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COMMAND OF A NON-LINEAR MODEL IN DISCRETE-TIME OF THE RECIDIVISM PHENOMENON IN MOROCCO

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Abstract. The phenomenon of recidivism is considered to be one of the significant characteristics of the performance of the penitentiary and reintegration system, as well as the effectiveness of the judicial and penal system of such a society. In this paper, we define a mathematical model of a dynamic system of non-linear differential equations in discrete time which describes the observation conceptually and exposes this phenomenon by taking into account four types of variables named; Susceptible Prisoners, Infected Prisoners (Recidivists), Exposed Prisoners and Recovred Prisoners. Our ambition behind this study is to characterise an optimal control that minimises the number of recidivist prisoners after their release, by defining two strategic controls which are respectively, an awareness programme through education in detention centres and socio-economic support with follow-up after release, with the aim of guiding public policy makers in their implementation to determine effective conditions aimed at combating this scourge. The characterisation of the optimal control analysis sought is based on the principle of Pontryagin's maximum, the objective of which is to minimise the numbers of recidivist individuals and inmates potentially susceptible to recidivate, in order to maximise the number of post-releases. The numerical

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simulation was performed using Python 3.12.3. Consequently, numerical illustrations of the results obtained are given, confirming the performance of the optimisation strategy followed.

Keywords: optimal control; recidivism; re-offending; rehabilitation programs; Pontryagin maximum principle.

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1. INTRODUCTION

There is no doubt that crime is a social dilemma that has confronted all human societies since they first existed, and throughout history man's efforts have failed to eliminate or even reduce it. Imprisonment was one of those methods supposed to dissuade criminals from committing their crimes, or at least to ensure that they did not repeat them, but the situation and state of prisons around the world suggests the opposite of what is intended, with the staggering number of prisoners in the world not suggesting that imprisonment has been a deterrent, either to crime nor for recidivism, since the number of prisoners in the world has reached almost 11 million [1], with the United States taking the lead when calculating the ratio of prisoners per 100.000 inhabitants, followed by China, despite the fact that the latter has a population of around one and a half billion, while in Morocco the number of prisoners reached more than 100.000 in 2023 [[2]; Table1], as stated by the High Commission for Penitentiary Administration and Rehabilitation through the voice of its Director General, and that Moroccan penitentiary institutions exceed their capacity by almost two times, which makes it necessary to find new penitentiary institutions and request a substantial budget for their construction as an urgent solution until long-term strategies are found for this scourge [2].

However, what is increasing the concern of the authorities concerned is not only the overcrowding of prisons in addition to the high rates of crime and inmates, but also the rate of recidivism after completion of the prison sentence, which in 2023 reached around 24% [2], while the same rate exceeded , 81.38% in 2019 [3], and what perhaps justifies this huge difference between the two rates, is that the period between these two years was marked by the large-scale spread of the Co-Vid19 pandemic [4], which was the subject of strict administrative and sanitary measures, including quarantine in most parts of the world, contributing significantly to the fall in the crime rate and therefore in the rate of recidivism [5].

Countries	**Prison Popultaion**	**Institutions**	**Capacity**	**Prisoners per 10^5 person (female %)**	**Rate of Recidiv**
USA	1.767.200 (2021)	4.455 (2014)	2.163.235 (2019)	531 (8.8%)	66%
China	1.690.000 (2018)	683 (– –)	Not known	119 (8.6%)	(6% - 8%)
Russia	433.006 (2023)	872 (2022)	714.253 (2021)	300 (8.9%)	54%
France	74.342 (2023)	187 (2021)	60.850 (2023)	109 (4.4%)	38%
UK	87.973 (2024)	120 (2023)	79.597 (2024)	146 (4.1%)	26,1%
Canada	33.833 (2022)	216 (2015)	38.771 (2015)	88 (7.0%)	43%
Spain	54.197 (2023)	82 (2018)	73.152 (2023)	113 (7.1%)	30%
Morocco	103.302 (2023)	78 (2020)	64.600 (2023)	270 (2.4%)	18.4%
Tunisia	23.484 (2021)	32 (2020)	18.577 (2021)	196 (3.3%)	40%
Algeria	94.749 (2021)	162 (2020)	79.620 (2021)	217 (1.5%)	40%
Egypt	120.000 (2022)	78 (2021)	Overcrowding	116 (3.7%)	34.6%
Saudi Arabia	68.056 (2017)	110 (2016)	Not Available	207 (1.9%)	14%
Brazil	839.672 (2022)	1.384 (2022)	482.875 (2022)	390 (5.4%)	70%
El Salvador	71.000 (2022)	25 (2025)	30.000 (2022)	1.086 (7.4%)	60%
Singapore	9.536 (2022)	13 (2019)	16.249 (2013)	156 (10.2%)	22%
Malaysia	72.437 (2023)	52 (2019)	65.762 (2023)	217 (41.7%)	17.06%
Finland	2.839 (2023)	26 (2022)	2.991 (2023)	51 (7.1%)	20%
Sweeden	8.635 (2023)	79 (2015)	8.220 (2023)	82 (5.9%)	30%

