Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2024, 2024:62
https://doi.org/10.28919/cmbn/8575
ISSN: 2052-2541

STABILIZING ECOSYSTEM DYNAMICS: THE TOXICITY EFFECT ON AN ECOSYSTEM IN THE PRESENCE OF SELF-DEFENCE

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Abstract: One of the important problems in environmental biology is the effect of toxins on the ecosystems focusing on trophic interactions with the planning of self-defence. This study presents a new eco-toxicant dynamical model to describe the interactions between prey, middle predator, and top predator. In this model, the Lotka-Volterra is used to devour the prey by the middle predator, which is affected by surrounding toxins. The middle predators, in turn, are subject to predation by top predators with Holling-IV functional. However, they spread toxins as a way of defensive strategy. The dynamic behavior of the proposed system is extensively explored by analyzing the boundedness of solutions and equilibrium points, followed by analyzing their stability under various ecological conditions. This study also contributes to a refined understanding of the trophic interplay under toxic stress. Determining key factors that lead to the systems' stability by exploiting the variation in parameter space is a primary objective. Important parameter ranges that potentially determine the stabilities of parameters used in the models are identified. These variations vital factors for stabilizing the ecosystem correspond to environmental

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Received March 30, 2024

toxicity levels and the extent to which species positively or negatively interact. These theoretical results are supported by numerical simulations which provide insights into the complex interactions of the ecosystem under the influence of toxic. It is also supported by a practical comprehensive model of an aquatic ecosystem that focuses on the lionfish species due to its importance in affecting the parrotfish and the moray eel given its predatory tendencies and associated toxicity.

Keywords: dynamical systems; toxicity; ecosystem; Holling-IV; stabilization; self-defence.

2020 AMS Subject Classification: 37B35, 34C23.

1. INTRODUCTION

In ecology systems, the relationship that is used to maintain health is called prey-predator. The interactions between these fundamental components lie in the effect of predators on the prey, and also in the movement of resources and energy that influence the structure of the ecosystems [1-3]. Studying the interactions between prey-predators is not only essential from a scientific perspective, but it is also important in maintaining sustainable coexistence with nature.

The environmental toxins are considered one of the important topics recently due to their important effecting that threaten ecosystem stability [4-5]. The toxins effect adds an additional layer of complexity to these interactions [6]. Many researchers have focused on this important issue including Cogan [7] studied the toxin/antitoxin hypothesis of bacteria in a chemist in three species mathematical model with results that include the ageing hypothesis of biofilms. Huang et al. [8] studied direct and indirect toxic effects on the prey model. Kadhum and Majeed [9] suggested a prey-predator model that consists of a stage structure in both species with harvesting in the prey population and toxicity in all species. Further, more studies in this field have been contributed by many researchers, such as: Lafta and Majeed [10], Al-Joubouri et al. [11], Lemnaouar et al. [12], Liu et al. [13], Zhang et al. [14], Ziem et al. [15], and many others. They have introduced different models to explore the implications of toxins on various ecological systems. These models vary from complex food webs with varying toxicity degrees to systems that undergo periodic detoxification, which highlights the multifaceted impact of toxins on ecosystems.

In addition to their impact on such biological interactions, these systems have long fascinated researchers due to their intricate and often unpredictable behavior. Understanding the dynamics of these systems is crucial for various scientific disciplines, including biology, ecology, engineering, and medicine [16-19].

Based on this background, the current study introduced a new eco-toxicant model by including prey, middle predator and top predator. It integrates the theory of traditional and advanced ecological systems by integrating Lotka-Volterra and Holling-IV to realise the toxic environmental interactions and consuming functional response. The proposed model offers a thorough understanding of ecosystem dynamics under toxic stress by focusing on the adaption strategies of middle predators, such as self-defence and toxin usage as well, as the consequent responses to top predators. The existence of the equilibrium points with their stability and assessing the solution boundedness leads to complementing the theoretical founding with numerical simulation analysis, which provides a deep understanding of the adaptability of ecosystems to environmental toxins.

For the practical application, we present characters about the lionfish that are of interest due to its economic value including its high predation rate; therefore, is likely to significantly impact its prey species population such as the parrotfish. The fact that it is a food source and an ornamental fish suggests that it is commercially harvested but also must be well managed to ensure that the economic benefits accrued are not at the expense of ecological benefits. Involvement of this species includes its incorporation in the eco-toxicant dynamical model and offers potential learning on this species in relation to the ecological and economic.

The remaining parts of this work are organized as follows: Section 2, presents a new eco-toxicant dynamical model to describe the interactions between prey, middle predator and top predator. It also presents an example of the lionfish that reflect the proposed model behavior. The existence of the equilibrium points is studied in Section 3. Section 4 discusses the stabilization of the equilibrium point with the local stability analysis, whereas, the global stability analysis is discussed in Section 5. The theoretical results are supported by numerical simulations in Section 6, which provide insights into the complex interactions of the ecosystem under the influence of toxins. Finally, Section 6 provides some analysis and conclusions to the presented work.

2. THE MODEL FORMULATION

The suggested model consists of three species prey–middle predator-top predator model, which presents as follows:

$$\left.\begin{aligned}
\frac{dZ_1}{dT} &= a_1 Z_1\left(1-\frac{Z_1}{a_2}\right) - a_3 Z_1 Z_2 - a_4 Z_1^2 = f_1(Z_1, Z_2, Z_3)\\
\frac{dZ_2}{dT} &= a_5 Z_1 Z_2 - \frac{a_6 Z_2 Z_3}{a_7 + a_8 Z_2^2} - a_9 Z_2 - a_{10} Z_2^2 Z_3 = f_2(Z_1, Z_2, Z_3)\\
\frac{dZ_3}{dT} &= \frac{a_{11} Z_2 Z_3}{a_7 + a_8 Z_2^2} - a_{12} Z_3 - a_{13} Z_2 Z_3^2 = f_3(Z_1, Z_2, Z_3)
\end{aligned}\right\} \quad (1)$$

Where $Z_1(T), Z_2(T) and\ Z_3(T)$ represent the prey's population density, the middle predator's population density and the top predator's population at time T respectively. All the assumptions of the model are explained in Table 1.

Table 1 The parameters' biological meaning

parameters	Biological meaning
a_1	The prey's population growth rate.
a_2	The prey's carrying capacity.
a_3, a_6	The middle and top predators' maximum predation rate over the prey and middle predator respectively.
a_4	The toxicant environment rate on the prey.
a_{10}, a_{13}	The toxicity rate in the middle predator, and the top predator respectively.
a_5, a_{11}	The conversion rate of food from the prey to the middle predator and from the middle predator to the top predator respectively.
a_7	The middle predator's half-saturation constant.
a_8	The middle predator's defence efficiency.
a_9	The middle predator's harvesting rate.
a_{12}	The top predator's mortality rate.

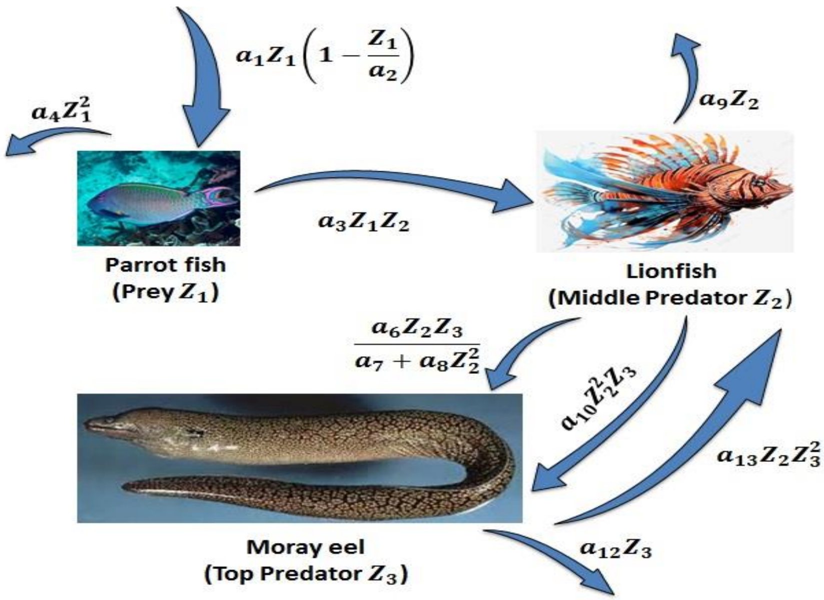

Figure 1. The model's block diagram for system (1)

One of the important applications of the system (1) is the lionfish lives in the Pacific Ocean and is very important economically because of its positive impact. It is one of the most sought-after fish because of its high protein, omega-3, low fat, and carbohydrate content. It is also desirable for decoration because it is a very beautiful fish, but it also hurts the environment in which it is found. The lionfish is a voracious fish and may eat more than twice the size of its stomach. Therefore, it is considered one of the economically influential fish in the species on which the parrotfish feeds. It is a dangerous fish, as the thorns in its fins contain a deadly poison for any organism, even humans. Therefore, it is harvested for these reasons. Therefore, this model has been studied as an application of the proposed system because of its economic and environmental importance. To interpret the parameters of the system (1) in this example as presented in Figure 1, it is as follows:

1. Z_1 represents the parrotfish: The phrase $a_1Z_1(1-Z_1/a_2)$, where a1 is the intrinsic growth rate and a_2 is the environment's carrying capacity, refers to the logistic growth of the parrotfish population. The parrotfish-lionfish interaction $a_3Z_1Z_2$ shows how the lionfish prey on the parrotfish, with a_3 representing the predation rate coefficient. The term of the parrotfish with $a_4Z_1Z_2$ may show how toxicity or intraspecific competition has affected the parrotfish

population, with a4 denoting how strong an effect it has.

