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ON THE FRACTIONAL-ORDER SMOKING EPIDEMIC MODEL

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Abstract. In this study, we introduce a fractional dynamic model to explain how the smoking habit spreads across people when active smokers are given nicotine gum parameters to consume. It was demonstrated that the basic reproduction number affects the stability of both the smoking-endemic equilibrium and the smoking-free equilibrium. These results are in accordance with the epidemic theory. A numerical example is given to demonstrate the validity of the results. The results show that the giving the consumption of the nicotine gum parameter to the active smoker able to reduce the number of people who are actively smoking, increase the number of people who have a risk of smoking ones, and increase the permanent smoker quitter, thus the model gives adequate information about the spread of the smoking habit.

Keywords: Caputo fractional-order derivative; PSR model; basic reproduction number; equilibrium.

2020 AMS Subject Classification: 26A33, 34C60, 34D23.

1. Introduction

Epidemiology is concerned with the spread of diseases within a population and the general factors that influence the spread of diseases [1]. One of the most fascinating topics in epidemiology is the smoking topic.

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When someone smokes, they burn or heat a substance, which produces smoke that is inhaled by the nose or mouth and enters the lungs. The active ingredients in smoke are quickly transported from the lungs to the circulation and, ultimately, to the brain, where they function. Almost 1.3 billion smokers worldwide use tobacco. The burden of smoking is a global health priority because it accounts for over 9% of all fatalities [2]. Several noncommunicable diseases, including diabetes, cancer, cardiovascular disease, and chronic respiratory conditions, have smoking as one of their primary risk factors. Therefore, it is necessary to prevent smoking habits to reduce the various risks that will arise.

Mathematical modeling is a method to better understand the dynamics of smoking habit transmission and evaluate the effectiveness of various control and prevention strategies. Several studies on the use of mathematical models to study the smoking habit can be seen in [3, 4, 5]. These models, known as compartmental models, are split up into population-filled compartments. The idea of moving data from one compartment to another relies on the type and speed of the data. The underlying premise of these models is that individuals will begin to lead healthy lives in communities. Diseases can affect healthy individuals, yet those who are affected can recover and rejoin the community healthily [5].

One of the well-known models of the spread of smoking habit is the PSR compartment model where the model is given in the form of a nonlinear differential equation [6]. We considered a region with total population N at any time t. In this PSR model, the observed population (N) is divided into three epidemiological compartments denoted potential smokers (P), the active smokers (S) and permanent smoker quitters (R), thus the total population at the time t is given by $N(t) = P(t) + S(t) + R(t)$. The potential smoker is the number of people who have a risk of smoking, the active smoker is the number of people who are actively smoking and the permanent quitter is individuals who have quit smoking. The assumption made in developing this model can be found in [6] and the involve various parameters in (1) are described in Table 1.

TABLE 1. Parameter with biological meaning occuring in the model (1).

Parameter	Biological meaning
Λ	Inflow rate of individuals in potential smoker class
μ	natural per capita death rate
β	the rate of transmission of smoking habit
α	the recovery rate of the active smokers
δ	the awareness rate of potential smokers
ε	the measure of determination

The dynamics of PSR model for smoking habit spread in human population are governed by the following system of coupled nonlinear differential equation [6],

$$\begin{cases} \dot{P}(t) = \Lambda - \beta P(t)S(t) + \alpha(1-\varepsilon)S(t) - (\mu+\delta)P(t) \\ \dot{S}(t) = \beta P(t)S(t) - (\mu+\alpha)S(t) \\ \dot{R}(t) = \alpha\varepsilon S(t) - \mu R(t) + \delta P(t), \end{cases} \tag{1}$$

with the initial conditions $P(0) = P_0 \geq 0, S(0) = S_0 \geq 0, R(0) = R_0 \geq 0$.

Currently, several epidemiological model was formulated in the form of fractional order differential equations and widely discussed by many researchers, see [7, 8, 9, 10, 11, 12]. It is known that fractional order derivatives are generalizations of integer order derivatives, so modeling using fractional differential equations is a powerful method for studying the overall spread of the disease.