TABLE 1. Statistics of the Prisoners of some countries around the world between 2015 and 2023 [1, 2]

The definitions and concepts of recidivism vary because of the different angles from which this phenomenon is viewed, and between a sociologist an a jurist, we can assume that the criminological definition of recidivism is considered one of the most comprehensive definitions of this phenomenon and the most consistent with sociology, because it does not restrict it to the case of those who have been convicted or those who have been punished more than once, because it goes beyond the scope of crimes established by a judicial decision to indicate the state of persistence in committing the crime, regardless of whether these crimes were tried for the first time, and therefore the definition of recidivism in this concept focuses on one fact, which is the repeated committing of crimes [6].

Nevertheless, recidivism is a psychological, social and juridical phenomenon by excellence, which has had and continues to have a significant impact on societies and which is one of the indicators of the effectiveness of a society's justice and penal system. Although recidivism is an aggravating circumstance in most international laws and legislation, especially in Morocco, which defines recidivism under Moroccan law and according to Article 154 of the Moroccan criminal code, as " A person who, after having been irrevocably convicted of a previous infraction, commits another infraction is deemed to be a repeat offender under the terms set out in the following articles" [7].

However, the causes and factors of re-offending (recidivism) can not be reduced to what is primarily punitive and punishing and only requires the strengthening of the legal and legislative arsenal to curb the phenomenon or the creation of prison institutions to accommodate the growing number of criminals and prisoners, but they go beyond that, they can include poverty, lack of economic opportunities, poor education, an unstable social environment, as well as psychological factors such as the absence of a sense of social belonging and poor socialization [8, 9].

One of the measures that can be taken to tackle the phenomenon of recidivism, based on scientific research and previous experiences, is to invest in the structure of the education system and vocational training in order to offer employment opportunities to those who are at risk of becoming criminals. Social and rehabilitation programs for potential and re-offending individuals should be reinforced with the aim of rehabilitating them and reintegrating them into society [10].

These strategies include cooperation between different government agencies and non-governmental organizations to provide the necessary support to individuals at risk of becoming involved in crime.

In this context, this article highlights the importance of understanding the causes and effects of recidivism, by adopting a mathematical model of differential equations for mathematical analysis and providing an insight into the effects of certain optimal controls, in order to orientate the efforts of public authorities towards the development of comprehensive and sustainable solutions that reduce this phenomenon and build a safer and more stable society [11, 12, 13, 14, 15].

2. PRESENTATION OF THE MODEL

2.1. The description of the compartments of the model. In this section, we propose a discrete mathematical model $\mathscr{S}\mathscr{I}\mathscr{E}\mathscr{R}$ taken from [14, 15, 16, 17, 18] by adapting it to the phenomenon of recidivism, by describing the possible evolution of a prisoner from his first arrest, his release, being a recidivist or remitted (never re-arrested). The population targeted by this study is divided into four sets. Using the concept of compartments, four have been considered in this model, described as follows:

Compartiment $\mathscr{S}$: This compartment increases with individuals arrested for the first time (Λ), and by social interactions as well as influences whether by simple will or coercion of previous detainees at the arrest centres, we speak of an influence between the two compartments of the susceptible and the Infectees (Recidivists) who will decrease from this compartment S by a rate alpha after interaction with the infectees ($\alpha\mathscr{S}\mathscr{I}$), a decrease by the released with a rate β, by taking into account the natural mortality rate μ.

