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QUALITATIVE BEHAVIOR OF A TWO-DIMENSIONAL DISCRETE-TIME PLANT-HERBIVORE MODEL

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Abstract. An important area of study in mathematical ecology is the coexistence of populations of herbivores and plants. The plant-herbivore model is a dynamic relationship where plants serve as a basic food source for herbivores, whereas herbivores are essential in the formation of plant populations and communities. In this paper, we study the dynamical behavior of a discrete-time plant-herbivore model. Existence and stability of equilibria are studied. Moreover, we analyse the transcritical bifurcation of the proposed system using bifurcation theory. The Neimark-Sacker bifurcation is also shown. The bifurcation diagrams are presented. Finally, some numerical examples are provided to support our theoretical results. The used approaches are powerful and useful to be applied for more nonlinear models.

Keywords: discrete plant-herbivore model; stability analysis; center manifold theorem; Neimark-Sacker bifurcation; transcritical bifurcation.

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1. Introduction

Mathematical modelling has recently attracted the attention of theoretical ecologists and mathematical biologists as a result of its rich dynamics and varied applications [3, 10, 14, 18,

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32]. Discrete-time and continuous-time models are two popular types of mathematical models used to simulate population dynamics. In contrast to differential equations, which are used to describe continuous-time models, difference equations are used to describe discrete-time models. In the case of non-overlapping generations, plant-herbivore interactions are frequently described using discrete-time models: such generations have a specific life span and their old generations are replaced by new generations after a regular interval of time. In addition, compared to continuous-time models, discrete-time models offer richer dynamics. For instance, a discrete-time model with a single species can exhibit chaos and more complex dynamical behaviour, but chaos in a continuous-time model requires at least three species [29].

The interaction between plants and herbivores has been studied by many researchers. Din [13] investigated the dynamical behaviour of a plant-herbivore model including weak predator functional response. Period-doubling and Neimark-Sacker bifurcations were studied by Li et al.[28] for a plant-herbivore model that included plant toxicity in the functional response of plant-herbivore interactions. The interested reader is further directed to [1, 4, 7, 8, 9, 11, 12, 15, 16, 17, 21, 27] for some other discussions related to qualitative behaviour of plant-herbivore models.

A difference equation model for the plant-herbivore system was developed and analyzed by Allen et al. [2] in 1993. The population at the three states that are most important for the management and control of the apple twig borer and grape vine systems were taken into consideration when developing the model. After that, it was simplified to a discrete two-dimensional model with two forms: the exponential function form and the rational function form. In this study, we take into consideration the following discrete two-dimensional model.

$$\begin{cases} x_{n+1} = \dfrac{x_n}{\alpha(1+y_n)+\beta x_n}, \\ y_{n+1} = \gamma y_n(1+x_n), \end{cases} \tag{1.1}$$

where $x_n \geq 0$ and $y_n \geq 0$, respectively, represent the population biomass of the herbivore and the plant. The parameters α, β, and γ are all positive.

This research aims to find the main equilibrium points of system (1.1) and to investigate the dynamics and bifurcations of system (1.1) in the closed first quadrant R_+^2. The stability and bifurcation diagrams are presented. The paper is organized as follows. In section 2, we analyses the presence of steady-states and their regional asymptotic behaviour. Section 3 focuses on examining transcritical bifurcation about the boundary equilibrium point of system (1.1). Moreover, Neimark-Sacker bifurcation at positive equilibrium point of system (1.1) is covered in Section 3. In Section 4, numerical simulations are carried out to verify the theoretical discussion. A brief conclusion is given in the last section.

Definition 1.1. *Let $M = (x, y)$ be a fixed point of the system (1.1) with multipliers λ_1 and λ_2.*

(1) M is called a sink (locally asymptotic stable) if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.
(2) M is called a saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$, or if $|\lambda_1| > 1$ and $|\lambda_2| < 1$.
(3) M is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$.
(4) M is called a non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma 1.2. *Let $\rho(\lambda) = \lambda^2 - T\lambda + D$, where $\rho(1) > 0$. Moreover, λ_1 and λ_2 are the two roots of $\rho(\lambda) = 0$. Then,*

(1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $\rho(-1) > 0$ and $\rho(0) < 1$.
(2) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $\rho(-1) < 0$.
(3) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $\rho(-1) > 0$ and $\rho(0) > 1$.
(4) λ_1 and λ_2 are complex numbers and $\lambda_1 = |\lambda_2| = 1$ if and only if $|T| < 2$ and $\rho(0) = 1$.

2. Local Stability of Steady-States

In this section, we study the existence and stability of fixed points of the system (1.1) in R^2_+. The equilibrium points that represent the steady states of the system (1.1) can be derived by resolving the algebraic system:

$$x = \frac{x}{\alpha(1+y)+\beta x}, \quad y = \gamma y(1+x). \tag{2.1}$$

The algebraic system (2.1) accepts three solutions namely $O = (0,0)$, $E = (\frac{1-\alpha}{\beta}, 0)$ and $P = \left(\frac{1-\gamma}{\gamma}, \frac{\gamma(1+\beta)-(\alpha\gamma+\beta)}{\gamma\alpha}\right)$. These solutions represent the equilibrium points of the system (1.1). Results about the presence of equilibrium points according to the parameters' values are outlined as follows:

Lemma 2.1. *For system (1.1), we can have at most three equilibrium points:*

(1) The trivial equilibrium point O always exists;

(2) The boundary equilibrium point E exists if $0 < \alpha < 1$*;*

(3) The interior equilibrium point P exists if $\beta/(1+\beta-\alpha) < \gamma < 1$*, and* $0 < \alpha < 1$.

The biological interruption for the three equilibrium points is: The fixed point O depicts an environment devoid of both plant and herbivore. The circumstance where there is plant but no herbivore is represented by the equilibrium point E. The coexistence of a fixed nonzero number of plant and herbivore is referred to as P.

The first step in studying the stability of the equilibrium points of the system (1.1) is to find the Jacobian matrix. The Jacobian matrix J of the system (1.1) evaluated at any equilibrium point (x,y) is given by

$$J(x,y) = \begin{pmatrix} \dfrac{\alpha(1+y)}{(\alpha(1+y)+\beta x)^2} & -\dfrac{\alpha x}{(\alpha(1+y)+\beta x)^2} \\ \gamma y & \gamma(1+x) \end{pmatrix}. \tag{2.2}$$

The characteristic polynomial of J is

$$\rho(\lambda) = \lambda^2 - T\lambda + D,$$

where

$$\begin{aligned} T &= Tr(J(x,y)) = \frac{\alpha(1+y)}{(\alpha(1+y)+\beta x)^2} + \gamma(1+x), \\ D &= Det(J(x,y)) = \frac{\alpha\gamma(1+y)(1+x)+\alpha\gamma xy}{(\alpha(1+y)+\beta x)^2}. \end{aligned}$$

The stability of the equilibrium points of the system (1.1) is confirmed by 0the next Theorems under some conditions.

Theorem 2.2. *For the equilibrium point O, the following results are true:*

- *O is locally asymptotically stable (a sink point) if* $\alpha > 1$ *and* $0 < \gamma < 1$.
- *O is unstable saddle if* $0 < \alpha < 1$ *and* $0 < \gamma < 1$*, or if* $\alpha > 1$ *and* $\gamma > 1$.
- *O is a source if* $0 < \alpha < 1$ *and* $\gamma > 1$.
- *O is a non-hyperbolic if* $\alpha = 1$*, or if* $\gamma = 1$.

Proof. The Jacobian matrix (2.2) evaluated at equilibrium point $O = (0,0)$ is given by

$$J(O) = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \gamma \end{pmatrix},$$

the eigenvalues of $J(O)$ are $\lambda_1 = \frac{1}{\alpha}$ and $\lambda_2 = \gamma$. By using Definition 1.1, we can easily obtain the results of this Theorem. □

Theorem 2.3. *For the equilibrium point E of the system (1.1), the following statements are true:*

- *E is a stable sink if* $0 < \gamma < \beta/(1+\beta-\alpha)$.
- *E is never a source.*
- *E is a saddle point if* $\gamma > \beta/(1+\beta-\alpha)$.
- *E is a non-hyperbolic point* $\gamma = \beta/(1+\beta-\alpha)$.

