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THE CONTRIBUTION OF AMENSALISM AND PARASITISM IN THE THREE-SPECIES ECOLOGICAL SYSTEM'S DYNAMIC

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Abstract: Amensalism is an ecological relationship in which one species experiences harm while the other remains unaffected. When one organism stops another from proliferating or living without causing harm to itself, this type of symbiotic relationship takes place. This research is interested in developing and examining a mathematical model including three species that integrate amensalism and parasitism because amensalism can involve a range of organisms and occur in a variety of settings. The qualitative properties of their solution are analyzed. Every possible point of equilibrium has been located. We look at both global and local stability. The requirements needed for the system to continue operating have been identified. The possibility of local bifurcations is investigated. Lastly, a numerical simulation is used to describe how parameter perturbations impact the system's behavior.

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1. INTRODUCTION

Ecology is the study of how living things interact with their environment. It is typical for two

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or more species to interact differently when they share a territory. Over the past 9 decades, mathematical modeling has been crucial in describing several natural phenomena involving people and populations. Lotka [1] and Volterra [2], who ushered in new eras in studying life and biological sciences, made significant advances in theoretical ecology. Amensalism, competition, commensalism, neutralism, mutualism, predation, and parasitism are some general categories under which ecological interactions might be categorized. The interactions between these categories define the ecosystem's structure. Interactions across species will produce diverse, dynamic, and interesting biological species that are complicated and varied [3-4]. Meyer [5], Cushing [6], and Kapur's [7] treatises introduce the general idea of modeling.

Most researchers have focused on the study of the main interactions that affect both sides of the interaction such as competition, mutualism, and predation. Ecological interactions between organisms, such as competition, mutualism, and predation, are important in forming ecosystems and affecting species populations. Competition arises from conflicts among individuals or groups of the same species or other species that share limited resources. It can also limit the expansion and survival of some populations by causing a fight for survival and reproduction. On the other hand, mutualism is a beneficial ecological relationship in which two distinct species gain from one another's existence. Both of the species in this symbiotic connection offer services, resources or help to one another, which produces a win-win situation. Conversely, predation refers to the interaction between two species in which the predator pursues, catches, and consumes the victim. An ecosystem's ability to regulate species numbers and preserve population balance is largely dependent on predatory behavior. Because they have an impact on species distribution, population levels, and evolutionary processes, these ecological interactions are essential to the dynamics and operation of ecosystems. Predation, mutualism, and competition are all related to one another and support the diversity and general balance of life in a particular ecosystem. Many scientific researchers have dealt with the study of the three above interactions, including [8-12] for competition, [13-18] for mutualism, and, [19-29] for predation between two species and three species.

However, studies of interactions like amensalism and commensalism that impact only one side of the involved species have received minimal attention. In biology, amensalism is a kind of ecological interaction in which one organism suffers harm while another is left untouched. The organism that suffers harm in this relationship is called the amensal, while the unaffected organism is called the enemy. The other organism's habitat may be physically harmed, certain chemicals or substances may be released, or there may be competition for resources leading to this contact. A big tree that shades smaller plants and prevents their growth without directly benefiting them is an example of amensalism. Nonetheless, a particular kind of ecological interaction involving two animals of different species is known as commensalism. It happens when one organism gains from the relationship without the other suffering any negative effects. The beneficial creature in this interaction is called the commensal, and the other organism is called the host. The coexisting relationship between cattle egrets and cattle is an illustration of commensalism. As the cattle roam around, they mix up a variety of insects and parasites that the cow egret feeds on. The cattle are neither adversely affected nor greatly helped by the egret's presence, but the egret gains by having a meal.

Numerous species of amensalism and commensalism have been studied [30–33]. Hari [34] studied a typical three-species syn-eco system with a commensal mortality rate analytically and numerically. The stability of three common species syn-ecosystems, which consist of one commensal and two hosts, has been investigated by Srinivas et al. [35]. Mougi [36] used a theoretical technique to show how unilateral interactions, such as amensalistic and commensalistic relationships, significantly improve community stability. According to the acquired data, symmetrical interactions like mutualism and competition were less stabilizing than asymmetric oppositional interactions, although amensalism and commensalism were more stabilizing than those types of interactions. A three-species dynamical system is investigated, whereby the third species serves as both a host and a predator, and the system is made up of two logistically competing species that are growing at the same time [37]. They noted the Hopf bifurcation's existence. In contrast, a three-species ecosystem with three pairings of species is thought to be modeled to assess stability [38]. One of the three species is thought to fulfill two roles that of a host and an opponent with a monod reaction. On the first species, Zhao and Du [39] suggested and investigated a novel amensalism system with the Allee effect. On the other hand, the Beddington-DeAngelis amensalism model with a substantial Allee effect on the second species was the subject of global dynamics research by Luo and Wang [40].

Keeping the above in mind, in this paper, a mathematical model that describes the interaction among three species involving amensalism and parasitism relationships is formulated mathematically and their dynamics are investigated. Gaining further insight into the interactions between various species and their effects on biodiversity and ecosystem stability is the aim of research on the dynamics of ecological systems incorporating amensalism and parasitism. An investigation into the effects of amensalistic and parasitic interactions on population dynamics and community structure, as well as an evaluation of the effects of these interactions on ecosystem resilience and functioning, are the objectives of this type of study. Through the investigation of amensalism and parasitism dynamics in ecological systems, scientists can expand on our comprehension of the complex web of interactions that form ecosystems and aid in their better management and conservation.

2. Mathematical Model

The mathematical model that describes the interaction among three species involves amensalism and parasitism relationships can be described using the following set of first-order differential equations

$$\begin{aligned}\frac{dX}{dT} &= r_1X - a_{11}X^2 - a_{12}XY,\\ \frac{dY}{dT} &= r_2Y - a_{22}Y^2 - a_{23}YZ,\\ \frac{dZ}{dT} &= r_3Z - a_{33}Z^2 + ea_{23}YZ,\end{aligned} \tag{1}$$

where $X(T)$, $Y(T)$, and $Z(T)$ represent the density at time T for the first, second, and third species, where the relationship between the first and second species represents amensal-enemy, while that is between the second and third species describes host-parasite respectively, with

$X(0) \geq 0$, $Y(0) \geq 0$, and $Z(0) \geq 0$. All the parameters are positive constants and are described in Table (1).

It is assumed in the system (1) that the three species are expanding logistically. The presence of the enemy at the second level harms the amensal at the first level. On the other hand, the third-level parasite species uses the second-level species as a host. The law of mass action is followed by the amensalism and parasitism processes.

Table (1): Description of parameters

Parameters	Description
$r_i; i = 1,2,3$	The intrinsic growth rate of X, Y, and Z respectively.
$a_{ii}; i = 1,2,3$	The intraspecific competition rate of X, Y, and Z respectively.
a_{12}	The amensalism rate
a_{23}	The parasite rate
$e \in (0,1)$	The conversion rate

By using the following dimensionless variables and parameters, the dimensionless form of system (1) can be written the form of system (3).

$$\begin{gathered} r_1 T = t, \frac{a_{11}}{r_1} X = x, \frac{a_{12}}{r_1} Y = y, \frac{a_{23}}{r_1} Z = z \\ w_1 = \frac{r_2}{r_1}, w_2 = \frac{a_{22}}{a_{12}}, w_3 = \frac{r_3}{r_1}, w_4 = \frac{a_{33}}{a_{23}}, w_5 = e\frac{a_{23}}{a_{12}} \end{gathered} \tag{2}$$

$$\begin{aligned} \frac{dx}{dt} &= x - x^2 - xy = x(1 - x - y) = xf_1(x, y, z), \\ \frac{dy}{dt} &= w_1 y - w_2 y^2 - yz = y(w_1 - w_2 y - z) = yf_2(x, y, z), \\ \frac{dz}{dt} &= w_3 z - w_4 z^2 + w_5 yz = z(w_3 - w_4 z + w_5 y) = zf_3(x, y, z), \end{aligned} \tag{3}$$

where $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$.