Compartiment $\mathscr{I}$: This compartment increases with a rate δ_1 of the outgoing portion of compartment S per $\alpha.\mathscr{S}\mathscr{I}$, thus a rate $\delta 2$ of the outgoing proportion per θ of Exhibitors (Released) by ($\delta_2\theta\mathscr{E}$), we take into consideration the rate of those released from the Infected ($\gamma\mathscr{I}$) and the natural mortality rate μ.

Compartiment $\mathscr{E}$: The peoples who have become liberated here, whether by the β and γ rates respectively of the two susceptible individuals and the infected (recidivists), will decrease by a θ rate and μ the natural mortality rate.

Compartiment $\mathscr{R}$: We consider in this compartment the remitted individuals who think never to return to prison, will obviously be composed by the remaining parts of the two portions $(1-\delta_1)\alpha\mathscr{S}\mathscr{I}$ the complement of the outgoing proportion of the susceptible in relation with the infected and the remaining proportion $(1-\delta_2)\theta\mathscr{E}$ of the individuals who chose not to return to the prison, and that it will decrease by the natural mortality rate μ.

The variables $\mathscr{S}, \mathscr{I}, \mathscr{E}$ and $\mathscr{R}$ are respectively, the numbers of the individuals in the four compartments at time k.

The unit k may correspond to years, phases or periods, it depends on the country or the frequency of the survey study as needed.

The graphical representation in Figure 1, show the model has been proposed.

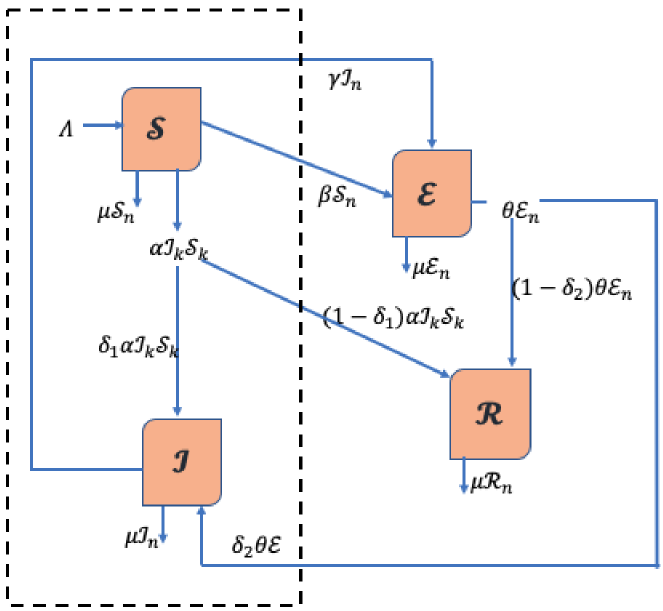

FIGURE 1. A descriptive diagram of our mathematical model

The total population size at time k is denoted by N_k with $N_k = \mathscr{S}_k + \mathscr{I}_k + \mathscr{E}_k + \mathscr{R}_k$ and the dynamic system represents this model is governed by the following nonlinear system of differences equations (1), defined by:

$$
(1) \qquad \begin{cases}
\mathscr{S}_{k+1} = \Lambda - \alpha \mathscr{I}_k \mathscr{S}_k + (1-\beta-\mu)\mathscr{S}_k \\
\mathscr{I}_{k+1} = \delta_1 \alpha \mathscr{I}_k \mathscr{S}_k + \delta_2 \theta \mathscr{E}_k + (1-\gamma-\mu)\mathscr{I}_k \\
\mathscr{E}_{k+1} = \beta \mathscr{S}_k + \gamma \mathscr{I}_k + (1-\theta-\mu)\mathscr{E}_k \\
\mathscr{R}_{k+1} = (1-\delta_1)\alpha \mathscr{I}_k \mathscr{S}_k + (1-\delta_2)\theta \mathscr{E}_k + (1-\mu)\mathscr{R}_k
\end{cases}
$$

Where $\mathscr{S}_0 \geq 0$, $\mathscr{I}_0 \geq 0$, $\mathscr{E}_0 \geq 0$ and $\mathscr{R}_0 \geq 0$ are considered in initial states.