Motivated by the current study, in this manuscript, we modified model (1) by replacing the first-order derivative with fractional-order derivatives and giving the consumption of the nicotine gum parameter to the active smoker with the rate ρ where $\rho \in [0,1)$. In [13], it is stated that consumption of nicotine gum can reduce dependence on smoking. Therefore, by embedding the nicotine gum parameter, denoted by ρ, into the model (1), we have the following new model:

$$\begin{cases} \mathscr{D}^{(\gamma)}P(t) = \Lambda - \beta P(t)S(t) + \alpha(1-\varepsilon)S(t) - (\mu+\delta)P(t) \\ \mathscr{D}^{(\gamma)}S(t) = \beta P(t)S(t) - (\mu+\alpha+\rho)S(t) \\ \mathscr{D}^{(\gamma)}R(t) = \alpha\varepsilon S(t) - \mu R(t) + \delta P(t) + \rho S(t), \end{cases} \tag{2}$$

where $\mathscr{D}^{(\gamma)}$ is the fractional-order derivative operator. In this study, we use the fractional-order derivative operator of Caputo type of order γ with $0 < \gamma < 1$ as defined in [14].

Examining the stability behavior of the model's equilibrium points (2), we investigate how parameters ρ affect each compartment. This problem hasn't been resolved as of yet, as far as the authors are aware. Therefore, the findings of this study represent a novel advancement in the field of the dynamics of fractional-order epidemics.

2. Some Useful Results

This section summarizes some mathematical instruments utilized in the research. Let $\mathbf{h} : [0,\infty) \to \mathbb{R}^n$ is an integrable vector function and $\gamma \in (k-1,k)$, $k \in \mathbb{N}$. The Caputo fractional-order derivative of order γ is defined by

$$\mathscr{D}^{(\gamma)}\mathbf{h}(t) = \frac{1}{\Gamma(k-\gamma)} \int_0^t (t-\tau)^{k-\gamma-1}\mathbf{h}^{(k)}(\tau)d\tau \tag{3}$$

where $\Gamma(.)$ is the Euler Gamma function [14]. It is easy to observe that if C is a contant then $\mathscr{D}^{(\gamma)}C = 0$.

Our focus will be on the Caputo derivative in a general fractional-order dynamic system

$$\mathscr{D}^{(\gamma)}\mathbf{h}(t) = \mathbf{g}(t,\mathbf{h}(t)) \tag{4}$$

with suitable initial conditions $\mathbf{h}(t_0) = \mathbf{h}_0$, where $\mathbf{h}(t)$ is the state at time t of the system (13), $\mathbf{g} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$. Note that the system (13) may be non-linear, or vice versa. If $\mathbf{g}$ is linear, the system (13) can be written as

$$\mathscr{D}^{(\gamma)}\mathbf{h}(t) = \mathscr{A}\mathbf{h}(t)), \tag{5}$$

where $\mathscr{A}$ is a n by n matrix.

A crucial aspect of system (4) is the equilibrium point's stability. When talking about stability, one is interested in the behavior of the solutions of (4) for $t \to \infty$ [15]. The point $\mathbf{h}^* \in \mathbb{R}^n$ is said the equilibrium point of the system (4) if $\mathbf{g}(t,\mathbf{h}^*) = \mathbf{0}$. Note that the equilibrium point is a constant solution to the dynamic system (4).

Definition 2.1. [14, 15] *Let $\mathbf{h}^*$ is an equilibrium point of the fractional-order system (13).*

(1). $\mathbf{h}^*$ *is said to be stable if for* $\eta > 0$, *there exists a* $\vartheta_\eta > 0$ *such that* $\|\mathbf{h}(t_0) - \mathbf{h}^*\| < \vartheta_\eta$ *implies* $\|\mathbf{h}(t) - \mathbf{h}^*\| < \eta$ *for* $t \geq t_0$.

(2). $\mathbf{h}^*$ *is said to be asymtotically stable if it is stable and* $\lim_{t\to\infty} \mathbf{h}(t) = \mathbf{h}^*$.

