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STABILITY AND HOPF BIFURCATION ANALYSIS OF CRYPTOSPORIDIOSIS TRANSMISSION DYNAMICS WITH TIME DELAYS

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Abstract. Cryptosporidiosis, caused by a protozoan parasite known as cryptosporidium, is a zoonotic disease that affects animals and humans. In this paper we have developed a mathematical model with a non linear incidence called Beddington-DeAngelis function to describe the spread of Cryptosporidiosis from the animals to human and from the animals to animal. We also consider the incubation periods of cryptosporidium in this model with different time delay in the infective animal and human populations. by analyzing behavior of the model, we calculate the basic reproduction number ($\mathscr{R}_{ha}$) and investigate the local and global stability of the disease-free equilibrium of the system. We also establish the sufficient conditions for the stability of the endemic equilibrium in presence of delays and we investigate the occurrence of Hopf bifurcation when certain conditions are satisfied. Finally, numerical simulations are performed and displayed graphically to support the analytical results.

Keywords: cryptosporidium; time delays; basic reproduction number; Beddington-Deangelis incidence; local and global asymptotical stability; Hopf bifurcation.

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1. INTRODUCTION

Cryptosporidiosis is an infection caused by an apicomplexan protozoan known as Cryptosporidium. Cryptosporidium common parasites of vertebrates have recently attracted increasing interest due to several serious waterborne outbreaks, and the life-threatening nature of infection in immunocompromised patients, children, the elderly, and patients on chemotherapy, pregnant women; and also the realization of economic losses caused by these pathogens in livestock. It is a common enteric pathogen in humans and domestic animals worldwide with a very low infective dose of one to ten ooysts (Pereira[1]). The sporulated ooysts are immediately infectious when excreted in faeces as there is no intermediate host. Cattle are reared throughout in Cameroon but the major production areas are in the West and North West Regions and from the Adamawa Province [2]. The cattle are transported on foot to the cattle market and the dung they pass along the road is likely to contaminate the environment and the oocysts possibly end up in streams after torrential rains. In time past, following the description of Cryptosporidium in mice by Ernest Edward Tyzzer [3], the genus Cryptosporidium has been studied, and now discovered to contain numerous species and genotypes adapted to parasitic life in almost all classes of vertebrates. Over the years, our knowledge has expanded from microscopic observations of infection and environmental contamination to the knowledge obtained from large application spread of molecular techniques to taxonomy and epidemiology. Although, the medical and veterinary significance of this protozoan was not fully appreciated for an- other 70 years. The interest in Cryptosporidium escalated tremendously over the last two and half decades [4, 5]. It was later recognized as a cause of disease in 1976. As several methods were developed to analyze stool samples, the protozoa was increasingly reported as the cause of human disease [6]. At first, Crypto was categorized as a veterinary problem because, majority of the early cases were diagnosed due to individuals rearing farm animals such as cows. Furthermore, 155 species of animals specifically mammals have been reported to be infected with Cryptosporidium parvum which is also known as C. parvum [7]. Among the 15 named species of Cryptosporidium infectious to non-human vertebrate hosts C. Baileyi, C. canis, C. felis, C. hominis, C meleagridis, C. muris, and C. parvum have been reported to also infect humans. The primary hosts for C. hominis are Humans, except for C. parvum, which is widespread in non-human hosts and is

the most frequently reported zoonotic species, the remaining species left have been reported primarily in immunocompromised or immunosuppressed humans [7]. The first Cryptosporidiosis outbreak that was widely known occurred in 1987[8] in Carrollton, Georgia. About 13,000 persons became sick as a result of the outbreak the disease. The main cause was traced to a large contaminated water system. In 1993, in Milwaukee area, Wisconsin, a massive outbreak of the disease occurred, causing approximately 400,000 people to fell sick as a result of contaminated drinking water in one of the two treatment plants serving the Milwaukee area [6]. Many biological processes such as gestation and infection take time to complete and in the case of persistently transmitted diseases, it takes time for a virus to invade a cell and spread throughout the host and the process is called time delays [9]. To the best of our knowledge, no model has been considered to systematically analyze the dynamics of Cryptosporidiosis disease with Beddington-DeAngelis incidence and time delays in terms of incubations periods. Therefore, motivated by the above discussion into account, in this paper, Motivated by references [10, 11] and the above discussion, in this paper, we extend the model in [12] by introducing two time delays accounting for incubation periods to explore their effect on the dynamics of the disease. So we propose and analyze a mathematical delayed model of Cryptosporidiosis disease dynamics in humans and animals population with the Beddington-DeAngelis incidence. The rest of the paper is organized as follows. In section 2, the mathematical model with two delays is described and formulated. In section 3, the positivity and boundedness of solutions are investigated. Section 4 deals with the computation of the threshold parameter ($\mathscr{R}_{ha}$) and the mathematical analysis of the model including the local stability of the disease-free and endemic equilibrium points together with the existence of Hopf-bifurcation. In section 5, numerical simulations are performed to support theoretical results. The last section presents a brief discussion and conclusions of the paper.

2. Assumption and Formulation of the Model

It is assumed that Cryptosporidium does not spread from person to person. The human population is classified into three subclasses: susceptible, infective and recovered represented by $H_S(t)$, $H_I(t)$ and $H_R(t)$, respectively, and the animal population is classified into three subclasses: susceptible, infective and recovered, represented by $A_S(t)$, $A_I(t)$ and $A_R(t)$ respectively.

In order to construct the model with time delays the following assumptions have been made:

(i) All new recruitment and new borns of the human population (the animal population) are susceptible with rate Λ_H (Λ_A respectively).

(ii) The cryptosporidium protozoan is only contagious from an infected animal to a susceptible human, and from an infected animal to a susceptible animal. But it is not contagious from an infected human to a susceptible human.

(iii) The incidence rate between the susceptible human an infective animal depends not only on their numbers at the previous moment $(t-\tau_1)$ but also on the probability which the infective human survived natural death (having the death rate μ_H); Similarly the incidence rate between the susceptible animal and the infective animal depends not only on their numbers at previous moment $(t-\tau_2)$ but also on the probability which the infective animal population survived natural death (having the death rate μ_A). Here $\tau_1 \geq 0$ ($\tau_2 \geq 0$) is a time delay describing the incubation period of cryptosporidium protozoan on human population (animal population respectively).

(iv) The incidence rate between the susceptible human and infective animal, and between the susceptible animal and infective animal is assumed to be the Beddington-DeAngelis reponse.

(v) Infected human(infected animal) can be recovered and the recovered human(recovered animal) has permanent immunity.

On the basis of these assumptions, the following mathematical model of Cryptosporidiosis with incubation periods has been formulated:

$$
(1)\qquad \begin{cases}
\frac{dH_S(t)}{dt} = \Lambda_H - \mu_H H_S(t) - \frac{\delta_1 H_S(t) A_I(t)}{1+m_1 H_S(t)+m_2 A_I(t)} \\
\frac{dH_I(t)}{dt} = \frac{\delta_1 e^{-\mu_H \tau_1} H_S(t-\tau_1) A_I(t-\tau_1)}{1+m_1 H_S(t-\tau_1)+m_2 A_I(t-\tau_1)} - (\alpha_H + \mu_H + \beta_H) H_I(t) \\
\frac{dH_R(t)}{dt} = \beta_H H_I(t) - \mu_H H_R(t) \\
\frac{dA_S(t)}{dt} = \Lambda_A - \mu_A A_S(t) - \frac{\delta_2 A_S(t) A_I(t)}{1+m_3 A_S(t)+m_4 A_I(t)} \\
\frac{dA_I(t)}{dt} = \frac{\delta_2 e^{-\mu_A \tau_2} A_S(t-\tau_2) A_I(t-\tau_2)}{1+m_3 A_S(t-\tau_2)+m_4 A_I(t-\tau_2)} - (\alpha_A + \mu_A + \beta_A) A_I(t) \\
\frac{dA_R(t)}{dt} = \beta_A A_I(t) - \mu_A A_R(t)
\end{cases}
$$

Here $\delta_1(\delta_2)$ is the contact rate between the susceptible human and the infective animal (between the susceptible animal and the infective animal respectively); $\mu_H(\mu_A)$ is the natural death rate of the human population(the animal population respectively); $\beta_H(\beta_A)$ is the human recovery rate(the animal recovery rate respectively); The time $\tau_1(\tau_2)$ is the incubation period of cryptosporidium protozoan on human population(animal population respectively); $m_i(i=1,2,3,4)$ are the parameters that measure the inhibitory effect;$H_S(t-\tau_1)$ and $A_I(t-\tau_1)$ denotes the numbers of susceptible human population and infected animal population respectively at time $t-\tau_1$; $A_S(t-\tau_2)$ and $A_I(t-\tau_2)$ stands for the numbers of the susceptible(infectious respectively) animal at the time $t-\tau_2$; $e^{-\mu_H\tau_1}(e^{-\mu_A\tau_2})$ is the probability which the infected human(animal) survives to time t(with the death rate $-\mu_H,-\mu_A$ respectively). All other parameters are assumed to be positive. The initial conditions for system (1) take the form:

$$H_S(\theta)=\phi_1, H_I(\theta)=\phi_2, H_R(\theta)=\phi_3, A_S(\theta)=\phi_4, A_I(\theta)=\phi_5, A_R(\theta)=\phi_6, \tag{2}$$

$\theta\in[-\tau,0]$ where τ= $max\{\tau_1,\tau_2\}$, $\phi_i(\theta)\geq 0$, $i=1,2,3,4,5,6$. Here

$$\phi=(\phi_1(\theta),\phi_2(\theta),\phi_3(\theta),\phi_4(\theta),\phi_5(\theta),\phi_6(\theta))\in \mathbf{C}\left([-\tau,0],\mathbf{R}^6_+\right), \tag{3}$$

the banach space of continuous functions mapping from the interval $[-\tau,0]$ to $\mathbf{R}^6_+$,

$$\mathbf{R}^6_+=\{(H_S,H_I,H_R,A_S,A_I,A_R): H_S\geq 0, H_I\geq 0, H_R\geq 0, A_S\geq 0, A_I\geq 0, A_R\geq 0\}. \tag{4}$$

For a biological meaning, we futher assume that $\phi_i(0)>0$, $i=1,2,3,4,5,6$.