2.2. The description of the model. Here, we can summarize the definitions of differents variables and parameters in the following table 2.

Λ	New inmates
$\mathscr{S}$	Susceptible inmates
$\mathscr{I}$	Infected prisoners (Recidivists)
$\mathscr{E}$	Detainees Released-Exposed (to be infected)
$\mathscr{R}$	Released Remitted
α	Rate of contact between a susceptible person and an infected person (recidivist) $(0 \leq \alpha \leq 1)$
β	Rate of release (ex-prisoners) ($0 \leq \beta \leq 1$)
γ	Rate of infected (recidivists) released ($0 \leq \gamma \leq 1$)
θ	Rate of exposed (ex-liberated) exits from the compartment $\mathscr{E}$ ($0 \leq \theta \leq 1$)
δ_1	Proportion rate $\alpha\mathscr{S}\mathscr{I}$ of individuals who decide to commit more crimes in prison ($0 \leq \delta_1 \leq 1$)
δ_2	Proportion rate $\theta\mathscr{E}$ of individuals who decide to commit more crimes ($0 \leq \delta_2 \leq 1$)
μ	Natural mortality rate ($0 \leq \mu \leq 1$)

TABLE 2. Description of the different compartments

3. THE OPTIMAL CONTROL PROBLEM

The strategies of the control that we adopt here, consist of an awareness program of education in prison and financial support with follow-up after releasing of detains. Our aim goal in considering those strategies is to minimize the number of Infected (Recidivists) individuals and Susceptibles persons in detains ceters, and prepare them for the social life; during the time step $k = 0$ to the final instant T.

In this model, we include two controls $u_{1,k}$ and $u_{2,k}$, those represent respectively the awareness program of education in detains centers and financial support with follow-up after releasing of detains as measures at time k. So, the controlled mathematical system is given by the following system of difference equations:

$$
(2)\qquad \begin{cases}
\mathscr{S}_{k+1} = \Lambda - \alpha \mathscr{I}_k \mathscr{S}_k + (1-\beta-\mu)\mathscr{S}_k - \varepsilon_1 u_{1,k} \mathscr{S}_k \\
\mathscr{I}_{k+1} = \delta_1 \alpha \mathscr{I}_k \mathscr{S}_k + \delta_2 \theta \mathscr{E}_k + (1-\gamma-\mu)\mathscr{I}_k - \varepsilon_2 u_{2,k} \mathscr{I}_k \\
\mathscr{E}_{k+1} = \beta \mathscr{S}_k + \gamma \mathscr{I}_k + (1-\theta-\mu)\mathscr{E}_k \\
\mathscr{R}_{k+1} = (1-\delta_1)\alpha \mathscr{I}_k \mathscr{S}_k + (1-\delta_2)\theta \mathscr{E}_k + (1-\mu)\mathscr{R}_k + \varepsilon_1 u_{1,k} \mathscr{S}_k + \varepsilon_2 u_{2,k} \mathscr{I}_k
\end{cases}
$$

Where $\mathscr{S}_0 \geq 0$, $\mathscr{I}_0 \geq 0$, $\mathscr{E}_0 \geq 0$ and $\mathscr{R}_0 \geq 0$ are considered in initial states, and

$$\varepsilon_i = \begin{cases} 1 \\ \\ 0 \end{cases} \quad for\ i = 1,2. \tag{3}$$

From the system of difference equations (2), the controlled model achieves the objective of decreasing the Susceptibles and Recidivists individuals when the rate $\varepsilon_k = 1$ for $k = 1,2$. Then the problem that we faced here is to minimize the objective functional given by

$$J(u_1,u_2) = A_T\mathscr{S}_T + B_T\mathscr{I}_T + \sum_{k=0}^{T-1}\left(A_k\mathscr{S}_T + B_k\mathscr{I}_T + \frac{C_k}{2}\varepsilon_1 u_{1,k}^2 + \frac{D_k}{2}\varepsilon_2 u_{2,k}^2\right) \tag{4}$$

Where the parameters $A_k > 0$, $B_k > 0$, $C_k > 0$ and $D_k > 0$ are the cost coefficients; they are selected to weigh the relative importance of $\mathscr{S}_k$, $\mathscr{I}_k$, $u_{1,k}$ and $u_{2,k}$ at time k. T represents the final time.