The interaction functions $f_i(x, y, z)$ for $i = 1,2,3$ are continuous and have continuous partial derivatives, so system (3) has a unique solution following the existence and uniqueness theorem for the solution of the system of ordinary differential equations. Furthermore, the subsequent theorems established the fundamental characteristics of the system's (3) solution.

Theorem 1. System (3) with any positive initial values is positively invariant for all $t \geq 0$.

Proof. System (3)'s form suggests that it is a Kolmogorov system, with growth rates $f_i(x, y, z)$

for $i = 1,2,3$, being functions that are continuously differentiable. As a result, we may use the positive conditions $\left(x(0), y(0), z(0)\right)$ to solve (3) and get:

$$x(t) = x(0)\exp\left[\int_0^1 f_1(x(s), y(s), z(s))\, ds\right] = x(0)\exp\left[\int_0^1 \left(1 - x(s) - y(s)\right) ds\right] > 0.$$

$$\begin{aligned} y(t) &= y(0)\exp\left[\int_0^1 f_2(x(s), y(s), z(s))\, ds\right] \\ &= y(0)\exp\left[\int_0^1 (w_1 - w_2 y(s) - z(s)) ds\right] > 0 \end{aligned}$$

$$\begin{aligned} z(t) &= z(0)\exp\left[\int_0^1 f_3(x(s), y(s), z(s))\, ds\right] \\ &= z(0)\exp\left[\int_0^1 \left(w_3 - w_4 z(s) + w_5 y(s)\right) ds\right] > 0 \end{aligned}$$

Because of the aforementioned equations and the definition of the exponential function, any solution that starts with positive initial circumstances $\left(x(0), y(0), z(0)\right)$ stays there indefinitely. Hence the proof is finished.

Theorem 2. All solutions of system (3) are uniformly bounded

Proof. From the first equation of system (3), it is observed that

$$\frac{dx}{dt} < x(1 - x).$$

Then according to the lemma (2.2) by Chen [41], it is obtained that

$$x(t) < \left[1 + \left(\frac{1}{x(0)} - 1\right) e^{-t}\right]^{-1}.$$

Therefore for $t \to \infty$, it is obtained $x(t) < 1$, also the following inequality is obtained from the second equation of system (3).

$$\frac{dy}{dt} < w_1 y - w_2 y^2$$

Similarly, it is obtained $y(t) < \frac{w_1}{w_2}\ as\ t \to \infty$. Then by using the resulting upper bound of the variable y in the third equation gives:

$$\frac{dz}{dt} < z\left(w_3 + \frac{w_5 w_1}{w_2}\right) - w_4 z^2.$$

Again, as $t \to \infty$, the following result is obtained:

$$z(t) < \left(\frac{w_2 w_3 + w_1 w_5}{w_2 w_4}\right).$$

Accordingly, all solutions of system (3) with positive initial values are uniformly bounded in the region:

$$\Omega = \left\{x(t), y(t), z(t): 0 \leq x(t) < 1, 0 \leq y(t) < \frac{w_1}{w_2}, 0 \leq z(t) < \frac{w_2 w_3 + w_1 w_5}{w_2 w_4}\right\}.$$

3. Stability Analysis

The equilibria and stability analysis of system (3) are examined in the following. The system (3) is shown to have the following equilibrium points:

The vanishing equilibrium point $q_1 = (0,0,0)$ along with the first, second, and third axial equilibrium points, which are denoted by $q_2 = (1,0,0)$, $q_3 = (0, \frac{w_1}{w_2}, 0)$, and $q_4 = (0,0,\frac{w_3}{w_4})$ respectively, always exist.

The parasite-free equilibrium point $q_5 = (\bar{x}, \bar{y}, 0)$, where $\bar{x} = \frac{w_2 - w_1}{w_2}, \bar{y} = \frac{w_1}{w_2}$ exists if and only if

$$w_1 < w_2. \tag{4}$$

While the enemy-host-free equilibrium point $q_6 = (\hat{x}, 0, \hat{z})$, where $\hat{x} = 1, \hat{z} = \frac{w_3}{w_4}$ always exists.

Moreover, the amensal-free equilibrium point $q_7 = (0, \breve{y}, \breve{z})$, where $\breve{y} = \frac{w_1 w_4 - w_3}{w_2 w_4 + w_5}, \breve{z} = \frac{w_2 w_3 + w_1 w_5}{w_2 w_4 + w_5}$ exists if and only if

$$w_3 < w_1 w_4. \tag{5}$$

Finally, the co-existing equilibrium point $q_8 = (x^*, y^*, z^*)$, where $x^* = \frac{w_3 - w_1 w_4 + w_2 w_4 + w_5}{w_2 w_4 + w_5}$, $y^* = \frac{w_1 w_4 - w_3}{w_2 w_4 + w_5}$, and $z^* = \frac{w_2 w_3 + w_1 w_5}{w_2 w_4 + w_5}$ exists uniquely in the $\text{int.}\mathbb{R}^3_+ = \{(x,y,z): x > 0, y > 0, z > 0\}$ if and only if

$$w_3 < w_1 w_4 < w_3 + w_2 w_4 + w_5. \tag{6}$$

Now, to study the local stability analysis of system (3), the variational matrix at the point (x,y,z) can be determined as:

$$\mathcal{V}(x,y,z) = \begin{pmatrix} 1 - 2x - y & -x & 0 \\ 0 & -z + w_1 - 2y w_2 & -y \\ 0 & z w_5 & w_3 - 2z w_4 + y w_5 \end{pmatrix} \tag{7}$$

Hence, the variational matrix at the vanishing point becomes

$$\mathcal{V}(q_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w_1 & 0 \\ 0 & 0 & w_3 \end{pmatrix}.$$

The eigenvalues of $\mathcal{V}(q_1)$ can be written as $\ell_{11} = 1$, $\ell_{21} = w_1$, and $\ell_{31} = w_3$, which are positive eigenvalues. Hence q_1 is an unstable point.

The variational matrix at the first axial equilibrium point becomes

$$\mathcal{V}(q_2) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & w_1 & 0 \\ 0 & 0 & w_3 \end{pmatrix}.$$

The eigenvalues of $\mathcal{V}(q_2)$ can be written by $\ell_{12} = -1$, $\ell_{22} = w_1$, and $\ell_{32} = w_3$, which are complain both positive and negative values. Hence, q_2 is a saddle point with an unstable manifold in the yz $-$plane.

The variational matrix at the second axial equilibrium point becomes

$$\mathcal{V}(q_3) = \begin{pmatrix} 1 - \frac{w_1}{w_2} & 0 & 0 \\ 0 & -w_1 & -\frac{w_1}{w_2} \\ 0 & 0 & w_3 + \frac{w_1 w_5}{w_2} \end{pmatrix}.$$

The eigenvalues of $\mathcal{V}(q_3)$ are given by $\ell_{13} = 1 - \frac{w_1}{w_2}$, $\ell_{23} = -w_1$, and $\ell_{33} = w_3 + \frac{w_1 w_5}{w_2}$. Therefore, the point q_3 is a saddle with a stable manifold xy $-$plane when $w_2 < w_1$ and a saddle point with a stable manifold in the y $-$direction when $w_1 < w_2$.

The variational matrix at the third axial equilibrium point becomes

$$\mathcal{V}(q_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w_1 - \frac{w_3}{w_4} & 0 \\ 0 & \frac{w_3 w_5}{w_4} & -w_3 \end{pmatrix}.$$

Clearly, $\mathcal{V}(q_4)$ have the following eigenvalues $\ell_{14} = 1$, $\ell_{24} = w_1 - \frac{w_3}{w_4}$, and $\ell_{34} = -w_3$. Therefore, the point q_4 is a saddle with a stable manifold yz $-$plane when $w_1 w_4 < w_3$ and a saddle point with a stable manifold in the z $-$direction when $w_3 < w_1 w_4$.

