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STATIONARY DISTRIBUTION OF A DELAYED HIV/AIDS EPIDEMIC MODEL WITH VERTICAL TRANSMISSION AND ENVIRONMENTAL FLUCTUATIONS

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Abstract. In this work we look at the dynamics of HIV/AIDS model with discrete delay influenced by stochastic perturbation. We prove the positivity of unique global solution of the underlying perturbed system. Furthermore, we set a threshold value $\widetilde{\mathfrak{R}}_e^d$ for the delay model with perturbation which is higher than basic reproduction rate of underlying non-perturbed model. We derive the necessary condition for the disease's eradication and we also establish a threshold value $\widetilde{\mathfrak{R}}_p^d$ which governs the presence of a unique stationary distribution. Our results demonstrate that the time delay and stochastic perturbation serve as crucial for reducing disease spread. When the noise is really high, the disease dies off and there are periodic outbreaks due to time delay. Numerical simulations corroborated our analytical findings.

Keywords: Lyapunov function; extinction; stationary distribution; persistence; vertical transmission.

2020 AMS Subject Classification: 37N25, 92D25.

1. Introduction

Everyone knows that the Human Immunodeficiency Virus (HIV) is a severe threat to human mortality. Without medication, an HIV infection may spread and progress to Acquired Immunodeficiency Syndrome (AIDS) which is a condition where the body's immunological or defense system is so weakened that it cannot fight against other diseases, within a decade. The

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weakened immune system makes people with AIDS are more susceptible to diseases including cancer and tuberculosis.

Researchers have developed different mathematical models to safeguard human life and hinder the spread of diseases that are transmissible. Following the essential work of Kermack and Mckendrick [9], a Susceptible-Infected framework, the study of disease transmission has shown to be an intriguing field of research. Since then, different models for describing various types of epidemics have been developed, and the dynamics of these models have been researched. The dynamical behavior of different HIV model and their numerous extensions is investigated by several researchers [2, 3, 7, 16, 17].

Everyone acknowledges that biological processes do not occur instantly and that time is required for interaction between them. Delay differential equations (DDEs) have been widely useful in control theory and more recently connected to biological and mathematical models. The last four decades have seen an enormous amount of interest in the stability of systems with time delays. The delayed epidemic models have been examined in numerous studies [10, 20, 23] for various types of epidemics particularly for HIV/AIDS.

Persistent or severe symptoms of being infected with HIV can remain undetected for a decade or longer in adults, and as well as within couple of years in young babies born with HIV infection. Every infected person experiences this asymptomatic infection duration in a completely unique way. The disease can be controlled and its extinction may be assisted by the delay tactics.

In this regard, Sharma et.al. [22] probed the influence of discrete delay in a HIV/AIDS model described by the following DDEs.

$$
\begin{aligned}
\frac{dS_h(t)}{dt} &= r_0 - (\beta_1 I_l(t) + \beta_2 I_e(t))S_h(t) - \mu_n S_h(t) \\
\frac{dI_l(t)}{dt} &= (\beta_1 I_l(t) + \beta_2 I_e(t))S_h(t) + \eta(I_l(t) + I_e(t)) - \delta I_l(t-\omega) - \mu_n I_l(t) \\
\frac{dI_e(t)}{dt} &= \delta I_l(t-\omega) - (\kappa + \sigma + \mu_n)I_e(t)
\end{aligned} \tag{1}
$$

where, $S_h(t)$, $I_l(t)$, $I_e(t)$ denote susceptible population, infective population without symptoms, infective population with symptoms respectively; ω is the limit superior of the time delay (10 - 11 years) which depicts the delay between the onset of the infection and the appearance of symptoms and other parameters are given in TABLE 1. The term $\delta I_l(t-\omega)$, the force of transmission from $I_l(t)$ to $I_e(t)$ at time t is greater than zero, when there is an association between the classes $I_l(t)$ and $I_e(t)$. It is assumed that the natural death rate is not less than the vertical transmission rate, that is $\mu_n > \eta$ and $(\kappa+\sigma+\mu_n)(\delta+\mu-\eta) > \delta\eta$, as in [22].

For $\rho \in [-\omega, 0]$, DDE system (1) has the initial values

$$S_h(\rho) = d_1(\rho),\ I_l(\rho) = d_2(\rho),\ E(\rho) = d_3(\rho). \tag{2}$$

We consider $d_i(\rho) \geq 0$ for all $i = 1,2,3$.

Fluctuating environmental conditions have a considerable influence on the population dynamics of the natural environment. To some extent, the deterministic model can describe disease spread, but in reality, infectious disease spread is also affected by many random elements. For example, the unpredictable nature of interactions between individuals, which means that contact between individuals is not always uniform. In terms of biology, random factors inevitably influence the mechanism of disease transmission. The stochastic model outperforms the deterministic model in this aspect.

The latest developments in stochastic differential equations facilitate many researchers to incorporate randomness into deterministic models of phenomena as a way to explain the impacts of environmental fluctuations, whether they are noise in differential equation system or environmental variation in parameters [4, 5].

The investigation of the dynamics of various models of the HIV/AIDS with stochastic consequences has grown rapidly during the past decades. To make the model (1) more realistic, we intend to establish a delayed HIV/AIDS model with perturbation, focusing on how noise and delay influence disease extinction and persistence.

We now extend the delay model described in (1) to include random environmental variations on it. Transmission rate of any disease is a crucial metric and thus we add randomness into the parameter β to see how changes in the environment affect the system (1).

The system (1) gets the transformation via stochastic delayed differential equations (SDDE) as:

$$dS_h(t) = [r_0 - (\beta_1 I_l(t) + \beta_2 I_e(t))S_h(t) - \mu_n S_h(t)]dt - \lambda S_h(t) I_l(t) d\mathscr{B}(t) \tag{3}$$

$$dI_l(t) = [(\beta_1 I_l(t) + \beta_2 I_e(t))S_h(t) + \eta(I_l(t) + I_e(t)) - \delta I_l(t-\omega) - \mu_n I_l(t)]dt$$
$$+ \lambda S_h(t) I_l(t) d\mathscr{B}(t)$$
$$dI_e(t) = [\delta I_l(t-\omega) - (\kappa + \sigma + \mu_n) I_e(t)]dt$$

where λ is the intensity of white noise which represented by the derivative of Brownian motion $\mathscr{B}(t)$. Equation (3) is often referred to as the delayed HIV/AIDS model with perturbation.

Many authors researched how environmental factors with time delays affect the survival and the extinction of diseases. The long term behavior of delayed model with perturbation have also been investigated [1, 14, 19, 21]. In 2016, Liu et.al. [11] introduced temporary immunity as the time lag and investigated the disappearance and persistence of a disease. In [12], the authors examined a HIV-1 infection model with delay influenced by stochastic perturbation with non-linear incidence rate. In 2018, Hattaf et.al. [5] probed the delayed stochastic SIR model by considering the time delay as temporary immunity duration and Liu et.al. [13] probed the dynamics of stochastic delayed behavior of SVEIR model and obtain the criteria for the persistence of the infection. Most recently, Zhang et.al. [24] examined the survival of the disease for a delayed model with perturbation by considering the incidence rate in the form of specific functional response.

The main objective of the present study is to examine the delayed stochastic HIV model (3) in order to find thresholds under which we can identify the eradication of the disease, future occurrence of the disease and furthermore, figure out how noise intensity and delay parameter affect the behavior of the delayed HIV system (3) with perturbation.

This work follows the following structure. We investigate the presence of positive solution which is unique and global for the system (3) in section 2. In section 3, we provide sufficient conditions under which the disease dies off. In section 4, we prove that the system (3) is ergodic under some conditions by using the appropriate Lyapunov function. In section 5, several numerical examples are provided to illustrate the analytical results of our model. Section 6 outlines the conclusions of this paper.

2. Positivity of Unique Global Solution

To be plausible, any disease model must be adequately posed in the sense that the number of individuals in every compartment must remain non-negative. So far biological reasoning, this section elucidates the positivity of solution of SDDE system (3).

Theorem 2.1. *System* (3) *has a unique and positive solution* $(S_h(t), I_l(t), I_e(t))$ *globally with the initial value* $(S_h(\rho), I_l(\rho), I_e(\rho))$ *for all* $t \geq 0$ *and also* $(S_h(t), I_l(t), I_e(t)) \in \mathbb{R}^3_+$ *for all* $t \geq 0$ *almost surely.*

Proof:

We notice that the right hand side of (3) is locally Lipschitz continuous. Thus there exists a unique maximum solution $(S_h(t), I_l(t), I_e(t))$ locally on $t \in [0, \upsilon_f)$ where υ_f is the explosion time. This solution becomes global, if $\upsilon_f = \infty$ almost surely.

If not, there is a finite period for which the solution $(S_h(t), I_l(t), I_e(t))$ does not reach infinity.

Let $\mathfrak{p}_0 > 0$ be sufficiently large such that, for each $\rho \in [-\omega, 0]$, $(S_h(\rho), I(\rho), E(\rho)) \in [\frac{1}{\mathfrak{p}_0}, \mathfrak{p}_0]^3$.

For every integer $\mathfrak{p} \geq \mathfrak{p}_0$, set the stopping time as

$$\upsilon_{\mathfrak{p}} = \inf\{t \in [-\omega, \upsilon_f)/S_h(t) \wedge I_l(t) \wedge I_e(t) \leq \frac{1}{\mathfrak{p}} \text{ or } S_h(t) \vee I_l(t) \vee I_e(t) \leq \mathfrak{p}\}$$

It is clear that $\upsilon_{\mathfrak{p}}$ is increasing and $\upsilon_\infty = \lim_{\mathfrak{p}\to\infty} \upsilon_{\mathfrak{p}} \leq \upsilon_f$ almost surely.

Now assume that $\upsilon_f < \infty$, then there exists $\xi > 0$ and $0 < \varepsilon' < 1$ satisfying $\mathscr{P}(\upsilon_\infty \leq \xi) \geq \varepsilon'$.