2. Z_2 represents the Lionfish: The term $a_5Z_1Z_2$ shows how the lionfish grow depending on the parrotfish, with a_5 representing the conversion rate of food from the parrotfish to the Lionfish. The expression $a_6Z_2Z_3/(a_7+a_8Z_2^2)$ may denote a Holling type-II functional response, in which a_6 representing the Moray Eel predators' maximum predation rate over the Lionfish, while a_8 adjusts the response to the middle predator's (lionfish) self-defence and a_7 is the half-saturation constant. Because of its commercial value, a_9Z_2 may represent the lionfish's harvesting rate at which people harvest them.

3. Z_3 represents the Moray Eel: The expression $a_{11}Z_2Z_3/(a_7+a_8Z_2^2)$ may show how the Moray Eel grows depending on the Lionfish, in which a_{11} adjusts The conversion rate of food from the Lionfish to the Moray Eel. The moray eel's natural mortality is represented by $a_{12}Z_2Z_3$.

4. Inter-Species Interactions: The terms $a_{10}Z_1Z_2$ and $a_{13}Z_2Z_3$ refer to intricate interactions that may include defence mechanisms, toxicity effects, or competition among food chain members. For instance, a13 might represent how the lionfish's toxicity influences the moray eel's rate of predation.

The directional arrows show how biomass is transferred from prey to predator and how energy flows in both directions. It also shows the direction of the poison's effect on predators. The diagram captures the dynamics of predation and the potential non-linear effects of interactions, such as the poisoning of lionfish by its predators or the over-predation of parrotfish.

Theorem 2.1. The solutions of system (1) which are in R_+^3 are uniformly bounded.

Proof: To confirm that system's (1) solutions are uniformly bounded we have to suppose a function [2]: $H(T) = Z_1(T) + Z_2(T) + Z_3(T)$.

Let $(Z_1(T), Z_2(T), Z_3(T))$ be a solution for system (1) with an initial condition $H(0) = H_0 = (Z_1(0), Z_2(0), Z_3(0)) \in R_+^3$. Now differentiate $H(T)$ with respect to time and the biological conditions $a_3 > a_5, a_6 > a_{11}$ we get:

$\frac{dH}{dT} \leq 2a_1Z_1 - SH < 2a_1Z_1(0)$, where $= min\{a_1, a_9, a_{12}\}$.

Since from Eq. (1) of system (1) we have,

$\frac{dZ_1}{dT} < a_1 Z_1$, by using Gronwall Lemma [20] and $Z_1(0)$ as an initial point we get

$Z_1(T) < Z_1(0)e^{-a_1 T}$, thus, $\lim_{T\to\infty} Z_1(T) = sup. Z_1(T) < Z_1(0), \forall T > 0$.

So, $\frac{dH}{dT} + SH < 2a_1 Z_1(0)$,

Now, by using Gronwall Lemma forH_0 we get:

$H(T) < \frac{S+(2a_1 Z_1(0)H_0 - S)e^{-ST}}{2a_1 Z_1(0)}$, thus, $\lim_{T\to\infty} H(T) = sup. H(T) < \frac{S}{2a_1 Z_1(0)} \quad \forall T > 0$.

So, $0 \le H(T) < \frac{S}{2a_1 Z_1(0)}$, $\forall T > 0$.

3. THE EXISTENCE OF THE EQUILIBRIUM POINTS

In this section, all the probable equilibrium points of system (1) and all the conditions of their existence have been found in the following:

1. $M_0 = (0,0,0)$ always exists.

2. $M_1 = (\bar{Z}_1, 0, 0)$, where $\bar{Z}_1 = \frac{a_1 a_2}{a_1 - a_2 a_4}$ exists if $a_1 > a_2 a_4$. (2)

3. $M_2 = (\bar{\bar{Z}}_1, \bar{\bar{Z}}_2, 0)$, where $\bar{\bar{Z}}_1 = \frac{a_9}{a_5}$ and $\bar{\bar{Z}}_2 = \frac{a_1 a_2 a_5 - a_9(a_1 - a_2 a_4)}{a_2 a_3 a_5}$ exists if condition (2) holds and $a_1 a_2 a_5 > a_9(a_1 - a_2 a_4)$.

$M_3 = (Z_1^*, Z_2^*, Z_3^*)$ exists if the three following equations have a positive solution:

$$a_1\left(1 - \frac{Z_1}{a_2}\right) - a_3 Z_2 - a_4 Z_1 = 0, \tag{3}$$

$$a_5 Z_1 - \frac{a_6 Z_3}{a_7 + a_8 Z_2^2} - a_9 - a_{10} Z_2 Z_3 = 0, \tag{4}$$

$$\frac{a_{11} Z_2}{a_7 + a_8 Z_2^2} - a_{12} - a_{13} Z_2 Z_3 = 0, \tag{5}$$

From Eq. (3) get,

$$Z_1 = \frac{a_2(a_1 - a_3 Z_2)}{a_1 - a_2 a_4} , \tag{6}$$

From Eq. (5) get

$$Z_3 = \frac{1}{a_{13} Z_2}\left(\frac{a_{11} Z_2}{a_7 + a_8 Z_2^2} - a_{12}\right), \tag{7}$$

By substituting Eqs. (6)and(7) in (4) we obtain:

$$\omega_1 Z_2^6 + \omega_2 Z_2^5 + \omega_3 Z_2^4 + \omega_4 Z_2^3 + \omega_5 Z_2^2 + \omega_6 Z_2 + \omega_7 = 0, \tag{8}$$

where

$\omega_1 = -a_2 a_3 a_5 a_8^2 < 0,$

$\omega_2 = a_8^2[a_1 a_2 a_3 a_5 a_{13} + (a_1 - a_2 a_4)(a_{10} a_{12} - a_9 a_{13})],$

$\omega_3 = -2a_2 a_3 a_5 a_7 a_8 a_{13} < 0,$

$\omega_4 = 2a_7 a_8[(a_1 - a_2 a_4)(a_{10} a_{12} - a_9 a_{13}) - a_1 a_2 a_5] - a_8 a_{10} a_{11}(a_1 - a_2 a_4),$

$\omega_5 = -[(a_1 - a_2 a_4)(a_7 a_{10} a_{11} - a_6 a_8 a_{12}) + a_2 a_3 a_5 a_7^2 a_{13}],$

$\omega_6 = (a_1 - a_2 a_4)[a_7^2(a_{10} a_{12} - a_9 a_{13}) - a_6 a_{11}] + a_1 a_2 a_5 a_7^2,$

$\omega_7 = a_6 a_7 a_{12}(a_1 - a_2 a_4),$

So, Eq. (8) by Discarte rule has three roots or a unique positive root say Z_2^* if condition (2) and the following conditions hold.

$a_{10} a_{12} < a_9 a_{13},$

$a_1 a_2 a_3 a_5 a_{13} > (a_1 - a_2 a_4)(a_9 a_{13} - a_{10} a_{12}).$

So, $M_3 = (Z_1^*, Z_2^*, Z_3^*)$, where $Z_1(Z_2^*) = Z_1^* > 0$ and $Z_3(Z_2^*) = Z_3^* > 0$ if in addition to condition (2) the next condition holds:

$$\frac{a_1}{a_3} > Z_2^* > \frac{a_{12}(a_7 + a_8 Z_2^2)}{a_{11}}.$$

4. STABILIZATION AND LOCAL STABILITY ANALYSIS

Controlling the dynamical of the eco-toxicant model is performed by modifying its parameters to reach a stable state, such as a stable equilibrium point. However, we need to specify the parameters that influence the stability of system (1) (e.g. growth rate, predation rates, toxicity level, etc.), and then adjust them to reach the desired stability. However, to enable us to stabilize the system, we first need to know the parameters that give this stable state, and then adjust them accordingly to a stable state. We start by analyzing the local stability of the equilibrium points, which requires linearizing the system around these points to analyze the resulting eigenvalues.