The variational matrix at the parasite-free equilibrium point becomes

$$\mathcal{V}(q_5)=\begin{pmatrix}\frac{w_1-w_2}{w_2} & \frac{w_1-w_2}{w_2} & 0\\ 0 & -w_1 & -\frac{w_1}{w_2}\\ 0 & 0 & w_3+\frac{w_1 w_5}{w_2}\end{pmatrix}.$$

Hence, the eigenvalues of $\mathcal{V}(q_5)$ are given by $\ell_{15}=\frac{w_1-w_2}{w_2}<0$ under the existence condition of q_5, with $\ell_{25}=-w_1$, and $\ell_{35}=w_3+\frac{w_1 w_5}{w_2}$. Therefore, the point q_5 is a saddle point with a stable manifold xy −plane.

The variational matrix at the enemy-host-free equilibrium point becomes

$$\mathcal{V}(q_6)=\begin{pmatrix}-1 & -1 & 0\\ 0 & w_1-\frac{w_3}{w_4} & 0\\ 0 & \frac{w_3 w_5}{w_4} & -w_3\end{pmatrix}. \tag{8}$$

So, the eigenvalues of $\mathcal{V}(q_6)$ can be written by:

$$\ell_{16}=-1,\ \ell_{26}=w_1-\frac{w_3}{w_4},\ \ell_{36}=-w_3. \tag{9}$$

Accordingly, the point q_6 is a stable node, a saddle point with a stable manifold xz −plane, and a non-hyperbolic point if and only if the following conditions are met respectively.

$$w_1 w_4<w_3. \tag{10}$$

$$w_3<w_1 w_4. \tag{11}$$

$$w_1 w_4=w_3. \tag{12}$$

The variational matrix at the amensal-free equilibrium point becomes

$$\mathcal{V}(q_7)=\begin{pmatrix}1+\frac{w_3-w_1 w_4}{w_2 w_4+w_5} & 0 & 0\\ 0 & \frac{w_2(w_3-w_1 w_4)}{w_2 w_4+w_5} & \frac{w_3-w_1 w_4}{w_2 w_4+w_5}\\ 0 & \frac{w_5(w_2 w_3+w_1 w_5)}{w_2 w_4+w_5} & -\frac{w_4(w_2 w_3+w_1 w_5)}{w_2 w_4+w_5}\end{pmatrix}. \tag{13}$$

Therefore, the characteristic equation of $\mathcal{V}(q_7)$ can be written in the form

$$\left[\left(1+\frac{w_3-w_1 w_4}{w_2 w_4+w_5}\right)-\ell\right][\ell^2-T_1\ell+D_1]=0, \tag{14}$$

where

$$T_1=\frac{w_2(w_3-w_1 w_4)}{w_2 w_4+w_5}-\frac{w_4(w_2 w_3+w_1 w_5)}{w_2 w_4+w_5}.$$

$$D_1 = \left(\frac{w_2(w_3-w_1w_4)}{w_2w_4+w_5}\right)\left(-\frac{w_4(w_2w_3+w_1w_5)}{w_2w_4+w_5}\right) - \left(\frac{w_3-w_1w_4}{w_2w_4+w_5}\right)\left(\frac{w_5(w_2w_3+w_1w_5)}{w_2w_4+w_5}\right).$$

Clearly, under the existence condition of the point q_7, it is obtained that $T_1 < 0$ and $D_1 > 0$. Hence the quadratic term in equation (14) has two eigenvalues with negative real parts due to the Routh-Hurwitz criterion. However, the first eigenvalue is given by $\ell_{17} = 1 + \frac{w_3-w_1w_4}{w_2w_4+w_5}$. Therefore, the amensal-free equilibrium point is a sink, a saddle with a stable manifold given by yz −plane, and a non-hyperbolic if and only if the following conditions are met respectively.

$$w_2w_4 + w_5 + w_3 < w_1w_4. \tag{15}$$

$$w_1w_4 < w_2w_4 + w_5 + w_3. \tag{16}$$

$$w_2w_4 + w_5 + w_3 = w_1w_4. \tag{17}$$

Note that, by comparing the existence condition (6) of the co-existing equilibrium point with condition (16), it is concluded that the co-existing equilibrium point exists only when the first-species free equilibrium point is unstable.

The variational matrix at the co-existing equilibrium point becomes

$$\mathcal{V}(q_8) = \begin{pmatrix} \frac{-w_3+w_1w_4-w_2w_4-w_5}{w_2w_4+w_5} & \frac{-w_3+w_1w_4-w_2w_4-w_5}{w_2w_4+w_5} & 0 \\ 0 & \frac{w_2(w_3-w_1w_4)}{w_2w_4+w_5} & \frac{w_3-w_1w_4}{w_2w_4+w_5} \\ 0 & \frac{w_5(w_2w_3+w_1w_5)}{w_2w_4+w_5} & -\frac{w_4(w_2w_3+w_1w_5)}{w_2w_4+w_5} \end{pmatrix}. \tag{18}$$

Similarly, the characteristic equation of $\mathcal{V}(q_8)$ can be written in the form

$$\left[\left(\frac{-w_3+w_1w_4-w_2w_4-w_5}{w_2w_4+w_5}\right) - \ell\right][\ell^2 - T_2\ell + D_2] = 0, \tag{19}$$

where

$$T_2 = \frac{w_2(w_3-w_1w_4)}{w_2w_4+w_5} - \frac{w_4(w_2w_3+w_1w_5)}{w_2w_4+w_5} \equiv (T_1).$$

$$D_2 = \left(\frac{w_2(w_3-w_1w_4)}{w_2w_4+w_5}\right)\left(-\frac{w_4(w_2w_3+w_1w_5)}{w_2w_4+w_5}\right) - \left(\frac{w_3-w_1w_4}{w_2w_4+w_5}\right)\left(\frac{w_5(w_2w_3+w_1w_5)}{w_2w_4+w_5}\right) \equiv (D_1).$$

Clearly, under the left side of the existence condition of the point q_8, it is obtained that $T_2 < 0$ and $D_2 > 0$. Hence the quadratic term in equation (19) has two eigenvalues with negative real parts due to the Routh-Hurwitz criterion. However, the first eigenvalue is given by $\ell_{18} = \frac{-w_3+w_1w_4-w_2w_4-w_5}{w_2w_4+w_5}$, which is negative due to the right side of the existence condition of the point

q_8. Hence, the co-existing equilibrium point is a sink whenever it exists.

4. PERSISTENCE

It is an essential subject in dynamic systems due to its importance in proving that the species in the system (3) are permanent. It has a biological meaning and the other is mathematical. Biologically it means the continued existence of all species of system over a long time, as it guarantees their non-extinction. Mathematically it means that the paths of the system (3) are eventually moved away from the border planes. This becomes clear when $\lim_{t\to\infty}\inf x_i(t) > 0$ for each species of $x_i(t)$. Therefore, the first step is to verify the existence of periodic dynamics in the boundary planes.

Observed that the system (3) has three subsystems, the first is located in the positive quadrant of $xy-$ plane, while the second sub-system is located in the positive quadrant of $xz-$plane, and the third sub-system is located in the positive quadrant of $yz-$plane.