Proof. The evaluation of $J(x,y)$ at $E = (\frac{1-\alpha}{\beta}, 0)$ gives

$$J(E) = \begin{pmatrix} \alpha & \dfrac{\alpha(1-\alpha)}{\beta} \\ 0 & \dfrac{\gamma(1+\beta-\alpha)}{\beta} \end{pmatrix}.$$

The roots of the characteristic polynomial of $J(E)$ are

$$\lambda_1 = \alpha < 1, \quad \lambda_2 = \frac{\gamma(1+\beta-\alpha)}{\beta}.$$

The findings of this Theorem are easily obtained by using Definition 1.1. □

Theorem 2.4. *For the equilibrium point P of the system (1.1), the following statements are true:*

(1) P is locally asymptotically stable (stable sink) if and only if

$$\gamma < \frac{2\beta}{1+\beta-\alpha}.$$

(2) P is unstable (source) if and only if

$$\gamma > \frac{2\beta}{1+\beta-\alpha}.$$

(3) The roots of equation $\rho(\lambda) = 0$ *are complex numbers with modulus one if and only if*

$$\alpha + \beta < 1 \text{ and } \gamma = \frac{2\beta}{1+\beta-\alpha}.$$

Proof. The Jacobian matrix $J(x,y)$ evaluated at the positive equilibrium $P = \left(\frac{1-\gamma}{\gamma}, \frac{(\gamma(1+\beta)-(\alpha\gamma+\beta)}{\gamma\alpha}\right)$ is given by

$$J(P) = \begin{pmatrix} 1+\beta-\dfrac{\beta}{\gamma} & -\dfrac{\alpha(1-\gamma)}{\gamma} \\ \dfrac{\gamma(1+\beta)-(\alpha\gamma+\beta)}{\alpha} & 1 \end{pmatrix}. \tag{2.3}$$

Hence, the characteristic polynomial of $J(P)$ is

$$\rho(\lambda) = \lambda^2 - \left(2+\beta-\frac{\beta}{\gamma}\right)\lambda + 2 + 3\beta - \alpha(1-\gamma) - \frac{2\beta}{\gamma} - \gamma(1+\beta). \tag{2.4}$$

From Equation (2.4), we can get

$$\rho(1) = \frac{(1-\gamma)(\gamma(1+\beta-\alpha)-\beta)}{\gamma}, \quad \rho(0) = 2+3\beta-\alpha(1-\gamma)-\frac{2\beta}{\gamma}-\gamma(1+\beta),$$

and

$$\rho(-1) = 5+4\beta-\frac{3\beta}{\gamma}-\alpha(1-\gamma)-\gamma(1+\beta).$$

From the existence conditions of the equilibrium points P, it is clear $\rho(1) > 0$ is always satisfied. Moreover, one can prove that $\rho(-1) > 0$ as follows:

$$\begin{aligned} \rho(-1) &= 5+4\beta-\frac{3\beta}{\gamma}-\alpha(1-\gamma)-\gamma(1+\beta), \\ &= 1+4\beta-\frac{3\beta}{\gamma}-\alpha(1-\gamma)-\gamma(1+\beta)+4, \\ &= (1-\gamma)(1-\alpha+\beta)-3(1-\gamma)(\frac{\beta}{\gamma})+4, \\ &> (1-\gamma)(\frac{\beta}{\gamma})-3(1-\gamma)(\frac{\beta}{\gamma})+4, \\ &= 2(2+\beta-\frac{\beta}{\gamma}) > 2(1+\alpha) > 0. \end{aligned}$$

Now, we need to investigate the sign of $\rho(0)-1$

$$\begin{aligned} \rho(0)-1 &= 2+3\beta-\frac{2\beta}{\gamma}-\alpha(1-\gamma)-\gamma(1+\beta)-1, \\ &= \frac{1}{\gamma}\left(2\gamma+3\gamma\beta-2\beta-\alpha\gamma(1-\gamma)-\gamma^2(1+\beta)-\gamma\right), \\ &= \frac{1}{\gamma}\left(-\gamma^2(1+\beta-\alpha)+\gamma(1+3\beta-\alpha)-2\beta\right), \end{aligned}$$

$$= \frac{-(1+\beta-\alpha)}{\gamma}\left(\gamma - \frac{2\beta}{1+\beta-\alpha}\right)(\gamma-1).$$

Based on the preceding discussion, we conclude that the type of the equilibrium point P is solely determined by the sign of $\rho(0)-1$. Thus, using Lemma 1.2 we obtain the results of this Theorem. □

3. Bifurcation Analysis

This section, which is based on theoretical investigations in Section 2 , examines the possibility of bifurcations around equilibria.

3.1. Transcritical Bifurcation of System (1.1). We first discuss the Transcritical bifurcation of the system (1.1) about equilibrium point E. Consider γ as a bifurcation parameter in the vicinity of the boundary equilibrium point $E = (\frac{1-\alpha}{\beta}, 0)$. From Theorem 2.3, we can see that if $\gamma_1 = \frac{\beta}{1-\alpha+\beta}$ holds then one of the eigenvalues about E is 1. We define the set $\mathscr{T}_B$ as follows

$$\mathscr{T}_B := \left\{(\alpha,\beta,\gamma_1) \in \mathrm{R}_+^{*^3} : \gamma_1 = \frac{\beta}{1-\alpha+\beta} > 0, \beta > 0, 0 < \alpha < 1\right\}.$$

Let $u_n = x_n - \left(\frac{1-\alpha}{\beta}\right)$ and $v_n = y_n$, then the equilibrium E of system (1.1) transforms into $O = (0,0)$. By calculating we get

$$\tag{3.1} \begin{cases} u_{n+1} = \dfrac{u_n + \frac{1-\alpha}{\beta}}{\alpha(1+v_n)+\beta(u_n+\frac{1-\alpha}{\beta})} - \left(\dfrac{1-\alpha}{\beta}\right), \\ v_{n+1} = \gamma v_n(1+u_n+\dfrac{1-\alpha}{\beta}). \end{cases}$$

Consider a small perturbation γ^* to the parameter γ (i.e. $\gamma = \gamma^* + \gamma_1$), with $0 <| \gamma^* |\ll 1$, then system (3.1) is perturbed into

$$\tag{3.2} \begin{cases} u_{n+1} = \dfrac{u_n + \frac{1-\alpha}{\beta}}{\alpha(1+v_n)+\beta(u_n+\frac{1-\alpha}{\beta})} - \left(\dfrac{1-\alpha}{\beta}\right), \\ v_{n+1} = (\gamma^*+\gamma_1)v_n(1+u_n+\dfrac{1-\alpha}{\beta}). \end{cases}$$

Letting $\gamma_{n+1}^* = \gamma_n^* = \gamma^*$, system (3.2) can be written as

$$\begin{cases} u_{n+1} = \dfrac{u_n + \frac{1-\alpha}{\beta}}{\alpha(1+v_n) + \beta(u_n + \frac{1-\alpha}{\beta})} - \left(\dfrac{1-\alpha}{\beta}\right), \\ v_{n+1} = (\gamma_n^* + \gamma_1) v_n (1 + u_n + \dfrac{1-\alpha}{\beta}), \\ \gamma_{n+1}^* = \gamma_n^*. \end{cases} \tag{3.3}$$

Expanding system (3.3) up to third order about $(u_n, v_n, \gamma_n^*) = (0,0,0)$ by Taylor series, we get

$$\begin{cases} u_{n+1} = a_{11}u_n + a_{12}v_n + a_{13}u_n^2 + a_{14}u_n v_n + a_{15}v_n^2 + a_{16}u_n^3 + \\ \qquad a_{17}u_n^2 v_n + a_{18}u_n v_n^2 + a_{19}v_n^3 + \mathscr{R}_4(u_n, v_n, \gamma_n^*), \\ v_{n+1} = a_{21}u_n + a_{22}v_n + a_{23}u_n^2 + a_{24}u_n v_n + b_{24}v_n\gamma_n^* + a_{25}v_n^2 + a_{26}u_n^3 + \\ \qquad a_{27}u_n^2 v_n + a_{28}u_n v_n^2 + b_{28}u_n v_n \gamma_n^* + a_{29}v_n^3 + \mathscr{R}_4(u_n, v_n, \gamma_n^*), \\ \gamma_{n+1}^* = \gamma_n^*, \end{cases} \tag{3.4}$$

where

$$a_{11} = \alpha, \quad a_{12} = \frac{\alpha(1-\alpha)}{\beta}, \quad a_{13} = -\alpha\beta, \quad a_{14} = -\alpha(1-2\alpha), \quad a_{15} = \frac{\alpha^2(1-\alpha)}{\beta},$$

$$a_{16} = \alpha\beta^2, \quad a_{17} = \alpha\beta(3\alpha-1), \quad a_{18} = \alpha^2(3\alpha-2), \quad a_{19} = -\frac{\alpha^3(1-\alpha)}{\beta},$$