In other words, we are looking for an optimal control ($u_{1,k}$,$u_{2,k}$) such that

$$J(u_1^*,u_2^*) = \min_{(u_1,u_2)\in\mathscr{U}_{ad}^2} J(u_1,u_2), \tag{5}$$

Where

$$\mathscr{U}_{ad} = \{(u_{1,k},u_{2,k})\,|\,0 \leq u_{1,k} \leq 1 \text{ and } 0 \leq u_{2,k} \leq 1 \text{ for } k = 0,1,2,\dots,T-1\} \tag{6}$$

The sufficient condition for the existence of the optimal controls u_1^* and u_2^* for problem (2) and (4) comes from the following theorem.

Theorem 3.1. *There exists an optimal control* (u_1^*,u_2^*) *such that:*

$$J(u_1^*,u_2^*) = \min_{(u_1,u_2)\in\mathscr{U}_{ad}} J(u_1,u_2) \tag{7}$$

subject to the control system (2) *with initial conditions.*

Proof. Since the coefficients of the state equations are bounded and there are a finite number of time steps, $\mathscr{S} = (\mathscr{S}_0,\mathscr{S}_1,\dots,\mathscr{S}_T)$, $\mathscr{I} = (\mathscr{I}_0,\mathscr{I}_1,\dots,\mathscr{I}_T)$, $\mathscr{E} = (\mathscr{E}_0,\mathscr{E}_1,\dots,\mathscr{E}_T)$, and $\mathscr{R} = (\mathscr{R}_0,\mathscr{R}_1,\dots,\mathscr{R}_T)$ are uniformly bounded for all (u_1,u_2) in the control set $\mathscr{U}_{ad}^2$ and thus (u_1,u_2) is bounded for all $(u_1,u_2) \in \mathscr{U}_{ad}^2$. Since $J(u_1,u_2)$ is bounded, $\inf_{(u_1,u_2)\in\mathscr{U}_{ad}^2} J(u_1,u_2)$ is finite, and there exists a sequence $(u_1^j,u_2^j) \in \mathscr{U}_{ad}^2$ such that $\lim_{j\to+\infty} J(u_1^j,u_2^j) = \inf_{(u_1,u_2)\in\mathscr{U}_{ad}} J(u_1,u_2)$, with corresponding sequences of states $\mathscr{S}^j$, $\mathscr{I}^j$, $\mathscr{E}^j$, and $\mathscr{R}^j$. Since there is a finite number of uniformly bounded sequences, there exist $(u_1^*, u_2^*) \in \mathscr{U}_{ad}^2$ and $\mathscr{S}^*, \mathscr{I}^*, \mathscr{E}^*, \mathscr{R}^* \in \mathbb{R}^{T+1}$ such that, on a subsequence, (u_1^j, u_2^j) converges to (u_1^*, u_2^*) and $(\mathscr{S}^j, \mathscr{I}^j, \mathscr{E}^j, \mathscr{R}^j)$ converges to $(\mathscr{S}^*, \mathscr{I}^*, \mathscr{E}^*, \mathscr{R}^*)$.