These sub-systems can be written as follows:

$$\begin{aligned}\frac{dx}{dt} &= x(1-x-y) = M_1(x,y)\\ \frac{dy}{dt} &= y(w_1 - w_2 y) = M_2(x,y)\end{aligned} \tag{20}$$

and

$$\begin{aligned}\frac{dx}{dt} &= x(1-x) = M_3(x,z)\\ \frac{dz}{dt} &= z(w_3 - w_4 z) = M_4(x,z)\end{aligned} \tag{21}$$

Finally

$$\begin{aligned}\frac{dy}{dt} &= y(w_1 - w_2 y - z) = M_5(y,z)\\ \frac{dz}{dt} &= z(w_3 - w_4 z + w_5 y) = M_6(y,z)\end{aligned} \tag{22}$$

Define the Dulac functions $L_1(x,y) = \frac{1}{xy}$, $L_2(x,z) = \frac{1}{xz}$, and $L_3(y,z) = \frac{1}{yz}$, which satisfy that $L_i > 0$, for $i = 1,2,3$ and C^1 functions in the int.$\mathbb{R}_+^2$ of the $xy-$, $xz-$, and $yz-$ planes respectively. Then direct computation gives that

$$D_1(x,y) = \frac{\partial(L_1M_1)}{\partial x} + \frac{\partial(L_1M_2)}{\partial y} = -\frac{1}{y} - \frac{w_2}{x} < 0.$$

$$D_2(x,z) = \frac{\partial(L_2M_3)}{\partial x} + \frac{\partial(L_2M_4)}{\partial z} = -\frac{1}{y} - \frac{w_4}{x} < 0.$$

$$D_3(y,z) = \frac{\partial(L_3M_5)}{\partial y} + \frac{\partial(L_3M_6)}{\partial z} = -\frac{w_2}{z} - \frac{w_4}{y} < 0$$

The expressions $D_1(x,y)$, $D_2(x,z)$, and $D_3(y,z)$ are not equal to zero and their sign does not change in the interior of the respective positive quadrant. Hence, there are no periodic dynamics of these subsystems, and their unique positive equilibrium points that are given by $q_{xy} = (\bar{x}, \bar{y})$, $q_{xz} = (\hat{x}, \hat{z})$, and $q_{yz} = (\breve{y}, \breve{z})$ are globally stable.

Now the persistence requirements of the system (3) are established in the following theorem.

Theorem 3. System (3) is uniformly persist if and only if the following conditions are met:

$$w_1 w_4 > w_3 \tag{23}$$

$$1 - \frac{w_1\, w_4 - w_3}{w_2\, w_4 + w_5} > 0 \tag{24}$$

Proof. Define the function $\varphi(x,y,z) = x^{p_1} y^{p_2} z^{p_3}$, where p_i represent the positive constants $\forall i = 1,2,3$. Hence, $\varphi(x,y,z) > 0$ for all $(x,y,z) \in \text{int.}\mathbb{R}^3_+$ and $\varphi(x,y,z) \to 0$ if any one of their variables approaches zero.

Consequently, it is obtained that

$$\rho(x,y,z) = \frac{\varphi'(x,y,z)}{\varphi(x,y,z)} = p_1\, f_1 + p_2\, f_2 + p_3 f_3$$

Thus

$$\rho(x,y,z) = p_1\,(1 - x - y) + p_2(w_1 - w_2 y - z) + p_3(w_3 - w_4 z - w_5 y)$$

Now, if $\rho(q) > 0$ for every attractor point q on the border planes and axes with a suitable choice of constants $p_i > 0$, $\forall i = 1,2,3$ holds, then due to the average Lyapunov function [42] the system is a uniform persist. Therefore,

$\rho\,(q_1) = p_1 + w_1 p_2 + p_3 w_3 > 0$, for a suitable choice of $p_i > 0, i = 1,2,3$.

$\rho\,(q_2) = p_2 w_1 + p_3 w_3 > 0$, for all $p_i > 0, i = 1,2,3$.

$\rho\,(q_3) = p_1\left(1 - \frac{w_1}{w_2}\right) + p_3\left(w_3 + w_5 \frac{w_1}{w_2}\right) > 0$, for a suitable choice of $p_i > 0, i = 1,3$.

$\rho(q_4) = p_1 + p_2\left(w_1 - \frac{w_3}{w_4}\right) > 0$, for all $p_i > 0, i = 1,2$.

$\rho(q_5) = p_3 w_3 + p_3 w_5 \bar{y} > 0$, for all $p_3 > 0$.

$\rho(q_6) = p_2 w_1 - p_2 \frac{w_3}{w_4} > 0$, for all $p_2 > 0$ under the condition (23).

$\rho(q_7) = p_1 - p_1 \frac{w_1 w_4 - w_3}{w_2 w_4 + w_5} > 0$, for all $p_1 > 0$ under the condition (24).

Then the system (3) is uniformly persistent.

5. Global Stability

In this part, the set of points corresponding to point q, which is known as the basin of attraction of point q and denoted $B(q)$, is investigated. This set of points is where the solution of system (3) starts and eventually approaches the equilibrium points. When $B(q)$ covers the whole domain of the system, it is referred to as being globally stable. Lyapunov functions can be used to conduct such an investigation. The following theorems demonstrate that the primary objective of investigating global stability is to demonstrate that all pathways eventually gravitate toward the system's attractor.

Theorem 4. Assume that condition (10) holds, then the enemy-host-free equilibrium point is globally asymptotically stable if the following condition is met.

$$w_1 + \hat{x} < w_5 \hat{z} \tag{25}$$

Proof. Consider the function $R_1 = \left(x - \hat{x} - \hat{x} \ln \frac{x}{\hat{x}}\right) + y + \left(z - \hat{z} - \hat{z} \ln \frac{z}{\hat{z}}\right)$, which is a positive definite function on the set $\Gamma_1 = \{(x, y, z) \in \mathbb{R}_+^3 : x > 0, y \geq 0, z > 0\}$. The derivative of this function can be computed as

$$\frac{dR_1}{dt} = \left(\frac{x-\hat{x}}{x}\right)\frac{dx}{dt} + \frac{dy}{dt} + \left(\frac{z-\hat{z}}{z}\right)\frac{dz}{dt}.$$

That gives after some algebraic steps

$$\frac{dR_1}{dt} \leq -(x - \hat{x})^2 - w_4(z - \hat{z})^2 - (w_5 \hat{z} - w_1 - \hat{x})y$$

Therefore the function $\frac{dR_1}{dt}$ is negative definite under the condition (25) and then the enemy-host-free equilibrium point is globally asymptotically stable.

Theorem 5. Assume that condition (15) holds. Then the amensal-free equilibrium point is globally asymptotically stable.

Proof. Consider the function $R_2 = \left(y - \check{y} - \check{y}\ln\frac{y}{\check{y}}\right) + \frac{1}{w_5}\left(z - \check{z} - \check{z}\ln\frac{z}{\check{z}}\right)$, which is a positive semi-definite function on the set $\Gamma_2 = \{(x, y, z) \in \mathbb{R}_+^3 : x \geq 0, y > 0, z > 0\}$. The derivative of this function can be computed as

$$\frac{dR_2}{dt} = \left(\frac{y-\check{y}}{y}\right)\frac{dy}{dt} + \frac{1}{w_5}\left(\frac{z-\check{z}}{z}\right)\frac{dz}{dt}$$

That gives after some algebraic steps

$$\frac{dR_2}{dt} = -w_2(y - \check{y})^2 - \frac{w_4}{w_5}(z - \check{z})^2$$

It is clear that $\frac{dR_2}{dt}$ is negative semi-definite then q_7 is a stable point since $\frac{dR_2}{dt} = 0$ at many points from which $\{q_7\}$ is the only invariant set then by using LaSalle Invariance Principle q_7 is attracting point. Hence, the amensal-free equilibrium point (q_7) is asymptotically stable. Now, because R_2 is a radially unbounded function, the point q_7 is globally asymptotically stable.