3. Model Properties

3.1. Nonnegativity of solutions. In this subsection, we must prove that at $t\geq 0$ all solutions of the model system (1) are positive.

Theorem 3.1 Each component of the solution of the delay system (1) with initial conditions (2) is positive for all $t\geq 0$.

Proof. From the first equation of the delay system (1), we get:

$$\begin{aligned} H_S(t) = {} & H_S(0)exp\left[\int_0^t -\left\{\mu_H+\frac{\delta_1 A_I(u)}{1+m_1H_S(u)+m_2A_I(u)}du\right\}\right] \\ & +\Lambda_H\int_0^t exp\left[\int_\theta^t -\left\{\mu_H+\frac{\delta_1 A_I(u)}{1+m_1H_S(u)+m_2A_I(u)}du\right\}\right]d\theta \end{aligned} \tag{5}$$

then $H_S(t) > 0$. From the second equation of delay system (1) we get:

$$H_I(t) = e^{-(\alpha_H+\mu_H+\beta_H)t}\left[H_I(0) + \int_0^t \frac{\delta_1 e^{-\mu_H\tau_1} H_S(u-\tau_1)A_I(u-\tau_1)}{1+m_1H_S(u-\tau_1)+m_2A_I(u-\tau_1)}du\right] \tag{6}$$

then $H_I(t) > 0$. From the third equation of delay system (1) we get:

$$H_R(t) = e^{-\mu_H t}\left[H_R(0) + \beta_H\int_0^t H_I(u)e^{\mu_H u}du\right] \tag{7}$$

then $H_R(t) > 0$. From the fourth equation of delay system (1) we get:

$$\begin{aligned} A_S(t) = A_S(0)exp\left[\int_0^t -\left\{\mu_A + \frac{\delta_2 A_I(u)}{1+m_3A_S(u)+m_4A_I(u)}du\right\}\right] \\ + \Lambda_A\int_0^t exp\left[\int_\theta^t -\left\{\mu_A + \frac{\delta_2 A_I(u)}{1+m_3A_S(u)+m_4A_I(u)}du\right\}\right]d\theta \end{aligned} \tag{8}$$

then $A_S(t) > 0$. From the fifth equation of delay system (1) wet get:

$$A_I(t) = e^{-(\alpha_A+\mu_A+\beta_A)t}\left[A_I(0) + \int_0^t \frac{\delta_2 e^{-\mu_A\tau_2} A_S(u-\tau_2)A_I(u-\tau_2)}{1+m_3H_S(u-\tau_2)+m_4A_I(u-\tau_2)}du\right] \tag{9}$$

then $A_I(t) > 0$. From the last equation of delay system (1) we get:

$$A_R(t) = e^{-\mu_A t}\left[A_R(0) + \beta_A\int_0^t A_I(u)e^{\mu_A u}du\right] \tag{10}$$

then $A_R(t) > 0$. Therefore $H_S > 0$, $H_I > 0$, $H_I > 0$, $A_S > 0$, $A_I > 0$, $A_R > 0$, for all $t \geq 0$. □

3.2. Boundedness of solutions. In this subsection, we must prove that at $t \geq 0$ all solutions of the model system (1) are bounded.

Theorem 3.2 Each component of the solution of the delay system (1) with initial condition (2) are bounded for all $t \geq 0$.

Proof. From the delay system (1), we define

$$H(t) = H_S(t-\tau_1) + H_I(t) + H_R(t) \tag{11}$$

and

$$A(t) = A_S(t-\tau_2) + A_I(t) + A_R(t). \tag{12}$$

Therefore,

$$\begin{aligned}\frac{dH(t)}{dt} &= \Lambda_H - \mu_H H_S(t-\tau_1) - \frac{\delta_1 H_S(t-\tau_1)A_I(t-\tau_1)}{1+m_1 H_S(t-\tau_1)+m_2 A_I(t-\tau_1)} \\ &+ \frac{\delta_1 e^{-\mu_H \tau_1} H_S(t-\tau_1)A_I(t-\tau_1)}{1+m_1 H_S(t-\tau_1)+m_2 A_I(t-\tau_1)} - (\alpha_H+\mu_H+\beta_H)H_I(t) \\ &+ \beta_H H_I(t) - \mu_H H_R(t)\end{aligned} \tag{13}$$

so,

$$\begin{aligned}\frac{dH(t)}{dt} &\leq \Lambda_H - \mu_H H_S(t-\tau_1) - \frac{\delta_1 H_S(t-\tau_1)A_I(t-\tau_1)}{1+m_1 H_S(t-\tau_1)+m_2 A_I(t-\tau_1)} \\ &+ \frac{\delta_1 H_S(t-\tau_1)A_I(t-\tau_1)}{1+m_1 H_S(t-\tau_1)+m_2 A_I(t-\tau_1)} - (\alpha_H+\mu_H)H_I(t) - \mu_H H_R(t) \\ &\leq \Lambda_H - \mu_H H_S(t-\tau_1) - \mu_H H_I(t) - \mu_H H_R(t) \\ &= \Lambda_H - \mu_H H(t)\end{aligned} \tag{14}$$

By standard comparison theorem, we get:

$$\lim_{t\to\infty} SupH(t) \leq \frac{\Lambda_H}{\mu_H}. \tag{15}$$

Again

$$\begin{aligned}\frac{dA(t)}{dt} &= \Lambda_A - \mu_A A_S(t-\tau_2) - \frac{\delta_2 A_S(t-\tau_2)A_I(t-\tau_2)}{1+m_3 A_S(t-\tau_2)+m_4 A_I(t-\tau_2)} \\ &+ \frac{\delta_2 e^{-\mu_A \tau_2} A_S(t-\tau_2)A_I(t-\tau_2)}{1+m_3 A_S(t-\tau_2)+m_4 A_I(t-\tau_2)} - (\alpha_A+\mu_A+\beta_A)A_I(t) \\ &+ \beta_A A_I(t) - \mu_A A_R(t)\end{aligned} \tag{16}$$

so,

$$\begin{aligned}\frac{dA(t)}{dt} &\leq \Lambda_A - \mu_A A_S(t-\tau_2) - \frac{\delta_2 A_S(t-\tau_2)A_I(t-\tau_2)}{1+m_3 A_S(t-\tau_2)+m_4 A_I(t-\tau_2)} \\ &+ \frac{\delta_2 A_S(t-\tau_2)A_I(t-\tau_2)}{1+m_3 A_S(t-\tau_2)+m_4 A_I(t-\tau_2)} - (\alpha_A+\mu_A)A_I(t) - \mu_A A_R(t) \\ &\leq \Lambda_A - \mu_A A_S(t-\tau_2) - \mu_A A_I(t) - \mu_A A_R(t) \\ &= \Lambda_A - \mu_A A(t)\end{aligned} \tag{17}$$

By standard comparison theorem, we get:

$$\lim_{t\to\infty} SupA(t) \leq \frac{\Lambda_A}{\mu_A}. \tag{18}$$

This completes the proof of the theorem 3.2 and then each component of the solution of the delay system (1) with initial condition (2) is bounded for all $t \geq 0$. □

Therefore the feasible solution set of the delayed system (1) is

(19)
$$\Omega = \Omega_1 \times \Omega_2 = \left\{ (H_S(t), H_I(t), H_R(t), A_S(t), A_I(t), A_R(t)) \in \mathbf{R}^6_+ : 0 \leq H(t) \leq \frac{\Lambda_H}{\mu_H}; 0 \leq A(t) \leq \frac{\Lambda_A}{\mu_A} \right\}$$

where

$$\Omega_1 = \left\{ (H_S(t), H_I(t), H_R(t)) \in \mathbf{R}^3_+ : 0 \leq H_S(t) + H_I(t) + H_R(t) \leq \frac{\Lambda_H}{\mu_H} \right\} \tag{20}$$

and

$$\Omega_2 = \left\{ (A_S(t), A_I(t), A_R(t)) \in \mathbf{R}^3_+ : 0 \leq A_S(t) + A_I(t) + A_R(t) \leq \frac{\Lambda_A}{\mu_A} \right\} \tag{21}$$