Theorem 2.2. [14, 15] *The equilibrium point* $\mathbf{h}^*$ *of the fractional-order linear system (5) with* $\gamma \in (0,1)$ *is asymptotically stable if*

$$|\arg(X_i)| > \frac{1}{2}\gamma\pi, \tag{6}$$

where $X_i,\ i = 1,2,\cdots,n$ *are eigenvalues of the matrix* $\mathscr{A}$.

Theorem 2.3. [14, 15] *The equilibrium point* $\mathbf{h}^*$ *of the the fractional-order nonlinear system (13) with* $\gamma \in (0,1)$ *is asymptotically stable if*

$$|\arg(X)| > \frac{1}{2}\gamma\pi, \tag{7}$$

for all roots X *of the characteristic equation*

$$|J_{\mathbf{h}^*} - XI| = 0 \tag{8}$$

where $J_{\mathbf{h}^*}$ *is the Jacobian matrix of system (13) around the equilibrium* $\mathbf{h}^*$ *and* I *is the identity matrix of suitable size.*

3. Equilibria and Their Asymptotic Stability

By following the procedure in [15], it is easy to show that the solution of the model under consideration is restricted in the region given by

$$\mathscr{U} = \left\{ (P,S,R) \in \mathbb{R}^3_+ \ : \ 0 \leq N(t) \leq \frac{\Lambda}{\mu} \right\}.$$

If the initial conditions $P(0) = P_0 \geq 0,\ S(0) = S_0 \geq 0,\ R(0) = R_0 \geq 0$. It is well-known in epidemiology that the dynamical behavior of the model (13) depends on the basic reproduction number. By using the next generation method, the basic reproduction number for the model (2) is given by

$$\mathscr{R}_0 = \frac{\Lambda\beta}{(\mu+\delta)(\mu+\alpha+\rho)}. \tag{9}$$

In order to find the equilibrium point of the model (2), we must solve the following equations:

$$\mathscr{D}^{(\gamma)}P(t) = \mathscr{D}^{(\gamma)}S(t) = \mathscr{D}^{(\gamma)}R(t) = 0.$$

By assuming $S = 0$, one finds the smoker free equilibrium, denoted by $\mathscr{E}_0$, of the fractional-order model (2), that is

$$\mathscr{E}_0 = \left(\frac{\Lambda}{\mu+\delta}, 0, \frac{\Lambda\delta}{\mu(\mu+\delta)}\right).$$

To find the smoker endemic equilibrium point (denoted by $\mathscr{E}^*$) of the fractional-order model (2), we solve the model (2) at steady state for P, S and R. One can observe that $\mathscr{E}^* = (P^*, S^*, R^*)$, where

$$P^* = \frac{1}{\beta}(\mu+\alpha+\rho), \tag{10}$$

$$S^* = \frac{\beta\Lambda - (\mu+\delta)(\mu+\alpha+\rho)}{\beta(\mu+\alpha\varepsilon+\rho)}, \tag{11}$$

$$R^* = \frac{\beta\Lambda(\alpha\varepsilon+\rho) + \mu(\mu+\alpha+\rho)(\delta-\alpha\varepsilon-\rho)}{\mu\beta(\mu+\alpha\varepsilon+\rho)}, \tag{12}$$

is the smoker endemic equilibrium point of the fractional-order model (2).

We will analyze the stability of these two equilibrium points. First of all, the Jacobian matrix of the vector field corresponding to model (2) around $\mathscr{E}_0$ is

$$J_{\mathscr{E}_0} = \begin{bmatrix} -\mu-\delta & -\dfrac{\beta\Lambda}{\mu+\delta}+\alpha(1-\varepsilon) & 0 \\ 0 & \dfrac{\beta\Lambda}{\mu+\delta}-\mu-\alpha-\rho & 0 \\ \delta & \alpha\varepsilon & -\mu \end{bmatrix}. \tag{13}$$