Theorem 6. Assume that condition (6) holds. Then the co-existing equilibrium point is globally asymptotically stable if the following condition is met

Proof. Consider the function

$$R_3 = \left(x - x^* - x^*\ln\frac{x}{x^*}\right) + \left(y - y^* - y^*\ln\frac{y}{y^*}\right) + \frac{1}{w_5}\left(z - z^* - z^*\ln\frac{z}{z^*}\right),$$

which is a positive definite function on the set $\Gamma_3 = \{(x, y, z) \in \mathbb{R}_+^3 : x > 0, y > 0, z > 0\}$. The derivative of this function can be computed as

$$\frac{dR_3}{dt} = \left(\frac{x-x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y-y^*}{y}\right)\frac{dy}{dt} + \frac{1}{w_5}\left(\frac{z-z^*}{z}\right)\frac{dz}{dt}.$$

That gives after some algebraic steps

$$\frac{dR_3}{dt} = -(x - x^*)^2 - (x - x^*)(y - y^*) - w_2(y - y^*)^2 - \frac{w_4}{w_5}(z - z^*)^2.$$

Since for any real numbers a and b, the following is satisfy $ab < \frac{a^2}{2} + \frac{b^2}{2}$. Then it is obtained that

$$\frac{dR_3}{dt} < -\frac{3}{2}(x - x^*)^2 - \left(\frac{1}{2} + w_2\right)(y - y^*)^2 - \frac{w_4}{w_5}(z - z^*)^2$$

The function $\frac{dR_3}{dt}$ is negative definite, and then the co-existing equilibrium point is globally asymptotically stable.

6. BIFURCATION ANALYSIS

By performing a thorough analysis of the function of the system's parameters that achieves the non-hyperbolic property of equilibrium points, which is regarded as a necessary but not sufficient condition to prove bifurcation, bifurcation is a significant technique that describes the qualitative change in the behavior of dynamic systems. By applying the Sotomayor theorem [43], it is possible to observe when a little variation in these parameters results in a notable change in the behavior of the system's solution.

Rewrite system (3) in the vector form as follows:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}),\ \mathbf{X} = (x, y, z)^{\mathrm{T}} \text{ and } \mathbf{F} = (xf_1, yf_2, zf_3)^{\mathrm{T}},$$

where $f_i; i = 1,2,3$ are given in system (3). Then the second derivative of $\mathbf{F}$ concerning $\mathbf{X}$ can be written as:

$$D^2\mathbf{F}(\mathbf{X},\theta)(\mathbf{V},\mathbf{V})\begin{pmatrix} -2v_1^2 - 2v_1v_2 \\ -2w_2v_2^2 - 2v_2v_3 \\ 2w_5v_2v_3 - 2w_4v_3^2 \end{pmatrix} \tag{26}$$

here $\mathbf{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is anon zero general vector and θ is a parameter..

Theorem 7. Assume that condition (12) holds, then the system (3) undergoes a transcritical bifurcation near the enemy-host-free equilibrium point.

Proof of Theorem 7. For the value $w_1 = w_1^* = \frac{w_3}{w_4}$, the variational matrix (8) becomes

$$J_1 = \mathcal{V}(q_6, w_1^*) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & \dfrac{w_3w_5}{w_4} & -w_3 \end{pmatrix}$$

So, J_1 has the following eigenvalues $\ell_{16}^* = -1$, $\ell_{26}^* = 0$, $\ell_{36}^* = -w_3$. Then the necessary condition but not sufficient for bifurcation is met. Let $\mathbf{V_1} = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix}$ and $\mathbf{\Psi_1} = \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{13} \end{pmatrix}$ be the eigenvectors related with $\ell_{26}^* = 0$ of J_1 and ${J_1}^T$ respectively. Direct computation gives that:

$$\mathbf{V_1} = \begin{pmatrix} -1 \\ 1 \\ \frac{w_5}{w_4} \end{pmatrix} \text{ and } \mathbf{\Psi_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Now it is acquired that

$$\frac{\partial \mathbf{F}}{\partial w_1} = \mathbf{F}_{w_1} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \quad \text{then} \quad \mathbf{\Psi_1}^{\mathrm{T}}\, \mathbf{F}_{w_1}(q_6, w_1^*) = 0$$

$$D\mathbf{F}_{w_1}(q_6\,, w_1^*)\mathbf{V_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{then} \quad \mathbf{\Psi_1}^{\mathrm{T}}\left[D\mathbf{F}_{w_1}(q_6, w_1^*)\mathrm{V}_1\right] = 1 \neq 0,$$

where $\mathrm{D}\mathbf{F}_{w_1}$ represents the derivative of $\mathbf{F}_{w_1}$ concerning $\mathbf{X}$. Moreover, using equation (27) gives

$$\mathbf{\Psi_1}^{\mathrm{T}}[D^2\mathbf{F}(q_6, w_1^*)(\mathbf{V_1}, \mathbf{V_1})] = -2(w_2 + \frac{w_5}{w_4}) \neq 0$$

According to the Sotomayor theorem, system (3) near the point q_6 when $w_1 = w_1^*$ undergoes a transcritical bifurcation.

Theorem 8. Assume that condition (17) holds, and then although the amensal-free equilibrium point is non-hyperbolic, the system (3) does not undergo any type of local bifurcation near the first species-free equilibrium point.

Proof. For the value $w_3 = w_3^* = w_1 w_4 - w_2 w_4 - w_5$, the variational matrix (13) becomes

$$J_2 = \mathcal{V}(q_7, w_3^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -w_2 & -1 \\ 0 & w_5(w_1 - w_2) & -w_4(w_1 - w_2) \end{pmatrix}$$

So, J_2 has the following eigenvalues $\ell_{17}^* = 0$, $\ell_{27}^* = \frac{T}{2} + \frac{\sqrt{T^2 - 4D}}{2}$, $\ell_{37}^* = \frac{T}{2} - \frac{\sqrt{T^2 - 4D}}{2}$, where $T = -w_2 - w_4(w_1 - w_2) < 0$, and $D = (w_2 w_4 + w_5)(w_1 - w_2) > 0$ due to condition (17). Then there is zero eigenvalue with two negative real parts eigenvalues.

Let $\mathbf{V_2} = \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix}$ and $\mathbf{\Psi_2} = \begin{pmatrix} \psi_{21} \\ \psi_{22} \\ \psi_{23} \end{pmatrix}$ be the eigenvectors related with $\ell_{17}^* = 0$ of J_2 and $J_2{}^T$ respectively. Direct computation gives that:

$$\mathbf{V_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{\Psi_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Now it is acquired that

$$\frac{\partial \mathbf{F}}{\partial w_3} = \mathbf{F}_{w_3} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, \text{ then } \quad \mathbf{\Psi_2}^{\mathrm{T}}\, \mathbf{F}_{w_3}(q_7, w_3^*) = 0$$

$$D\mathbf{F}_{w_3}(q_7\,, w_3^*)\mathbf{V_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ then } \quad \mathbf{\Psi_2}^{\mathrm{T}}\big[D\mathbf{F}_{w_3}(q_7\,, w_3^*)\mathbf{V_2}\big] = 0.$$

Therefore, the second condition of local bifurcation fails to satisfy, and hence the system (1) does not undergo any type of local bifurcation near q_7. Hence the proof is complete.

Finally, since the co-existing equilibrium point $q_8 = (x^*, y^*, z^*)$, is a sink whenever it exists It turns into a structural stable, meaning that little perturbations do not affect the trajectories' qualitative behavior. Thus, there isn't a bifurcation close to it.

7. Numerical Simulation

This section uses the fictitious set of parameters listed below to solve system (3) numerically. Our goals are to verify our theoretical findings and comprehend the impact of every parameter value.

$$w_1 = 2, w_2 = 1, w_3 = 0.3, w_4 = 0.4, w_5 = 0.2 \tag{27}$$

It is observed that under set (2) the solution of system (3) approaches asymptotically to $q_8 = (0.16, 0.83, 1.16)$ starting from different sets of initial points as shown in Figure 1.