4. Model analysis

4.1. Disease-free equilibrium point (DFE). Assume that $\lim_{t\to+\infty} H_S(t-\tau_1) = \lim_{t\to+\infty} H_S(t)$, $\lim_{t\to+\infty} H_I(t-\tau_1) = \lim_{t\to+\infty} H_I(t)$, $\lim_{t\to+\infty} A_S(t-\tau_2) = \lim_{t\to+\infty} A_S(t)$, $\lim_{t\to+\infty} A_I(t-\tau_1) = \lim_{t\to+\infty} A_I(t)$. The DFE E_0 is obtained by setting the right hand side of equations in the delayed system (1) to zero and letting

$$H_I(t) = H_R(t) = A_I(t) = A_R(t) = 0 \tag{22}$$

that is

$$\frac{dH_S(t)}{dt} = \frac{dH_I(t)}{dt} = \frac{dH_R(t)}{dt} = \frac{dA_S(t)}{dt} = \frac{dA_I(t)}{dt} = \frac{dA_R(t)}{dt} \tag{23}$$

we obtain:

$$E_0 = \left(\frac{\Lambda_H}{\mu_H}, 0, 0, \frac{\Lambda_A}{\mu_A}, 0, 0 \right) \tag{24}$$

4.2. Basic reproduction number($\mathscr{R}_{ha}$). As we know, the basic reproduction number,($\mathscr{R}_{ha}$) is the expected number of secondary infections arising from one newly infected individual introduced into a healthy population that determines the dynamic behaviour of the model. There are several methods for calculating ($\mathscr{R}_{ha}$), but for this work, we follow the idea of [13]. Firstly, we

rewrite the delayed model system (1) considering only the equations for diseased population. That is

$$(25)\quad \begin{cases} \dfrac{dH_I(t)}{dt} = \dfrac{\delta_1 e^{-\mu_H \tau_1} H_S(t-\tau_1) A_I(t-\tau_1)}{1+m_1 H_S(t-\tau_1)+m_2 A_I(t-\tau_1)} - (\alpha_H + \mu_H + \beta_H) H_I(t) \\ \dfrac{dA_I(t)}{dt} = \dfrac{\delta_2 e^{-\mu_A \tau_2} A_S(t-\tau_2) A_I(t-\tau_2)}{1+m_3 A_S(t-\tau_2)+m_4 A_I(t-\tau_2)} - (\alpha_A + \mu_A + \beta_A) A_I(t) \end{cases}$$

Next, the linearization of the system (25) at the disease-free equilibrium $E_0 = \left(\frac{\Lambda_H}{\mu_H}, 0, 0, \frac{\Lambda_A}{\mu_A}, 0, 0\right)$ is

(26)
$$\begin{pmatrix} \frac{dH_I(t)}{dt} \\ \frac{dA_I(t)}{dt} \end{pmatrix} = \begin{pmatrix} -(\alpha_H + \mu_H + \beta_H) & 0 \\ 0 & -(\alpha_A + \mu_A + \beta_A) \end{pmatrix} \begin{pmatrix} H_I(t) \\ A_I(t) \end{pmatrix} + \begin{pmatrix} 0 & \frac{\delta_1 e^{-\mu_H \tau_1} \Lambda_H}{\mu_H + m_1 \Lambda_H} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} H_I(t-\tau_1) \\ A_I(t-\tau_1) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{\mu_A + m_3 \Lambda_A} \end{pmatrix} \begin{pmatrix} H_I(t-\tau_2) \\ A_I(t-\tau_2) \end{pmatrix}$$

that is

$$(27)\quad \begin{pmatrix} \frac{dH_I(t)}{dt} \\ \frac{dA_I(t)}{dt} \end{pmatrix} = \begin{pmatrix} -(\alpha_H + \mu_H + \beta_H) H_I(t) \\ -(\alpha_A + \mu_A + \beta_A) A_I(t) \end{pmatrix} + \begin{pmatrix} \frac{\delta_1 e^{-\mu_H \tau_1} \Lambda_H}{\mu_H + m_1 \Lambda_H} A_I(t-\tau_1) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{\mu_A + m_3 \Lambda_A} A_I(t-\tau_2) \end{pmatrix}$$

that is

$$(28)\quad \begin{cases} \dfrac{dH_I(t)}{dt} = -(\alpha_H + \mu_H + \beta_H) H_I(t) + \dfrac{\delta_1 e^{-\mu_H \tau_1} \Lambda_H}{\mu_H + m_1 \Lambda_H} A_I(t-\tau_1) \\ \dfrac{dA_I(t)}{dt} = -(\alpha_A + \mu_A + \beta_A) A_I(t) + \dfrac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{\mu_A + m_3 \Lambda_A} A_I(t-\tau_2) \end{cases}$$

Let $H_I(0)$ and $A_I(0)$ be the numbers of each diseased at $t = 0$, $H_I(t)$ and $A_I(t)$ be the remaining population of each class at time t, respectively. equation one and two of system (28) give respectively

$$(29)\quad H_I(t) = H_I(0) e^{-(\alpha_H + \mu_H + \beta_H) t}$$

and

$$(30)\quad A_I(t) = A_I(0) e^{-(\alpha_A + \mu_A + \beta_A) t}.$$

Then, from the system (28), the total number of newly infected in diseased populations H_I and A_I is obtained as follows:

$$\begin{aligned}\bar{H}_I &= \int_{\tau_1}^{\infty} \frac{\delta_1 e^{-\mu_H \tau_1}\Lambda_H}{\mu_H + m_1\Lambda_H} A_I(t-\tau_1)dt \\ &= \int_{\tau_1}^{\infty} \frac{\delta_1 e^{-\mu_H \tau_1}\Lambda_H}{\mu_H + m_1\Lambda_H} A_I(0)e^{-(\alpha_A+\mu_A+\beta_A)t}dt \\ &= \lim_{q\to\infty}\int_{\tau_1}^{q} \frac{\delta_1 e^{-\mu_H \tau_1}\Lambda_H}{\mu_H + m_1\Lambda_H} A_I(0)e^{-(\alpha_A+\mu_A+\beta_A)t}dt \\ &= \frac{\delta_1 e^{-\mu_H \tau_1}\Lambda_H}{(\mu_H + m_1\Lambda_H)(\alpha_A+\mu_A+\beta_A)} A_I(0)\end{aligned} \tag{31}$$

$$\begin{aligned}\bar{A}_I &= \int_{\tau_2}^{\infty} \frac{\delta_2 e^{-\mu_A \tau_2}\Lambda_A}{\mu_A + m_3\Lambda_A} A_I(t-\tau_2)dt \\ &= \int_{\tau_2}^{\infty} \frac{\delta_2 e^{-\mu_A \tau_2}\Lambda_A}{\mu_A + m_3\Lambda_A} A_I(0)e^{-(\alpha_A+\mu_A+\beta_A)t}dt \\ &= \lim_{q\to\infty}\int_{\tau_2}^{q} \frac{\delta_2 e^{-\mu_A \tau_2}\Lambda_A}{\mu_A + m_3\Lambda_A} A_I(0)e^{-(\alpha_A+\mu_A+\beta_A)t}dt \\ &= \frac{\delta_2 e^{-\mu_A \tau_2}\Lambda_A}{(\mu_A + m_3\Lambda_A)(\alpha_A+\mu_A+\beta_A)} A_I(0)\end{aligned} \tag{32}$$

then, it follows that

$$\begin{pmatrix}\bar{H}_I \\ \bar{A}_I\end{pmatrix} = \begin{pmatrix}0 & \frac{\delta_1 e^{-\mu_H \tau_1}\Lambda_H}{(\mu_H+m_1\Lambda_H)(\alpha_A+\mu_A+\beta_A)} \\ 0 & \frac{\delta_2 e^{-\mu_A \tau_2}\Lambda_A}{(\mu_A+m_3\Lambda_A)(\alpha_A+\mu_A+\beta_A)}\end{pmatrix}\begin{pmatrix}H_I(0) \\ A_I(0)\end{pmatrix} \tag{33}$$

the following 2×2 matrix denoted by

$$M = \begin{pmatrix}0 & \frac{\delta_1 e^{-\mu_H \tau_1}\Lambda_H}{(\mu_H+m_1\Lambda_H)(\alpha_A+\mu_A+\beta_A)} \\ 0 & \frac{\delta_2 e^{-\mu_A \tau_2}\Lambda_A}{(\mu_A+m_3\Lambda_A)(\alpha_A+\mu_A+\beta_A)}\end{pmatrix} \tag{34}$$

is the next infection operator and the basic reproduction number $\mathscr{R}_{ha}$ is the spectral radius of the matrix M. Letting λ be the eigenvalue of the matrix M leads to

$$\begin{vmatrix}-\lambda & \frac{\delta_1 e^{-\mu_H \tau_1}\Lambda_H}{(\mu_H+m_1\Lambda_H)(\alpha_A+\mu_A+\beta_A)} \\ 0 & -\lambda + \frac{\delta_2 e^{-\mu_A \tau_2}\Lambda_A}{(\mu_A+m_3\Lambda_A)(\alpha_A+\mu_A+\beta_A)}\end{vmatrix} = 0 \tag{35}$$

that is

$$-\lambda\left(-\lambda + \frac{\delta_2 e^{-\mu_A \tau_2}\Lambda_A}{(\mu_A+m_3\Lambda_A)(\alpha_A+\mu_A+\beta_A)}\right) = 0 \tag{36}$$

Hence, the basic reproduction number $\mathscr{R}_{ha}$ of the delayed model system (1) is

$$\mathscr{R}_{ha} = \frac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{(\mu_A + m_3 \Lambda_A)(\alpha_A + \mu_A + \beta_A)} \tag{37}$$

4.3. Local stability of disease-free equilibrium point. The local stability of the disease-free equilibrium point E_0 is investigated by adopting the same technique as in [12].