Thus there exists a $\mathfrak{p}_1 \in \mathbb{N}$ such that $\mathfrak{p}_1 \geq \mathfrak{p}_0$ and

$$\mathscr{P}(\upsilon_{\mathfrak{p}} \leq \xi) \geq \varepsilon' \quad \text{for all} \quad \mathfrak{p} \geq \mathfrak{p}_1. \tag{4}$$

In the meanwhile, for $t \leq \upsilon_p$

$$\begin{aligned} d(S_h(t) + I_l(t) + I_e(t)) &= [r_0 - \mu_n S_h(t) - \mu_n I_l(t) + \eta(I_l(t) + \eta I_e(t)) - (\delta + \mu_n) I_e(t)]\, dt \\ &\leq [r_0 - (\mu_n - \eta)(S_h(t) + I_l(t) + I_e(t))]\, dt \end{aligned}$$

with

$$S_h(t)+I_l(t)+I_e(t) \leq \begin{cases} \dfrac{r_0}{\mu_n-\eta}, & \text{when} \quad S_h(0)+I_l(0)+I_e(0) \leq \dfrac{r_0}{\mu_n-\eta} \\ S_h(0)+I_l(0)+I_e(0), & \text{when} \quad S_h(0)+I_l(0)+I_e(0) > \dfrac{r_0}{\mu_n-\eta} \end{cases} : \mathfrak{K}$$

Define $\mathscr{V}:\mathbb{R}^3_+ \to \mathbb{R}_+$ as a C^2-function by

$$\begin{aligned}\mathscr{V}(S_h(t),I_l(t),I_e(t)) =& (S_h(t)-1-\ln S_h(t))+(I_l(t)-1-\ln I_l(t))+(I_e(t)-1-\ln I_e(t)) \\ &+\int_{t-\omega}^{t} \delta I_l(u)du\end{aligned}$$

In view of Itô's formula,

$$\begin{aligned}d\mathscr{V}(S_h(t),I_l(t),I_e(t)) = \mathscr{L}\mathscr{V}(S_h(t),I_l(t),I_e(t))\ dt + \lambda(I_l(t)(S_h(t)-1) \\ +S_h(t)(I_l(t)-1))d\mathscr{B}(t)\end{aligned} \tag{5}$$

in which

$$\begin{aligned}\mathscr{L}\mathscr{V}(S_h(t),I_l(t),I_e(t)) \;=\;& \left(1-\frac{1}{S_h(t)}\right)[r_0-(\beta_1 I_l(t)+\beta_2 I_e(t))S_h(t)-\mu_n S_h(t)] \\ &+\left(1-\frac{1}{I_l(t)}\right)[(\beta_1 I_l(t)+\beta_2 I_e(t))S_h(t)+\eta(I_l(t)+I_e(t)) \\ &-\delta I_l(t-\omega)-\mu_n I_l(t)]+\left(1-\frac{1}{I_e(t)}\right)[\delta I_l(t-\omega) \\ &-(\kappa+\sigma+\mu_n)I_e(t)]+\frac{\lambda^2}{2}S_h^2(t)+\frac{\lambda^2}{2}I_l^2(t)+\delta I_l(t)-\delta I_l(t-\omega) \\ \leq\;& r_0+(\beta_1+\beta_2+\eta+\delta)(S_h(t)+I_l(t)+I_e(t))+(\kappa+\sigma+\mu_n) \\ &+\frac{\lambda^2}{2}(I_l^2(t)+S_h^2(t)) \\ \leq\;& r_0+(\beta_1+\beta_2+\eta+\delta)\mathfrak{K}+(\kappa+\sigma+\mu_n)+\frac{\lambda^2}{2}\mathfrak{K}^2 := \widetilde{U}\end{aligned}$$

This together with (5) yields

$$d\mathscr{V}(S_h(t),I_l(t),I_e(t)) = \widetilde{U}dt+\lambda(I_l(t)(S_h(t)-1)+S_h(t)(I_l(t)-1))d\mathscr{B}(t) \tag{6}$$

The rest of the argument is excluded here as it follows that in [15].

3. Extinction of the Disease

Our objective herein is to set a sufficient criteria for the elimination of the disease in a population. Let us first provide a few lemmas before we get to the main thrust of this section.

Define

$$\tilde{\mathfrak{R}}_e^d = \frac{(\beta_1+\beta_2)\frac{r_0}{\mu_n}+\frac{\lambda^2}{2}}{\mu_n-\eta}$$

Lemma 3.1. *Let* $(S_h(t), I_l(t), I_e(t))$ *be the solution of* (3) *with any initial solution* $(S_h(0), I_l(0), I_e(0)) \in \mathbb{R}_+^3$. *Then*

$$\lim_{t\to\infty}\frac{S_h(t)}{t}=0,\ \lim_{t\to\infty}\frac{I_l(t)}{t}=0,\ \lim_{t\to\infty}\frac{I_e(t)}{t}=0 \quad \textit{almost surely}.$$

Moreover,

$$\lim_{t\to\infty}\frac{1}{t}\int_{t-\omega}^{t}\delta I_l(u)du=0,\ \lim_{t\to\infty}\frac{1}{t}\int_{0}^{t}\delta I_l(u-\omega)du=0 \quad \textit{almost surely}.$$

Proof:

Let $\mathscr{J}(t) = S_h(t)+I_l(t)+I_e(t)+\int_{t-\omega}^{t}\delta I_l(u)du+\delta\int_0^t I_l(u-\omega)du$.

Set $Z(\mathscr{J}(t)) = (1+\mathscr{J}(t))^{\varsigma}$ and consequently $dZ(\mathscr{J}(t)) = \mathscr{L}(Z(\mathscr{J}(t)))dt$

where

$$\begin{aligned}
\mathscr{L}(Z(\mathscr{J}(t))) &= \varsigma(1+\mathscr{J}(t))^{\varsigma-1}\left[r_0-\mu_n S_h(t)-\mu_n I_l(t)-\mu_n I_e(t)+\eta(S_h(t)+I_l(t)+I_e(t))\right.\\
&\quad \left.-\eta S_h(t)-(\kappa+\sigma)I_e(t)+\delta I_l(t)\right]+\varsigma(\varsigma-1)(1+\mathscr{J}(t))^{\varsigma-2}\lambda^2 S_h^2(t)I_l^2(t)\\
&\le \varsigma(1+\mathscr{J}(t))^{\varsigma-1}\left[r_0\left(1+\frac{\mu\delta}{\mu-\eta}\right)\right.\\
&\quad \left.-(\mu_n-\eta-\delta)\left(S_h(t)+I_l(t)+I_e(t)+\int_{t-\omega}^{t}\delta I_l(u)du+\delta\int_0^t I_l(u-\omega)du\right)\right]\\
&\quad +\varsigma(\varsigma-1)(1+\mathscr{J}(t))^{\varsigma-2}\lambda^2 S_h^2(t)I_l^2(t)\\
&\le \varsigma(1+\mathscr{J}(t))^{\varsigma-2}\left[r^*+(r^*-\mathfrak{a})\mathscr{J}(t)-(\mu_n-\eta-\delta-(\varsigma-1)\lambda^2)\mathscr{J}^2(t)\right]
\end{aligned}$$

where $\mathfrak{a} = \mu_n-\eta-\delta$ and $r^* = r_0\left(1+\frac{\mu\delta}{\mu-\eta}\right)$.

Choose $\varsigma > 0$ such that $\mu_n-\eta-\delta-(\varsigma-1)\lambda^2 > 0$.

Let $\mathfrak{b} = \mu_n-\eta-\delta-(\varsigma-1)\lambda^2$.

Thus

$$\mathscr{L}(Z(\mathscr{J}(t))) \leq \varsigma(1+\mathscr{J}(t))^{\varsigma-2}\left[-\mathfrak{b}\mathscr{J}^2(t)+(r^*-\mathfrak{a})\mathscr{J}(t)+r^*\right]$$

which yields

$$d(Z(\mathscr{J}(t))) \leq \varsigma(1+\mathscr{J}(t))^{\varsigma-2}\left[-\mathfrak{b}\mathscr{J}^2(t)+(r^*-\mathfrak{a})\mathscr{J}(t)+r^*\right]dt \tag{7}$$

For any $0<\mathfrak{h}<\varsigma\mathfrak{b}$, we can see that

$$d(e^{\mathfrak{h}t}Z(\mathscr{J}(t))) = \mathscr{L}(e^{\mathfrak{h}t}Z(\mathscr{J}(t)))dt \tag{8}$$

Consequently,

$$E\left[e^{\mathfrak{h}t}Z(\mathscr{J}(t))\right] = Z(\mathscr{J}(0)) + E\int_0^t \mathscr{L}(e^{\mathfrak{h}u}Z(\mathscr{J}(u)))\,du \tag{9}$$

where

$$\begin{aligned}
\mathscr{L}(e^{\mathfrak{h}t}Z(\mathscr{J}(t))) &\leq \mathfrak{h}e^{\mathfrak{h}t}Z(\mathscr{J}(t)) + e^{\mathfrak{h}t}\mathscr{L}(Z(\mathscr{J}(t)))\\
&\leq \mathfrak{h}e^{\mathfrak{h}t}(1+\mathscr{J}(t))^{\varsigma} + e^{\mathfrak{h}t}\varsigma(1+\mathscr{J}(t))^{\varsigma-2}\left[-\mathfrak{b}\mathscr{J}^2(t)+(r^*-\mathfrak{a})\mathscr{J}(t)+r^*\right]\\
&= \varsigma e^{\mathfrak{h}t}(1+\mathscr{J}(t))^{\varsigma-2}\left[\frac{\mathfrak{h}(1+\mathscr{J}(t))^2}{\varsigma}+(-\mathfrak{b}\mathscr{J}^2(t)+(r^*-\mathfrak{a})\mathscr{J}(t)+r^*)\right]\\
&= \varsigma e^{\mathfrak{h}t}(1+\mathscr{J}(t))^{\varsigma-2}\left[-(\mathfrak{b}-\frac{\mathfrak{h}}{\varsigma})\mathscr{J}^2(t)+(r^*-\mathfrak{a}+\frac{2\mathfrak{h}}{\varsigma})\mathscr{J}(t)+(r^*+\frac{\mathfrak{h}}{\varsigma})\right]\\
&\leq \varsigma e^{\mathfrak{h}t}\mathbb{M}
\end{aligned}$$

where $\mathbb{M} := \sup_{\mathscr{J}(t)\in\mathbb{R}_+}\left\{(1+\mathscr{J}(t))^{\varsigma-2}\left[-(\mathfrak{b}-\frac{\mathfrak{h}}{\varsigma})\mathscr{J}^2(t)+(r^*-\mathfrak{a}+\frac{2\mathfrak{h}}{\varsigma})\mathscr{J}(t)+(r^*+\frac{\mathfrak{h}}{\varsigma})\right]+1\right\}$.

In view of (9),

$$\begin{aligned}
E[e^{\mathfrak{h}t}Z(\mathscr{J}(t))] = E[e^{\mathfrak{h}t}(1+\mathscr{J}(t))^{\varsigma}] &\leq (1+\mathscr{J}(0))^{\varsigma}+\frac{\varsigma e^{\mathfrak{h}t}}{\mathfrak{h}}\mathbb{M}\\
\text{That is, } E[(1+\mathscr{J}(t))^{\varsigma}] &\leq \frac{(1+\mathscr{J}(0))^{\varsigma}}{e^{\mathfrak{h}t}}+\frac{\varsigma\mathbb{M}}{\mathfrak{h}}
\end{aligned}$$

Taking limit superior on both sides,

$$\limsup_{t\to\infty} E[(1+\mathscr{J}(t))^{\varsigma}] \leq \frac{\varsigma\mathbb{M}}{\mathfrak{h}} := \mathbb{M}_0. \tag{10}$$

This together with the continuity of $\mathscr{J}(t)$ gives that there exists $\mathbb{M}_1>0$ such that $E[(1+\mathscr{J}(t))^{\varsigma}] \leq \mathbb{M}_1$ for $t\geq 0$.