$$a_{22} = 1, \quad a_{24} = \frac{4\beta}{1-\alpha+\beta}, \quad a_{21} = a_{23} = a_{25} = a_{26} = a_{27} = a_{28} = a_{29} = 0,$$

$$b_{24} = 2, \quad b_{28} = \frac{1}{2}.$$

Next, we create an invertible matrix $J'(E)$ as follows

$$J'(E) = \begin{pmatrix} \alpha & \frac{\alpha(1-\alpha)}{\beta} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $J'(E)$ are

$$\lambda_1 = \alpha, \quad \lambda_{2,3} = 1,$$

while the corresponding eigenvectors

$$V_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad V_2=\begin{pmatrix}\alpha\\\beta\\0\end{pmatrix},\quad V_3=\begin{pmatrix}0\\0\\1\end{pmatrix},$$

respectively. Let $T=(V_1,V_2,V_3)$, i.e.,

$$T=\begin{pmatrix}1&\alpha&0\\0&\beta&0\\0&0&1\end{pmatrix}, \text{ then } T^{-1}=\begin{pmatrix}1&-\frac{\alpha}{\beta}&0\\0&\frac{1}{\beta}&0\\0&0&1\end{pmatrix}.$$

The transformation $(u_n,v_n,\gamma_n^*)^T=T(X_n,Y_n,\delta_n)^T$ converts system (3.4) to

$$\begin{cases}X_{n+1}=\alpha X_n+F(X_n,Y_n,\delta_n)+\mathscr{R}_4(X_n,Y_n,\delta_n),\\Y_{n+1}=Y_n+G(X_n,Y_n,\delta_n)+\mathscr{R}_4(X_n,Y_n,\delta_n),\\\delta_{n+1}=\delta_n,\end{cases}\tag{3.5}$$

where

$$F(X_n,Y_n,\delta_n)=m_{13}X_n^2+m_{14}X_nY_n-\alpha m_{26}Y_n\delta_n+m_{15}Y_n^2+m_{16}X_n^3+m_{17}X_n^2Y_n+$$
$$m_{18}X_nY_n^2-\alpha m_{27}X_nY_n\delta_n-\alpha m_{28}Y_n^2\delta_n+m_{19}Y_n^3,$$
$$G(X_n,Y_n,\delta_n)=m_{24}X_nY_n+m_{25}Y_n^2+m_{26}Y_n\delta_n+m_{27}X_nY_n\delta_n+m_{28}Y_n^2\delta_n,$$

and

$$m_{13}=-\alpha\beta,\ m_{14}=\frac{2\alpha\beta((\alpha-1)(1-\alpha+\beta)-2)}{1-\alpha+\beta},\ m_{15}=\frac{\alpha^2\beta((2\alpha-1)(1-\alpha+\beta)-4)}{1-\alpha+\beta},$$
$$m_{16}=\alpha\beta^2,\ m_{17}=\alpha\beta^2(6\alpha-1),\ m_{18}=4\alpha^2\beta^2(3\alpha-1), m_{19}=4\alpha^3\beta^2(2\alpha-1),$$
$$m_{24}=\frac{4\beta}{1-\alpha+\beta},\ m_{25}=\frac{4\alpha\beta}{1-\alpha+\beta},\ m_{26}=2,\ m_{27}=\frac{1}{2},\ m_{28}=\frac{1}{2}\alpha.$$

By the center manifold theorem, there exists a center manifold that can be represented as follows:

$$X_n=h(Y_n,\delta_n)=s_1Y_n^2+s_2Y_n\delta_n+s_3\delta_n^2+\mathscr{R}_4(X_n,Y_n,\delta_n).\tag{3.6}$$

therefore,

$$X_{n+1}=\alpha X_n+F(h(Y_n,\delta_n),Y_n,\delta_n)+\mathscr{R}_4(X_n,Y_n,\delta_n).$$

Hence,

$$\begin{aligned} h(Y_{n+1},\delta_{n+1}) &= s_1 Y_{n+1}^2 + s_2 Y_{n+1}\delta_{n+1} + s_3\delta_{n+1}^2 + \mathscr{R}_4(X_n,Y_n,\delta_n) \\ &= s_1(Y_n + G(X_n,Y_n,\delta_n))^2 + s_2(Y_n + G(X_n,Y_n,\delta_n))\delta_{n+1} \\ &\quad + s_3\delta_{n+1}^2 + \mathscr{R}_4(X_n,Y_n,\delta_n). \end{aligned}$$

Next, we obtain the center manifold equation as follows

$$\begin{aligned} \alpha X_n + F(h(Y_n,\delta_n),Y_n,\delta_n) = s_1(Y_n + G(X_n,Y_n,\delta_n))^2 \\ + s_2(Y_n + G(X_n,Y_n,\delta_n))\delta_{n+1} + s_3\delta_{n+1}^2. \end{aligned} \tag{3.7}$$

By comparing the coefficients of the same order terms in equation (3.7), we end up with

$$s_1 = \frac{\alpha^2\beta((2\alpha-1)(1-\alpha+\beta)-4)}{1-\alpha+\beta}, \quad s_2 = -2\alpha, \quad s_3 = 0.$$

Therefore, we consider system (3.5) restricted to the center manifold as follows:

$$\begin{aligned} X_{n+1} &= f(Y_n,\delta_n) = Y_n + G(X_n,Y_n,\delta_n) + \mathscr{R}_4(X_n,Y_n,\delta_n), \\ &= Y_n + 2Y_n\delta_n + \frac{4\alpha\beta}{1-\alpha+\beta}Y_n^2 + \left(\frac{-8\alpha\beta}{1-\alpha+\beta} - \frac{1}{2}\alpha\right)Y_n^2\delta_n + \frac{4\beta}{1-\alpha+\beta}Y_n^3. \end{aligned}$$

In addition,

$$f(0,0) = 0, \quad \frac{\partial f(0,0)}{\partial Y_n} = 1, \quad \frac{\partial f(0,0)}{\partial \delta_n} = 0$$

$$\frac{\partial^2 f(0,0)}{\partial Y_n \partial \delta_n} = 2 \neq 0, \quad \frac{\partial^2 f(0,0)}{\partial^2 Y_n} = \frac{8\alpha\beta}{1-\alpha+\beta} \neq 0.$$

From the above analysis and theorem in [26, 25], we have the following Theorem.

Theorem 3.1. *System (1.1) undergoes a Transcritical bifurcation about the equilibrium point E when the parameter γ varies in a small neighborhood of γ_1.*

3.2. Neimark-Sacker Bifurcation of System (1.1). In this section, we will examine the Neimark-Sacker bifurcation of system (1.1) around the positive equilibrium point

$$P = \left(\frac{1-\gamma}{\gamma}, \frac{\gamma(1+\beta)-(\alpha\gamma+\beta)}{\gamma\alpha}\right).$$

The bifurcation is triggered by the parameter γ. We will determine conditions under which system (1.1) will have an equilibrium point with non-hyperbolic properties, consisting of a pair of complex conjugate eigenvalues with magnitude equal to one. The equilibrium point P will experience a Neimark-Sacker bifurcation if the parameters (α,β,γ) vary within the vicinity of the following set

$$\mathscr{N}_B = \left\{(\alpha,\beta,\gamma) \in \mathrm{R}_+^{*^3} : 0<\alpha<1, \beta>0, \alpha+\beta<1 \text{ and } \gamma = \frac{2\beta}{1+\beta-\alpha} > 0\right\}.$$

Now, we consider the change of variables $u_n = x_n - x_0$, $v_n = y_n - y_0$, which transforms fixed point $P = \left(\frac{1-\gamma}{\gamma}, \frac{\gamma(1+\beta)-(\alpha\gamma+\beta)}{\gamma\alpha}\right)$ to the origin $(u_n, v_n) = (0,0)$, and system (1.1) into

$$\begin{cases} u_{n+1} = \dfrac{u_n + x_0}{\alpha(1+v_n+y_0)+\beta(u_n+x_0)} - x_0, \\ v_{n+1} = \gamma(v_n+y_0)(1+u_n+x_0) - y_0. \end{cases} \tag{3.8}$$

