.P. Tripathi, D. Jana, V. Tiwari, A Beddington–DeAngelis type one-predator two-prey competitive system with help, Nonlinear Dyn. 94 (2018), 553-573. https://doi.org/10.1007/s11071-018-4377-9.

[48] A.K. Umrao, P.K. Srivastava, Bifurcation analysis of a predator-prey model with allee effect and fear effect in prey and hunting cooperation in predator, Differ. Equ. Dyn. Syst. (2023). https://doi.org/10.1007/s12591-023-00663-w.

[49] P. Waltman, S.P. Hubbell, S.B. Hsu, Theoretical and experimental investigations of microbial competition in continuous culture, in: T.A. Burton, ed., Modeling and differential equations in biology, 1st ed., Routledge, 2017.

[50] Y.N. Xiao, L.S. Chen, Global stability of a predator-prey system with stage structure for the predator, Acta Math. Sinica. 20 (2004), 63-70. https://doi.org/10.1007/s10114-002-0234-2.