It follows from (7) that, we can obtain that for infinitesimal $\nu > 0$, $\mathfrak{h} \in \mathbb{N}$,

$$\begin{aligned} E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(t))^{\varsigma}\right] &\le E\left[(1+\mathscr{J}(\mathfrak{h}\nu))^{\varsigma}\right] + \mathsf{l}_1 \\ &\le \mathbb{M}_1 + \mathsf{l}_1 \end{aligned} \tag{11}$$

where

$$\begin{aligned} \mathsf{l}_1 &= E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} \left| \int_{\mathfrak{h}\nu}^{t} \varsigma (1+\mathscr{J}(u))^{\varsigma-2} \left[-\mathfrak{b}\mathscr{J}^2(u) + (r_0 - \mathfrak{a})\mathscr{J}(u) + r_0\right] du \right|\right] \\ &\le \mathscr{C}_1 E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} \left| \int_{\mathfrak{h}\nu}^{t} (1+\mathscr{J}(u))^{\varsigma} du \right|\right] \\ &\le \mathscr{C}_1 E\left[\int_{\mathfrak{h}\nu}^{(\mathfrak{h}+1)\nu} (1+\mathscr{J}(u))^{\varsigma} du\right] \\ &\le \mathscr{C}_1 \mathfrak{a}_1 E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(u))^{\varsigma}\right] \end{aligned}$$

Equation (11) implies

$$E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(t))^{\varsigma}\right] \le E\left[(1+\mathscr{J}(\mathfrak{h}\nu))^{\varsigma}\right] + \mathscr{C}_1 \mathfrak{a}_1 E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(u))^{\varsigma}\right] \tag{12}$$

Particularly, choose $\mathfrak{a}_1 > 0$ such that $\mathscr{C}_1 \mathfrak{a}_1 \le \frac{1}{2}$ and so

$$E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(t))^{\varsigma}\right] \le E\left[(1+\mathscr{J}(\mathfrak{h}\nu))^{\varsigma}\right] + \frac{1}{2} E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(t))^{\varsigma}\right] \tag{13}$$

That is, $E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(t))^{\varsigma}\right] \le 2E\left[(1+\mathscr{J}(\mathfrak{h}\nu))^{\varsigma}\right] \le 2\mathbb{M}_1$.

For an arbitrary $\varkappa > 0$ Chebychev's inequality yields

$$\begin{aligned} \mathscr{P}\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(t))^{\varsigma} > (\mathfrak{h}\nu)^{1+\varkappa}\right] &\le \frac{E\left[\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(t))^{\varsigma}\right]}{(\mathfrak{h}\nu)^{1+\varkappa}} \\ &\le \frac{2\mathbb{M}_1}{(\mathfrak{h}\nu)^{1+\varkappa}} \text{ for } \mathfrak{h} \in \mathbb{N}. \end{aligned}$$

According to Borel-Cantelli lemma, for any $\tau' \in \Omega$, $\sup_{\mathfrak{h}\nu \le t \le (\mathfrak{h}+1)\nu} (1+\mathscr{J}(t))^{\varsigma} \le (\mathfrak{h}\nu)^{1+\varkappa}$ holds for finitely many $\mathfrak{h}$.

Therefore there is a $K_0(\tau')$ such that the above holds for almost $\tau' \in \Omega$ whenever $\mathfrak{h} \geq \mathfrak{h}_0$.

Consequently, for almost $\tau' \in \Omega$ if $\mathfrak{h} \geq \mathfrak{h}_0$ and $\mathfrak{h}\nu \leq t \leq (\mathfrak{h}+1)\nu$,

$$\begin{aligned} \frac{\ln(1+\mathscr{J}(t))^{\varsigma}}{\ln t} &\leq \frac{(1+\varkappa)\ln(\mathfrak{h}\nu)}{\ln(\mathfrak{h}\nu)} = 1+\varkappa \\ \therefore \limsup_{t\to\infty} \frac{\ln(1+\mathscr{J}(t))^{\varsigma}}{\ln t} &\leq 1+\varkappa \text{ almost surely.} \end{aligned}$$

Allowing $\varkappa \to 0$, $\limsup_{t\to\infty} \frac{\ln(1+\mathscr{J}(t))^{\varsigma}}{\ln t} \leq 1$ almost surely.

For $\varsigma > 1$,

$$\limsup_{t\to\infty} \frac{\ln \mathscr{J}(t)}{\ln t} \leq \limsup_{t\to\infty} \frac{\ln(1+\mathscr{J}(t))}{\ln t} \leq \frac{1}{\varsigma} \text{ almost surely.} \tag{14}$$

i.e. to say, for arbitrary small $0 < \vartheta < 1 - \frac{1}{\varsigma}$, there is a constant $\mathscr{A} = \mathscr{A}(\tau')$ and set Ω with

$$\mathscr{P}(\Omega_{\vartheta}) \geq 1 - \vartheta$$

and for $t \geq \mathscr{A},\ \tau' \in \Omega_{\vartheta},\ \ln \mathscr{J}(t) \leq (\frac{1}{\varsigma} + \vartheta)\ln t$.

This leads to

$$\limsup_{t\to\infty} \frac{\mathscr{J}(t)}{t} = 0 \text{ almost surely.}$$

This together with the fact that the solutions are positive implies

$$\lim_{t\to\infty} \frac{S_h(t)}{t} = 0,\ \lim_{t\to\infty} \frac{I_l(t)}{t} = 0,\ \lim_{t\to\infty} \frac{I_e(t)}{t} = 0,$$

$$\lim_{t\to\infty} \frac{\int_{t-\omega}^{t} \delta I_l(u)du}{t} = 0,\ \lim_{t\to\infty} \frac{\int_{0}^{t} \delta I_l(u-\omega)du}{t} = 0 \text{ almost surely.}$$

Theorem 3.2. *Assume that the system* (3) *possesses the solution* $(S_h(t), I_l(t), I_e(t))$ *with any given initial value* $(S_h(0), I_l(0), I_e(0)) \in \mathbb{R}^3_+$, $I(\rho) \geq 0,\ E(\rho) \geq 0$ *for all* $\rho \in [-\omega, 0]$. *If* $\widetilde{\mathfrak{R}^d_e} < 1$, *then*

$$\limsup_{t\to\infty} \frac{\ln(I_l(t)+I_e(t))}{t} \leq -(1-\widetilde{\mathfrak{R}^d_e})(\mu_n - \eta) < 0 \text{ almost surely.}$$

That is, the paths of infective populations will reach zero exponentially with probability one.

Proof: system (3) yields,

$$\begin{aligned} & d\left(S_h(t)+I_l(t)+I_e(t)+\int_{t-\omega}^{t} \delta I_l(u)du\right) \\ &= [r_0 - \mu_n S_h(t) - (\mu_n - \eta - \delta)I_l(t) - (\kappa+\sigma+\mu_n-\eta)I_e(t) - \delta I_l(t-\omega)]dt \end{aligned}$$

which we can integrate immediately on $(0,t)$ and divide by t to get

$$\frac{S_h(t)+I_l(t)+I_e(t)+\int\limits_{t-\omega}^{t}\delta I_l(u)du}{t}-\frac{S_h(0)+I(0)+E(0)+\delta\int\limits_{-\omega}^{0}I_l(u)du}{t}$$

$$\begin{aligned} &= r_0-\frac{\mu_n}{t}\int_0^t S_h(u)du-\frac{\mu_n-\eta-\delta}{t}\int_0^t I_l(u)du \\ &\quad -\frac{\kappa+\sigma+\mu_n-\eta}{t}\int_0^t I_e(u)du-\frac{\delta}{t}\int_0^t I_l(u-\omega)du \\ &= r_0-\mu_n\langle S_h(t)\rangle-(\mu_n-\eta-\delta)\langle I_l(t)\rangle-(\kappa+\sigma+\mu_n-\eta)\langle I_e(t)\rangle \\ &\quad -\frac{\delta}{t}\int_0^t I_l(u-\omega)du \end{aligned}$$

So,

$$\mu_n\langle S_h(t)\rangle=r_0-(\mu_n-\eta-\delta)\langle I_l(t)\rangle-(\kappa+\sigma+\mu_n-\eta)\langle I_e(t)\rangle-\frac{\delta}{t}\int_0^t I_l(u-\omega)du-\Theta(t) \tag{15}$$

where

$$\Theta(t)=\frac{S_h(t)+I_l(t)+I_e(t)+\int\limits_{t-\omega}^{t}\delta I_l(u)du}{t}-\frac{S_h(0)+I(0)+E(0)+\delta\int\limits_{-\omega}^{0}I_l(u)du}{t} \tag{16}$$

In terms of Lemma 3.1, as $t\to\infty$, $\Theta(t)\to 0$ almost surely.

In view of Itô's formula, the third equation of system (3) renders,

$$\frac{I_e(t)-I_e(0)}{t}=-\frac{\delta}{t}\int_0^t I_l(u-\omega)du-(\kappa+\sigma+\mu_n)\langle I_e(t)\rangle$$

which implies

$$(\kappa+\sigma+\mu_n)\langle I_e(t)\rangle=\frac{\delta}{t}\int_0^t I_l(u-\omega)du-\frac{I_e(t)-E(0)}{t} \tag{17}$$

Notice that $\lim\limits_{t\to\infty}\langle I_e(t)\rangle=0$ almost surely.

Let $\mathscr{V}_1(I_l(t),I_e(t))=\ln(I_l(t)+I_e(t))$.