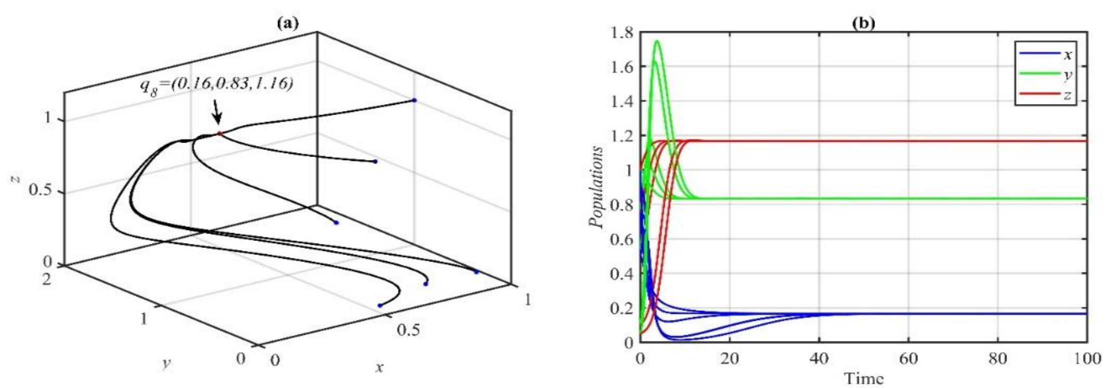

Figure 1. The trajectories of system (3) start from different initial points using the set (27). (a) Global asymptotic stable co-existing equilibrium point. (b) Trajectories of all populations as a function of time.

As Figure 2 explains, system (3) is global asymptotic stable at q_6, q_8, and q_7, respectively, for the parameter w_1 in the ranges $(0,0.75]$, $(0.75,2.25)$, and $w_1 \geq 2.25$ with the rest of parameters as given by the set (27). However, it approaches q_7 and q_8 respectively, when the parameter w_2 falls in the ranges $(0,0.75]$, and $w_1 > 0.75$, see Figure 3 at the selected values with the other parameters as (27).

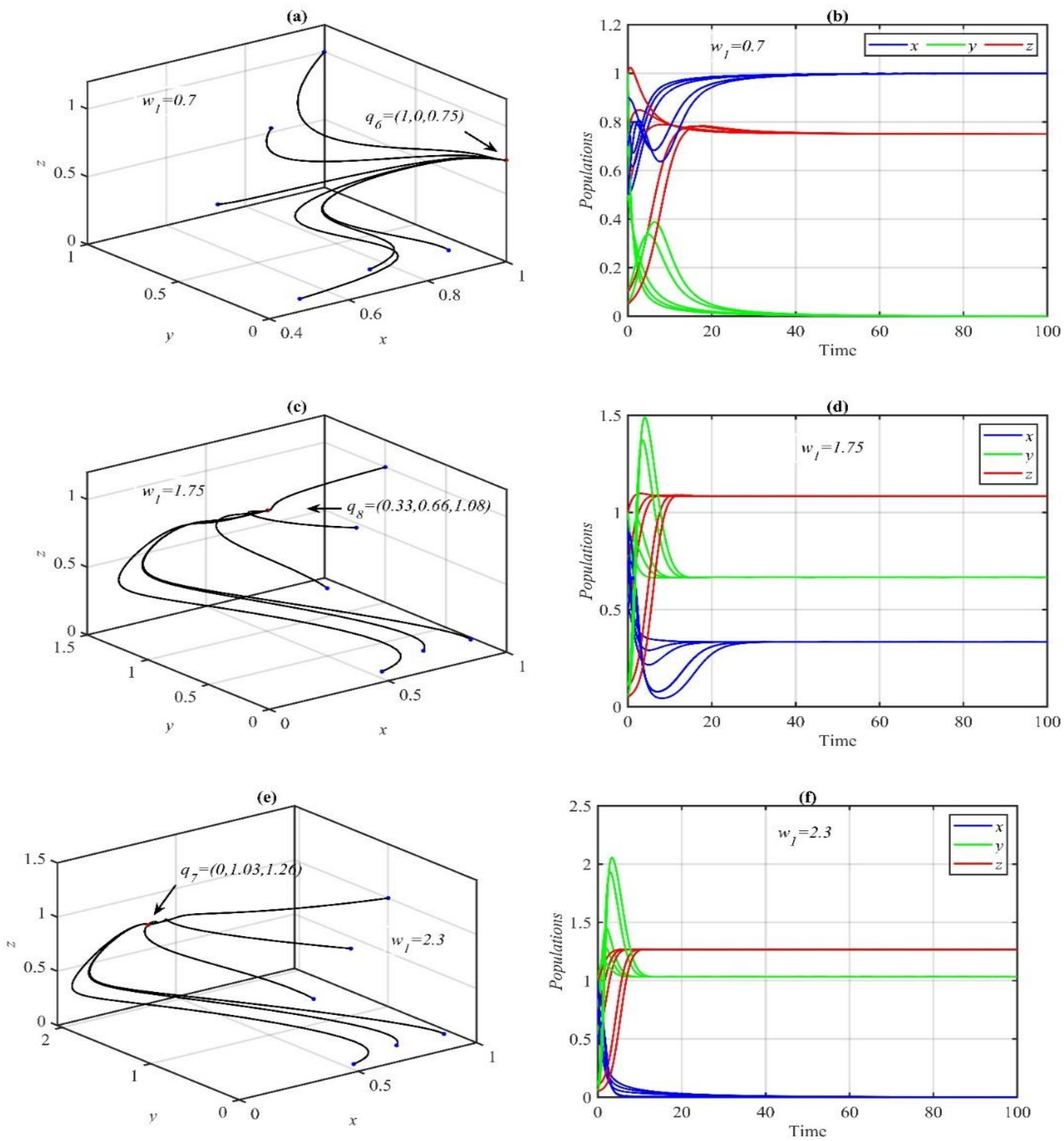

Figure 2. The trajectories of system (3) start from different initial points using the set (27). (a) Global asymptotic stable at the enemy-host-free equilibrium point $q_6 = (1,0,0.75)$ when $w_1 = 0.7$. (b) Time series when $w_1 = 0.7$. (c) Global asymptotic stable at the co-existing equilibrium

point $q_8 = (0.33,0.66,1.08)$ when $w_1 = 1.75$. (d) Time series when $w_1 = 1.75$. (e) Global asymptotic stable at the amensal-free equilibrium point $q_7 = (0,1.03,1.26)$ when $w_1 = 2.3$. (f) Time series when $w_1 = 2.3$.