Consider the parameter γ in a neighborhood of γ_0, i.e. $\gamma = \gamma_0 + \bar{\gamma}$. Next, we get

$$\begin{cases} u_{n+1} = \dfrac{u_n + x_0}{\alpha(1+v_n+y_0)+\beta(u_n+x_0)} - x_0 = F(u_n, v_n, \bar{\gamma}), \\ v_{n+1} = (\gamma_0+\bar{\gamma})(v_n+y_0)(1+u_n+x_0) - y_0 = G(u_n, v_n, \bar{\gamma}). \end{cases} \tag{3.9}$$

the characteristic equation associated with the linearization of the model (3.9) at $(u_n, v_n) = (0,0)$ is given by

$$\lambda^2 - T(\bar{\gamma})\lambda + D(\bar{\gamma}) = 0, \tag{3.10}$$

where

$$\begin{aligned} T(\bar{\gamma}) = &\ \left(2+\beta-\frac{\beta}{(\gamma+\bar{\gamma})}\right), \\ D(\bar{\gamma}) = &\ \left(2+3\beta-\alpha(1-(\gamma+\bar{\gamma}))-\frac{2\beta}{(\gamma+\bar{\gamma})}-(\gamma+\bar{\gamma})(1+\beta)\right). \end{aligned}$$

Equation (3.10) has a pair of complex conjugate roots with unit modulus since $(\alpha,\beta,\gamma) \in \mathscr{N}_B$. These roots are given by

$$\lambda_{1,2}(\bar{\gamma}) = \frac{T(\bar{\gamma}) \pm i\sqrt{4D(\bar{\gamma}) - T(\bar{\gamma})^2}}{2},$$

it follows that

$$|\lambda_{1,2}(\bar{\gamma})| = \sqrt{D(\bar{\gamma})} = \sqrt{\left(2+\beta-\alpha+(\alpha-1-\beta)\bar{\gamma}-\frac{2\beta(1+\beta-\alpha)}{2\beta+(1+\beta-\alpha)\bar{\gamma}}\right)},$$

and

$$|\lambda_{1,2}(0)| = \sqrt{D(0)} = 1,$$

which implies

$$\left(\frac{d\mid\lambda_1\mid}{d\bar{\gamma}}\right)_{\bar{\gamma}=0}=\left(\frac{d\mid\lambda_2\mid}{d\bar{\gamma}}\right)_{\bar{\gamma}=0}=\frac{(1+\beta-\alpha)(1-\beta-\alpha)}{2\beta}>0. \tag{3.11}$$

Moreover, we required that when $\bar{\gamma}=0$, $\lambda_{1,2}^{j}\neq 1$, $j=1,2,3,4$ which is equivalent to $T(0)\neq 0,1$, lead to

$$\gamma\neq\frac{\beta}{2+\beta},\quad \gamma\neq\frac{\beta}{1+\beta}. \tag{3.12}$$

Expanding F and G by Taylor series at $(u_n,v_n)=(0,0)$ to the third order allows us to deduce the normal form of system (3.9), which is represented by

$$\begin{cases} u_{n+1}=a_{11}u_n+a_{12}v_n+f(u_n,v_n),\\ v_{n+1}=a_{21}u_n+a_{22}v_n+g(u_n,v_n), \end{cases} \tag{3.13}$$

where

$$\begin{aligned} f(u_n,v_n)=\ & a_{13}u_n^2+a_{14}u_nv_n+a_{15}v_n^2+a_{16}u_n^3+a_{17}u_n^2v_n+a_{18}u_nv_n^2+a_{19}v_n^3+\\ & \mathscr{R}_{f,4}(u_n,v_n),\\ g(u_n,v_n)=\ & a_{23}u_n^2+a_{24}u_nv_n+a_{25}v_n^2+a_{26}u_n^3+a_{27}u_n^2v_n+a_{28}u_nv_n^2+a_{29}v_n^3+\\ & \mathscr{R}_{g,4}(u_n,v_n). \end{aligned}$$

and

$$a_{11}=\left(\frac{1+\beta+\alpha}{2}\right), a_{12}=\alpha\left(\frac{\alpha+\beta-1}{2\beta}\right), a_{13}=-\frac{\beta}{2}\left(1+\beta+\alpha\right),$$

$$a_{14}=\frac{-\alpha(3\beta+3\beta^2-\alpha+\alpha^2)}{\beta},$$

$$a_{15}=\frac{\alpha^2}{2}\left(\frac{1-\beta-\alpha}{\beta}\right),\quad a_{16}=\frac{\beta^2}{2}\left(1+\beta+\alpha\right),\quad a_{17}=\frac{\alpha}{2}(3\beta+2\beta^2+2\alpha\beta+\alpha-1),$$

$$a_{18}=\frac{\alpha^2}{2}(3\beta+2\beta^2+2\alpha\beta+\alpha-1), a_{19}=\frac{\alpha^3}{2}\left(\frac{1-\beta-\alpha}{\beta}\right), a_{21}=\left(\frac{\beta}{\alpha}\right), a_{22}=1,$$

$$a_{24}=\frac{4\beta}{1+\beta-\alpha}, a_{23}, a_{25}, a_{26}, a_{27}, a_{28}, a_{29}=0.$$

Now, let

$$J^{'}(P)=\begin{pmatrix} \frac{1+\beta+\alpha}{2} & \frac{\alpha(\alpha+\beta-1)}{2\beta}\\ \frac{\beta}{\alpha} & 1 \end{pmatrix}.$$

The eigenvalues of matrix $J^{'}(P)$ are

$$\lambda_{1,2}=\frac{3+\beta+\alpha}{4}\pm i\frac{\sqrt{(1-\beta-\alpha)(7+\beta+\alpha)}}{4}.$$

Next, we study the normal form of (3.13). Let $\kappa=\Im(\lambda_{1,2})$ and $\omega=\Re(\lambda_{1,2})$, we define

$$\mathscr{T}=\begin{pmatrix} a_{11} & 0 \\ \kappa-a_{11} & -\omega \end{pmatrix}=\begin{pmatrix} \frac{1+\beta+\alpha}{2} & 0 \\ \frac{1-\beta-\alpha}{4} & -\omega \end{pmatrix},$$

and consider the following transformation:

$$\begin{pmatrix} u \\ v \end{pmatrix}=\mathscr{T}\begin{pmatrix} X \\ Y \end{pmatrix}. \tag{3.14}$$

Hence, by applying the transformation (3.14) on Equation (3.13), we obtain

$$\begin{cases} X_{n+1}=\kappa X_n-\omega Y_n+\bar{f}(X_n,Y_n), \\ Y_{n+1}=\omega X_n+\kappa Y_n+\bar{g}(X_n,Y_n), \end{cases} \tag{3.15}$$

where

$$\begin{aligned} \bar{f}(X_n,Y_n)= \quad & \tfrac{a_{13}}{2a_{12}}u_n^2+\tfrac{2a_{14}}{a_{12}}u_nv_n+\tfrac{a_{15}}{2a_{12}}v_n^2+\tfrac{a_{16}}{6a_{12}}u_n^3+\tfrac{a_{17}}{2a_{12}}u_n^2v_n+\tfrac{a_{18}}{2a_{12}}u_nv_n^2+ \\ & \tfrac{a_{19}}{6a_{12}}v_n^3+\mathscr{R}_{\bar{f},4}(u_n,v_n), \\ \bar{g}(X_n,Y_n)= \quad & \tfrac{a_{13}(\kappa-a_{11})}{2a_{12}}u_n^2+\left(\tfrac{2a_{14}(\kappa-a_{11})}{a_{12}}-\tfrac{2a_{24}}{\omega}\right)u_nv_n+\tfrac{a_{15}(\kappa-a_{11})}{2a_{12}}v_n^2+\tfrac{a_{16}(\kappa-a_{11})}{6a_{12}}u_n^3 \\ & +\tfrac{a_{17}(\kappa-a_{11})}{2a_{12}}u_n^2v_n+\tfrac{a_{18}(\kappa-a_{11})}{2a_{12}}u_nv_n^2+\tfrac{a_{19}(\kappa-a_{11})}{6a_{12}}v_n^3+\mathscr{R}_{\bar{g},4}(u_n,v_n). \end{aligned}$$