In this regard, Itô's formula yields,

$$\begin{aligned} d\mathscr{V}_1(I_l(t),I_e(t)) &= \frac{1}{I_l(t)+I_e(t)}\left[(\beta_1 I_l(t)+\beta_2 I_e(t))S(t)+\eta(I_l(t)+I_e(t))-\delta I_l(t-\omega)\right. \\ &\quad \left.-\mu_n I_l(t)+\delta I_l(t-\omega)-(\kappa+\sigma+\mu_n)I_e(t)\right]dt \end{aligned}$$

$$-\frac{1}{2}\frac{1}{(I_l(t)+I_e(t))^2}\lambda^2 I_l^2(t)S_h^2(t)+\lambda I_l(t)S_h(t)d\mathscr{B}(t)$$

which implies

$$\begin{aligned} d\mathscr{V}_1(I_l(t),I_e(t)) &\leq \frac{1}{I_l(t)+I_e(t)}\left[(\beta_1+\beta_2)(I_l(t)+I_e(t))S(t)-(\mu_n-\eta)(I_l(t)+I_e(t))\right. \\ &\quad \left.-(\kappa+\sigma)I_e(t)\right]dt+\frac{\lambda^2}{2}dt+\lambda I_l(t)S_h(t)d\mathscr{B}(t) \\ &\leq \left[(\beta_1+\beta_2)S_h(t)-\left((\mu_n-\eta)-\frac{\lambda^2}{2}\right)\right]dt+\lambda I_l(t)S_h(t)d\mathscr{B}(t) \end{aligned}$$

Integrating on $(0,t)$,

$$\begin{aligned} \mathscr{V}_1(I_l(t),I_e(t))-\mathscr{V}_1(I_l(0),I_e(0)) \leq (\beta_1+\beta_2)\int_0^t S_h(u)du-\left[(\mu_n-\eta)-\frac{\lambda^2}{2}\right]t \\ +\int_0^t \lambda I_l(u)S_h(u)d\mathscr{B}(u) \end{aligned}$$

Dividing by t,

$$\frac{\mathscr{V}_1(I_l(t),I_e(t))-\mathscr{V}_1(I_l(0),I_e(0))}{t} \leq (\beta_1+\beta_2)\langle S_h(t)\rangle-\left[(\mu_n-\eta)-\frac{\lambda^2}{2}\right]+\frac{\mathscr{M}_1(t)}{t} \tag{18}$$

where $\mathscr{M}_1(t)=\lambda\int_0^t I_l(u)S_h(u)d\mathscr{B}(u)$, which is continuous and also local martingale with $\mathscr{M}_1(0)=0$ and obeys

$$\begin{aligned} \langle\mathscr{M}_1,\mathscr{M}_1\rangle_t &= \int_0^t I_l^2(u)S_h^2(u)du \\ &\leq \int_0^t \left(\frac{r_0}{\mu_n-\eta}\right)^2 du \leq \left(\frac{r_0}{\mu_n-\eta}\right)^2 t \end{aligned}$$

Thus $\lim_{t\to\infty}\frac{\langle\mathscr{M}_1,\mathscr{M}_1\rangle_t}{t}<\infty$ almost surely.

By the strong law of large numbers, $\lim_{t\to\infty}\frac{\mathscr{M}_1(t)}{t}=0$ almost surely.

This together with (18) renders

$$\begin{aligned} \frac{\mathscr{V}_1(I_l(t),I_e(t))}{t} &\leq \frac{\mathscr{V}_1(I_l(0),I_e(0))}{t}+\frac{\beta_1+\beta_2}{\mu_n}\left[r_0-(\mu_n-\eta-\delta)\langle I_l(t)\rangle\right. \\ &\quad \left.-(\kappa+\sigma+\mu_n-\eta)\langle I_e(t)\rangle-\frac{1}{t}\int_0^t \delta I_l(u-\omega)du-\Theta(t)\right] \end{aligned}$$

$$-\left[(\mu_n-\eta)-\frac{\lambda^2}{2}\right]+\frac{\mathscr{M}(t)}{t}$$

$$i.e. \quad \frac{\mathscr{V}_1(I_l(t),I_e(t))}{t} \leq \frac{\mathscr{V}_1(I_l(0),I_e(0))}{t}+(\beta_1+\beta_2)\frac{r_0}{\mu_n}-\frac{1}{\mu_n t}\int_0^t \delta I_l(u-\omega)du-\frac{\Theta(t)}{\mu_n}$$
$$-\left[(\mu_n-\eta)-\frac{\lambda^2}{2}\right]+\frac{\mathscr{M}(t)}{t}$$

which results in

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln(I_l(t)+I_e(t))}{t} &\leq (\beta_1+\beta_2)\frac{r_0}{\mu_n}-(\mu_n-\eta)+\frac{\lambda^2}{2}\\ &\leq -\left[1-\frac{(\beta_1+\beta_2)\frac{r_0}{\mu_n}+\frac{\lambda^2}{2}}{\mu_n-\eta}\right](\mu_n-\eta)\\ &\leq -(1-\widetilde{\mathfrak{R}_e^d})(\mu_n-\eta)<0 \quad \text{almost surely.}\end{aligned}$$

This completes the proof.

Remark 3.3. As a result of Theorem 3.2, the infective populations extinct over time but, the susceptible population $S_h(t)$ is stable in distribution in the sense that it stabilized around the mean value $\frac{r_0}{\mu_n}$ (see Figure 2).

4. Existence of Stationary Distribution

Here we have examined the the existence of ergodic property, which indicates that the system (3) has a stationary distribution in a unique way that predicts the future prevalence of the disease under some criteria related to white noise intensity.

Lemma 4.1. [8] *If there exists a bounded domain $\mathscr{H}\subset\mathbb{R}^q$ with regular boundary Ω^* such that*

$\mathfrak{A}1$*: There exists a number $\mathscr{A}>0$ satisfying $\sum_{i,j=1}^{q} a_{ij}(x)\psi_i\psi_j\geq\mathscr{A}|\psi|^2$ for $x\in\mathscr{H},\ \psi\in\mathbb{R}^q$.*

$\mathfrak{A}2$*: There exists a non-negative C^2 -function F with the property that $\mathscr{L}F\leq 1$ for all $x\in\mathbb{R}^q\setminus\mathscr{H}$,*

then the Markov process $X(t)$ has a unique ergodic stationary distribution $\varpi(.)$, and

$$\mathscr{P}\left\{\lim_{t_0\to\infty}\frac{1}{t_0}\int_0^{t_0} f(X(t))dt=\int_{\mathbb{R}^q} f(x)\varpi(dx)=1\right\},$$

holds for all $x \in \mathbb{R}^q$, where $f(.)$ is an integrable function with respect to the measure ϖ.

Define

$$\widetilde{\mathfrak{R}}_p^d = \frac{r_0\beta_2}{\left(\delta\hat{q}+\mu_n-\eta+\bar{\lambda}\right)\left(\hat{q}+\kappa+\mu_n+\sigma\right)\left(\mu_n+\bar{\lambda}\right)}$$

where $\hat{q} = 1+\dfrac{r_0}{\mu_n-\eta}$ and $\bar{\lambda} = \dfrac{\lambda^2}{2}\left(\dfrac{r_0}{\mu_n-\eta}\right)^2$

Theorem 4.2. *Assume that $\widetilde{\mathfrak{R}}_p^d > 1$. Then for any initial value $(S_h(0), I_l(0), I_e(0)) \in \mathbb{R}^3_+$, system* (3) *admits a unique ergodic stationary distribution $\varpi(.)$.*

Proof: Based on Lemma 4.1, we will show that conditions $(\mathfrak{A}1)$and $(\mathfrak{A}2)$ hold.

For the system (3) the diffusion matrix is

$$\mathfrak{B}(S_h(t),I_l(t),I_e(t)) = \begin{pmatrix} \lambda^2 I_l^2(t)S_h^2(t) & 0 & 0 \\ 0 & \lambda^2 I_l^2(t)S_h^2(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

For any bounded domain U in $\mathbb{R}^3_+$, there exists a constant

$$\mathfrak{B}_0 = \min_{(S(t),I_l(t),I_e(t))\in\bar{U}_\sigma}(\lambda^2 I_l^2(t)S_h^2(t), \lambda^2 I_l^2(t)S_h^2(t), 0) > 0$$

such that

$$\sum_{i,j=1}^{3} a_{ij}(S_h(t),I_l(t),I_e(t))\psi_i\psi_j = \lambda^2 I_l^2(t)S_h^2(t)\psi_1^2 + \lambda^2 I_l^2(t)S_h^2(t)\psi_2^2 \geq \mathfrak{B}_0|\psi|^2, \tag{20}$$

for any $(S_h(t),I_l(t),I_e(t)) \in \bar{U}_\sigma;\ \psi = (\psi_1,\psi_2,\psi_3) \in \mathbb{R}^3_+$.

Thus the condition $(\mathfrak{A}1)$ of Lemma 4.1 is verified.

For $(\mathfrak{A}2)$, we set a non-negative C^2-function $\mathscr{G} : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$\begin{aligned}\mathscr{G}(S_h(t),I_l(t),I_e(t)) &= \mathscr{Q}\left[-\ln S_h(t) - c_1\ln I_l(t) - c_2\ln I_e(t) + c_1\delta\int_{t-\omega}^{t} I_l(u)du\right] - \ln S_h(t) \\ &\quad - \ln I_l(t) + \int_{t-\omega}^{t}\delta I_l(u)du + \frac{1}{\mathfrak{m}+1}(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}+1},\end{aligned}$$

where $c_1 > 0$and $c_2 > 0$ are constants to be evaluated in this sequel.

Denote

$$\begin{aligned}
\mathscr{G}_1(S_h(t),I_l(t),I_e(t)) &= -\ln S_h(t)-c_1\ln I_l(t)-c_2\ln I_e(t)+c_1\delta\int_{t-\omega}^{t} I_l(u)du\\
\mathscr{G}_2(S_h(t),I_l(t),I_e(t)) &= -\ln S_h(t)\\
\mathscr{G}_3(S_h(t),I_l(t),I_e(t)) &= -\ln I_l(t)+\int_{t-\omega}^{t}\delta I_l(u)du\\
\mathscr{G}_4(S_h(t),I_l(t),I_e(t)) &= \frac{1}{\mathfrak{m}+1}(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}+1}
\end{aligned}$$

Noting that $\mathscr{G}(S_h(t),I_l(t),I_e(t))$ is continuous and approaches to ∞ since $(S_h(t),I_l(t),I_e(t))$ reaches to the boundary of $\mathbb{R}^3_+$ and the Euclidean norm of $(S_h(t),I_l(t),I_e(t))$ tends to ∞.

Therefore $(S_h(0),I(0),E(0))$ is a minimum point of $\mathscr{G}$ in the interior of $\mathbb{R}^3_+$.

Define $\bar{\mathscr{G}}:\mathbb{R}^3_+\to\mathbb{R}_+$ a C^2-function as

$$\begin{aligned}
\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t)) &= \mathscr{Q}\mathscr{G}_1(S_h(t),I_l(t),I_e(t))+\mathscr{G}_2(S_h(t),I_l(t),I_e(t))+\mathscr{G}_3(S_h(t),I_l(t),I_e(t))\\
&\quad+\mathscr{G}_4(S_h(t),I_l(t),I_e(t))-\mathscr{G}(S_h(0),I_l(0),I_e(0)) \qquad (21)
\end{aligned}$$

where $(S_h(t),I_l(t),I_e(t))\in\left(\frac{1}{n_0},n_0\right)^3$ and n_0 is sufficiently large integer exceeds 1. Besides, $\mathfrak{m}>1$ is a constant such that $\mu_n-\eta-\mathfrak{m}\bar{\lambda}>0$.