One can play around the values of say the rate of toxin spread or the defence efficiency of the middle predator and determine how much this affects the stability of the points. How large can these parameters be increased such that the system becomes unstable? This kind of linear analysis is what has been utilized in most literature in stabilizing the models [20-22]. The eigenvalues of the Jacobian matrix $J_i = J(M_i)$ can be represented by $\lambda_{iZ_1}, \lambda_{iZ_2}\ and\ \lambda_{iZ_3}; i = 0,1,2,3$.

$$J_i = \begin{bmatrix} \frac{\partial f_1}{\partial Z_1} & \frac{\partial f_1}{\partial Z_2} & \frac{\partial f_1}{\partial Z_3} \\ \frac{\partial f_2}{\partial Z_1} & \frac{\partial f_2}{\partial Z_2} & \frac{\partial f_2}{\partial Z_3} \\ \frac{\partial f_3}{\partial Z_1} & \frac{\partial f_3}{\partial Z_2} & \frac{\partial f_3}{\partial Z_3} \end{bmatrix}, \tag{9}$$

where $\frac{\partial f_1}{\partial Z_1} = a_1 - \frac{2a_1 Z_1}{a_2} - a_3 Z_2 - 2a_4 Z_1, \frac{\partial f_1}{\partial Z_2} = -a_3 Z_1, \frac{\partial f_1}{\partial Z_3} = 0, \frac{\partial f_2}{\partial Z_1} = a_5 Z_2,$

$$\frac{\partial f_2}{\partial Z_2} = a_5 Z_1 - a_9 - 2a_{10} Z_2 Z_3 - \frac{a_6 Z_3 (a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2}, \frac{\partial f_2}{\partial Z_3} = -Z_2 \left(\frac{a_6}{a_7 + a_8 Z_2^2} + a_{10} Z_2 \right),$$

$$\frac{\partial f_3}{\partial Z_1} = 0, \frac{\partial f_3}{\partial Z_2} = Z_3 \left[\frac{a_{11}(a_7 - a_8 Z_2^2)}{\left(a_7 + a_8 Z_2^2\right)^2} - a_{13} Z_3 \right], \frac{\partial f_3}{\partial Z_3} = \frac{a_{11} Z_2}{a_7 + a_8 Z_2^2} - a_{12} - 2a_{13} Z_2 Z_3.$$

A. LOCAL STABILITY FOR $M_0 = (0,0,0)$

$$J_0 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & -a_9 & 0 \\ 0 & 0 & -a_{12} \end{bmatrix}.$$

Then M_0 is unstable since the eigenvalues of the characteristic equation of J_0 are $\lambda_{0Z_1} = a_1 > 0, \lambda_{0Z_2} = -a_9 < 0, \lambda_{0Z_3} = -a_{12} > 0.$

B. LOCAL STABILITY FOR $M_1 = (\bar{Z}_1, 0,0)$.

$$J_1 = J(M_1) = \begin{bmatrix} a_1 - 2\bar{Z}_1(\frac{a_1}{a_2} + a_4) & -a_3\bar{Z}_1 & 0 \\ 0 & a_5\bar{Z}_1 - a_9 & 0 \\ 0 & 0 & -a_{12} \end{bmatrix},$$

The characteristic equation of J_2 can be given by:

$$\lambda^3 + O_1\lambda^2 + O_2\lambda + O_3 = 0. \tag{10}$$

where

$O_1 = a_9 + a_{12} + 2\bar{Z}_1\left(\frac{a_1}{a_2} + a_4\right) - a_1 - a_5\bar{Z}_1,$

$O_2 = \left[a_1 - 2\bar{Z}_1(\frac{a_1}{a_2} + a_4)\right][a_5\bar{Z}_1 - a_9 - a_{12}] - a_{12}(a_5\bar{Z}_1 - a_9)\ ,$

$O_3 = a_{12}\left[a_1 - 2\bar{Z}_1(\frac{a_1}{a_2} + a_4)\right][a_5\bar{Z}_1 - a_9].$

By Routh-Hurwitz principle [23] the roots of Eq.(16), have negative real parts if and only if $O_1 > 0, O_3 > 0$ and $\Delta = O_4 - O_3 > 0$, where $O_4 = O_1O_2$ if the next conditions hold:

$a_9 + a_{12} + 2\bar{Z}_1\left(\frac{a_1}{a_2} + a_4\right) > a_1 + a_5\bar{Z}_1\ ,$

$a_1 > 2\bar{Z}_1(\frac{a_1}{a_2} + a_4)$

$a_5\bar{Z}_1 > a_9$

$O_4 > O_3$

If the above condition is not satisfied to make the point stable, the stability conditions of the point can be controlled to achieve our aim.

C. LOCAL STABILITY FOR $M_2 = (\bar{\bar{Z}}_1, \bar{\bar{Z}}_2, 0)$

$$J_2 = \begin{bmatrix} a_1 - 2\bar{\bar{Z}}_1(\frac{a_1}{a_2} + a_4) - a_3\bar{\bar{Z}}_2 & -a_3\bar{\bar{Z}}_1 & 0 \\ a_5\bar{\bar{Z}}_2 & a_5\bar{\bar{Z}}_1 - a_9 & -\bar{\bar{Z}}_2\left(\frac{a_6}{a_7 + a_8\bar{\bar{Z}}_2^2} + a_{10}\bar{\bar{Z}}_2\right) \\ 0 & 0 & \frac{a_{11}\bar{\bar{Z}}_2}{a_7 + a_8\bar{\bar{Z}}_2^2} - a_{12} \end{bmatrix},$$

The characteristic equation of J_2 can be given by:

$$\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 = 0. \tag{11}$$

where

$P_1 = \bar{\bar{Z}}_1\left(2\left(\frac{a_1}{a_2} + a_4\right) - a_5\right) + \bar{\bar{Z}}_2\left(a_3 - \frac{a_{11}}{a_7 + a_8\bar{\bar{Z}}_2^2}\right) + a_9 + a_{12} - a_1,$

$P_2 = \left[\frac{a_{11}\bar{\bar{Z}}_2}{a_7 + a_8\bar{\bar{Z}}_2^2} - a_{12}\right][a_5\bar{\bar{Z}}_1 - a_9 - a_3\bar{\bar{Z}}_2] + a_3a_9\bar{\bar{Z}}_2 + \left[a_1 - 2\bar{\bar{Z}}_1\left(\frac{a_1}{a_2} + a_4\right)\right][\frac{a_{11}\bar{\bar{Z}}_2}{a_7 + a_8\bar{\bar{Z}}_2^2}$

$\quad + a_5\bar{\bar{Z}}_1 - (a_9 + a_{12})]\ ,$

$$P_3 = \left[\frac{a_{11}\bar{\bar{Z}}_2}{a_7 + a_8\bar{\bar{Z}}_2^2} - a_{12}\right]\left[\left(a_9 - a_5\bar{\bar{Z}}_1\right)\left[a_1 - 2\bar{\bar{Z}}_1\left(\frac{a_1}{a_2} + a_4\right)\right] - a_3a_9\bar{\bar{Z}}_2\right].$$

By the Routh-Hurwitz principle the roots of Eq.(11), *have negative real parts if and only if* $P_1 > 0, P_3 > 0$ *and* $\Delta = P_4 - P_3 > 0$, *where* $P_4 = P_1P_2$ *if the next conditions hold:*

$$\frac{a_9}{\bar{\bar{Z}}_1} \leq a_5 < 2\left(\frac{a_1}{a_2} + a_4\right) < \frac{a_1}{\bar{\bar{Z}}_1},$$

$$\frac{a_{11}}{a_7 + a_8\bar{\bar{Z}}_2^2} < min\left\{a_3, \frac{a_{12}}{\bar{\bar{Z}}_2}\right\},$$

$$\bar{\bar{Z}}_1\left(2\left(\frac{a_1}{a_2} + a_4\right) - a_5\right) + \bar{\bar{Z}}_2\left(a_3 - \frac{a_{11}}{a_7 + a_8\bar{\bar{Z}}_2^2}\right) + a_9 + a_{12} > a_1,$$

$$P_4 > P_3,$$

Thus, M_2 becomes local asymptotically stable. If the above, conditions are not met. However, to stabilize this point, the stability conditions can be controlled to achieve the conduct of the stability condition.