The Jacobian matrix (13) gives the following characteristic polynomial:

$$p(X) = (-\mu-\delta-X)\left(\frac{\beta\Lambda}{\mu+\delta}-\mu-\alpha-\rho-X\right)(-\mu-X).$$

Clearly $J_{\mathscr{E}_0}$ has the following three eigenvalues given by

$$\begin{aligned} X_1 &= -\mu-\delta, \\ X_2 &= -\mu, \\ X_3 &= \frac{\beta\Lambda}{\mu+\delta}-\mu-\alpha-\rho = (\mathscr{R}_0-1)(\mu+\alpha+\rho), \end{aligned}$$

where $\mathscr{R}_0$ is given by (9). One can see that eigenvalues X_i, for $i = 1,2$, is negative, thus they satisfy $|\arg(X_i)| > \frac{\gamma\pi}{2}$, whereas $|\arg(X_3)| > \frac{\gamma\pi}{2}$ if $\mathscr{R}_0 < 1$, and it implies $|\arg(X_3)| < \frac{\gamma\pi}{2}$ when $\mathscr{R}_0 > 1$. Hence, based on the Theorem 2.3, $\mathscr{E}_0$ is asymptotically stable if $\mathscr{R}_0 < 1$ and becomes unstable if $\mathscr{R}_0 > 1$.

Next, the Jacobian matrix of the vector field corresponding to model (2) around $\mathscr{E}^*$ is given by

$$J_{\mathscr{E}^*} = \begin{bmatrix} \frac{\beta\Lambda-(\mu+\alpha+\rho)(\mu+\delta)}{(\mu+\alpha\varepsilon+\rho)}-\mu-\delta & -(\mu+\rho+\alpha\varepsilon) & 0 \\ \frac{\beta\Lambda-(\mu+\alpha+\rho)(\mu+\delta)}{(\mu+\alpha\varepsilon+\rho)} & 0 & 0 \\ \delta & \alpha\varepsilon+\rho & -\mu \end{bmatrix}$$

The characteristic polynomial of $J_{\mathscr{E}^*}$ is given by the following equation:

$$p(X) = (X+\mu)\Big(X^2+\Big(\frac{\beta\Lambda-(\mu+\alpha+\rho)(\mu+\delta)}{(\mu+\alpha\varepsilon+\rho)}+\mu+\delta\Big)X+\beta\Lambda-(\mu+\alpha+\rho)(\mu+\delta)\Big).$$

Based on this last equation, one finds that one of the eigenvalues of $J_{\mathscr{E}^*}$ is negative, namely, $X_1 = -\mu$. The second and third eigenvalues are determined from the roots of the following polynomial,

$$\Big(X^2+\Big(\frac{\beta\Lambda-(\mu+\alpha+\rho)(\mu+\delta)}{(\mu+\alpha\varepsilon+\rho)}+\mu-\delta\Big)X+\beta\Lambda-(\mu+\alpha+\rho)(\mu+\delta)\Big). \tag{14}$$

Using the Routh-Hurwitz condition, all the roots of the polynomial (14) have a negative real part if and only if

$$\frac{\beta\Lambda(\mathscr{R}_0-1)}{\mathscr{R}_0(\mu+\alpha\varepsilon+\rho)}+\mu+\delta > 0 \tag{15}$$

and

$$\frac{\beta\Lambda}{\mathscr{R}_0}\Big(\frac{\beta\Lambda(\mathscr{R}_0-1)}{\mathscr{R}_0(\mu+\alpha\varepsilon+\rho)}+\mu+\delta\Big)(\mathscr{R}_0-1) > 0. \tag{16}$$

where $\mathscr{R}_0$ is given by (9). It obvious that the both quantities (15) and (16) are positive if and only if $\mathscr{R}_0 > 1$. It is simple to verify that the negativity of real parts of each eigenvalue of $J_{\mathscr{E}^*}$ implies $|\arg(X_i)| > \frac{1}{2}\gamma\pi$ for i = 1, 2, 3. Therefore, if $\mathscr{R}_0 > 1$, then $\mathscr{E}^*$ is asymptotically stable; if $\mathscr{R}_0 < 1$, then $\mathscr{E}^*$ becomes unstable.