Finally, due to the finite-dimensional structure of system (2) and the objective function $J(u_1, u_2)$, (u_1^*, u_2^*) is an optimal control with corresponding states $\mathscr{S}^*$, $\mathscr{I}^*$, $\mathscr{E}^*$, and $\mathscr{R}^*$. Therefore, $\inf_{(u_1,u_2)\in\mathscr{U}_{ad}^2} J(u_1, u_2)$ is achieved. □

In order to derive the necessary condition for optimal control, the Pontryagin's maximum principle in discrete time given in [19, 20, 21, 22, 23] was used. This principle converts the problem of seeking the optimal control of the objective functional subject **to** state difference equations with initial conditions into minimizing the Hamiltonian $\mathscr{H}_k$ at time step k. The Hamiltonian $\mathscr{H}_k$ is defined by

$$\mathscr{H}_k = A_k \mathscr{S}_T + B_k \mathscr{I}_T + \varepsilon_1 \frac{C_k}{2} u_{1,k}^2 + \varepsilon_2 \frac{D_k}{2} u_{2,k}^2 + \sum_{j=1}^{4} \lambda_{j,k+1} f_{j,k+1}, \tag{8}$$

where $f_{j,k+1}$ is the right side of the system of difference equations (2) for the j-th state variable at time step $k+1$.

Theorem 3.2. *Given an optimal control $(u_1^*, u_2^*) \in \mathscr{U}_{ad}^2$ and the solutions $\mathscr{S}_k^*, \mathscr{I}_k^*, \mathscr{E}_k^*$, and $\mathscr{R}_k^*$ of the corresponding state system (2), there exist adjoint functions $\lambda_{1,k}, \lambda_{2,k}, \lambda_{3,k}$, and $\lambda_{4,k}$ satisfying*

(9)

$$\lambda_{1,k} = A_k + \lambda_{1,k+1}(1-\beta-\mu-\alpha\mathscr{I}_k-\varepsilon_1 u_{1,k}) + \lambda_{2,k+1}\delta_1\alpha\mathscr{I}_k + \lambda_{3,k+1}\beta + \lambda_{4,k+1}((1-\delta_1)\alpha\mathscr{I}_k + \varepsilon_1 u_{1,k})$$

$$\lambda_{2,k} = B_k - \lambda_{1,k+1}\alpha\mathscr{S}_k + \lambda_{2,k+1}(1-\gamma-\mu+\delta_1\alpha\mathscr{S}_k-\varepsilon_2 u_{2,k}) + \lambda_{3,k+1}\gamma + \lambda_{4,k+1}((1-\delta_1)\alpha\mathscr{S}_k + \varepsilon_2 u_{2,k})$$

$$\lambda_{3,k} = \lambda_{2,k+1}\delta_2\theta + \lambda_{3,k+1}(1-\theta-\mu) + \lambda_{4,k+1}(1-\delta_2)\theta$$

$$\lambda_{4,k} = \lambda_{4,k+1}(1-\mu)$$

with the transversality conditions at time T,

$$\lambda_{1,T} = A_T, \qquad \lambda_{2,T} = B_T, \qquad \lambda_{3,T} = 0, \tag{10}$$

$$\lambda_{4,T} = 0$$

Furthermore, for $k \in \{0, 1, 2, \ldots, T-1\}$ and $\varepsilon_1 = \varepsilon_2 = 1$, the optimal controls u_1^* and u_2^* are given by

$$u_{1,k+1} = \min\left[1, \max\left(0, \frac{1}{C_k}(\lambda_{1,T-k+1} - \lambda_{4,T-k+1})S_k\right)\right] \tag{11}$$
$$u_{2,k+1} = \min\left[1, \max\left(0, \frac{1}{D_k}(\lambda_{2,T-k+1} - \lambda_{4,T-k+1})I_k\right)\right]$$