D. LOCAL STABILITY FOR $M_3 = (Z_1^*, Z_2^*, Z_3^*)$

$$J_3 = \begin{bmatrix} -Z_1^*\left(\frac{a_1}{a_2} + a_4\right) & -a_3Z_2^* & 0 \\ a_5Z_2^* & a_5Z_1^* - a_9 - 2a_{10}Z_2^*Z_3^* - \frac{a_6Z_3^*(a_7 - a_8Z_2^{*2})}{(a_7 + a_8Z_2^{*2})^2} & -Z_2^*\left(\frac{a_6}{a_7 + a_8Z_2^{*2}} + a_{10}Z_2^*\right) \\ 0 & Z_3^*\left(\frac{a_{11}(a_7 - a_8Z_2^{*2})}{(a_7 + a_8Z_2^{*2})^2} - a_{13}Z_3^*\right) & -a_{13}Z_2^*Z_3^* \end{bmatrix},$$

The characteristic equation of J_3 is given by:

$$\lambda^3 + \mu_1\lambda^2 + \mu_2\lambda + \mu_3 = 0, \qquad (12)$$

where

$$\mu_1 = Z_1^*\left(\frac{a_1}{a_2} + a_4 - a_5\right) + a_9 + Z_2^*Z_3^*[2a_{10} + a_{13}] + \frac{a_6Z_3^*(a_7 - a_8Z_2^{*2})}{(a_7 + a_8Z_2^{*2})^2},$$

$$\mu_2 = Z_1^*Z_2^*\left[a_3a_5 + a_{13}Z_3^*\left(\frac{a_1}{a_2} + a_4\right)\right] + (a_{13}Z_2^*Z_3^* + Z_1^*(\frac{a_1}{a_2} + a_4))(a_9 + 2a_{10}Z_2^*Z_3^* + \frac{a_6Z_3^*(a_7 - a_8Z_2^{*2})}{(a_7 + a_8Z_2^{*2})^2} - a_5Z_1^*),$$

$$\mu_3 = Z_2^* Z_3^* \left(Z_1^* \left[a_3 a_5 Z_2^* + a_{13} \left(\frac{a_1}{a_2} + a_4 \right) \left(a_9 + 2a_{10} Z_2^* Z_3^* + \frac{a_6 Z_3^* (a_7 - a_8 Z_2^{*2})}{(a_7 + a_8 Z_2^{*2})^2} - a_5 Z_1^* \right) \right] - \left(\frac{a_6}{a_7 + a_8 Z_2^{*2}} + a_{10} Z_2^* \right) \left(\frac{a_{11}(a_7 - a_8 Z_2^{*2})}{(a_7 + a_8 Z_2^{*2})^2} - a_3 Z_3^* \right) \right)$$

The roots of Eq. (12), have negative real parts according to the Routh-Hurwitz principle if and only if $\mu_1 > 0, \mu_3 > 0$ *and* $\Delta = \mu_4 - \mu_3 > 0$ where $\mu_4 = \mu_1 \mu_2$. If the following conditions hold:

$a_5 < min\left\{\frac{a_1}{a_2} + a_4, \frac{1}{Z_1^*}\left[a_9 + 2a_{10}Z_2^*Z_3^* + \frac{a_6Z_3^*(a_7 - a_8Z_2^{*2})}{(a_7 + a_8Z_2^{*2})^2}\right]\right\}$,

$a_7 > a_8 Z_2^{*2}$,

$\frac{a_{11}(a_7 - a_8Z_2^{*2})}{(a_7 + a_8Z_2^{*2})^2} < a_3 Z_3^*$, *and* $\mu_4 > \mu_3$.

Thus, M_3 becomes local asymptotically stable. However, from the practical side, the lionfish is best known for its high rate of predation, which indeed can lead to significant influences on the prey species, particularly the parrotfish. I presumed that this fact is reflected in the local stability of your model. According to the local stability analysis in my package, changes in predation rates caused by varying the predation level can change the system's stability. Consequently, the high predation rate of the lionfish might mean the parrotfish is likely unstable or potentially stable when certain measures are introduced. Your model might have recommended using the information that the lionfish population is likely stable when appropriately managed by policies because the species are stable at a particular level without causing a disturbance in the whole ecosystem. Alternatively, the model might assume possible and serious future consequences of lionfish predation, which require more severe preventative measures or eradication programs.

5. THE GLOBAL STABILITY ANALYSIS

In this section, we discuss the global stability analysis of the ecotoxicological model, which is key to understanding the resilience and behavior of ecosystems in the long term when subjected to different environmental conditions. Both analyses approaches complement each other: global stability provides the most integrated vision and reveals system dynamics over the entire phase space, whereas local stability substantiates the behavior of the system in the vicinity of equilibrium points. The goal of this section is to find out whether the previously obtained equilibrium states are global, which implies that the systems tend to return to them from various initial conditions

$$G_3^{\circ}(Z_1, Z_2, Z_3) = \left(Z_1 - Z_1^{\circ} - Z_1^{\circ}\, ln\frac{Z_1}{Z_1^{\circ}}\right) + \left(Z_2 - Z_2^{\circ} - Z_2^{\circ}\, ln\frac{Z_2}{Z_2^{\circ}}\right) + \left(Z_3 - Z_3^{\circ} - Z_3^{\circ}\, ln\frac{Z_3}{Z_3^{\circ}}\right),$$

This function [1] has been used with the Lyapunov method to study the global stability for all local asymptotically equilibrium points.

Theorem 5.1. Suppose that $M_1 = (\bar{Z}_1, 0,0)$ of the system (1) is local asymptotically stable (LAS) in R_+^3. Then M_1 is globally asymptotically stable (GAS) under the condition

$$a_3 \geq a_2\ . \tag{13}$$

Proof: Let $\bar{G}_1(Z_1, Z_2, Z_3) = \left(Z_1 - \bar{Z}_1 - \bar{Z}_1\, ln\frac{Z_1}{\bar{Z}_1}\right) + Z_2 + Z_3$.

Now differentiating $\bar{G}_1$ for time T, where C^1 obviously is a positive definite function (PDF), $\bar{G}_1 : R_+^3 \rightarrow R$. Then by some algebraic calculation, we get:

$$\frac{d\bar{G}_1}{dT} = -(Z_1 - \bar{Z}_1)^2\left(\frac{a_1}{a_2} + a_4\right) - Z_1Z_2(a_3 - a_2) - a_3\bar{Z}_1Z_2 - \frac{Z_2Z_3}{a_7 + a_8Z_2^2}(a_6 - a_{11}) - a_9Z_2$$
$$- a_{12}Z_3 - a_{10}Z_2^2Z_3\ - a_{13}Z_2Z_3^2.$$

By the biological fact $a_6 > a_{11}$ and condition (13) get

$$\frac{d\bar{G}_1}{dT} < -(Z_1 - \bar{Z}_1)^2\left(\frac{a_1}{a_2} + a_4\right) - Z_1Z_2(a_3 - a_2) - a_3\bar{Z}_1Z_2 - Z_2(a_9 + a_{10}Z_2Z_3)$$
$$- Z_3(\ a_{12} - a_{13}Z_2Z_3) < 0$$

Theorem 5.2. Assume that the system's (1) equilibrium point $M_2 = (\bar{\bar{Z}}_1, \bar{\bar{Z}}_2, 0)$ is (LAS) in R_+^3. Then M_2 is (GAS) if and only if:

$$Z_1 > \bar{\bar{Z}}_1, \tag{14}$$

$$Z_2 > \bar{\bar{Z}}_2\ , \tag{15}$$

$$a_3 > a_5\ , \tag{16}$$

$$\left(Z_1-\bar{\bar{Z}}_1\right)^2\left(\frac{a_1}{a_2}+a_4\right)+\left(Z_1-\bar{\bar{Z}}_1\right)\left(Z_2-\bar{\bar{Z}}_2\right)(a_3-a_5)+Z_3[a_{12}+Z_2(\,a_{13}Z_3+a_{10}Z_2)]$$
$$>\bar{\bar{Z}}_2Z_3\left(\frac{a_6}{a_7+a_8Z_2^2}-a_{10}Z_2\right). \quad (17)$$

Proof: Consider $\bar{\bar{G}}_2(Z_1,Z_2,Z_3)=\left(Z_1-\bar{\bar{Z}}_1-Z_1\,ln\frac{Z_1}{\bar{\bar{Z}}_1}\right)+\left(Z_2-\bar{\bar{Z}}_2-Z_2\,ln\frac{Z_2}{\bar{\bar{Z}}_2}\right)+Z_3$