From equation (3.14), we obtain $u_n=a_{11}X_n$ and $v_n=MX_n-\omega Y_n$, where $M=\left(\frac{1-\beta-\alpha}{4}\right)$. Thus, we get the following form

$$\begin{aligned} \bar{f}(X_n,Y_n)= \quad & b_{13}X_n^2-b_{14}X_nY_n+b_{15}Y_n^2+b_{16}X_n^3+b_{17}X_n^2Y_n-b_{18}X_nY_n^2-b_{19}Y_n^3+ \\ & \mathscr{R}_{\bar{f},4}(X_n,X_n), \\ \bar{g}(X_n,Y_n)= \quad & M(b_{13}-\tfrac{2a_{24}a_{11}}{\omega})X_n^2-(Mb_{14}-2a_{24}a_{11})X_nY_n+Mb_{15}Y_n^2+Mb_{16}X_n^3+ \\ & Mb_{17}X_n^2Y_n-Mb_{18}X_nY_n^2-Mb_{19}Y_n^3+\mathscr{R}_{\bar{g},4}(X_n,X_n), \end{aligned}$$

where

$$b_{13}=\frac{a_{13}a_{11}^2+4Ma_{14}a_{11}+M^2a_{15}}{2a_{12}}, \quad b_{14}=\frac{2a_{14}a_{11}+\omega Ma_{15}}{a_{12}}, \quad b_{15}=\frac{\omega^2a_{15}}{2a_{12}},$$

$$b_{16}=\frac{a_{16}a_{11}^3+3Ma_{17}a_{11}^2+3M^2a_{18}a_{11}+M^3a_{19}}{6a_{12}}, \quad b_{17}=\frac{\omega^2a_{18}a_{11}+\omega^2Ma_{19}}{2a_{12}},$$

$$b_{18}=\frac{\omega a_{17}a_{11}^2+2\omega Ma_{18}+\omega M^2a_{19}}{2a_{12}}, \quad b_{19}=\frac{\omega^3a_{19}}{6a_{12}}.$$

Furthermore,

$$\bar{f}_{XX}=\frac{a_{13}a_{11}^2+4Ma_{14}a_{11}+M^2a_{15}}{a_{12}}, \quad \bar{f}_{XYY}=-\frac{\omega a_{17}a_{11}^2+2\omega Ma_{18}+\omega M^2a_{19}}{a_{12}},$$

$$\bar{f}_{XXX}=\frac{a_{16}a_{11}^3+3Ma_{17}a_{11}^2+3M^2a_{18}a_{11}+M^3a_{19}}{a_{12}}, \quad \bar{f}_{XY}=-\frac{2a_{14}a_{11}+\omega Ma_{15}}{a_{12}},$$

$$\bar{f}_{XXY}=\frac{\omega^2a_{18}a_{11}+\omega^2Ma_{19}}{a_{12}}, \quad \bar{f}_{YY}=\frac{\omega^2a_{15}}{a_{12}}, \quad \bar{f}_{YYY}=-\frac{\omega^3a_{19}}{a_{12}},$$

$$\bar{g}_{XX}=M\bar{f}_{XX}-\frac{4Ma_{14}a_{11}}{\omega}, \quad \bar{g}_{XY}=M\bar{f}_{XY}+2a_{14}a_{11}, \quad \bar{g}_{XXX}=M\bar{f}_{XXX},$$

$$\bar{g}_{XYY}=M\bar{f}_{XYY}, \quad \bar{g}_{YY}=M\bar{f}_{YY}, \quad \bar{g}_{YYY}=M\bar{f}_{YYY}, \quad \bar{g}_{XXY}=M\bar{f}_{XXY}.$$

In order for (3.15) to undergo a Neimark-Sacker bifurcation, It is necessary for the discriminatory value L to be nonzero

$$L=\left(\left[Re(\lambda_2\xi_{21})-Re\left(\frac{(1-2\lambda_1)\nu_2^2}{1-\lambda}\xi_{20}\xi_{11}\right)-\frac{1}{2}\mid\xi_{11}|^2-|\xi_{02}\mid^2\right]\right)_{\bar{\gamma}=0},$$

where

$$\begin{aligned}
\xi_{20} &= \frac{1}{8}\left[\bar{f}_{XX}-\bar{f}_{YY}+2\bar{g}_{XY}+i\left(\bar{g}_{XX}-\bar{g}_{YY}-2\bar{f}_{XY}\right)\right],\\
\xi_{11} &= \frac{1}{4}\left[\bar{f}_{XX}+\bar{f}_{YY}+i\left(\bar{g}_{XX}+\bar{g}_{YY}\right)\right],\\
\xi_{02} &= \frac{1}{8}\left[\bar{f}_{XX}-\bar{f}_{YY}-2\bar{g}_{XY}+i\left(\bar{g}_{XX}-\bar{g}_{YY}+2\bar{f}_{XY}\right)\right],\\
\xi_{21} &= \frac{1}{16}\left[\bar{f}_{XXX}+\bar{f}_{XYY}+\bar{g}_{XXY}+\bar{g}_{YYY}+i\left(\bar{g}_{XXX}+\bar{g}_{XYY}-\bar{f}_{XXY}-\bar{f}_{YYY}\right)\right].
\end{aligned}$$

By calculation, we get

$$\begin{aligned}
\xi_{20} &= \frac{1}{8}\left(\frac{a_{13}a_{11}^2+4Ma_{14}a_{11}+M^2a_{15}+\omega^2a_{15}-4a_{14}a_{11}-2\omega Ma_{15}}{a_{12}}+4a_{14}a_{11}\right)+\\
&\quad \frac{i}{8}\left(\frac{Ma_{13}a_{11}^2+4M^2a_{14}a_{11}+M^3a_{15}-M\omega^2a_{15}+4a_{14}a_{11}+2\omega Ma_{15}}{a_{12}}-\frac{4Ma_{14}a_{11}}{\omega}\right),
\end{aligned}$$

$$\begin{aligned}\xi_{11} &= \frac{1}{4}\left(\frac{a_{13}a_{11}^2+4Ma_{14}a_{11}+M^2a_{15}+\omega^2a_{15}}{a_{12}}\right)+\\ &\quad \frac{i}{4}\left(\frac{Ma_{13}a_{11}^2+4M^2a_{14}a_{11}+M^3a_{15}+M\omega^2a_{15}}{a_{12}}-\frac{4Ma_{14}a_{11}}{\omega}\right),\end{aligned}$$

$$\begin{aligned}\xi_{02} &= \frac{1}{8}\left(\frac{a_{13}a_{11}^2+4Ma_{14}a_{11}+M^2a_{15}-\omega^2a_{15}+4Ma_{14}a_{11}+2\omega M^2a_{15}}{a_{12}}-4a_{14}a_{11}\right)+\\ &\quad \frac{i}{8}\left(\frac{Ma_{13}a_{11}^2+4M^2a_{14}a_{11}+M^3a_{15}-M\omega^2a_{15}-4a_{14}a_{11}-2\omega Ma_{15}}{a_{12}}-\frac{4Ma_{14}a_{11}}{\omega}\right)\end{aligned}$$

and

$$\begin{aligned}\xi_{21} &= \frac{1}{16}\left(\frac{a_{16}a_{11}^3+3Ma_{17}a_{11}^2+3M^2a_{18}a_{11}+M^3a_{19}-\omega a_{17}a_{11}^2-2\omega Ma_{18}-\omega M^2a_{19}}{a_{12}}+\right.\\ &\quad \left.\frac{\omega^2Ma_{18}a_{11}+\omega^2M^2a_{19}-\omega^3Ma_{19}}{a_{12}}\right)+\frac{i}{16}\left(\frac{Ma_{16}a_{11}^3+3M^2a_{17}a_{11}^2+3M^3a_{18}a_{11}}{a_{12}}\right.\\ &\quad \left.+\frac{M^4a_{19}-\omega Ma_{17}a_{11}^2-2\omega M^2a_{18}-\omega M^3a_{19}-\omega^2a_{18}a_{11}-\omega^2Ma_{19}+\omega^3a_{19}}{a_{12}}\right).\end{aligned}$$

We have the following conclusion as a result of the analysis above.