Theorem 4.1 The disease-free equilibrium point is locally asymptotically stable if $\mathscr{R}_{ha} < 1$ and unstable if $\mathscr{R}_{ha} > 1$.

Proof. The linearization of the Jacobian matrix of delayed model system (1) at disease-free equilibrium point is given by:

(38)

$$J(E_0) = \begin{pmatrix} -\mu_H & 0 & 0 & 0 & \frac{-\delta_1 \Lambda_H}{\mu_H + m_1 \Lambda_H} & 0 \\ 0 & -(\alpha_H + \mu_H + \beta_H) & 0 & 0 & \frac{\delta_1 e^{-\mu_H \tau_1} \Lambda_H}{\mu_H + m_1 \Lambda_H} & 0 \\ 0 & \beta_H & -\mu_H & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_A & \frac{-\delta_2 \Lambda_A}{\mu_A + m_3 \Lambda_A} & 0 \\ 0 & 0 & 0 & 0 & \frac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{\mu_A + m_3 \Lambda_A} - (\alpha_A + \mu_A + \beta_A) & 0 \\ 0 & 0 & 0 & 0 & \beta_A & -\mu_A \end{pmatrix}$$

The characteristic equation of the Jacobian matrix (38) is given by:

$$(a - \lambda)(b - \lambda)(c - \lambda)(d - \lambda)(e - \lambda)(h - \lambda) = 0 \tag{39}$$

where $a = -\mu_H$, $b = -(\alpha_H + \mu_H + \beta_H)$, $c = -\mu_H$, $d = -\mu_A$, $e = \frac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{\mu_A + m_3 \Lambda_A} - (\alpha_A + \mu_A + \beta_A)$, $h = -\mu_A$.

From the characteristic equation (39), the eigenvalues of the Jacobian matrix of system (38) evaluated at E_0 are a, b, c, d,e and h. Clearly a, b, c,d and h are negative.

If $e < 0$, we get $\frac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{\mu_A + m_3 \Lambda_A} - (\alpha_A + \mu_A + \beta_A) < 0$, that is $\frac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{\mu_A + m_3 \Lambda_A} < \alpha_A + \mu_A + \beta_A$, that is $\mathscr{R}_{ha} < 1$. If $e > 0$ we have $\mathscr{R}_{ha} > 1$. Therefore E_0 is locally asymptotically stable if $\frac{\delta_2 e^{-\mu_A \tau_2} \Lambda_A}{\mu_A + m_3 \Lambda_A} < \alpha_A + \mu_A + \beta_A$ whenever $\mathscr{R}_{ha} < 1$, and unstable otherwise. □

Because the removed populations have no effect on the dynamics of H_S, H_I, A_S and A_I, the delayed system (1) can be decoupled to the following delayed system

$$
\begin{cases}
\frac{dH_S(t)}{dt} = \Lambda_H - \mu_H H_S(t) - \frac{\delta_1 H_S(t) A_I(t)}{1+m_1 H_S(t)+m_2 A_I(t)} \\
\frac{dH_I(t)}{dt} = \frac{\delta_1 e^{-\mu_H \tau_1} H_S(t-\tau_1) A_I(t-\tau_1)}{1+m_1 H_S(t-\tau_1)+m_2 A_I(t-\tau_1)} - (\alpha_H + \mu_H + \beta_H) H_I(t) \\
\frac{dA_S(t)}{dt} = \Lambda_A - \mu_A A_S(t) - \frac{\delta_2 A_S(t) A_I(t)}{1+m_3 A_S(t)+m_4 A_I(t)} \\
\frac{dA_I(t)}{dt} = \frac{\delta_2 e^{-\mu_A \tau_2} A_S(t-\tau_2) A_I(t-\tau_2)}{1+m_3 A_S(t-\tau_2)+m_4 A_I(t-\tau_2)} - (\alpha_A + \mu_A + \beta_A) A_I(t)
\end{cases} \tag{40}
$$

The initial condition of model (40) take the following form

$$
\begin{cases}
H_S(\theta) = \phi_1(\theta), H_I(\theta) = \phi_2(\theta), A_S(\theta) = \phi_3(\theta), A_I(\theta) = \phi_4(\theta) \\
\phi_i(\theta) \in \mathbf{C}\left([-\tau, 0], \mathbf{R}_+\right), i = 1,2,3,4
\end{cases} \tag{41}
$$

In what follows we are going to use the delayed system (40) with initial condition (41).

4.4. Global stability of the disease-free equilibrium point.

Theorem 4.2 The disease-free equilibrium E_0 of delayed system (1) is globally asymptotically stable when $\mathscr{R}_{ha} < 1$.

Proof. To show that the disease-free equilibrium point E_0 is globally asymptotically stable, it suffices to show that the disease-free equilibrium point $E_0^* = \left(\frac{\Lambda_H}{\mu_H}, 0, \frac{\Lambda_A}{\mu_A}, 0\right)$ of the delayed system (40) is globally asymptotically stable. To prove it, we firstly study the following animal-only system:

$$
\begin{cases}
\frac{dA_S(t)}{dt} = \Lambda_A - \mu_A A_S(t) - \frac{\delta_2 A_S(t) A_I(t)}{1+m_3 A_S(t)+m_4 A_I(t)} \\
\frac{dA_I(t)}{dt} = \frac{\delta_2 e^{-\mu_A \tau_2} A_S(t-\tau_2) A_I(t-\tau_2)}{1+m_3 A_S(t-\tau_2)+m_4 A_I(t-\tau_2)} - (\alpha_A + \mu_A + \beta_A) A_I(t)
\end{cases} \tag{42}
$$

In fact, the animal-only subsystem (42) are independent of the full system (40). system (42) always has a unique disease-free equilibrium $A_0 = \left(\frac{\Lambda_A}{\mu_A}, 0\right)$ is locally asymptotically stable if $\mathscr{R}_{ha} < 1$. Let now consider the global stability of A_0. To show that A_0 is globally asymptotically stable, we define a Lyapunov function as follows:

$$
L = \left(A_S - A_S^0 - A_S^0 ln \frac{A_S}{A_S^0}\right) e^{-\mu_A \tau_2} + A_I + \int_{t-\tau_2}^{t} \frac{\delta_2 e^{-\mu_A \tau_2} A_S A_I}{1 + m_3 A_S + m_4 A_I} du \tag{43}
$$

The derivative of L is

$$
\frac{dL}{dt} = \frac{-e^{-\mu_A \tau_2} \mu_A (A_S - A_S^0)^2}{A_S} - (\alpha_A + \mu_A + \beta_A)(1 - \mathscr{R}_{ha}) A_I \tag{44}
$$

since

$$\left\{(A_S,A_I)\in \mathbf{R}^2_+ : \frac{dL}{dt}=0\right\} = \left\{(A_S,A_I)\in \mathbf{R}^2_+ : A_S=A_S^0=\frac{\Lambda_A}{\mu_A}, A_I=A_I^0=0\right\}, \tag{45}$$

and by LaSalle'S Invariance Principle, we easily obtain that the disease-free equilibrium $A_0 = \left(\frac{\Lambda_A}{\mu_A},0\right)$ of subsystem (42) is globally asymptotically stable when $\mathscr{R}_{ha}<1$. To prove the global stability of E_0^*, we therefore only need to discuss model (40) at the disease-free steady state, that is the following system

$$\begin{cases} \frac{dH_S(t)}{dt} = \Lambda_H - \mu_H H_S(t) \\ \frac{dH_I(t)}{dt} = -(\alpha_H+\mu_H+\beta_H)H_I(t) \end{cases} \tag{46}$$

The solution of (46) is as follows

$$\begin{cases} H_S(t) = \frac{\Lambda_H}{\mu_H} + C_1 e^{-\mu_H t} \\ H_I(t) = C_2 e^{-(\alpha_H+\mu_H+\beta_H)t} \end{cases} \tag{47}$$

where C_1 and C_2 are constants. Obviously, we can see that $H_S \to \frac{\Lambda_H}{\mu_H}$ and $H_I \to 0$ when $t\to\infty$. Hence, the disease-free equilibrium E_0^* of system (40) is globally asymptotically stable. □