It can be chosen that $\mathscr{Q}>0$, a constant is large enough that

$$-\mathscr{Q}\psi+\pi\le -2 \qquad (22)$$

where $\psi=\dfrac{r_0\beta_2}{\left(\delta\hat{q}+\mu_n+\bar{\lambda}\right)(\hat{q}+\kappa+\mu_n+\sigma)}-(\mu_n+\bar{\lambda})>0$ since $\widetilde{\mathfrak{R}}^d_p>1$,

$$\pi=\sup_{(S(t),I_l(t),I_e(t))\in\mathbb{R}^3_+}\left\{-\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}]I^{m+1}(t)+\beta_1I_l(t)+\beta_2I_e(t)+\mu_n+\delta\hat{q}+\pi_0\right\}$$

Hence $\mathscr{L}\bar{\mathscr{G}}=\mathscr{Q}\,\mathscr{L}\mathscr{G}_1+\mathscr{L}\mathscr{G}_2+\mathscr{L}\mathscr{G}_3+\mathscr{L}\mathscr{G}_4$.

In this regard, Itô's formula yields,

$$\begin{aligned}
\mathscr{L}\mathscr{G}_1 &= -\frac{r_0}{S_h(t)}+\beta_1I_l(t)+\beta_2I_e(t)+\mu_n-c_1\beta_1S_h(t)-c_1\beta_2\frac{I_e(t)S_h(t)}{I_l(t)}-c_1\eta-c_1\eta\frac{I_e(t)}{I_l(t)}\\
&\quad+c_1\frac{\delta I_l(t-\omega)}{I_l(t)}+c_1\mu_n-c_2\frac{\delta I_l(t-\omega)}{I_e(t)}+c_2(\kappa+\sigma+\mu_n)+\frac{\lambda^2}{2}I_l^2(t)+c_1\frac{\lambda^2}{2}S_h^2(t)\\
&= -\frac{r_0}{S(t)}+\beta_1I_l(t)+\beta_2I_e(t)+\mu_n-c_1\beta_1S_h(t)-c_1\beta_2\frac{I_e(t)S_h(t)}{I_l(t)}-c_1\eta-c_1\eta\frac{I_e(t)}{I_l(t)}
\end{aligned}$$

$$+c_1\delta+c_1\mu_n-c_2\frac{\delta I_l(t-\omega)}{I_e(t)}+c_2(\kappa+\sigma+\mu_n)+\frac{\lambda^2}{2}I_l^2(t)+c_1\frac{\lambda^2}{2}S_h^2(t)$$

which results in

$$\begin{aligned}\mathscr{L}\mathscr{G}_1 &= -\frac{r_0}{S_h(t)}+\beta_1 I_l(t)+\beta_2 I_e(t)+\mu_n-c_1\beta_1 S_h(t)-c_1\beta_2\frac{I_e(t)S_h(t)}{I_l(t)}-c_1\eta-c_1\eta\frac{I_e(t)}{I_l(t)}\\ &\quad +c_1\delta+c_1\mu_n+c_2(\kappa+\sigma+\mu_n)+\frac{\lambda^2}{2}\left(\frac{r_0}{\mu_n-\eta}\right)^2+c_1\frac{\lambda^2}{2}\left(\frac{r_0}{\mu_n-\eta}\right)^2\\ &\quad +c_2\frac{r_0}{\mu_n-\eta}-c_2\frac{I_l(t)}{I_e(t)}\end{aligned}$$

With the identity $-(a+b+c)\leq -3\sqrt[3]{abc}$,

$$\begin{aligned}\mathscr{L}\mathscr{G}_1 &\leq -3\sqrt[3]{r_0\beta_2 c_1 c_2}+\beta_1 I_l(t)+\beta_2 I_e(t)+\mu_n+c_1\delta\hat{q}+c_1(\mu_n-\eta)+c_2(\kappa+\sigma+\mu_n)\\ &\quad +\bar{\lambda}+c_1\bar{\lambda}+c_2\hat{q}\\ &\leq -3\sqrt[3]{r_0\beta_2 c_1 c_2}+\beta_1 I_l(t)+\beta_2 I_e(t)+\mu_n+\bar{\lambda}+c_1\left[\delta\hat{q}+\mu_n-\eta+\bar{\lambda}\right]\\ &\quad +c_2\left[\hat{q}+\kappa+\sigma+\mu_n\right]\end{aligned}$$

Choose c_1 and c_2 such that

$$c_1\left[\delta\hat{q}+\mu_n-\eta+\bar{\lambda}\right]=\frac{r_0\beta_2}{\left(\delta\hat{q}+\mu_n-\eta+\bar{\lambda}\right)\left(\hat{q}+\kappa+\sigma+\mu_n\right)}=c_2\left[\hat{q}+\kappa+\sigma+\mu_n\right]$$

which renders

$$c_1=\frac{r_0\beta_2}{\left(\delta\hat{q}+\mu_n-\eta+\bar{\lambda}\right)^2\left(\hat{q}+\kappa+\mu_n+\sigma\right)};\ c_2=\frac{r_0\delta\beta_2}{\left(\delta\hat{q}+\mu_n-\eta+\bar{\lambda}\right)\left(\hat{q}+\kappa+\mu_n+\sigma\right)^2}$$

Thus

$$\begin{aligned}\mathscr{L}\mathscr{G}_1 &\leq -\left[\frac{r_0\beta_2}{\left(\delta\hat{q}+\mu_n-\eta+\bar{\lambda}\right)\left(\hat{q}+\kappa+\sigma+\mu_n\right)}-(\mu_n+\bar{\lambda})\right]+\beta_1 I_l(t)+\beta_2 I_e(t)\\ &\leq -(\mu_n+\bar{\lambda})\left[\frac{r_0\beta_2}{\left(\delta\hat{q}+\mu_n-\eta+\bar{\lambda}\right)\left(\hat{q}+\kappa+\sigma+\mu_n\right)(\mu_n+\bar{\lambda})}-1\right]+\beta_1 I_l(t)+\beta_2 I_e(t)\\ &\leq -\psi+\beta_1 I_l(t)+\beta_2 I_e(t) \qquad (23)\end{aligned}$$

From $\mathscr{G}_2$,

$$\mathscr{L}\mathscr{G}_2 = -\frac{1}{S_h(t)}\left[r_0-(\beta_1 I_l(t)+\beta_2 I_e(t))S_h(t)-\mu_n S_h(t)\right]+\frac{1}{2}\lambda^2 I_l^2(t)$$

$$\leq \quad -\frac{r_0}{S_h(t)}+\beta_1 I_l(t)+\beta_2 I_e(t)+\mu_n+\frac{1}{2}\lambda^2\left(\frac{r_0}{\mu_n-\eta}\right)^2$$

$$\leq \quad -\frac{r_0}{S_h(t)}+\beta_1 I_l(t)+\beta_2 I_e(t)+\mu_n+\frac{1}{2}\bar{\lambda} \tag{24}$$

Applying Itó's formula to $\mathscr{G}_3$,

$$\begin{aligned}
\mathscr{L}\mathscr{G}_3 \;=\;& -\beta_1 S_h(t)-\beta_2\frac{I_e(t)S_h(t)}{I_l(t)}-\eta-\eta\frac{I_e(t)}{I_l(t)}+\frac{\delta I_l(t-\omega)}{I_l(t)}+\mu_n+\frac{1}{2}\lambda^2 S_h^2(t)\\
& +\delta I_l(t)-\delta I_l(t-\omega)\\
\leq\;& -\beta_1 S_h(t)-\beta_2\frac{I_e(t)S_h(t)}{I_l(t)}-\eta-\eta\frac{I_e(t)}{I_l(t)}+\delta\frac{r_0}{\mu_n-\eta}+\mu_n+\frac{\lambda^2}{2}\left(\frac{r_0}{\mu_n-\eta}\right)^2\\
\leq\;& -\beta_1 S_h(t)-\beta_2\frac{I_e(t)S_h(t)}{I_l(t)}-\eta-\eta\frac{I_e(t)}{I_l(t)}+\delta\hat{q}+\mu_n+\bar{\lambda} \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{L}\mathscr{G}_4 \;=\;& (S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}}\{r_0-(\beta_1 I_l(t)+\beta_2 I_e(t))S_h(t)-\mu_n S_h(t)+(\beta_1 I_l(t)\\
& +\beta_2 I_e(t))S_h(t)+\eta(I_l(t)+I_e(t))-\delta I_l(t-\omega)-\mu_n I_l(t)+\delta I_l(t-\omega)\\
& -(\kappa+\sigma+\mu_n)I_e(t)\}+\frac{1}{2}\mathfrak{m}(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}-1}\lambda^2 I_l^2(t)S_h^2(t)\\
& +\frac{1}{2}\mathfrak{m}(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}-1}\lambda^2 I_l^2(t)S_h^2(t)\\
\leq\;& (S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}}\left[r_0-\mu_n S_h(t)+\eta I_l(t)+\eta I_e(t)-\mu_n I_l(t)-(\kappa+\sigma+\mu_n)I_e(t)\right]\\
& +\mathfrak{m}(S(t)+I_l(t)+I_e(t))^{\mathfrak{m}-1}\lambda^2 I_l^2(t)S_h^2(t)\\
\leq\;& r_0(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}}-(\mu_n-\eta)(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}+1}\\
& +\mathfrak{m}\frac{\lambda^2}{2}\left(\frac{r_0}{\mu_n-\eta}\right)^2(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}+1}\\
& =r_0(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}}-(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}+1}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\\
\leq\;& \pi_0-\frac{1}{2}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}+1}\\
\leq\;& \pi_0-\frac{1}{2}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](S_h^{\mathfrak{m}+1}(t)+I_l^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t)) \qquad (26)
\end{aligned}$$

where

(27)

$$\pi_0=\sup_{(S(t),I_l(t),I_e(t))\in\mathbb{R}_+^3}\left\{r_0(S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}}-\frac{1}{2}[\mu_n-\eta-m\bar{\lambda}](S_h(t)+I_l(t)+I_e(t))^{\mathfrak{m}+1}\right\}<\infty$$

In view of (23)-(27)

$$
\begin{aligned}
&\mathscr{L}\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t))\\
=\;& -\mathscr{Q}\psi+\mathscr{Q}(\beta_1 I_l(t)+\beta_2 I_e(t))-\frac{r_0}{S_h(t)}+\beta_1 I_l(t)+\beta_2 I_e(t)+\mu_n+\bar{\lambda}\\
&-\beta_1 S_h(t)-\beta_2\frac{I_e(t)S_h(t)}{I_l(t)}-\eta-\eta\frac{I_e(t)}{I_l(t)}+\delta\hat{q}+\mu_n+\bar{\lambda}\\
&+\pi_0-\frac{1}{2}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](S_h^{\mathfrak{m}+1}(t)+I_l^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t))\\
\leq\;& -\mathscr{Q}\psi+\mathscr{Q}(\beta_1 I_l(t)+\beta_2 I_e(t))-\frac{r_0}{S_h(t)}+\beta_1 I_l(t)+\beta_2 I_e(t)\\
&-\beta_1 S_h(t)-\beta_2\frac{I_e(t)S_h(t)}{I_l(t)}-\eta-\eta\frac{I_e(t)}{I_l(t)}+\delta\hat{q}+2(\mu_n+\bar{\lambda})\\
&+\pi_0-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](S_h^{\mathfrak{m}+1}(t)+I_l^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t))\\
&-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t))
\end{aligned}
$$

For $\varepsilon>0$, we can define a set

$$
\mathscr{D}=\left\{(S_h(t),I_l(t),I_e(t))\in\mathbb{R}_+^3|\varepsilon\leq S_h(t)\leq\frac{1}{\varepsilon},\ \varepsilon^2\leq I_l(t)\leq\frac{1}{\varepsilon^2},\ \varepsilon\leq I_e(t)\leq\frac{1}{\varepsilon}\right\} \tag{28}
$$

which is closed and bounded.