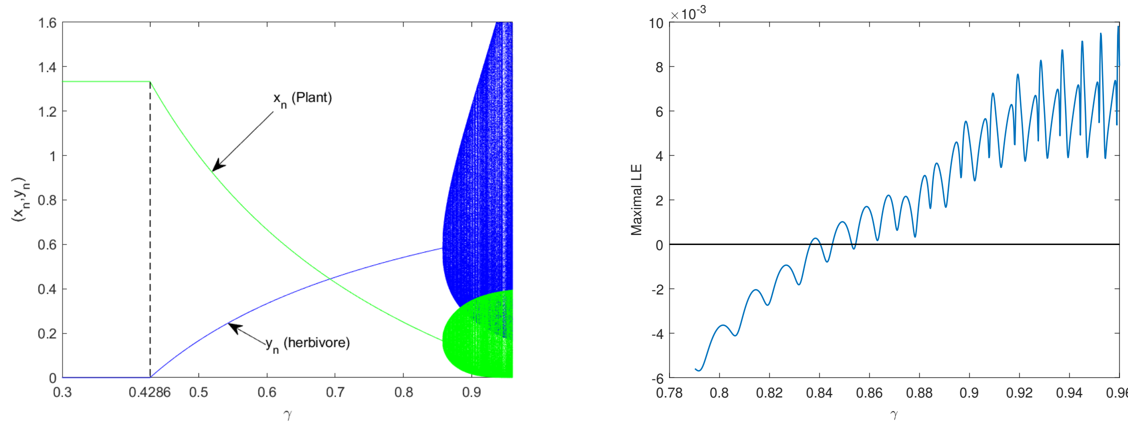
Theorem 3.2. *If $L \neq 0$ holds, then model (1.1) undergoes Neimark-Sacker Bifurcation about P as (α,β,γ) go through $\mathcal{N}_B$. Additionally, an attracting (resp. repelling) closed curve bifurcates from P if $L < 0$(resp. $L > 0$).*

4. Numerical Simulation

In this section, we present bifurcation diagrams, the Lyapunov exponents, and phase portraits generated by Matlab software for different parameter values. These diagrams support our theoretical findings and demonstrate some novel dynamical behaviours in system (1.1). Next, we consider some special cases of system (1.1). In first example, we verify the existance of the Transcritical bifurcation of system (1.1) at the equilibrium point E. Second example shows local asymptotic stability of the positive equilibrium point P of system (1.1) and its Neimark-Sacker bifurcation under certain parametric values.

Example 4.1. *Let us consider the particular case of system (1.1) defined by the following values of the parameters:*

$$\alpha = 0.6, \quad \beta = 0.3, \quad \gamma \in [0.3, 0.75],$$



(a) Bifurcation diagram for $\alpha = 0.6$, $\beta = 0.3$, $\gamma \in [0.3, 0.89]$

(b) Maximum Lyapunov exponent with $\gamma \in [0.75, 0.96]$, $(\alpha, \beta) = (0.6, 0.3)$ and initial conditions $(x_0, y_0) = (0.816, 0.583)$.

FIGURE 1. Bifurcation diagram and MLE for system (1.1).

where γ *is considered as the bifurcation parameter and the initial condition is* $(x_0, y_0) = (0.16666, 0.58331)$. *Then, as the bifurcation parameter* γ *passes through* $\gamma_1 \approx 0.428571$, *system (1.1) undergoes a Transcritical bifurcation at its boundary equilibrium point* $E \approx (1.3333, 0)$. *Figure 1 (a) shows the bifurcation diagram of system (1.1) at the equilibrium point E, it is clear that the equilibrium point E is asymptotically stable for* $0.3 < \gamma \leq \gamma_1$. *Then, E loses its stability when* $\gamma > \gamma_1$, *and a new asymptotically stable interior equilibrium* $P = \left(\frac{1-\gamma}{\gamma}, \frac{\gamma(1+\beta)-(\alpha\gamma+\beta)}{\gamma\alpha}\right)$ *is born.*

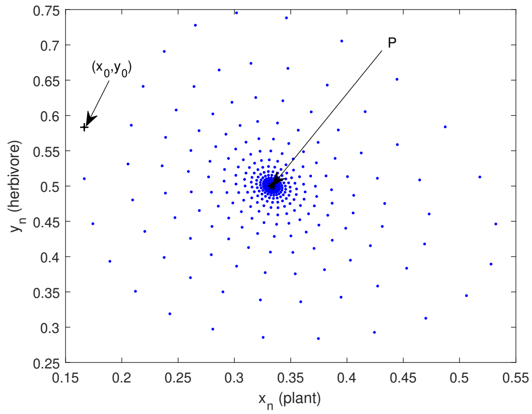
From the viewpoint of biology, the occurrence of a transcritical bifurcation corresponds to change in the system's dynamical behavior. When the parameter $\gamma < \gamma_1$ *the boundary equilibrium E is stable, that is, there is plant but the herbivore population goes to extinction. However, if* $\gamma > \gamma_1$ *the boundary equilibrium is unstable, then both the populations of plant and herbivore will coexist.*

Example 4.2. *Choosing the following values for the parameters:*

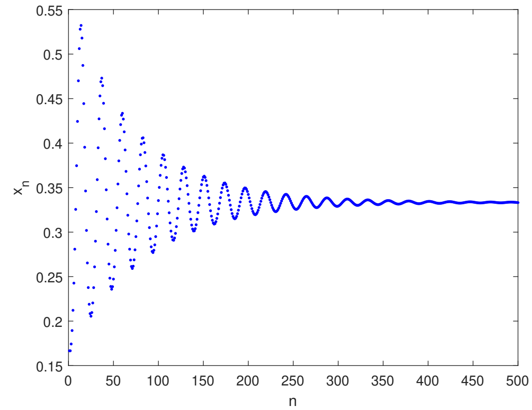
$$\alpha = 0.6, \quad \beta = 0.3, \quad \gamma \in [0.75, 0.96],$$

when $\gamma_1 < \gamma < \gamma_0 = 0.85714$, *system (1.1) has a unique positive stable equilibrium point (see Figure 2 (a) and (c)). At* γ_0 *the characteristic equation of (1.1) at the positive equilibrium is given by*

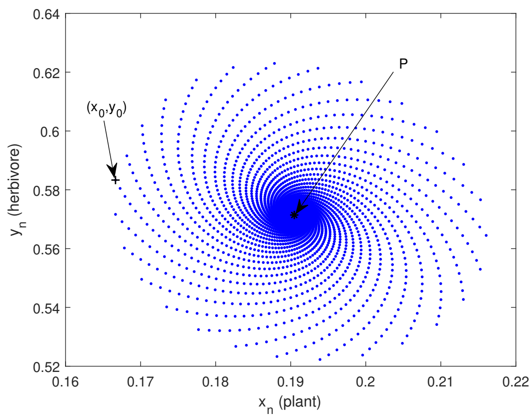
$$\lambda^2 - 1.9499\lambda + 0.9999 = 0,$$



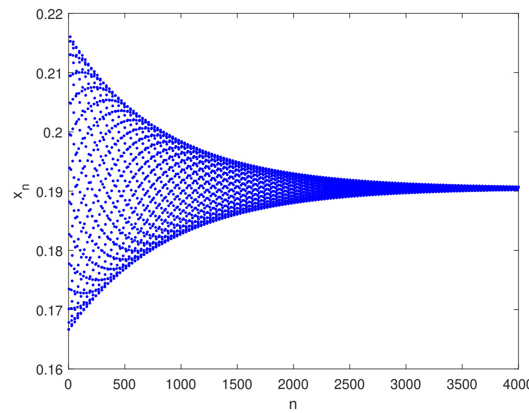
(a) $\gamma = 0.75$



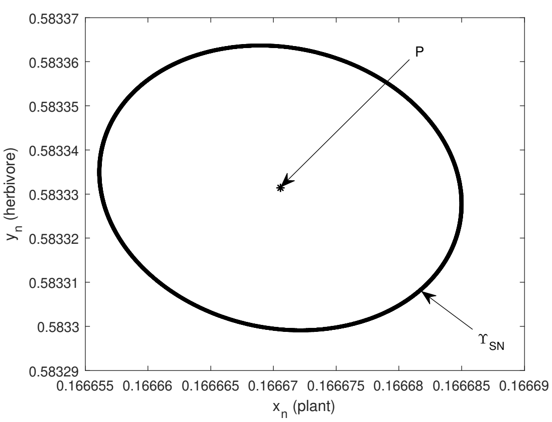
(b) Evolution of x_n for $\gamma = 0.75$