4.5. Existence of Endemic equilibrium point. Let $E_1 = (H_S^*,H_I^*,A_S^*,A_I^*)$ be the endemic equilibrium point of the model system (40). As in [14], the delay-dependency must vanish so that $\lim\limits_{t\to+\infty} H_S(t-\tau_1) = \lim\limits_{t\to+\infty} H_S(t) = H_S^*$, $\lim\limits_{t\to+\infty} H_I(t-\tau_1) = \lim\limits_{t\to+\infty} H_I(t) = H_I^*$, $\lim\limits_{t\to+\infty} A_S(t-\tau_2) = \lim\limits_{t\to+\infty} A_S(t) = A_S^*$, $\lim\limits_{t\to+\infty} A_I(t-\tau_2) = \lim\limits_{t\to+\infty} A_I(t) = A_I^*$, such that at equilibrium, we obtain

$$\begin{cases} \Lambda_H - \mu_H H_S - \frac{\delta_1 H_S A_I}{1+m_1 H_S+m_2 A_I} = 0 \\ \frac{\delta_1 e^{-\mu_H\tau_1} H_S A_I}{1+m_1 H_S+m_2 A_I} - (\alpha_H+\mu_H+\beta_H)H_I = 0 \\ \Lambda_A - \mu_A A_S - \frac{\delta_2 A_S A_I}{1+m_3 A_S+m_4 A_I} = 0 \\ \frac{\delta_2 e^{-\mu_A\tau_2} A_S A_I}{1+m_3 A_S+m_4 A_I} - (\alpha_A+\mu_A+\beta_A)A_I = 0 \end{cases} \tag{48}$$

If $\mathscr{R}_{ha}>1$, the system (40) has a unique endemic equilibrium point given by

$E_1 = (H_S^*,H_I^*,A_S^*,A_I^*)$, where

$$A_S^* = \frac{\Lambda_A(\delta_2+m_4\mu_A\mathscr{R}_{ha}+m_4\Lambda_A\mathscr{R}_{ha})}{\mathscr{R}_{ha}(m_4\mu_A+\delta_2)(\mu_A+m_3\Lambda_A)-\delta_2 m_3\Lambda_A} > 0 \tag{49}$$

since $\mathscr{R}_{ha} > 1$ this implies $\mathscr{R}_{ha}(m_4\mu_A + \delta_2)(\mu_A + m_3\Lambda_A) - \delta_2 m_3\Lambda_A > (m_4\mu_A + \delta_2)(\mu_A + m_3\Lambda_A) - \delta_2 m_3\Lambda_A = \mu_A(m_4\mu_A + \delta_2) + m_3 m_4\Lambda_A\mu_A > 0$,

$$A_I^* = \frac{\mathscr{R}_{ha}(\mu_A + m_3\Lambda_A)(\Lambda_A - \mu_A A_S^*)}{\delta_2\Lambda_A} > 0 \tag{50}$$

since $\frac{dA_S}{dt} < \Lambda_A - \mu_A A_S$,

$$H_S^* = \frac{-(\mu_H(1+m_2A_I^*) + \delta_1 A_I^* - m_1\Lambda_H) + \sqrt{(\mu_H(1+m_2A_I^*) + \delta_1 A_I^* - m_1\Lambda_H)^2 + 4m_1\mu_H\Lambda_H(1+m_2A_I^*)}}{2m_1\mu_H} > 0 \tag{51}$$

$$H_I^* = \frac{\delta_2 e^{-\mu_H\tau_1} H_S^* A_I^*}{(1+m_1H_S^* + m_2A_I^*)(\alpha_H + \mu_H + \beta_H)} > 0 \tag{52}$$

4.6. Stability of the Endemic equilibrium point and Hopf-bifurcation. In this section, we investigate the local stability of the endemic equilibrium E_1 and existence of the local Hopt-bifurcation occurring at E_1 by choosing τ_1,τ_2 and $\tau = \max\{\tau_1, \tau_2\}$ as bifurcation parameters. To this end, we follow the Hopf-bifurcation theory in [15] by first computing transcendental equation. The expression for the transcendental equation is obtained by linearizing the model system (40) around the endemic equilibrium point E_1. We get the following system

$$\frac{dY(t)}{dt} = FY(t) + GY(t-\tau_1) + QY(t-\tau_2) \tag{53}$$

where F, G and Q are 4×4 matrix given by

$$F = [f_{ij}]$$

$$= \begin{pmatrix} \mu_H - \frac{\delta_1 A_I^*(t)[1+m_2A_I^*(t)]}{(1+m_1H_S^*(t)+m_2A_I^*(t))^2} & 0 & 0 & -\frac{\delta_1 H_S^*(t)[1+m_1H_S^*(t)]}{(1+m_1H_S^*(t)+m_2A_I^*(t))^2} \\ 0 & -(\alpha_H+\mu_H+\beta_H) & 0 & 0 \\ 0 & 0 & \mu_A - \frac{\delta_2 A_I^*(t)[1+m_4A_I^*(t)]}{(1+m_3A_S^*(t)+m_4A_I^*(t))^2} & -\frac{\delta_2 A_S^*(t)[1+m_3A_S^*(t)]}{(1+m_3A_S^*(t)+m_4A_I^*(t))^2} \\ 0 & 0 & 0 & -(\alpha_A+\mu_A+\beta_A) \end{pmatrix} \tag{54}$$

$$G = [g_{ij}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\delta_1 e^{-\mu_H\tau_1} A_I^*(t-\tau_1)[1+m_2A_I^*(t-\tau_1)]}{(1+m_1H_S^*(t-\tau_1)+m_2A_I^*(t-\tau_1))^2} & 0 & 0 & \frac{\delta_1 e^{-\mu_H\tau_1} H_S^*(t-\tau_1)[1+m_2H_S^*(t-\tau_1)]}{(1+m_1H_S^*(t-\tau_1)+m_2A_I^*(t-\tau_1))^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{55}$$

$$Q = [q_{ij}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\delta_2 e^{-\mu_A \tau_2} A_I^*(t-\tau_2)[1+m_4 A_I^*(t-\tau_2)]}{(1+m_3 A_S^*(t-\tau_2)+m_4 A_I^*(t-\tau_2))^2} & \frac{\delta_2 e^{-\mu_A \tau_2} A_S^*(t-\tau_2)[1+m_3 A_S^*(t-\tau_2)]}{(1+m_3 A_S^*(t-\tau_2)+m_4 A_I^*(t-\tau_2))^2} \end{pmatrix} \tag{56}$$

Clearly,F is the matrix of partial derivatives of the model system (40) with respect to the state variables without delays, G is the matrix of the partial derivatives of the model system (40) with respect to the state variables with incubation period τ_1 of cryptosporidium protozoan on human population and Q is the matrix of the partial derivatives of the model system (40) with respect to the state variables with incubation period τ_2 of cryptosporidium protozoan on animal population both evaluate at E_1. Then the resulting Jacobian matrix is found to be

$$\begin{aligned} J &= F + Ge^{-\lambda\tau_1} + Qe^{-\lambda\tau_2} \\ &= \begin{pmatrix} d_1 & 0 & 0 & d_2 \\ d_3 e^{(-\mu_H+\lambda)\tau_1} & d_4 & 0 & d_5 e^{(-\mu_H+\lambda)\tau_1} \\ 0 & 0 & d_6 & d_7 \\ 0 & 0 & d_8 e^{(-\mu_A+\lambda)\tau_2} & d_9 + d_1 0 e^{(-\mu_A+\lambda)\tau_2} \end{pmatrix} \end{aligned} \tag{57}$$

with

$$d_1 = -\mu_H - \frac{\delta_1 A_I^*(t)[1+m_2 A_I^*(t)]}{(1+m_1 H_S^*(t)+m_2 A_I^*(t))^2},\ d_2 = -\frac{\delta_1 H_S^*(t)[1+m_1 H_S^*(t)]}{(1+m_1 H_S^*(t)+m_2 A_I^*(t))^2} \tag{58}$$

$$d_3 = \frac{\delta_1 A_I^*(t-\tau_1)[1+m_2 A_I^*(t-\tau_1)]}{(1+m_1 H_S^*(t-\tau_1)+m_2 A_I^*(t-\tau_1))^2},\ d_4 = -(\alpha_H+\mu_H+\beta_H) \tag{59}$$

$$d_5 = \frac{\delta_1 H_S^*(t-\tau_1)[1+m_1 H_S^*(t-\tau_1)]}{(1+m_1 H_S^*(t-\tau_1)+m_2 A_I^*(t-\tau_1))^2},\ d_6 = -\mu_A - \frac{\delta_2 A_I^*(t)[1+m_4 A_I^*(t)]}{(1+m_3 A_S^*(t)+m_4 A_I^*(t))^2} \tag{60}$$

$$d_7 = -\frac{\delta_2 A_S^*(t)[1+m_3 A_S^*(t)]}{(1+m_3 A_S^*(t)+m_4 A_I^*(t))^2},\ d_8 = \frac{\delta_2 A_I^*(t-\tau_2)[1+m_4 A_I^*(t-\tau_2)]}{(1+m_3 A_S^*(t-\tau_2)+m_4 A_I^*(t-\tau_2))^2} \tag{61}$$

$$d_9 = -(\alpha_A+\mu_A+\beta_A),\ d_{10} = \frac{\delta_2 A_S^*(t-\tau_2)[1+m_3 A_S^*(t-\tau_2)]}{(1+m_3 A_S^*(t-\tau_2)+m_4 A_I^*(t-\tau_2))^2} \tag{62}$$

Hence, after some calculation and after making some simplification and arrangement, the transcendental equation of the system (25) is explicitly written as

(63)
$$\psi(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0)e^{-(\mu_A+\lambda)\tau_2} = 0$$

where

(64) $$a_3 = -d_1 - d_4 - d_6 - d_9$$

(65) $$a_2 = d_1d_4 + d_1d_6 + d_4d_6 + d_1d_9 + d_4d_9 + d_6d_{10}$$

(66) $$a_1 = -d_1d_4d_6 - d_1d_4d_9 - d_1d_6d_9 - d_4d_6d_9$$

(67) $$a_0 = -d_1d_4d_6d_9$$

(68) $$b_3 = -d_{10}$$

(69) $$b_2 = d_1d_{10} + d_4d_{10} + d_6d_{10} - d_7d_8$$

(70) $$b_1 = -d_1d_4d_{10} - d_1d_6d_{10} - d_4d_6d_{10} + d_1d_7d_8 + d_4d_7d_8$$

(71) $$b_0 = -d_1d_4d_7d_8 - d_1d_4d_6d_{10}$$

Now the stability of the endemic equilibrium E_1 is investigated by considering two different cases which are: no delays and delay only for τ_2.