In the set $\mathbb{R}_+^3\setminus\mathscr{D}$, we select ε sufficiently small satisfying

$$
\begin{aligned}
-\frac{r_0}{\varepsilon}+\pi_1 &\leq -1\\
-\mathscr{Q}\psi+\mathscr{Q}\beta_1\varepsilon+\beta_1\varepsilon+\pi_2 &\leq -1\\
\mathscr{Q}\beta_1\varepsilon^2+\beta_1\varepsilon^2-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\varepsilon^{\mathfrak{m}+1}+\pi_3 &\leq -1\\
-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\frac{1}{\varepsilon^{\mathfrak{m}+1}}+\pi_4 &\leq -1\\
-\frac{1}{2}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\frac{1}{\varepsilon^{2\mathfrak{m}+2}}+\pi_5 &\leq -1\\
-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\frac{1}{\varepsilon^{\mathfrak{m}+1}}+\pi_6 &\leq -1
\end{aligned}
$$

where

$$
\begin{aligned}
\pi_1 =\;& -\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}]I_l^{m+1}(t)+\mathscr{Q}(\beta_1 I_l(t)+\beta_2 I_e(t))+\beta_1 I_l(t)+\beta_2 I_e(t)+2\mu_n+2\bar{\lambda}\\
&+\delta\hat{q}+\pi_0-\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}](S_h^{\mathfrak{m}+1}(t)+I_l^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t))
\end{aligned}
$$

$$\begin{aligned}\pi_2 &= -\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}]I_l^{m+1}(t)+\mathscr{Q}\beta_2 I_e(t)+\beta_2 I_e(t)+2\mu_n+2\bar{\lambda}+\delta\hat{q}+\pi_0\\ &\quad -\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}](S_h^{m+1}(t)+I_l^{m+1}(t)+I_e^{m+1}(t))\end{aligned}$$

$$\begin{aligned}\pi_3 &= -\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}]I_l^{m+1}(t)+\mathscr{Q}\beta_2 I_e(t)+\beta_2 I_e(t)+2\mu_n+2\bar{\lambda}\\ &\quad +\delta\hat{q}+\pi_0-\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}](I_l^{m+1}(t)+I_e^{m+1}(t))\\ \pi_4 &= -\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}]I_l^{m+1}(t)+\mathscr{Q}(\beta_1 I_l(t)+\beta_2 I_e(t))+\beta_1 I_l(t)+\beta_2 I_e(t)+2\mu_n+2\bar{\lambda}\\ &\quad +\delta\hat{q}+\pi_0-\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}](I_l^{m+1}(t)+I_e^{m+1}(t))\\ \pi_5 &= \mathscr{Q}(\beta_1 I_l(t)+\beta_2 I_e(t))+\beta_1 I_l(t)+\beta_2 I_e(t)+2\mu_n+2\bar{\lambda}+\delta\hat{q}+\pi_0\\ &\quad -\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}](S_h^{m+1}(t)+I_e^{m+1}(t))\\ \pi_6 &= -\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}]I_l^{m+1}(t)+\mathscr{Q}(\beta_1 I_l(t)+\beta_2 I_e(t))+\beta_1 I_l(t)+\beta_2 I_e(t)+2\mu_n+2\bar{\lambda}\\ &\quad +\delta\hat{q}+\pi_0-\frac{1}{4}[\mu_n-\eta-m\bar{\lambda}](S_h^{m+1}(t)+I_l^{m+1}(t))\end{aligned}$$

In the set $\mathbb{R}_+^3 \setminus \mathscr{D}$, consider

$$\begin{aligned}\mathscr{D}_1 &= \left\{(S_h(t),I_l(t),I_e(t)) \in \mathbb{R}_+^3 | 0 < S_h(t) < \varepsilon\right\}\\ \mathscr{D}_2 &= \left\{(S_h(t),I_l(t),I_e(t)) \in \mathbb{R}_+^3 | 0 < I_l(t) < \varepsilon\right\}\\ \mathscr{D}_3 &= \left\{(S_h(t),I_l(t),I_e(t)) \in \mathbb{R}_+^3 | S_h(t) \geq \varepsilon,\ 0 < I_l(t) < \varepsilon^2\right\}\\ \mathscr{D}_4 &= \left\{(S_h(t),I_l(t),I_e(t)) \in \mathbb{R}_+^3 | S_h(t) > \frac{1}{\varepsilon}\right\}\\ \mathscr{D}_5 &= \left\{(S_h(t),I_l(t),I_e(t)) \in \mathbb{R}_+^3 | I_l(t) > \frac{1}{\varepsilon^2}\right\}\\ \mathscr{D}_6 &= \left\{(S_h(t),I_l(t),I_e(t)) \in \mathbb{R}_+^3 | I_e(t) > \frac{1}{\varepsilon}\right\}\end{aligned}$$

Obviously, $\mathbb{R}_+^3 \setminus \mathscr{D} = \bigcup_{i=1}^{6} \mathscr{D}_i$.

We will prove that, $\mathscr{L}\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t)) \leq -1$ for any $(S_h(t),I_l(t),I_e(t)) \in \mathbb{R}_+^3 \setminus \mathscr{D}$, which is equivalent to proving it on the above six domains.

Case (i): For any $(S_h(t),I_l(t),I_e(t)) \in \mathscr{D}_1$, we have

$$\begin{aligned}&\mathscr{L}\bar{\mathscr{G}}(S(t),I_l(t),I_e(t))\\ \leq\ &\mathscr{Q}(\beta_1 I_l(t)+\beta_2 I_e(t))-\frac{r_0}{S_h(t)}+\beta_1 I_l(t)+\beta_2 I_e(t)\end{aligned}$$

$$+\delta\hat{q}+2(\mu_n+\bar{\lambda})+\pi_0-\frac{1}{4}\left[\mu_n-\eta-\mathfrak{m}\bar{\lambda}\right](S_h^{\mathfrak{m}+1}(t)+I_l^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t))$$
$$-\frac{1}{4}\left[\mu_n-\eta-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t))$$
$$\leq -\frac{r_0}{\varepsilon}+\pi_1$$
$$\leq -1 \tag{29}$$

Case (ii): For any $(S_h(t),I_l(t),I_e(t))\in\mathscr{D}_2$, we have

$$\begin{aligned}\mathscr{L}\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t)) \leq\ & -\mathscr{Q}\psi+\mathscr{Q}(\beta_1I_l(t)+\beta_2I_e(t))-\frac{r_0}{S_h(t)}+\beta_1I_l(t)+\beta_2I_e(t)\\ & +\delta\hat{q}+2(\mu_n+\bar{\lambda})+\pi_0\\ & -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](S_h^{\mathfrak{m}+1}(t)+I_l^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t))\\ & -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t))\\ \leq\ & -\mathscr{Q}\psi+\mathscr{Q}\beta_1\varepsilon+\beta_1\varepsilon+\pi_2\end{aligned}$$
$$\leq -1 \tag{30}$$

Case (iii): For any $(S(t),I_l(t),I_e(t))\in\mathscr{D}_3$, we obtain

$$\begin{aligned}\mathscr{L}\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t)) \leq\ & \mathscr{Q}\beta_1I_l(t)-\frac{r_0}{S_h(t)}+\beta_1I_l(t)+\beta_2I_e(t)\\ & +\delta\hat{q}+2(\mu_n+\bar{\lambda})+\pi_0-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]S_h^{\mathfrak{m}+1}(t)+\mathscr{Q}\beta_2I_e(t)\\ & -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t))\\ & -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t))\\ \leq\ & \mathscr{Q}\beta_1\varepsilon^2+\beta_1\varepsilon^2-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\varepsilon^{\mathfrak{m}+1}+\pi_3\end{aligned}$$
$$\leq -1 \tag{31}$$

Case (iv): For any $(S_h(t),I_l(t),I_e(t))\in\mathscr{D}_4$, we obtain

$$\begin{aligned}\mathscr{L}\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t)) \leq\ & \mathscr{Q}\beta_1I_l(t)-\frac{r_0}{S_h(t)}+\beta_1I_l(t)+\beta_2I_e(t)\\ & +\delta\hat{q}+2(\mu_n+\bar{\lambda})+\pi_0-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]S_h^{\mathfrak{m}+1}(t)+\mathscr{Q}\beta_2I_e(t)\\ & -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t))\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t)) \\
\leq\ & -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\frac{1}{\varepsilon^{\mathfrak{m}+1}}+\pi_4 \\
\leq\ & -1 \qquad (32)
\end{aligned}$$

Case (v): For any $(S(t),I_l(t),I_e(t))\in\mathscr{D}_5$, we obtain

$$\begin{aligned}
\mathscr{L}\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t)) \leq\ & \mathscr{Q}\beta_1 I_l(t)+\mathscr{Q}\beta_2 I_e(t)-\frac{r_0}{S_h(t)}+\beta_1 I_l(t)+\beta_2 I_e(t) \\
& +\delta\hat{q}+2(\mu_n+\bar{\lambda})+\pi_0-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]I_l^{\mathfrak{m}+1}(t) \\
& -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t)) \\
& -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](S_h^{\mathfrak{m}+1}(t)+I_e^{\mathfrak{m}+1}(t)) \\
\leq\ & -\frac{1}{2}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\frac{1}{\varepsilon^{2\mathfrak{m}+2}}+\pi_5 \\
\leq\ & -1 \qquad (33)
\end{aligned}$$

Case (vi): For any $(S_h(t),I_l(t),I_e(t))\in\mathscr{D}_6$, we have

$$\begin{aligned}
\mathscr{L}\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t)) \leq\ & -\mathscr{Q}\psi+\mathscr{Q}\beta_1 I_l(t)+\mathscr{Q}\beta_2 I_e(t)-\frac{r_0}{S_h(t)}+\beta_1 I_l(t)+\beta_2 I_e(t) \\
& +\delta\hat{q}+2(\mu_n+\bar{\lambda})+\pi_0-\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]I_e^{\mathfrak{m}+1}(t) \\
& -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](I_l^{\mathfrak{m}+1}(t)) \\
& -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right](S_h^{\mathfrak{m}+1}(t)+I_l^{\mathfrak{m}+1}(t)) \\
\leq\ & -\frac{1}{4}\left[(\mu_n-\eta)-\mathfrak{m}\bar{\lambda}\right]\frac{1}{\varepsilon^{\mathfrak{m}+1}}+\pi_6 \\
\leq\ & -1 \qquad (34)
\end{aligned}$$

Obviously from(29)-(34), we obtain that for sufficiently small ε, $\mathscr{L}\bar{\mathscr{G}}(S_h(t),I_l(t),I_e(t))\leq -1$ for any $(S_h(t),I_l(t),I_e(t))\in\mathbb{R}_+^3\setminus\mathscr{D}$.