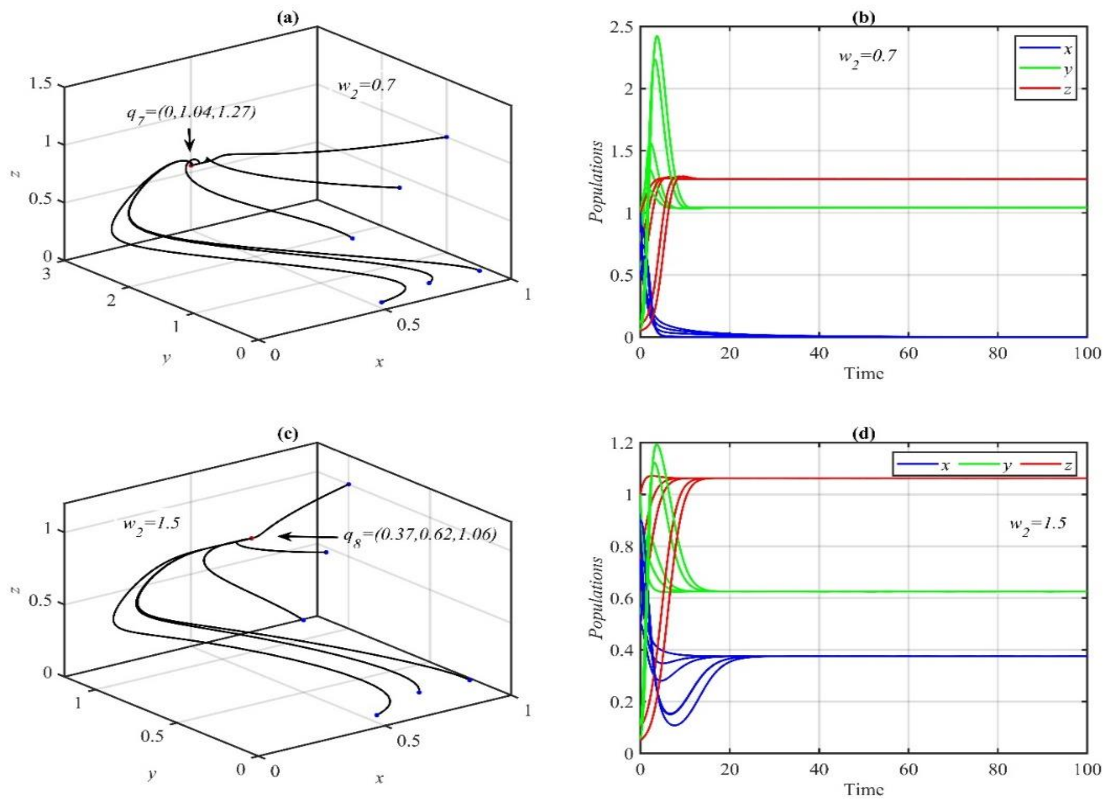

Figure 3. The trajectories of system (3) start from different initial points using the set (27). (a) Global asymptotic stable at the amensal-free equilibrium point $q_7 = (0,1.04,1.27)$ when $w_2 = 0.7$. (b) Time series when $w_2 = 0.7$. (c) Global asymptotic stable at the co-existing equilibrium point $q_8 = (0.37,0.62,1.06)$ when $w_2 = 1.5$. (d) Time series when $w_2 = 1.5$.

For the parameter w_3 in the ranges $(0,0.2]$, $(0.2,0.8)$, and $w_3 \geq 0.8$, system (3) is globally asymptotically stable at q_7, q_8, and q_6, respectively, as Figure 4 illustrates. The other parameters are provided by the set (27). For the parameter w_4, however, system (3) approaches q_6, q_8, and q_7, respectively, in the ranges $(0,0.15]$, $(0.15,0.5)$, and $w_4 \geq 0.5$; for some suggested values, refer to Figure 5, and the remaining parameters are provided by the set (27).

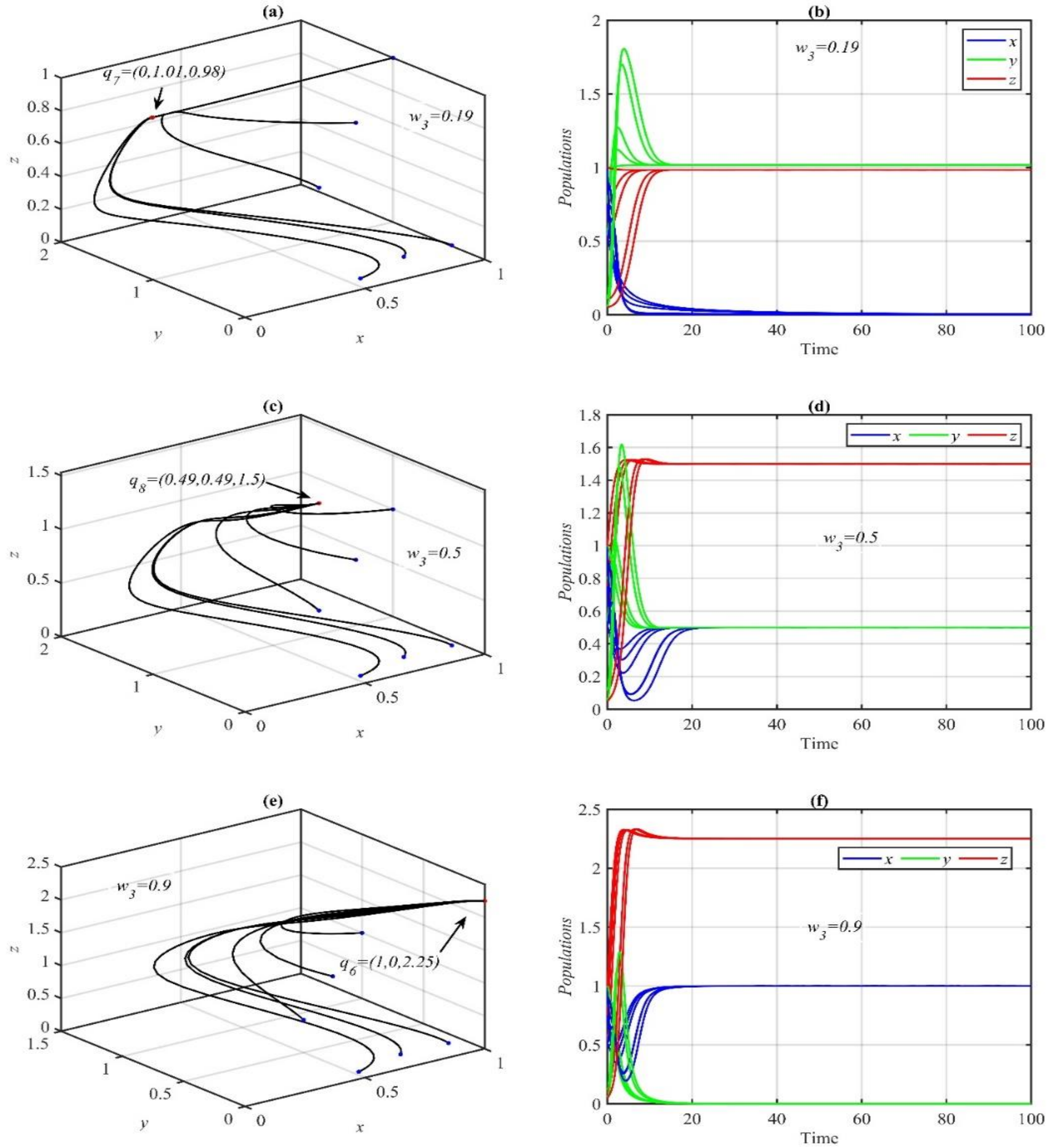

Figure 4. The trajectories of system (3) start from different initial points using the set (27). (a) Global asymptotic stable at the amensal-free equilibrium point $q_7 = (0,1.01,0.98)$ when $w_3 = 0.19$. (b) Time series when $w_3 = 0.19$. (c) Global asymptotic stable at the co-existing equilibrium point $q_8 = (0.49,0.49,1.5)$ when $w_3 = 0.5$. (d) Time series when $w_3 = 0.5$. (e) Global asymptotic stable at the enemy-host-free equilibrium point $q_6 = (1,0,2.25)$ when $w_3 = 0.9$. (f) Time series when $w_3 = 0.9$.