Based on these explanations, we have the following results.

Theorem 3.1. (i). *The smoker free equilibrium point* $\mathscr{E}_0$ *is asymptotically stable if* $\mathscr{R}_0 < 1$, *and it becomes unstable if* $\mathscr{R}_0 > 1$.

(ii). *The smoker endemic equilibrium point* $\mathscr{E}^*$ *is asymptotically stable if* $\mathscr{R}_0 > 1$, *and it becomes unstable when* $\mathscr{R}_0 < 1$.

In order to show the validity of the results, let us consider the following numerical example. For the model (2), we use an estimate of parameter value represented in Table 2 [6].

TABLE 2. Reasonable values of the parameters in the model (2).

Parameter	**Value**
Λ	0.2
μ	0.2
β	0.9
α	0.1
δ	0.1
ε	0.02

Based on these parameter values in Table 2 with $\rho = 0.17$, we find the basic reproduction number $\mathscr{R}_0 = 1.2766$ which shows that the endemic equilibrium is asymptotic stable.

Graphs of the potential smoker subpopulation, the active smoker subpopulation, and the permanent smoker quitter subpopulation under the effect of giving the the nicotine gum to the active smoker with $\rho = 0.17$, is given in Figure 1. The graphs show that the giving the the nicotine gum to the active smoker reduce the number of the active smoker, increase the number of permanent smoker quitter, and increase the number of the potential smoker.

Replacing ρ becomes $\rho = 0.24$ for the same parameter values, we find the basic reproduction number $\mathscr{R}_0 = 1.0227$ which shows that the endemic equilibrium is asymptotic stable. Graphs of the potential smoker subpopulation, the active smoker subpopulation, and the permanent smoker quitter subpopulation under the effect of giving the the nicotine gum to the active smoker is given in Figure 2. Again, the giving the the nicotine gum to the active smoker reduce the number of the active smoker, increase the number of permanent smoker quitter, and increase the number of the potential smoker.

4. Conclusion

We have found the fractional PSR model for the dynamic of the smoking epidemic. An example that illustrates the result has been presented. The analysis shows that giving the the nicotine gum to the active smoker increases the number of potential smoker subpopulations, reduce the active smoker subpopulation, and increases the permanent smoker quitters subpopulations, thus the PSR model gives adequate information about the spread of the smoking habit.