As a consequence, the condition $(\mathfrak{A}2)$ of Lemma 4.1 holds. In an application of Lemma 4.1, the solution of system (3) is ergodic and the system (3) admits a stationary distribution $\varpi(.)$ in a unique way, which completes the proof.

Remark 4.3. In view of Theorem 4.2, it is proved that if $\widetilde{\mathfrak{R}}_p^d > 1$, the SDDE system (3) is ergodic. We also determined that the persistence of the infective populations depend on the stochastic fluctuation intensity of the noise λ, from of the expression of $\widetilde{\mathfrak{R}}_p^d$.

TABLE 1. Description of Parameters

Parameter	Description	Value	Data Source
r_0	Recruitment rate	0.55/year	[7]
β_1	Horizontal transmission rate of $I_l(t)$	variable	
β_2	Horizontal transmission rate of $I_e(t)$	variable	
η	Rate of vertical transmission	0.05/year	[7]
δ	Progression rate from $I_l(t)$ to $I_e(t)$	variable	
κ	Proportion of $I_e(t)$ who enter into $T(t)$	0.35/year	[7]
σ	Progression rate of $A(t)$ from $I_e(t)$	0.15/year	[7]
μ_n	Natural death rate	variable	

5. NUMERICAL SIMULATIONS

Numerical simulations are required to study the behavior of systems which cannot able to solve analytically.

In this section, we provide the numerical simulations to support the theoretical predictions through Euler Maruyama method [6] for SDDE system (3) for which the discretization is

$$\begin{aligned}
x_{m+1} &= x_m + [r_0 - (\beta_1 y_m + \beta_2 z_m)x_m - \mu_n x_m]h - \lambda x_m y_m \sqrt{h}\, \xi_{1,m} \\
y_{m+1} &= y_m + [(\beta_1 y_m + \beta_2 z_m)x_m + \eta(y_m + z_m) - \delta y_{m-j} - \mu_n y_m]h + \lambda x_m y_m \sqrt{h}\, \xi_{1,m} \\
z_{m+1} &= z_m + [\delta y_{m-j} - (\kappa + \sigma + \mu_n) z_m]h
\end{aligned}$$

where $(x,y,z) = (S_h(t), I_l(t), I_e(t))$, $\xi_{1,m}$ represents the Gaussian random variable which follows the standard normal distribution, $\omega = jh$, $j \in \mathbb{Z}$ and $h = 0.1$. The parameter values are as Table 1.

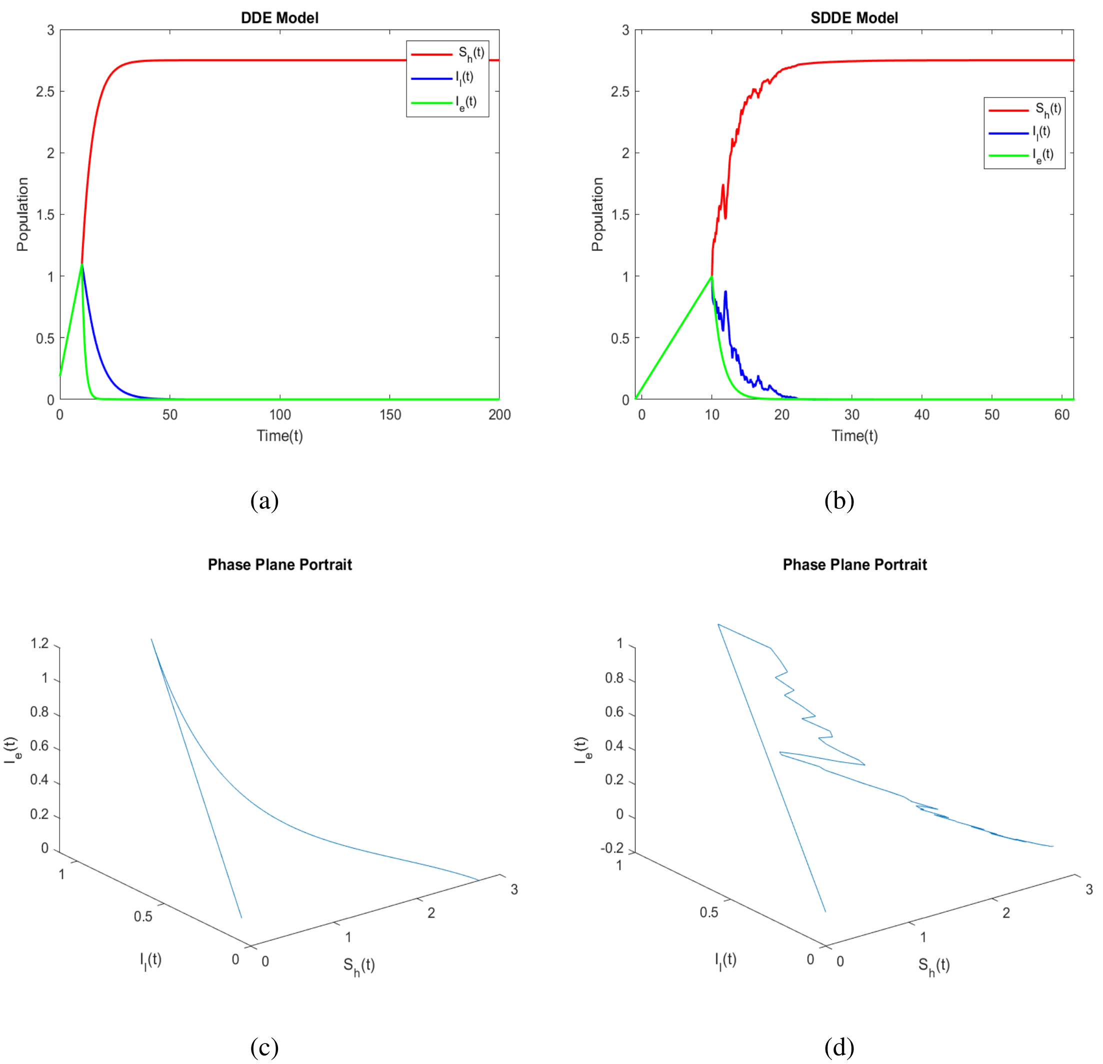

FIGURE 1. *The dynamical behavior of DDE model* (1) *and SDDE model* (3) *with the phase plane portrait. The parameter values are* $\beta_1 = 0.0001$; $\beta_2 = 0.006$; $\delta = 0.002$; *with* $\omega = 11$ *and* $\lambda = 0.2$. *In both DDE and SDDE models the solutions approaches the disease free equilibrium* $\mathfrak{E}_0 = (2.75, 0, 0)$.

If we choose $\beta_1 = 0.0001$; $\beta_2 = 0.006$; $\delta = 0.002$; $\mu = 0.2$; $\omega = 11; \lambda = 0.2$, $\widetilde{\mathfrak{R}}_e^d = 0.2452 < 1$. This shows that the hypothesis in Theorem 3.2 are hold. Figure 1 illustrates the analogy between the DDE model (1) and the SDDE model (3), as demonstrated by the conclusion reached in Theorem (3.2), i.e., the disease disappears with probability 1. In both cases, the sample paths of solutions are converge to the infection-free equilibrium $\mathfrak{E}^0 = (2.75, 0, 0)$ of deterministic model [22]. If we set $\beta_1 = 0.0001;\ \beta_2 = 0.006;\ \delta = 0.002;\ \mu = 0.2;\ \omega = 11; \lambda = 0.1$. The eradication of the infection is revealed in Figure 2. Direct calculation yields $\widetilde{\Re}_e^d = 0.1452 < 1$. Thus the parameter values fulfilled the hypothesis prescribed in Theorem 3.2 which leads to the conclusion that the disease dies out exponentially. To show the impact of white noise on SDDE model (3), we increase the intensity of the noise to $\lambda = 0.2$. We can see that the infection goes to zero exponentially very fast when the intensity is small.

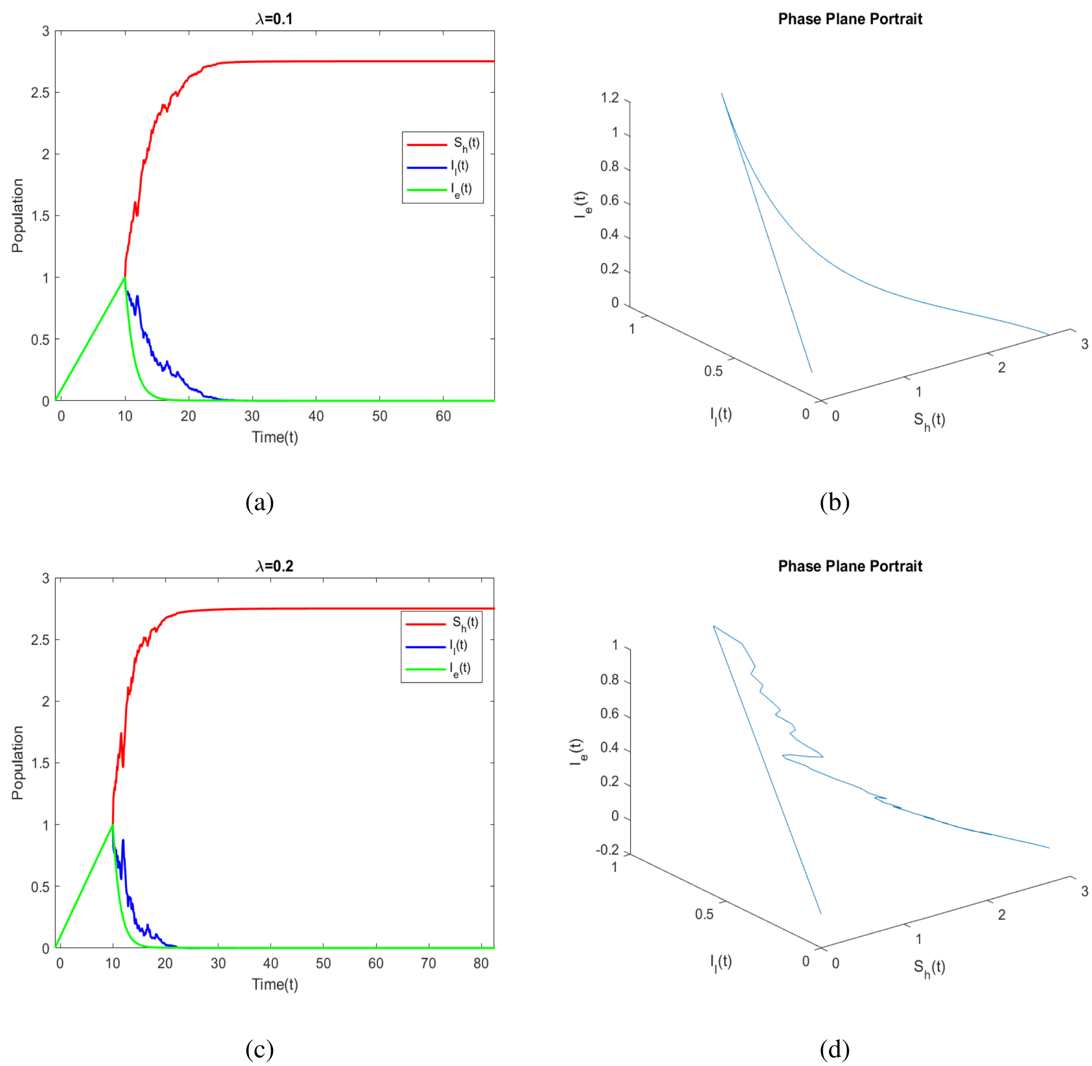

FIGURE 2. Plots of solutions of SDDE model (3) for different noise intensities $\lambda = 0.1$ and $\lambda = 0.2$ with $\omega = 11$, $\beta_1 = 0.0001; \beta_2 = 0.006;\ \delta = 0.002$.