Obviously $\bar{\bar{G}}_2\colon R_+^3\to R$ is a C^1 (PDF). Now $\bar{\bar{G}}_2$ could be differentiating for time T and with the help of some algebraic manipulation get:

$$\frac{d\bar{\bar{G}}_2}{dT}=-\left(Z_1-\bar{\bar{Z}}_1\right)^2\left(\frac{a_1}{a_2}+a_4\right)-\left(Z_1-\bar{\bar{Z}}_1\right)\left(Z_2-\bar{\bar{Z}}_2\right)(a_3-a_5)-\frac{Z_2Z_3}{a_7+a_8Z_2^2}(a_6-a_{11})$$
$$+\bar{\bar{Z}}_2Z_3\left(\frac{a_6}{a_7+a_8Z_2^2}+a_{10}Z_2\right)-Z_3[a_{12}+Z_2(\,a_{13}Z_3+a_{10}Z_2)].$$

By the biological fact $a_6>a_{11}$ get

$\frac{d\bar{\bar{G}}_2}{dT}<-\left(Z_1-\bar{\bar{Z}}_1\right)^2\left(\frac{a_1}{a_2}+a_4\right)-\left(Z_1-\bar{\bar{Z}}_1\right)\left(Z_2-\bar{\bar{Z}}_2\right)(a_3-a_5)+\bar{\bar{Z}}_2Z_3\left(\frac{a_6}{a_7+a_8Z_2^2}-a_{10}Z_2\right)$
$-Z_3[a_{12}+Z_2(\,a_{13}Z_3+a_{10}Z_2)].$

According to the conditions $(14)-(17)$ we have $\frac{d\bar{\bar{G}}_2}{dT}<0.$

Theorem 5.3. Suppose that the system's (1) equilibrium point $M_3=(Z_1^*,Z_2^*,Z_3^*)$ is (LAS) in R_+^3. Then M_3 is (GAS) under the conditions (18-20):

$Z_1>Z_1^*$, (18)

$Z_2>Z_2^*$, (19)

$V_1<2\sqrt{V_2a_{13}Z_2}$, (20)

Where $V_1=a_{10}Z_2-a_{13}Z_3^*-\frac{1}{(a_7+a_8Z_2^2)(a_7+a_8Z_2^{*2})}\left([a_7+a_8Z_2^{*2}][a_6-a_{11}^2]+a_8a_{11}^2Z_2(Z_2+Z_2^*)\right), V_2=\left(a_{10}Z_3^*-\frac{a_6a_8Z_3^*(Z_2+Z_2^*)}{(a_7+a_8Z_2^2)(a_7+a_8Z_2^{*2})}\right).$

Proof: Let

$$G_3^*(Z_1,Z_2,Z_3)=\left(Z_1-Z_1^*-Z_1^*\,ln\frac{Z_1}{Z_1^*}\right)+\left(Z_2-Z_2^*-Z_2^*\,ln\frac{Z_2}{Z_2^*}\right)+\left(Z_3-Z_3^*-Z_3^*\,ln\frac{Z_3}{Z_3^*}\right).$$

Clearly $G_3^*\colon R_+^3\to R$ is a C^1 (PDF). Now by differentiating G_3^* for time T get:

$$\frac{dG_3^*}{dT} = -(Z_1 - Z_1^*)^2\left(\frac{a_1}{a_2} + a_4\right) - (a_3 - a_5)(Z_1 - Z_1^*)(Z_2 - Z_2^*) - (Z_2 - Z_2^*)^2 V_2 - a_{13}Z_2(Z_3 - Z_3^*)^2 - (Z_2 - Z_2^*)(Z_3 - Z_3^*)V_1.$$

By the conditions $(18) - (20)$ get

$$\frac{dG_3^*}{dT} < -(Z_1 - Z_1^*)^2\left(\frac{a_1}{a_2} + a_4\right) - (a_3 - a_5)(Z_1 - Z_1^*)(Z_2 - Z_2^*) - \left[\sqrt{V_2}(Z_2 - Z_2^*) - \sqrt{a_{13}Z_2}(Z_2 - Z_2^*)\right]^2.$$

Now if conditions (16) holds. So, $\frac{\mathrm{dG}_3^*}{\mathrm{dT}} < 0$.

An extensive understanding of the system's behavior over time and under different initial conditions is provided by global stability analysis. This may entail knowing how the lionfish's population will fluctuate over time in relation to that of its prey, such as the parrotfish. If so, it implies that the lionfish population can grow to a size where it can live in harmony with its prey without endangering the ecosystem in the long run. The effects of varying harvesting intensities on the ecosystem's overall stability can be seen through the global stability analysis. It can provide answers to queries such as: How much can be harvested from lionfish populations without causing the dynamics between predators and prey to become unstable? Alternatively, what effect does overharvesting have on the ecosystem's long-term stability?

The findings can be used to illustrate how the presence of a poisonous predator such as lionfish affects the ecosystem's long-term dynamics and what needs to happen in order to lessen any negative effects.

6. NUMERICAL SIMULATION

In previous sections system (1) has been studied theoretically and now to prove the validity of the system, MATLAB [2] has been used to consider the system numerically. The effectiveness of parameters has been shown on the dynamics of the model by observing the parameters' set given in (21) which achieves the positive equilibrium point stability conditions, as seen in Fig.2 (a-d) the solution converges asymptotically to $M_3 = (0.059, 2.657, 0.093)$ starting from three initial points $(5,3,1)$, $(2,3,5)$ and $(1,2,3)$, which proves that the system is valid that we selected three randomly initial points and from all of them the solution converges to one positive equilibrium point M_3.

$$\left.\begin{array}{c} a_1 = 1, a_2 = a_3 = 0.3, a_4 = a_6 = 0.1, a_5 = 0.25, a_7 = a_8 = 0.5, \\ a_9 = a_{10} = a_{12} = 0.01, a_{11} = 0.09, a_{13} = 0.2 \end{array}\right\} \quad (21)$$

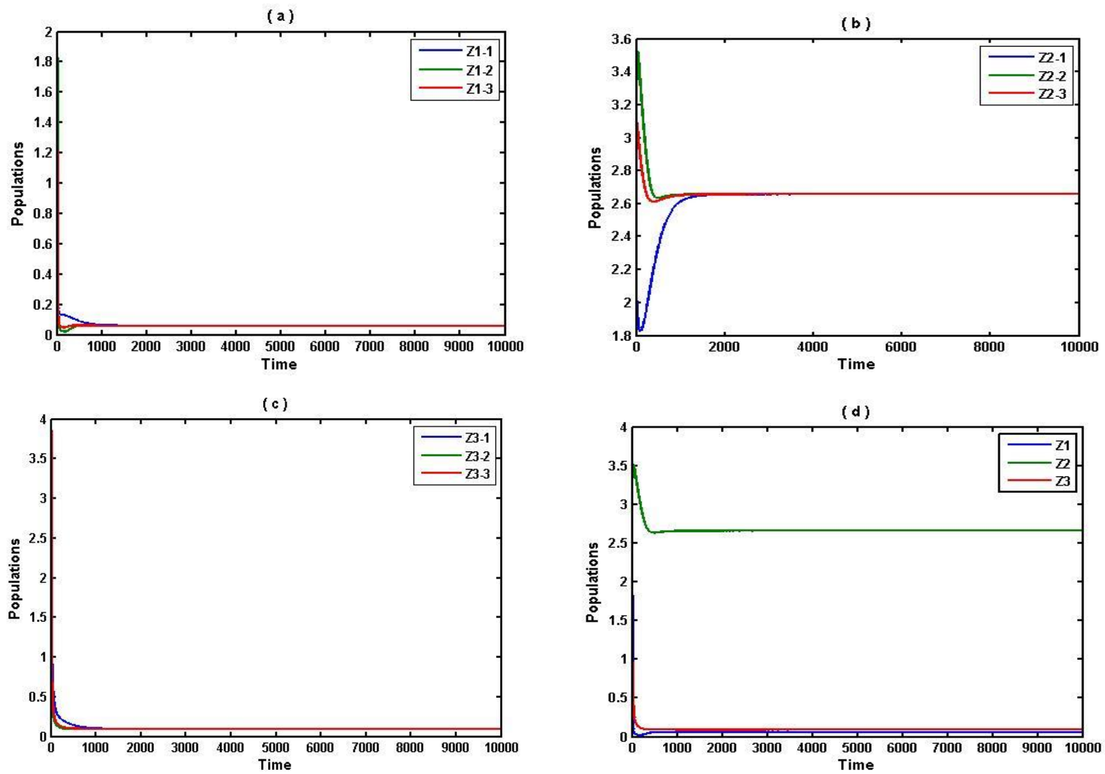

Figure.2 Time series of system's (1) solution starting from three different initial points $(5,3,1), (2,3,5)$ and $(1,2,3)$ for data in (21) (a) Trajectories of Z_1, (b) Trajectories of Z_2, (c) Trajectories of Z_3, (d) The solution converges asymptotically to $M_3 = (0.059, 2.657, 0.093)$.