Case 1: Delay-free system ($\tau_2 = 0$)

The characteristic equation corresponding to E_1 is of the form

(72) $$-(j-\lambda)(l-\lambda)[-\lambda^2 + (m+n)\lambda - mn + pq] = 0$$

where

(73) $$j = d_1,\ l = d_4,\ m = d_6,\ n = d_9 + d_{10}e^{(-\mu_A+\lambda)\tau_2},\ p = d_7,\ q = d_8e^{(-\mu_A+\lambda)\tau_2}$$

It is easily to know that two eigenvalues

$$\lambda_1 = -\mu_H - \frac{\delta_1 A_I^*(t)[1+m_2 A_I^*(t)]}{(1+m_1 H_S^*(t)+m_2 A_I^*(t))^2},\ and\ \lambda_2 = -(\alpha_H+\mu_H+\beta_H) \tag{74}$$

of (72) are negative. So we only need to examine the last two eigenvalues of (72), which are determined by

$$\lambda^2 + N_1\lambda + N_2\lambda e^{-\lambda\tau_2} + N_3 + N_4 e^{-\lambda\tau_2} = 0 \tag{75}$$

where

$$N_1 = d_6 + d_9,\ N_2 = d_{10},\ N_3 = d_6 d_9,\ N_4 = d_6 d_{10} + d_7 d_8. \tag{76}$$

If $\tau_2 = 0$, then (75) becomes

$$\lambda^2 + (N_1+N_2)\lambda + (N_3+N_4) = 0 \tag{77}$$

All the eigenvalues obviously have negative real parts if

$$m_4 > m_3\ and\ A_S^* > \frac{m_4[m_3(\mu_A+\alpha_A)-(m_4\mu_A+\delta_2)]}{(\mu_A+\alpha_A)(m_3\Lambda_A+m_4+\mu_A+\alpha_A)-\Lambda_A(m_4\mu_A+\delta_2)} \tag{78}$$

Theorem 4.3 In the absence of delays that is $\tau_2 = 0$, the endemic equilibrium point E_1 is locally asymptotically stable for $\mathscr{R}_{ha} > 1$ if and only if

$$m_4 > m_3\ and\ A_S^* > \frac{m_4[m_3(\mu_A+\alpha_A)-(m_4\mu_A+\delta_2)]}{(\mu_A+\alpha_A)(m_3\Lambda_A+m_4+\mu_A+\alpha_A)-\Lambda_A(m_4\mu_A+\delta_2)}. \tag{79}$$

Case 2: Delay for ($\tau_2 > 0$)

In this case, the transcendental equation (63) is

$$\psi(\lambda, e^{-\lambda\tau_2}) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0)e^{-(\mu_A+\lambda)\tau_2} = 0 \tag{80}$$

Multiplying both sides of equation (80) by $e^{(\mu_A+\lambda)\tau_2}$ we obtain

$$\psi(\lambda, e^{-\lambda\tau_2}) = b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 + (\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0)e^{\mu_A\tau_2}e^{\lambda\tau_2} = 0 \tag{81}$$

that is

$$\psi(\lambda, e^{-\lambda\tau_2}) = b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 + (c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0)e^{\lambda\tau_2} = 0 \tag{82}$$

where

$$\sigma = e^{\mu_A\tau_2},\ c_4 = \sigma,\ c_3 = \sigma a_3,\ c_2 = \sigma a_2,\ c_1 = \sigma a_1,\ c_0 = \sigma a_0. \tag{83}$$

Equation (82) is a transcendental equation in λ having infinitely many roots. Possible Hopf-bifurcation occurs if purely imaginary roots of the transcendental equation (82) exist. Now substituting $\lambda = iw$ into (82) and expressing the exponential in terms of trigonometric ratios, it follows that

$$-b_3w^3i-b_2w^2+b_1wi+b_0+(c_4w^4-c_3w^3i-c_2w^2+c_1wi+c_0)(cos(\tau_2w)+isin(\tau_2w)=0 \tag{84}$$

that is

$$\begin{aligned} &-b_2w^2+b_0+(c_4w^4-c_2w^2+c_0)cos(\tau_2w)+(c_3w^3-c_1w)sin(\tau_2w)\\ &+[-b_3w^3+b_1w+(c_1w-c_3w^3)cos(\tau_2w)+(c_4w^4-c_2w^2+c_0)sin(\tau_2w)]i=0 \end{aligned} \tag{85}$$

that is

$$\begin{cases} (c_4w^4-c_2w^2+c_0)cos(\tau_2w)+(c_3w^3-c_1w)sin(\tau_2w)=b_2w^2-b_0\\ (c_1w-c_3w^3)cos(\tau_2w)+(c_4w^4-c_2w^2+c_0)sin(\tau_2w)=b_3w^3-b_1w \end{cases} \tag{86}$$

solving system (86) we get

$$sin(\tau_2w)=\frac{(c_1w-c_3w^3)(b_2w^2-b_0)-(b_3w^3-b_1w)(c_4w^4-c_2w^2+c_0)}{(c_3w^3-c_1w)(-c_3w^3+c_1w)-(c_4w^4-c_2w^2+c_0)(c_4w^4-c_2w^2+c_0)} \tag{87}$$

$$cos(\tau_2w)=\frac{(c_4w^4-c_2w^2+c_0)(b_2w^2-b_0)-(c_3w^3-c_1w)(b_3w^3-b_1w)}{(c_4w^4-c_2w^2+c_0)(c_4w^4-c_2w^2+c_0)-(-c_3w^3+c_1w)(c_3w^3-c_1w)} \tag{88}$$

that is

(89)
$$sin(\tau_2w)=\frac{b_3c_4w^7+(c_3b_2-b_3c_2-b_1c_4)w^5+(c_3b_0-c_1b_2+b_3c_0+b_1c_2)w^3+(c_1b_0-b_1c_0)w}{c_4^2w^8+(c_3^2-2c_4c_2)w^6+(2c_4c_0-2c_3c_1+c_2^2)w^4+(c_1^2-2c_2c_0)w^2+c_0^2}$$

(90)
$$cos(\tau_2w)=\frac{(c_4b_2-c_3b_3)w^6+(-c_4b_0-c_2b_2+c_3b_1+c_1b_3)w^4+(c_2b_0+c_0b_2-c_1b_1)w^2-c_0b_0}{c_4^2w^8+(c_3^2-2c_4c_2)w^6+(2c_4c_0-2c_3c_1+c_2^2)w^4+(c_1^2-2c_2c_0)w^2+c_0^2}$$

that is

$$sin(\tau_2w)=\frac{e_7w^7+e_5w^5+e_3w^3+e_1w}{f_8w^8+f_6w^6+f_4w^4+f_2w^2+f_0} \tag{91}$$

and

$$cos(\tau_2w)=\frac{e_6w^6+e_4w^4+e_2w^2+e_0}{f_8w^8+f_6w^6+f_4w^4+f_2w^2+f_0} \tag{92}$$

where

$$e_7 = b_3c_4,\ e_6 = c_4b_2 - c_3b_3,\ e_5 = c_3b_2 - b_3c_2 - b_1c_4,\ e_4 = -c_4b_0 - c_2b_2 + c_3b_1 + c_1b_3 \tag{93}$$

$$e_3 = c_3b_0 - c_1b_2 + b_3c_0 + b_1c_2,\ e_2 = c_2b_0 + c_0b_2 - c_1b_1,\ e_1 = c_1b_0 - b_1c_0,\ e_0 = -c_0b_0 \tag{94}$$

$$f_8 = c_4^2,\ f_6 = c_3^2 - 2c_4c_2,\ f_4 = 2c_4c_0 + c_2^2 - 2c_3c_1,\ f_2 = c_1^2 - 2c_2c_0,\ f_0 = c_0^2. \tag{95}$$

Using the fundamental Pythagorean trigonometric identity, we obtain

$$M_{16}w^{16} + M_{14}w^{14} + M_{12}w^{12} + M_{10}w^{10} + M_8w^8 + M_6w^6 + M_4w^4 + M_2w^2 + M_0 = 0 \tag{96}$$

where

(97)

$$M_{16} = f_8^2,\ M_{14} = 2f_8f_6 - e_7^2,\ M_{12} = f_6^2 + 2f_8f_4 - e_6^2 - 2e_7e_5,\ M_{10} = 2(f_8f_2 + f_6f_4 - e_7e_3 - e_6e_4) - e_5^2$$

(98)

$$M_8 = 2(f_8f_0 + f_6f_2 - e_7e_1 - e_5e_3 - e_6e_2) + f_4^2 - e_4^2,\ M_6 = 2(f_8f_0 + f_4f_2 - e_5e_1 - e_6e_0 - e_4e_2) - e_3^2$$

$$M_4 = 2(f_4f_0 - e_3e_1 - e_4e_0) + f_2^2 - e_2^2,\ M_2 = 2(f_2f_0 - e_2e_0) - e_1^2,\ M_0 = f_0^2 - e_0^2. \tag{99}$$

Let $w^2 = \eta$, then equation (96) becomes

$$P(\eta) = M_{16}\eta^8 + M_{14}\eta^7 + M_{12}\eta^6 + M_{10}\eta^5 + M_8\eta^4 + M_6\eta^3 + M_4\eta^2 + M_2\eta + M_0 = 0 \tag{100}$$

which represents the Hopf frequency. Next, the following assumption is made for the Hopf frequency equation.