Next, let us choose $\beta_1 = 0.25$; $\beta_2 = 2.3$; $\delta = 0.2$; $\mu = 0.2$; $\omega = 11$, $\lambda = 0.01$. We obtain $\widetilde{\mathfrak{R}}_p^d = 1.0809 > 1$. Thus the hypothesis of Theorem 4.2 holds and thus we can conclude that the infection remains in the population and so the system (3) is ergodic (see Figure 3). If we increase the noise intensity to $\lambda = 0.02$, then $\widetilde{\mathfrak{R}}_p^d = 1.0605 > 1$. This implies that still the SDDE model (3) has a stationary distribution however the magnitude of fluctuations increase. Further, it can be noted that the solution of SDDE model (3) swing around the solution of DDE Model (1) (see Figure 4). The density functions of the state variables of the model (3) are presented in Figure 4.

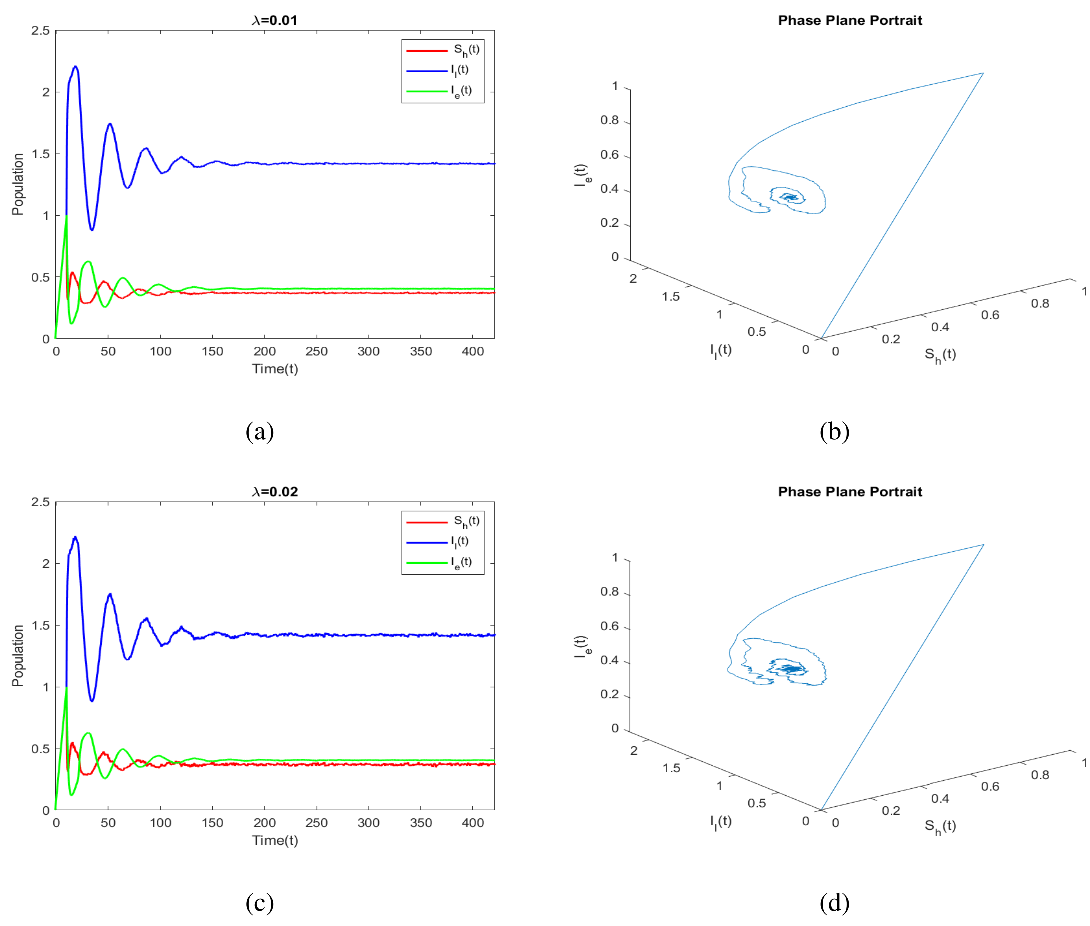

FIGURE 3. Plots of solution of SDDE model(3) with $\lambda = 0.01$ and $\lambda = 0.02$. When $\widetilde{\mathfrak{R}}_p^d > 1$, the disease remains and the model admits a stationary distribution. Notice that the magnitude of the oscillations become stronger when the noise intensity increases.

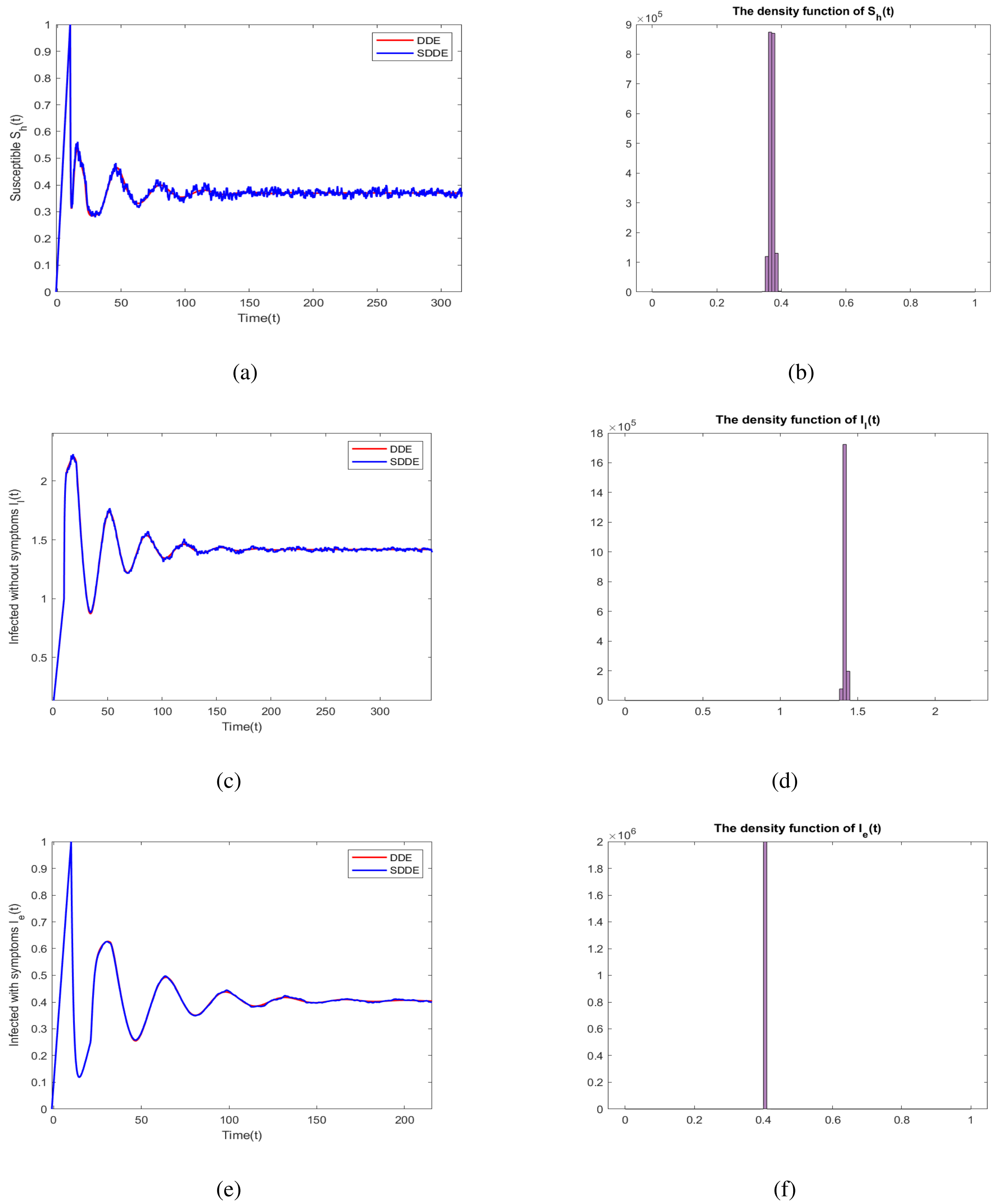

FIGURE 4. Existence of stationary distribution for $\beta_1 = 0.25;\ \beta_2 = 2.3;\ \delta = 0.2;\ \omega = 11,\ \lambda = 0.02$ with $\widetilde{\mathfrak{R}}_p^d > 1$.

6. Conclusion

In this paper, we have developed a delayed HIV/AIDS model with stochastic perturbation to examine the threshold behavior. We showed that the existence and uniqueness of positive global solution of the system (3) and the ergodic stationary distribution are responsible for how long the disease lasts.. In our study, We incorporate the stochastic perturbation into the horizontal transmission parameter β_1 from $S_h(t)$ to $I_l(t)$ to investigate the dynamics of HIV infection. With regard to the threshold value $\widetilde{\mathfrak{R}}_e^d$, we set up certain necessary criteria that ensure the extinction of the disease. It is proved that if $\widetilde{\mathfrak{R}}_e^d < 1$ then the population no longer suffers from the disease. We demonstrated that ergodic stationary distribution exists in a unique way for the system (3) under the threshold value $\widetilde{\mathfrak{R}}_p^d > 1$ by using suitable Lyapunov functions. The dynamics of SDDE model (3) are significantly impacted by the accumulation of environmental perturbations and time-based delay. This also affects how long the disease may last and if it will eventually become extinct.

Conflict of Interests

The authors declare that there is no conflict of interests.

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