To argue the impacts of the system's (1) parameters on the dynamic system behaviour. One parameter is changed each time for data given in (21). Changing the parameter a_1(the prey's growth rate), it is seen that in the range of $0.1 \leq a_1 < 1.5$, system's (1) path converges to M_3 and this means that changing this parameter did not cause the extinction of this food chain, see Fig.3, for the perfect value $a_1 = 0.5$.

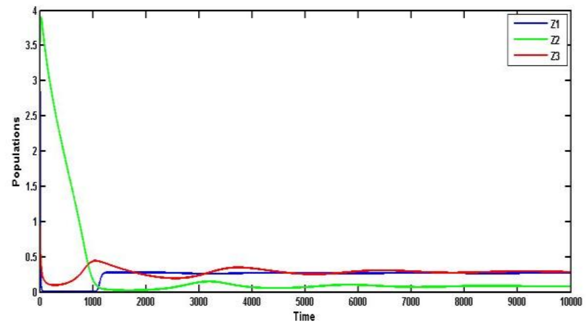

Figure.3: The time series (TS) of the system's (1) solution converges to $M_3 = (0.2696,0.0789,0.2848)$ with data (21) for perfect value $a_1 = 0.5$.

Changing the parameter a_2 (the prey's carrying capacity), it is seen that in the range of $0.2 \leq a_2 < 1$ the system's (1) path converges to M_3, so this parameter was unaffected and did not cause the extinction of this food chain, see Fig. 4, for the perfect value $a_2 = 0.6$.

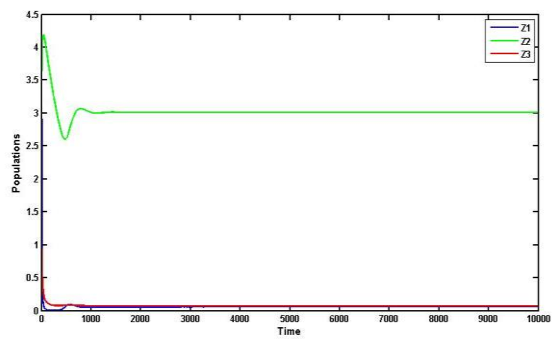

Figure 4: TS of the system's (1) solution converges to $M_3 = (0.0545,3.0121,0.0727)$ for given data in (21) for the perfect value $a_2 = 0.6$.

Now, by varying the parameter a_3 (the middle's maximum predation rate over the prey), it is seen that in the range $0.3 \leq a_3 < 1$, the system's (1) path converges to M_3, so it does not affect the extinction of these organisms see Fig.5, for the perfect value $a_3 = 0.7$.

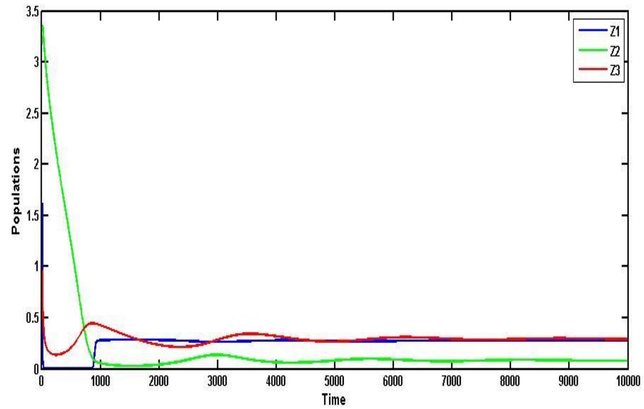

Figure 5: TS of the system's (1) solution converges to $M_3 = (0.2745,0.0821,0.291)$ for given data in (21) for perfect value $a_3 = 0.7$.

By changing the parameter a_4 (the toxicant environment rate on the prey), it is seen that system's (1) path converges to M_3 in the range $0.1 \leq a_4 < 1$, so it does not affect the extinction of these organisms see Fig.6, for the perfect value $a_4 = 0.3$.

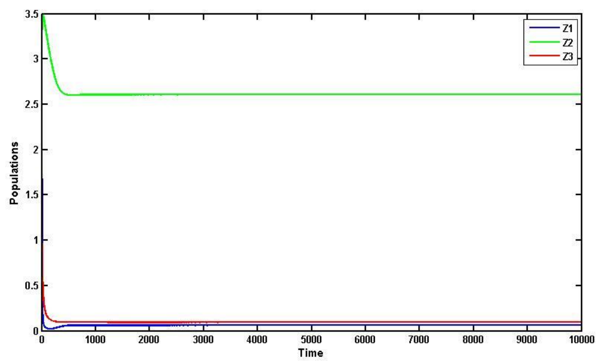

Figure 6: (TS) of the system's (1) solution converges to $M_3 = (0.0599,2.6079,0.0962)$ for the data in (21) and the perfect value $a_4 = 0.3$.

The effectiveness of changing the parameter a_5(the conversion rate of food from the prey to the middle predator). It is observed that the system's (1) solution approaches to M_1 for the range $0.01 < a_5 \leq 0.05$, the parameter was effective as only the parrot fish remained as seen in Fig.7(a) for perfect value $a_5 = 0.04$, whereas for $0.05 < a_5 \leq 0.08$, the solution approaches to M_2, the parameter was effective as only the parrot fish and the lionfish remained as seen in Fig.7(b) for perfect value $a_5 = 0.07$, and approaches to M_3 for the range $0.08 < a_5 \leq 0.3$.

the parameter restores the system to non-extinction as seen in Fig.7(c) for the perfect value $a_5 = 0.15$.

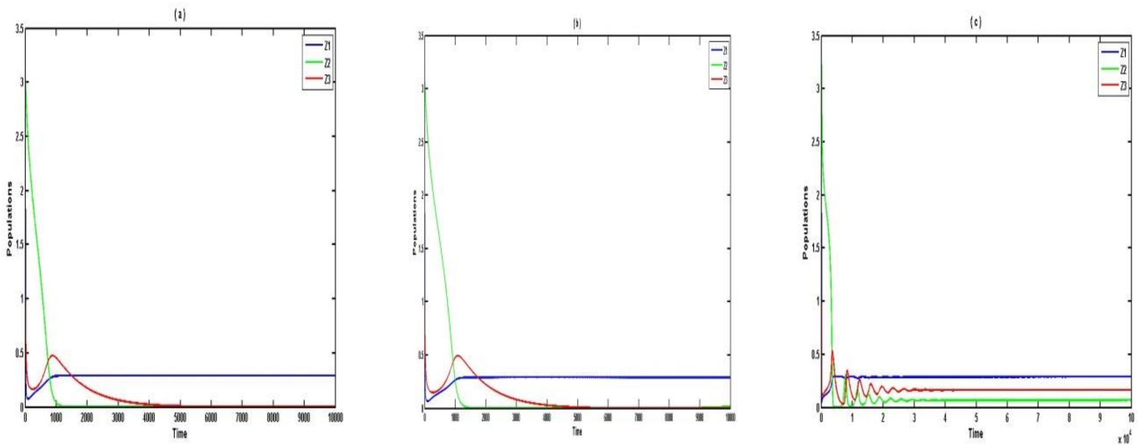

Figure.7: TS of system's (1) solution shows that it is approaches to: (a) $M_1 = (0.2913,0,0)$ for perfect value $a_5 = 0.04$, (b) $M_2 = (0.2896,0.0196,0)$ for $a_5 = 0.07$, (c) $M_3 = (0.2853,0.0683,0.1642)$ for $a_5 = 0.15$.

Changing parameter a_6 (the top predators' maximum predation rate over the middle predator), it is seen that the system's (1) path converges to M_3 in the range $0.01 \le a_6 < 0.5$, this means that changing this parameter did not cause the extinction of this food chain see Fig.8, for the perfect value $a_6 = 0.2$.