(T_1): Equation (100) has at least one positive root. Now, if condition (T_1) holds,equation (100) has a positive root η_0 and the equation (96) definitely has a pair of purely imaginary roots $\pm iw_0 = \pm i\sqrt{\eta_0}$. Consequently the corresponding critical value of τ_2 at which a Hopf-bifurcation occurs is found to be

$$\tau_{2_k} = \frac{1}{w_k} arccos\left(\frac{e_6w_k^6 + e_4w_k^4 + e_2w_k^2 + e_0}{f_8w_k^8 + f_6w_k^6 + f_4w_k^4 + f_2w_k^2 + f_0}\right) + \frac{2k\pi}{w_k},\ (k = 0,1,2,3,...) \tag{101}$$

Hence, (w_0, τ_{2_k}) is the solution of the equation (82) meaning that there exists a pair of purely imaginary roots $\lambda = \pm iw_0$ for equation (82) when $\tau_2 = \tau_{2_k}$. Especially, for $k = 0$ one get

$$\tau_{2_0} = \frac{1}{w_0} arccos\left(\frac{e_6 w_0^6 + e_4 w_0^4 + e_2 w_0^2 + e_0}{f_8 w_0^8 + f_6 w_0^6 + f_4 w_0^4 + f_2 w_0^2 + f_0}\right) \tag{102}$$

Differentiating both sides of equation (82) with respect to τ_2 it follows that

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{3b_3\lambda^2 + 2b_2\lambda + b_1 + (4c_4\lambda^3 + 3c_3\lambda^2 + 2c_2\lambda + c_1)e^{\lambda\tau_2}}{(c_4\lambda^5 + c_3\lambda^4 + c_2\lambda^3 + c_1\lambda^2 + c_0\lambda)e^{\lambda\tau_2}} + \frac{\tau_2}{\lambda} \tag{103}$$

Now, taking the real component of $\left(\frac{d\lambda}{d\tau_2}\right)^{-1}$ at $\tau_2 = \tau_{2_k}$ with $\lambda = iw$ we get

$$Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{X_1Y_1 + X_2Y_2}{Y_1^2 + Y_2^2} \tag{104}$$

where

$$X_1 = -3b_3w_0^3 + b_1 + (c_1 - 3c_3w_0^2)cos(w_0\tau_{2_0}) + (4c_4w_0^3 - 2c_2w_0)sin(w_0\tau_{2_0}) \tag{105}$$

$$X_2 = 2b_2w_0 + (2c_2w_0 - 4c_4w_0^3)cos(w_0\tau_{2_0}) + (c_1 - 3c_3w_0^2)sin(w_0\tau_{2_0}) \tag{106}$$

$$Y_1 = (-c_3w_0^4 - c_1w_0^2)cos(w_0\tau_{2_0}) + (c_2w_0^3 - c_4w_0^5 - c_0w_0)sin(w_0\tau_{2_0}) \tag{107}$$

$$Y_2 = (c_2w_0^3 - c_4w_0^5 - c_0w_0)cos(w_0\tau_{2_0}) + (-c_3w_0^4 - c_1w_0^2)sin(w_0\tau_{2_0}) \tag{108}$$

Obviously, if worth noting if the condition $(T_2) : X_1Y_1 + X_2Y_2 \neq 0$ holds, then $Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\tau_2 \equiv \tau_{2_0}} \neq 0$. Therefore, following the above analysis and the Hopf-bifurcation theory in (Hassard et al., 1981), one have the theorem below

Theorem 4.4 Suppose that the conditions $(T_1) - (T_2)$ hold, then the endemic equilibrium $E_1 = (H_S^*, H_I^*, A_S^*, A_I^*)$ of the delayed model system (40) is locally asymptotically stable when $\tau_2 \in [0, \tau_{2_0})$ and the delayed model system (40) undergoes a Hopf-bifurcation at $E_1 = (H_S^*, H_I^*, A_S^*, A_I^*)$ when $\tau_2 = \tau_{2_0}$.

In the following table of the parameters most are assumed (due to the lack of data) while few are taken from the literature.

Table 2. Parameter values for the model (1)

Parameter	Value	Source of data
Λ_H	2000/36500 per day	[16]
μ_H	5.48×10^{-5} per day	[18,19]
α_H	0.001 per day	[18,19]
β_H	0.1 per day	[18,19]
δ_1	2×10^{-6} per day	[16]
Λ_A	1000/245 per day	[16,17]
μ_A	1/245 per day	[16,17]
α_A	1/400 per day	[16]
β_A	0.1 per day	Assumed
δ_2	5.1×10^{-4} per day	[16]
m_1	0.01	Assumed
m_2	0.03	Assumed
m_3	0.01	Assumed
m_4	0.01	Assumed
τ_1	8	Assumed
τ_2	12	Assumed

5. NUMERICAL SIMULATIONS

We performed numerical simulations of our proposed model (1) to support some of the analytical results. We use the set of parameters values given in Table 2 and the initial values of the model are set as: $H_S(0) = 1000, H_I(0) = 10, H_R(0) = 5, A_S(0) = 50, A_I(0) = 500, A_R(0) = 10$. We set the final time as $t_f = 120$ days. This was chosen on the basis of the assumption that a period of four month is enough for the disease spread. All simulations are done using MATLAB dde23.

5.1. Simulations of the effects of incubation delay τ_1 on the dynamics of the disease. Figure 1 shows that,the incubation period τ_1 which does not appear in the formula of the basic reproduction number has an effect on the dynamics of the disease.The endemic equilibrium E_1

of system (1) is locally asymptotically stable when the basic reproduction number $\mathscr{R}_{ha}$ is greater than unity. In fact, from Figure 1(b) we observe that the susceptible animals decrease and converges to a steady state to acquire endemic equilibrium level while in Figure 1(d), there is a slight ascent in the number of infected animals to a certain endemic level and a steep descent and converges to a steady state to acquire endemic equilibrium level. From Figure 1 (a) and 1 (c), we observe that the magnitude of the amplitude of oscillations for susceptible and infected humans are weak but have an small effect on the dynamics of the disease. The biological interpretation of this is that whenever the incubation period is $\tau_1 = \tau < \tau_{1_0}$ the disease can be contolled with ease.

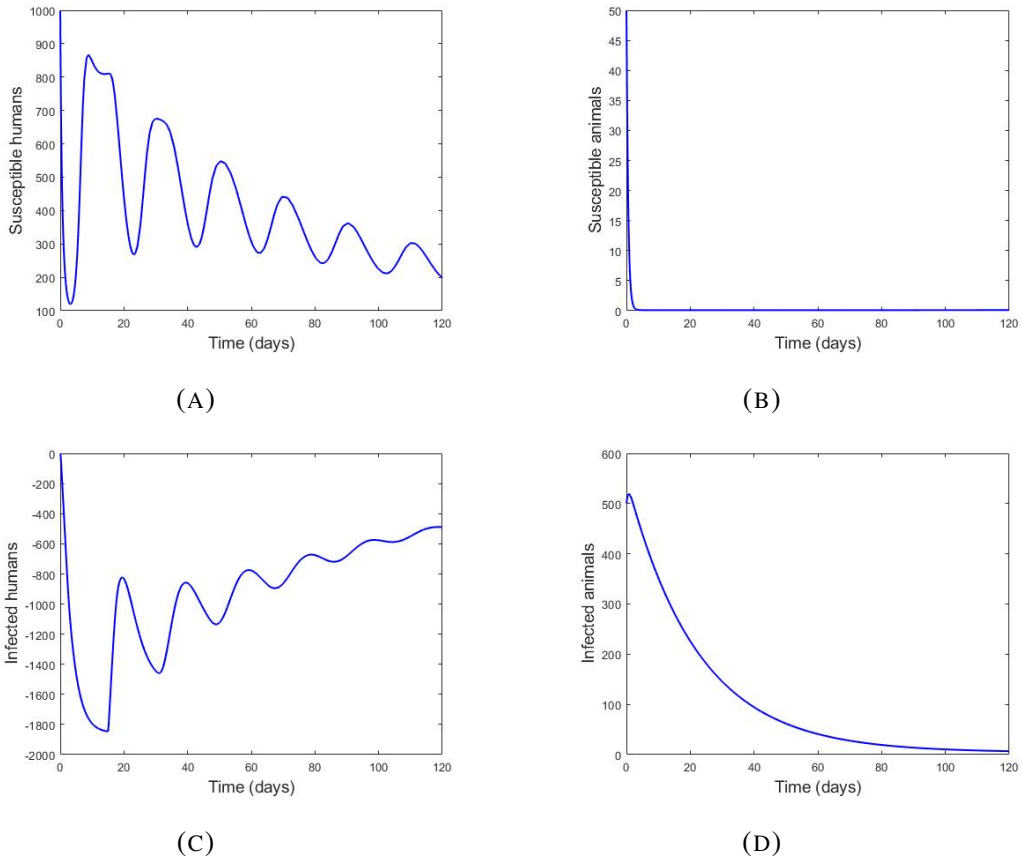
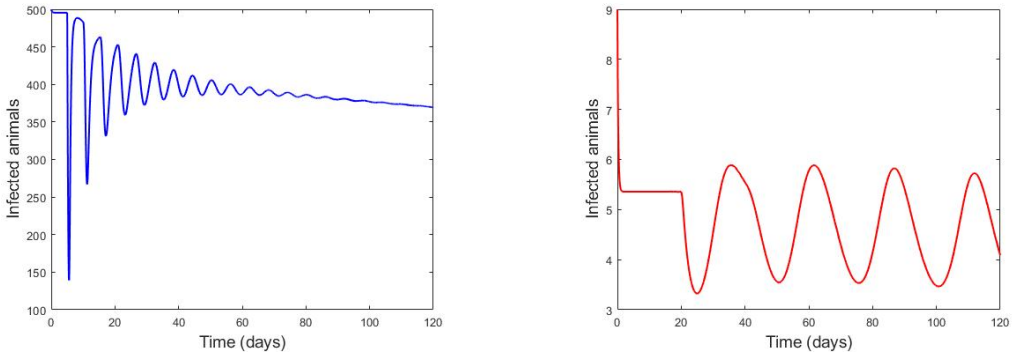