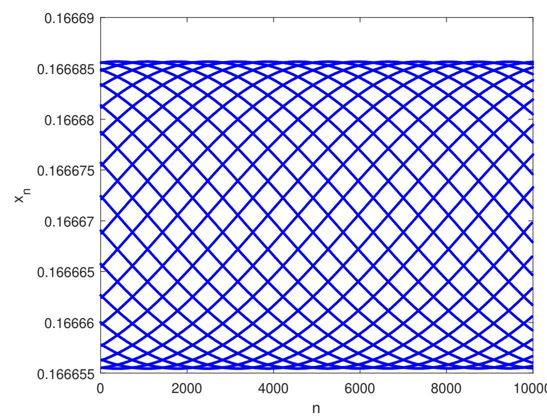
(c) $\gamma = 0.84$



(d) Evolution of x_n for $\gamma = 0.84$

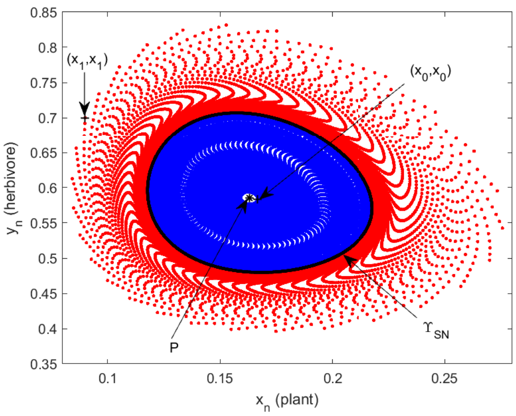


(e) Invariant closed curve Υ_{SN} for $\gamma = 0.85714$

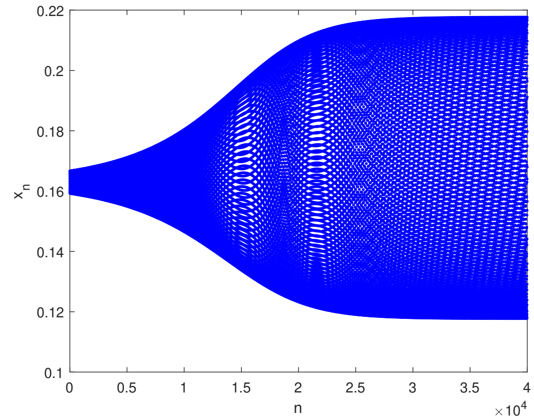


(f) Evolution of x_n for $\gamma = 0.85714$

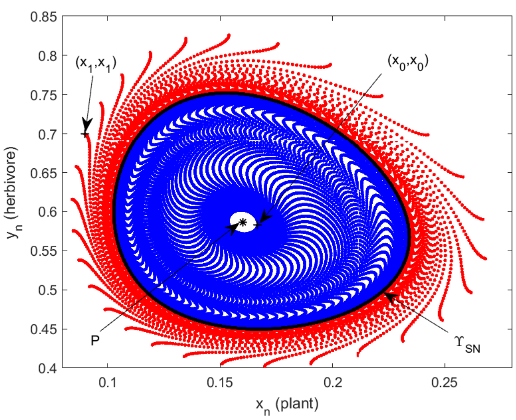
FIGURE 2. Phase portraits and orbits evolution of system (1.1) for $\gamma \in \{0.75, 0.84, 0.85714\}$.



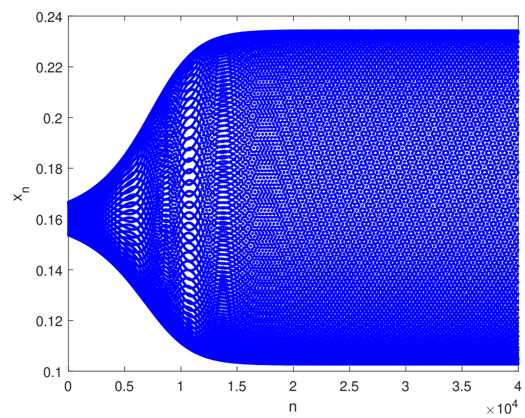
(a) $\gamma = 0.8599$



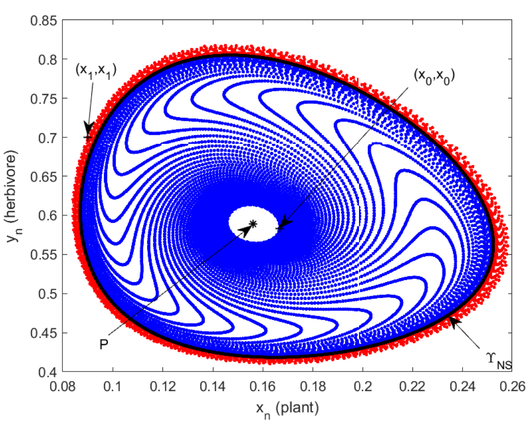
(b) Evolution of x_n for $\gamma = 0.8599$



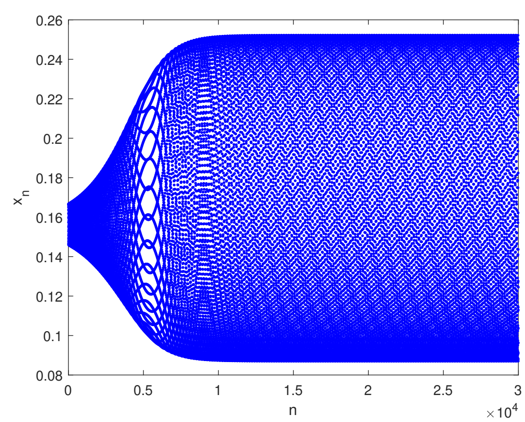
(a) $\gamma = 0.862$



(b) Evolution of x_n for $\gamma = 0.862$



(a) $\gamma) = 0.865$



(b) Evolution of x_n for $\gamma = 0.865$

FIGURE 3. Phase portraits and orbits evolution of system (1.1) for $\gamma \in \{0.8599, 0.862, , 0.865\}$.

with the following complex roots

$$\lambda_1 = 0.975 + 0.223i, \quad \lambda_2 = 0.975 - 0.223i.$$

Furthermore, it is easy to see that

$$\rho(1) = 0.05007, \quad \left(\frac{d|\lambda_{1,2}|}{d\bar{\gamma}}\right)_{\bar{\gamma}=0} = 0.1166, \quad T(0) = 1.9499,$$

which leads to $\mu_{1,2}^n \neq 1$, *for any* $n = 1,2,3,4$. f *and* g, *provided in* (3.13), *are given by*

$$f(u_n, v_n) = -0.285u_n^2 - 1.86u_nv_n + 0.06v_n^2 + 0.0855u_n^3 + 0.312u_n^2v_n + 0.1872u_nv_n^2$$
$$+0.036v_n^3 + \mathscr{R}_{f,4}(u_n, v_n),$$
$$g(u_n, v_n) = 1.7142u_nv_n + \mathscr{R}_{g,4}(u_n, v_n).$$

Moreover, the first Lyapunov exponent for these values of the parameters is given by $L = 0.0001898$, *which prove that system (1.1) undergoes Neimark-Sacker bifurcation. The bifurcation diagrams for* x_n *and* y_n *are depicted in Figure 1 (a), and the maximum Lyapunov exponent is plotted in Figure 1 (b). Figures 2 and 3 show that the equilibrium point P is a stable attractor when* $\gamma < 0.85714$. *When* $\gamma > 0.85714$ *the Neimark–Sacker bifurcation occurs. Figures 2 and 3 also illustrate that increasing the value of* γ *leads to the change of stability of the equilibrium point P. Figure 2 (e) depicts the invariant closed curve* Υ_{SN} *around P, which shows that Theorem 3.2 is correct.*

Remark 4.3. *From the information provided above, we can deduce that all orbits with initial conditions within the invariant curve* Υ_{SN}, *excluding the equilibrium point P, will asymptotically converge towards* Υ_{SN}. *Additionally, all orbits starting from outside the curve will also eventually approach* Υ_{SN}.
As γ *increases above* γ_1, *the invariant curve* Υ_{SN} *expands, but as* γ *decreases below* γ_1, *the curve narrows. (i.e The distance between the equilibrium point P and* Υ_{SN} *tends to decrease as* γ *approaches* γ_1 *(see Figure 4)).*

5. Concluding Remarks

We investigated a discrete-time plant-herbivore model in this study. We studied the equilibrium solutions, stability, and bifurcation of the model. We saw the transition from normal to chaotic behaviour. By using the centre manifold theorem and bifurcation theory of normal forms, it is demonstrated that the plant-herbivore model experiences transcritical bifurcation at its boundary equilibrium. At a positive steady-state of the plant-herbivore model, Neimark-Sacker bifurcation direction and existence criteria are investigated. When considering the bifurcating behaviour for biological populations, it should be