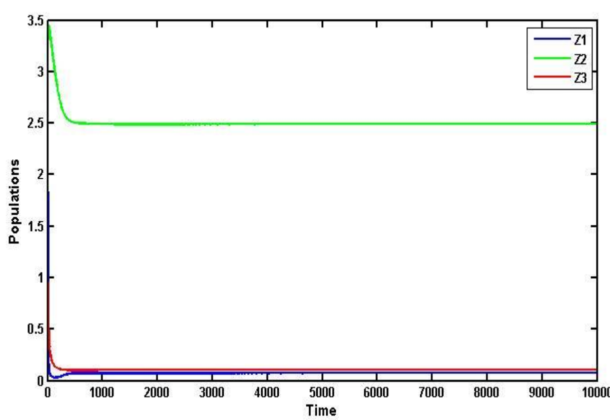

Figure. 8: TS of the system's (1) solution approaches to $M_3 = (0.0738,2.4885,0.1050)$ for the data in (21) for perfect value $a_6 = 0.2$.

Varying the parameter a_7 (the middle predator's half saturation constant), it is gotten that the system's (1) path converges to M_3 in the range $0.1 \leq a_7 \leq 1$, this means that changing this parameter did not cause the extinction of this food chain as you can see in Fig.9, for perfect value $a_7 = 0.3$.

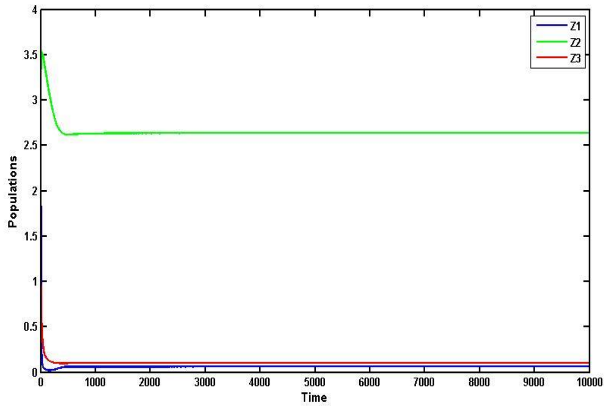

Figure 9: TS of system's (1) solution converges to $M_3 = (0.0613,2.6321,0.1006)$ for the data in (21) for perfect value $a_7 = 0.3$.

Changing the parameter a_8 (the middle predator's defence efficiency) in the range $0.1 \leq a_8 \leq 1$, the system's (1) path converges to M_3, which means changing this parameter did not cause the extinction of this food chain, see Fig.10, for the perfect value $a_8 = 0.7$.

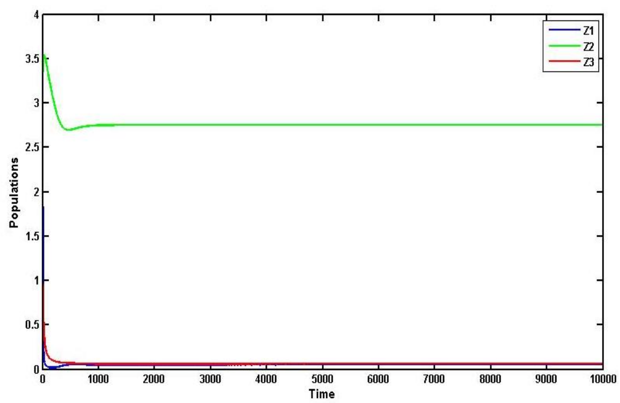

Figure 10: TS of system's (1) solution approaches to $M_3 = (0.0506,2.7541,0.0593)$ for the data in (21) for perfect value $a_8 = 0.7$.

The effect of varying the parameter a_9 (the middle predator's mortality rate) while keeping the other parameters as given in (21) has been studied. It is observed that the system's (1) solution converges to M_3 for $0.01 \leq a_9 < 0.07$, thus means that changing this parameter keeps this food chain free from extinction as seen in Fig.11(a) for perfect value $a_9 = 0.05$, whereas for $0.07 \leq a_9 < 1$, the solution converges to M_1, so the parameter was effective as only the parrot fish remained as seen in Fig.11(b) for perfect value $a_9 = 0.5$.

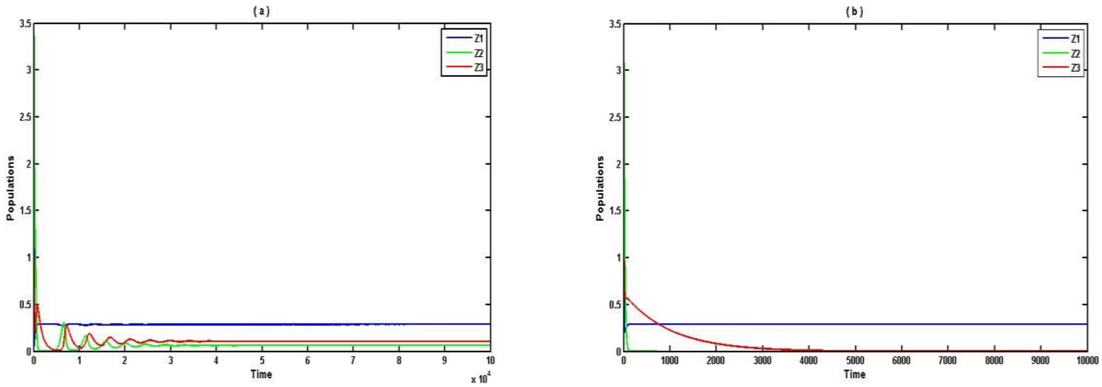

Figure 11: TS of system's (1) solution shows that the solution approaches to (a) $M_3 = (0.288,0.038,0.039)$ for perfect value $a_9 = 0.05$, (b) $M_1 = (0.2913,0,0)$ for perfect value $a_9 = 0.5$.

Changing the parameter a_{10} (the middle predator's toxicity), it's seen that the system's (1) path approaches to M_3 in the range $0.01 \leq a_{10} < 1$, this means that changing this parameter keeps this food chain free from extinction see Fig.12, for the perfect value $a_{10} = 0.5$.

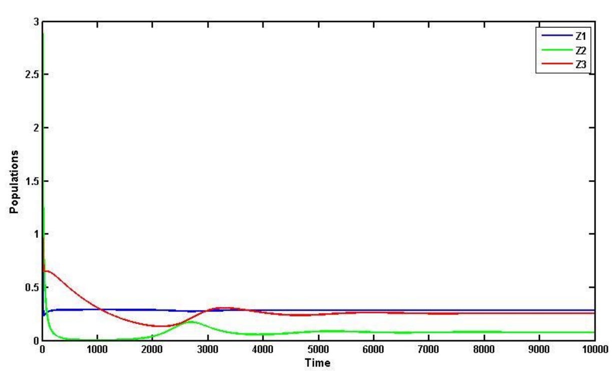

Figure 12: TS of the solution of system (1) converges to $M_3 = (0.2844,0.0785,0.2564)$ for the data in (21) for perfect value $a_{10} = 0.5$.

The effectiveness of changing the parameter a_{11} (the conversion rate of food from middle predator to top predator) in the range $0.01 \leq a_{11} \leq 0.1$ and the given parameters in (21) have been studied. It is observed that the system's (1) solution approaches to M_2 for $0.01 \leq a_{11} \leq 0.04$ the parameter was effective as only the parrot fish and the lionfish remained as seen in Fig.13(a) for perfect value $a_{11} = 0.01$ whereas for $0.04 < a_{11} \leq 0.1$, the solution approaches to M_3, thus changing this parameter in this range keeps this food chain free from extinction as seen in Fig.13(b) for perfect value $a_{11} = 0.05$.

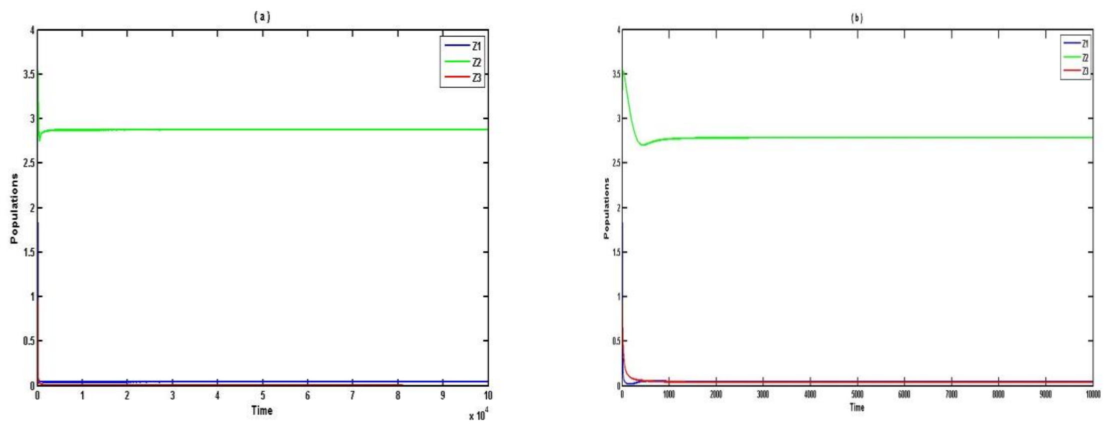

Figure 13: TS of system's (1) solution shows that it approaches to: (a) $M_2 = (0.04, 2.8756, 0)$ for perfect value $a_{11} = 0.01$, (b) $M_3 = (0.0479, 2.7847, 0.0392)$ for perfect value $a_{11} = 0.05$.