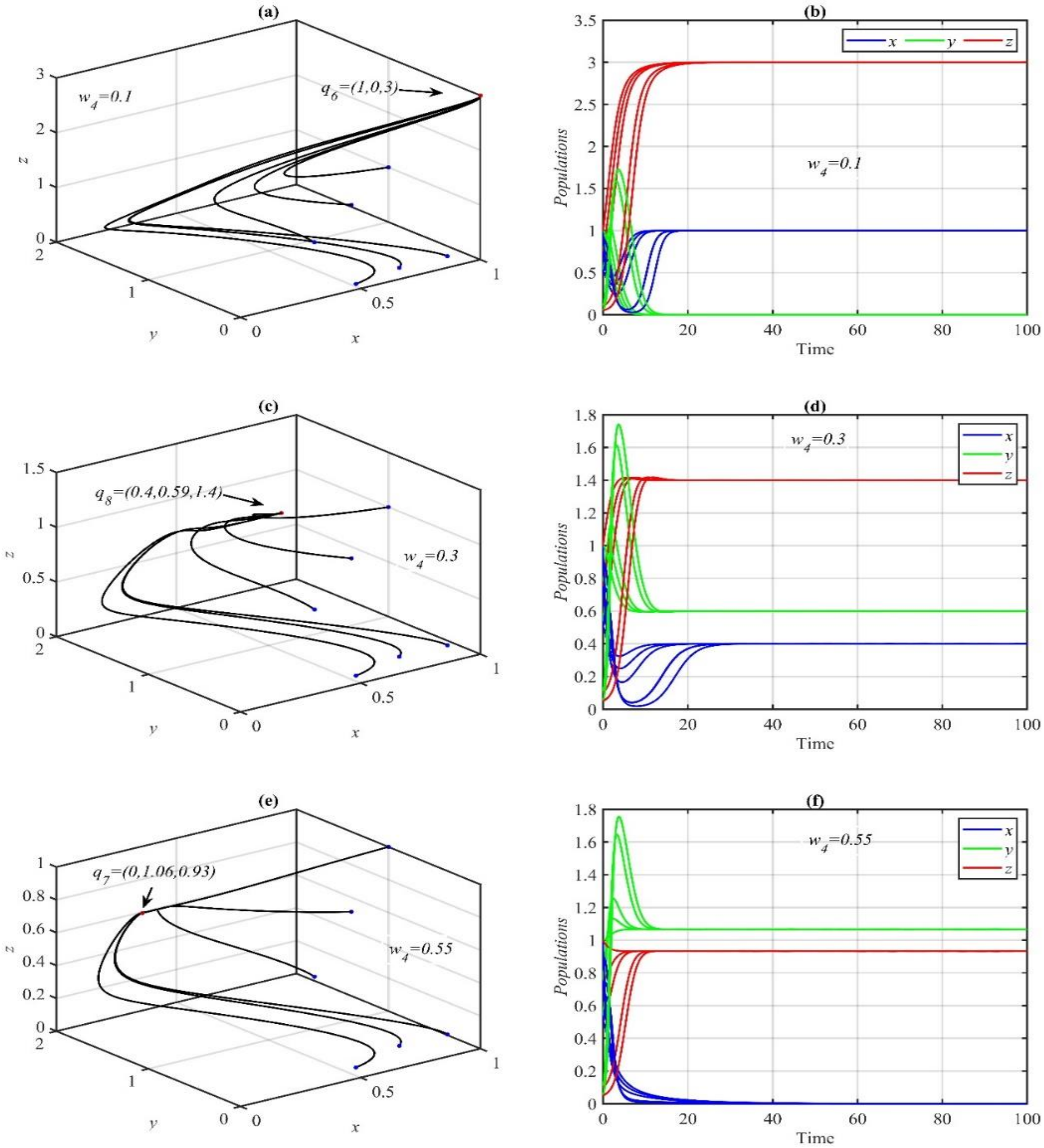

Figure 5. The trajectories of system (3) start from different initial points using the set (27). (a) Global asymptotic stable at the enemy-host-free equilibrium point $q_6 = (1,0,3)$ when $w_4 = 0.1$. (b) Time series when $w_4 = 0.1$. (c) Global asymptotic stable at the co-existing equilibrium point $q_8 = (0.4,0.59,1.4)$ when $w_4 = 0.3$. (d) Time series when $w_4 = 0.3$. (e) Global asymptotic stable at the amensal-free equilibrium point $q_7 = (0,1.06,0.93)$ when $w_4 = 0.55$. (f) Time series when $w_4 = 0.55$.

Finally, it is obtained that, when the parameter w_5 falls in the ranges $(0,0.1]$, and $w_5 > 0.1$

keeping the rest of parameters in the set (27) the system (3) approaches to q_7, and q_8 respectively, see Figure 6 for the selected values.

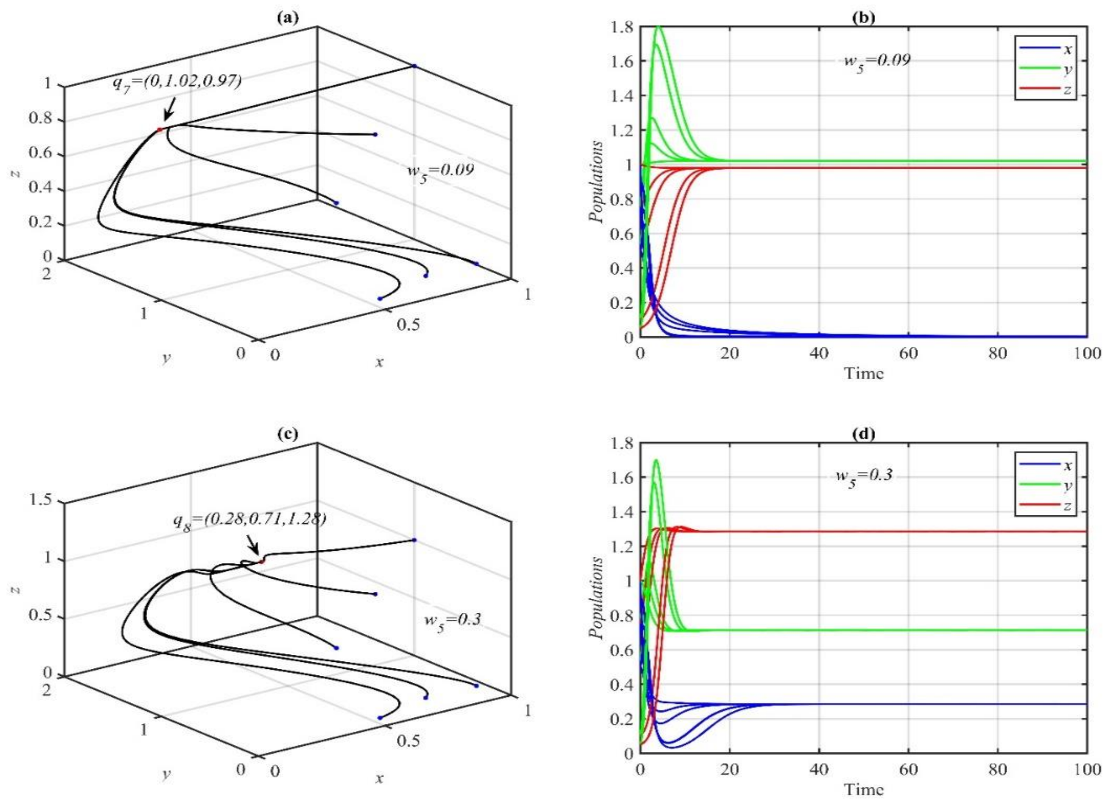

Figure 6. The trajectories of system (3) start from different initial points using the set (27). (a) Global asymptotic stable at the amensal-free equilibrium point $q_7 = (0,1.02,0.97)$ when $w_5 = 0.09$. (b) Time series when $w_5 = 0.09$. (c) Global asymptotic stable at the co-existing equilibrium point $q_8 = (0.28,0.71,1.28)$ when $w_5 = 0.3$. (d) Time series when $w_5 = 0.3$.

8. CONCLUSION

This work formulates and studies a mathematical model that describes the amensalism and parasitism dynamics of three species. Every qualitative attribute is examined, such as existence, uniqueness, positively invariant, and boundedness. Three axial equilibrium points, or saddle points, and the unstable vanishing equilibrium point make up the system (3). There may also be three additional planar equilibrium points, with or without conditions: parasite-free, amensal-free, and enemy-host-free. It is noticed that the enemy-host-free point is either stable when the amensal-free point does not exist or is a saddle point when the amensal-free point does. Similarly, the parasite-free point is a saddle point. Lastly, anytime the co-existing equilibrium point occurs, it is stable. The prerequisites for persistence are established. An appropriate Lyapunov function is used to analyze global stability. It is evidence of a transcritical bifurcation of the system near the enemy-host-free point. Lastly, the numerical analysis of the system (3) with the set of parameters (27) demonstrates that the system is stabilized at the co-existing point by the enemy-host species' intraspecific competition rate and the host biomass's rate of conversion to parasite biomass. Up to a critical number, both the intraspecific competition rate of parasites and the intrinsic growth rate of the enemy-host species contribute to survival. The system then reaches an amensal-free equilibrium point, meaning that system (3) no longer exhibits persistence. The intrinsic growth rate of the parasite species has a survival role up to a vital value. After that, the system settles at an enemy-host-free equilibrium point, which means the system (3) loses its persistence too.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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