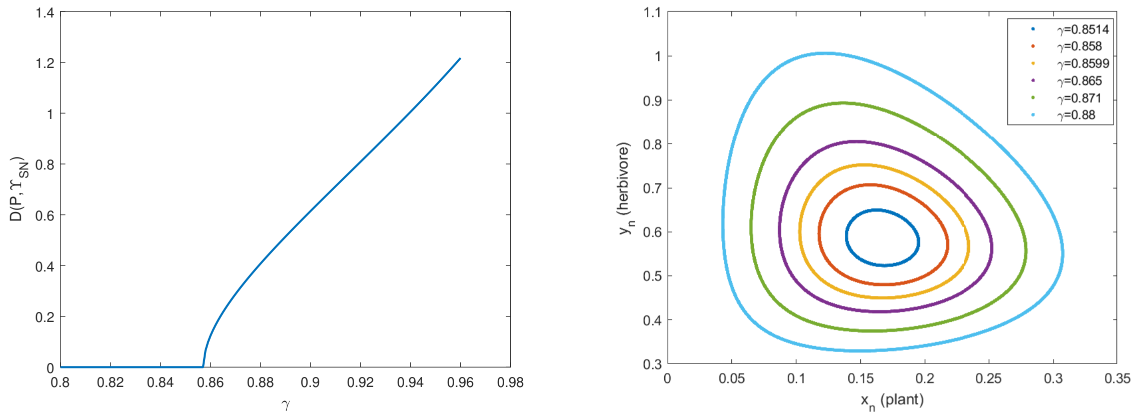


FIGURE 4. The distance between the equilibrium point P and the invariant closed curve Υ_{SN} as γ changes (left panel), and Invariant closed curves Υ_{SN} for different values of γ (right panel).

mentioned that these phenomena are crucial for competition between plants and herbivores. Chaos and bifurcating behaviour have long been seen negatively in biology. On the other hand, unpredictable behaviour in a population can increase the danger of extinction. These are therefore disastrous for the biological population's ability to reproduce. The plant and herbivore came close to having constant values for the various values of γ. The dynamics on the invariant curve showed periodic and quasi-periodic orbits as γ was changed. Additionally, the stable, closed, invariant curves lost stability and dispersed into chaotic attractors.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] M.A. Abbasi, Q. Din, Under the influence of crowding effects: Stability, bifurcation and chaos control for a discrete-time predator-prey model, Int. J. Biomath. 12 (2019), 1950044. https://doi.org/10.1142/s179352451950044x.

[2] L.J.S. Allen, M.K. Hannigan, M.J. Strauss, Mathematical analysis of a model for a plant-herbivore system, Bull. Math. Biol. 55 (1993), 847–864. https://doi.org/10.1007/bf02460676.

[3] E.S. Allman, J.A. Rhodes, Mathematical models in biology: an introduction, Cambridge University Press, Cambridge, 2004.

[4] M.B. Almatrafi, M. Berkal, Bifurcation analysis and chaos control for prey-predator model with Allee effect, Int. J. Anal. Appl. 21 (2023), 131. https://doi.org/10.28924/2291-8639-21-2023-131.

[5] M.B. Almatrafi, Construction of closed form soliton solutions to the space-time fractional symmetric regularized long wave equation using two reliable methods, Fractals. 31 (2023), 2340160. https://doi.org/10.1142/s0218348x23401606.

[6] M.B. Almatrafi, Solitary wave solutions to a fractional model using the improved modified extended Tanh-function method, Fractal Fract. 7 (2023), 252. https://doi.org/10.3390/fractalfract7030252.

[7] M. Berkal, J.F. Navarro, Qualitative behavior of a two-dimensional discrete-time prey-predator model, Comput. Math. Methods. 3 (2021), e1193. https://doi.org/10.1002/cmm4.1193.

[8] M. Berkal, J.F. Navarro, Qualitative study of a second order difference equation, Turk. J. Math. 47 (2023), 516–527. https://doi.org/10.55730/1300-0098.3375.

[9] M. Berkal, M.B. Almatrafi, Bifurcation and stability of two-dimensional activator-inhibitor model with fractional-order derivative, Fractal Fract. 7 (2023), 344. https://doi.org/10.3390/fractalfract7050344.

[10] A.A. Berryman, The orgins and evolution of predator-prey theory, Ecology. 73 (1992), 1530–1535. https://doi.org/10.2307/1940005.

[11] Q. Din, Complexity and chaos control in a discrete-time prey-predator model, Commun. Nonlinear Sci. Numer. Simul. 49 (2017), 113–134. https://doi.org/10.1016/j.cnsns.2017.01.025.

[12] Q. Din, T. Donchev, D. Kolev, Stability, bifurcation analysis and chaos control in chlorine dioxide-iodine-malonic acid reaction, MATCH Commun. Math. Comput. Chem. 79 (2018), 577–606.

[13] Q. Din, M.S. Shabbir, M.A. Khan, K. Ahmad, Bifurcation analysis and chaos control for a plant-herbivore model with weak predator functional response, J. Biol. Dyn. 13 (2019), 481-501. https://doi.org/10.1080/17513758.2019.1638976.

[14] L. Edelstein-Keshe . Mathematical models in biology, SIAM, Philadelphia, 2005.

[15] E.M. Elabbasy, A. Elsadany, Y. Zhang, Bifurcation analysis and chaos in a discrete reduced lorenz system, Appl. Math. Comput. 228 (2014), 184–194. https://doi.org/10.1016/j.amc.2013.11.088.

[16] E.M. Elsayed, Q. Din, Period-doubling and Neimark-Sacker bifurcations of plant-herbivore models, Adv. Differ. Equ. 2019 (2019), 271. https://doi.org/10.1186/s13662-019-2200-7.

[17] W. Feller, On the logistic law of growth and its empirical verifications in biology. Acta Biotheor. 5 (1940), 51–66. https://doi.org/10.1007/BF01602862.

[18] J. Guckenheimer, P. Holmes. Nonlinear oscillations, dynamical systems and bifurcations of vector fields, J. Appl. Mech. 51 (1984), 947. https://doi.org/10.1115/1.3167759.

[19] R. Herdiana, Z.A. Rafsanjani, R.H. Tjahjana, et al. Analysis of SIR model with optimal control strategy for a simple traffic congestion process, Commun. Math. Biol. Neurosci. 2024 (2024), 6. https://doi.org/10.28919/cmbn/8336.

[20] M. Imran, M.B. Almatrafi, R. Ahmed, Stability and bifurcation analysis of a discrete predator-prey system of Ricker type with harvesting effect, Commun. Math. Biol. Neurosci. 2024 (2024), 11. https://doi.org/10.28919/cmbn/8313.

[21] A.Q. Khan, J. Ma, D. Xiao, Bifurcations of a two-dimensional discrete time plant-herbivore system, Commun. Nonlinear Sci. Numer. Simul. 39 (2016), 185–198. https://doi.org/10.1016/j.cnsns.2016.02.037.

[22] A.Q. Khan, S.A.H. Bukhari, M.B. Almatrafi, Global Dynamics, Neimark-Sacker bifurcation and hybrid control in a Leslie's prey-predator model, Alexandria Eng. J. 61 (2022), 11391-11404. https://doi.org/10.1016/j.aej.2022.04.042.

[23] A.Q. Khan, F. Nazir, M.B. Almatrafi, Bifurcation analysis of a discrete Phytoplankton-Zooplankton model with linear predational response function and toxic substance distribution, Int. J. Biomath. 16 (2022), 2250095. https://doi.org/10.1142/s1793524522500954.

[24] A.Q. Khan, M. Tasneem, M.B. Almatrafi, Discrete-time COVID-19 epidemic model with bifurcation and control, Math. Biosci. Eng. 19 (2021), 1944–1969. https://doi.org/10.3934/mbe.2022092.

[25] Y.A. Kuznetsov, Elements of applied bifurcation theory, Springer-Verlag, New York, 2004.

[26] Y.A. Kuznetsov, H.G.E Meijer, Numerical bifurcation analysis of maps: From theory to software, Cambridge University Press, Cambridge, 2019.

[27] S. Kalabušić, E. Pilav, Bifurcations, permanence and local behavior of the plant-herbivore model with logistic growth of plant biomass, Qual. Theory Dyn. Syst. 21 (2022), 26. https://doi.org/10.1007/s12346-022-00561-6.

[28] Y. Li, Z. Feng, R. Swihart, et al. Modeling the impact of plant toxicity on plant-herbivore dynamics, J. Dyn. Differ. Equ. 18 (2006), 1021–1042. https://doi.org/10.1007/s10884-006-9029-y.

[29] R.M. May, Stability and complexity in model ecosystems, Princeton University Press, 2019.

[30] A. Suryanto, I. Darti, Trisilowati, Dynamic behavior of a harvested logistic model incorporating feedback control, Commun. Math. Biol. Neurosci. 2024 (2024), 4. https://doi.org/10.28919/cmbn/8227.

[31] Y.H. Wan, Computation of the stability condition for the hopf bifurcation of diffeomorphisms on R^2, SIAM J. Appl. Math. 34 (1978), 167–175. https://doi.org/10.1137/013401.

[32] S. Wiggins, Introduction to applied nonlinear dynamical systems and chaos, Springer, 2003.