The effectiveness of changing the parameter a_{12} (the top predator's mortality rate) in the range $0.01 \leq a_{12} < 1$ while keeping the other parameters in (21) has been studied. It is observed that the system's (1) solution approaches to M_3, thus changing this parameter in this range $0.01 \leq a_{12} < 0.06$ keeps this food chain free from extinction as seen in Fig.14(a) for perfect value $a_{12} = 0.03$, whereas for $0.06 \leq a_{12} < 1$, the solution converges to M_2 the parameter was effective as only the parrot fish and the lionfish remained as seen in Fig.14(b) for perfect value $a_{12} = 0.5$.

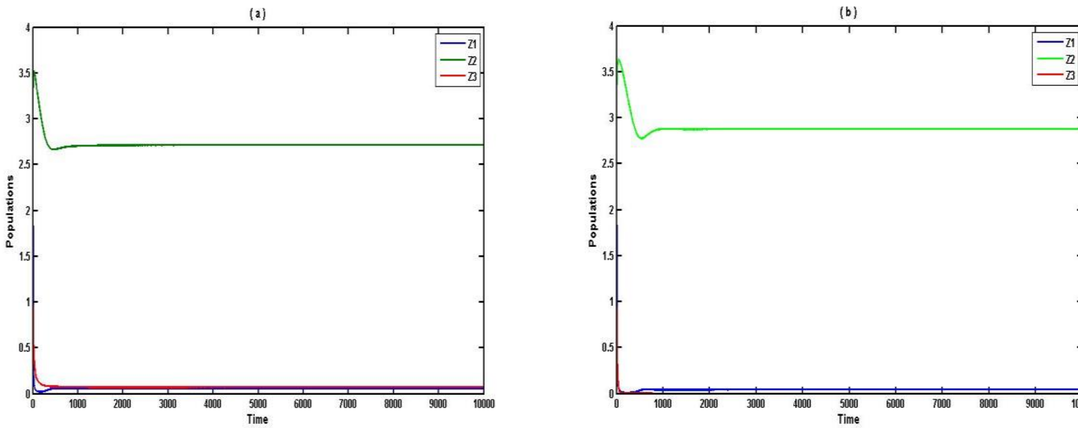

Figure 14: (TS) of system's (1) solution shows that it approaches: (a) $M_3 = (0.0545, 2.7096, 0.071)$ for perfect value $a_{12} = 0.03$, (b) $M_2 = (0.04, 2.8756, 0)$ for $a_{12} = 0.5$.

Varying the parameter a_{13} (the top predator's toxicity). It is seen that the system's (1) path converges to M_3 for $0.1 \leq a_{13} < 1$, this means that changing this parameter did not cause the extinction of this food chain see Fig.15, for the perfect value $a_{13} = 0.5$.

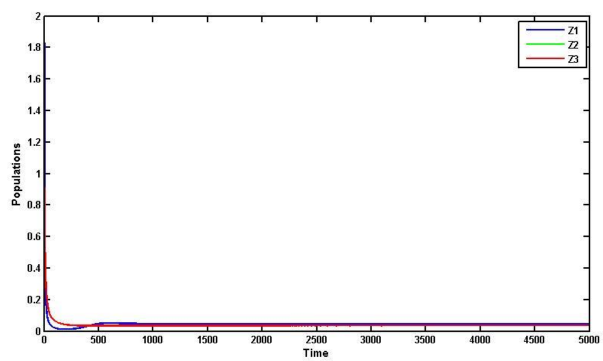

Figure 15: TS of the system's (1) solution converges to $M_3 = (0.0468, 2.7976, 0.0337)$ for the data in (21) for perfect value $a_{13} = 0.5$.

So, the most effective parameters are shown in Table 2. But Table 3 shows the ineffective parameters. Whereas Table 4 shows the parameters in which the bifurcation appeared.

Table.2: The most effective parameters

Range	Converge	Range	Converge
$0.01 < a_5 \leq 0.05$ $0.05 < a_5 \leq 0.08$ $0.08 < a_5 \leq 0.3$	M_1 M_2 M_3	$0.01 \leq a_{11} \leq 0.04$ $0.04 < a_{11} \leq 0.1$	M_2 M_3
$0.01 \leq a_9 < 0.07$ $0.07 \leq a_9 < 1$	M_3 M_1	$0.01 \leq a_{12} < 0.06$ $0.06 \leq a_{12} < 1$	M_3 M_2

Table 3. The ineffective parameters converge to M_3

Range	Range
$0.1 \leq a_1 < 1.5$	$0.1 \leq a_7 \leq 1$
$0.2 \leq a_2 < 1$	$0.1 \leq a_8 \leq 1$
$0.3 \leq a_3 < 1$	$0.01 \leq a_{10} < 1$
$0.1 \leq a_4 < 1$	$0.1 \leq a_{13} < 1$
$0.01 \leq a_6 < 0.5$	

Table 4. The bifurcation parameters

Parameters	Converges	Bifurcation
$0.1 \leq a_1 < 1.5$	M_3	
$0.2 \leq a_2 < 1$	M_3	
$0.3 \leq a_3 < 1$	M_3	
$0.1 \leq a_4 < 1$	M_3	
$0.01 < a_5 \leq 0.05$ $0.05 < a_5 \leq 0.08$ $0.08 < a_5 \leq 0.3$	M_1 M_2 M_3	$a_5 = 0.05$ $a_5 = 0.08$
$0.01 \leq a_6 < 0.5$	M_3	
$0.1 \leq a_7 \leq 1$	M_3	
$0.1 \leq a_8 \leq 1$	M_3	
$0.01 \leq a_9 < 0.07$ $0.07 \leq a_9 < 1$	M_3 M_1	 $a_9 = 0.07$
$0.01 \leq a_{10} < 1$	M_3	
$0.01 \leq a_{11} \leq 0.04$ $0.04 < a_{11} \leq 0.1$	M_2 M_3	$a_{11} = 0.04$
$0.01 \leq a_{12} < 0.06$ $0.06 \leq a_{12} < 1$	M_3 M_2	 $a_{12} = 0.06$
$0.1 \leq a_{13} < 1$	M_3	

7. Conclusions and Discussion

In this work, an eco-toxicant mathematical model consists of a food chain where the functional response (Lotka-Volterra) between the middle predator and the prey while (Holling-IV) between the top and the middle predator has been suggested. Theoretically, the uniformly boundedness of the system's solutions has been shown. The local and global stability analysis has been studied. Therefore, the model has been solved numerically for the given set of parameters in Eq.(21) and different three initial points and the observations below were obtained

1- The model has three global equilibrium points.

2- The model has one kind of attraction in Int. R_+^3 for data given in (21).

3- The solution of the model converges asymptotically to $M_3 = (0.059,2.657,0.093)$ for the data given in (21).

4- The most effective parameters a_5, a_{11}, a_9, a_{12}, which represent the conversion rate of food from the prey to the middle predator and from the middle predator to the top predator, and the mortality rate of the middle and top predator respectively.

5- The ineffective parameters $a_1, a_2, a_3, a_6, a_4, a_{10}, a_{13}, a_7, a_8$, which represents the prey's population growth rate, the prey's carrying capacity, and the middle and top predators' maximum predation rate over the prey and middle predator respectively, the toxicity represents the prey, middle and top predator, the middle predator's half saturation constant, the middle predator's self-defence efficiency respectively.

The implication of such approaches, which are modelled by various scientists, in preserving species in such environments gives us a step further in ecological stabilization. In conclusion, a detailed analysis of the eco-toxicant model has exposed much to its stabilization. System (1) can be stabilized through key changes in the parameter’s values, such as the predation rates and toxicant levels, as tackled in the model. The various factors identified can help in the real system of ensuring a much-stabilized ecosystem under a toxic environment. As an application to the proposed model, the lionfish is introduced as a key species both in terms of economic value and ecological impact. The latter is due to the lionfish’s status as a middle predator, indicating the difficulty of maintaining its population to achieve economic and ecological sustainability.

Conflict of Interests

The authors declare that there is no conflict of interests.

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