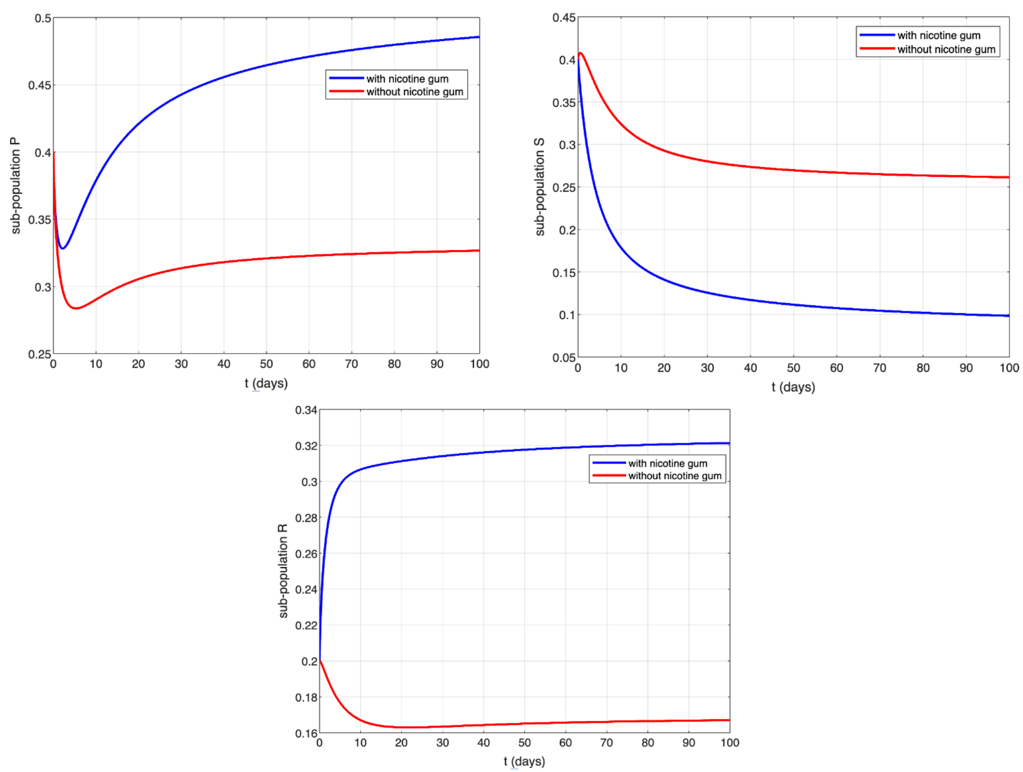

FIGURE 1. The curves of potential smoker, active smoker and permanent smoker quitter for $\gamma = 0.75,\ \rho = 0.17$

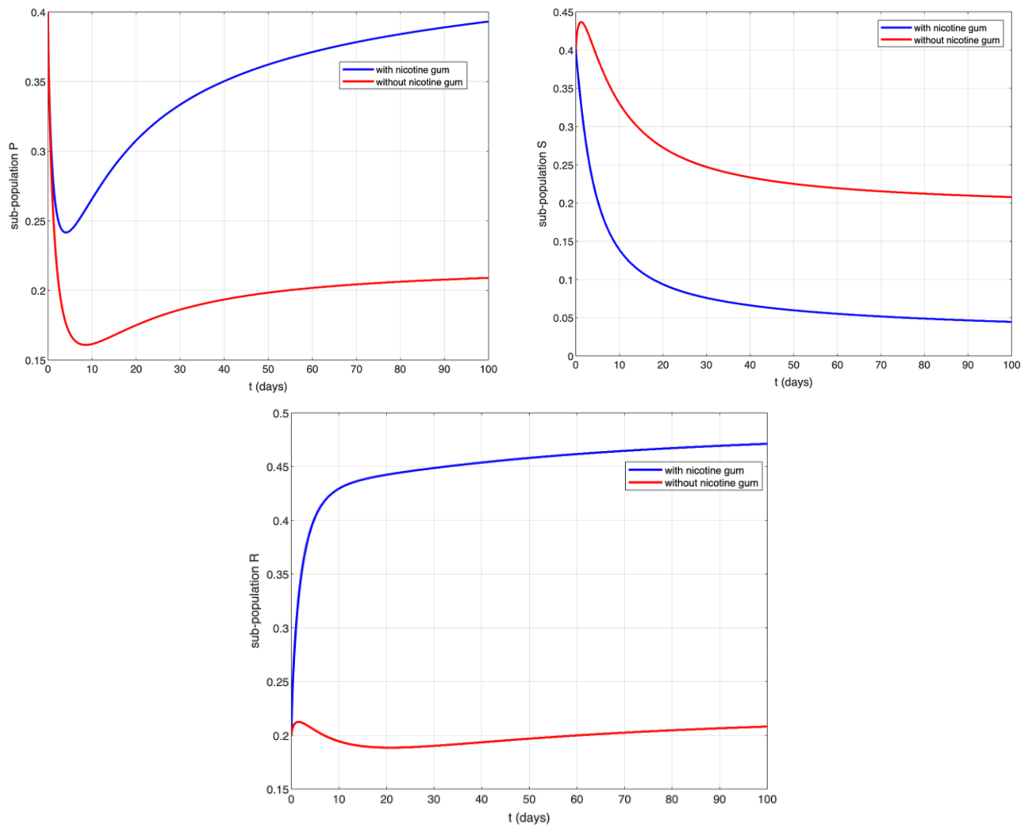

FIGURE 2. The curves of potential smoker, active smoker and permanent smoker quitter for $\gamma = 0.75,\ \rho = 0.24$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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