FIGURE 1. Graph showing the effects of transmission of incubation delay τ_1 on the dynamics of the disease.

5.2. Simulations for the local asymptotic stability of the endemic equilibrium and existence of Hopf-bifurcation with delay $\tau_2 > 0$. For $\tau_2 > 0$, a computation by means of Maple

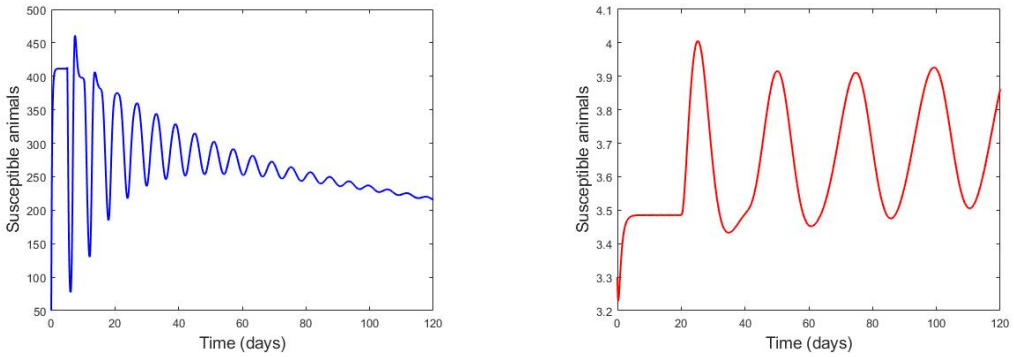
18 software gives $w_0 = 0.7139$ and $\tau_{2_0} = 15.8210$. Hence, according to Theorem 4.4, the positive endemic equilibrium E_1 is locally asymptotically stable when $\tau_{2_0} = 12 < 15.8210$ and this result is illustrated in Figure 2(a). However, when the incubation delay $\tau_{2_0} = 12$ passes through the critical value $\tau_{2_0} = 15.8210$; then the positive endemic equilibrium E_1 loses its stability and model (1) undergoes a Hopf-bifurcation at E_1: That is, the family of periodic solutions of the model (1) bifurcate from the positive endemic equilibrium E_1 and this property is illustrated in Figure 2(b). In this case, the disease will be out of control.



(A) (B)

FIGURE 2. Figure 2 (a) is the graph of solutions of the model (1) where the endemic equilibrium point E_1 is locally asymptotically stable when $\tau_{2_0} = 12 < 15.8210$. Figure 2 (b) shows the occurrence of Hopf-bifurcation at the endemic equilibrium point E_1 when $\tau_2 = 17 > 15.8210 = \tau_{2_0}$.

5.3. Simulations for the local asymptotic stability of the endemic equilibrium and existence of Hopf-bifurcation with delay $\tau_2 > 0$ in incubation period. For $\tau_2 > 0$, a computation by means of Maple 18 software gives $w_0 = 0.8239$ and $\tau_{2_0} = 14.8210$. Hence, according to Theorem 4.4, the positive endemic equilibrium E_1 is locally asymptotically stable when $\tau_{2_0} = 12 < 14.8210$ and this result is illustrated in Figure 3(a). However, when the incubation delay $\tau_{2_0} = 12$ passes through the critical value $\tau_{2_0} = 14.8210$; then the positive endemic equilibrium E_1 loses its stability and model (1) undergoes a Hopf-bifurcation at E_1: That is, the family of periodic solutions of the model (1) bifurcate from the positive endemic equilibrium E_1 and this property is illustrated in Figure 3(b). In this case, the disease will be out of control.



(A) (B)

FIGURE 3. Figure 3 (a) is the graph of solutions of the model (1) where the endemic equilibrium point E_1 is locally asymptotically stable when $\tau_{2_0} = 12 < 14.8210$. Figure 3 (b) shows the occurrence of Hopf-bifurcation at the endemic equilibrium point E_1 when $\tau_2 = 16 > 14.8210 = \tau_{2_0}$.

5.4. Simulations of the effects of incubation delay τ_2 on the dynamics of the disease. In Figure 5, the spiral-shaped graphical representation of the number of humans susceptible to and infected with animal-transmitted cryptosporidiosis suggests complex dynamics of disease emergence and control over time. The observed variations reflect distinct phases of introduction from the animal reservoir, with seasonal cycles potentially influencing transmission. Negative values on the y-axis indicate a reduction in the number of infected humans, suggesting effective interventions. However, the spiral shape highlights the need for continued monitoring, as resurgences or changes in transmission dynamics could occur.

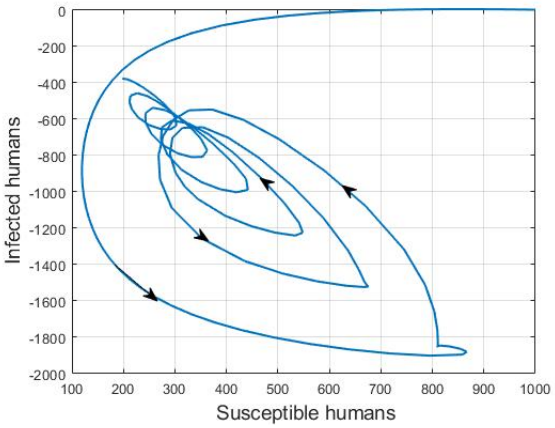


FIGURE 4

FIGURE 5. Graph showing the effects of incubation period τ_2 on the dynamics of the disease.

6. DISCUSSIONS AND CONCLUSIONS

In this paper, we have formulated a delayed model of Cryptosporidiosis disease in human and animal population extending the model proposed in [mon article], by introducing two delays times in terms of incubation periods. Compared with the Cryptosporidiosis disease model in [mon article], we mainly explore the effect of the time delays on its dynamic nature. Compared with other zoonotic disease models, we incorporate the incubation periods in the equations of infected humans and infected animals which appears to be consistent with the reality. In the numerical simulations, we have shown that when the value of the incubation delay τ_2 are below the corresponding critical values τ_{2_0}, the positive endemic equilibrium E_1 of model (1) is locally asymptotically stable under some certain conditions. The result in this case implies that the spread of Cryptosporidiosis disease in human an animal population can be controlled with ease. Conversely, once the value of the delay τ_2 passes through the critical value τ_{2_0}, the positive endemic equilibrium E_1 loses its stability and a Hopf–bifurcation occurs with a family of periodic solutions bifurcating from E_1. The result in this case means that Cryptosporidiosis disease will persist in the human and animal population and will be out of control. The Hopf bifurcation

phenomenon in figure 3 and figure 4 manifests that the human and animal population will remain in an oscillatory behavior as long as the incubation period is greater that its corresponding critical value. We can observe from equation (37) of the basic reproduction number $\mathscr{R}_{ha}$ that only delay τ_2 is present, this does not mean that τ_1 has no effect on the dynamics of the disease, we can see Figure 1, so delayed models are more realistic and mathematically, the delays affect the dynamics and the stability of disease-free equilibrium. Thus, if the incubation periods can be reduced to an level for which $\mathscr{R}_{ha<1}$, the Cryptosporidiosis disease can be eliminated from the human and animal population. Some authors considered that in zoonotic diseases control, taking into account the incubation periods is crucial to design appropriate policy towards controlling and preventing the disease. Hence, in the near future we will come with a paper to derive the optimal control problem for our proposed delayed model.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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