Proof. The Hamiltonian at time step k is given by

(12)
$$\begin{aligned}
\mathscr{H}_k &= A_k\mathscr{S}_k + B_k\mathscr{I}_k + \varepsilon_1\frac{C_k}{2}u_{1,k}^2 + \varepsilon_2\frac{D_k}{2}u_{2,k}^2 + \lambda_{1,k+1}f_{1,k+1} + \lambda_{2,k+1}f_{2,k+1} + \lambda_{3,k+1}f_{3,k+1} + \lambda_{4,k+1}f_{4,k+1} \\
&= A_k\mathscr{S}_k + B_k\mathscr{I}_k + \varepsilon_1\frac{C_k}{2}u_{1,k}^2 + \varepsilon_2\frac{D_k}{2}u_{2,k}^2 \\
&\quad + \lambda_{1,k+1}\left(\Lambda - \alpha\mathscr{I}_k\mathscr{S}_k + (1-\beta-\mu)\mathscr{S}_k - \varepsilon_1 u_{1,k}\mathscr{S}_k\right) \\
&\quad + \lambda_{2,k+1}\left(\delta_1\alpha\mathscr{I}_k\mathscr{S}_k + \delta_2\theta\mathscr{E}_k + (1-\gamma-\mu)\mathscr{I}_k - \varepsilon_2 u_{2,k}\mathscr{I}_k\right) \\
&\quad + \lambda_{3,k+1}\left(\beta\mathscr{S}_k + \gamma\mathscr{I}_k + (1-\theta-\mu)\mathscr{E}_k\right) \\
&\quad + \lambda_{4,k+1}\left((1-\delta_1)\alpha\mathscr{I}_k\mathscr{S}_k + (1-\delta_2)\theta\mathscr{E}_k + (1-\mu)\mathscr{R}_k + \varepsilon_1 u_{1,k}\mathscr{S}_k + \varepsilon_2 u_{2,k}\mathscr{I}_k\right)
\end{aligned}$$

From other hand,

$$\lambda_{1,k} = \frac{\partial\mathscr{H}_k}{\partial\mathscr{S}_k} \tag{13}$$
$$\lambda_{2,k} = \frac{\partial\mathscr{H}_k}{\partial\mathscr{I}_k}$$
$$\lambda_{3,k} = \frac{\partial\mathscr{H}_k}{\partial\mathscr{E}_k}$$
$$\lambda_{4,k} = \frac{\partial\mathscr{H}_k}{\partial\mathscr{R}_k}$$

Thus,

(14)
$$\lambda_{1,k} = A_k + \lambda_{1,k+1}(1-\beta-\mu-\alpha\mathscr{I}_k-\varepsilon_1 u_{1,k}) + \lambda_{2,k+1}\delta_1\alpha\mathscr{I}_k + \lambda_{3,k+1}\beta + \lambda_{4,k+1}\left((1-\delta_1)\alpha\mathscr{I}_k + \varepsilon_1 u_{1,k}\right)$$

$$\lambda_{2,k} = B_k - \lambda_{1,k+1}\alpha\mathscr{S}_k + \lambda_{2,k+1}\left(1-\gamma-\mu+\delta_1\alpha\mathscr{S}_k-\varepsilon_2 u_{2,k}\right) + \lambda_{3,k+1}\gamma + \lambda_{4,k+1}\left((1-\delta_1)\alpha\mathscr{S}_k + \varepsilon_2 u_{2,k}\right)$$

$$\lambda_{3,k} = \lambda_{2,k+1}\delta_2\theta + \lambda_{3,k+1}(1-\theta-\mu) + \lambda_{4,k+1}(1-\delta_2)\theta$$

$$\lambda_{4,k} = \lambda_{4,k+1}(1-\mu)$$

and for $k \in 0,1,\ldots,(T-1)$ the optimal controls u_1^* and u_2^* can be solved from the optimally condition,

$$\frac{\partial \mathscr{H}_k}{\partial u_{1,k}} = 0 \qquad \frac{\partial \mathscr{H}_k}{\partial u_{2,k}} = 0 \tag{15}$$

that are

$$\varepsilon_1 C_k u_{1,k} + \lambda_{1,k+1}\left(-\varepsilon_1 \mathscr{S}_k\right) + \lambda_{4,k+1}\left(\varepsilon_1 \mathscr{S}_k\right) = 0$$

$$\varepsilon_2 D_k u_{2,k} + \lambda_{2,k+1}\left(-\varepsilon_2 \mathscr{I}_k\right) + \lambda_{4,k+1}\left(\varepsilon_2 \mathscr{I}_k\right) = 0$$

Then, for $\varepsilon_1 = \varepsilon_2 = 1$ we have

$$u_{1,k} = \frac{\lambda_{1,k+1} - \lambda_{4,k+1}}{C_k}\mathscr{S}_k \qquad u_{2,k} = \frac{\lambda_{2,k+1} - \lambda_{4,k+1}}{D_k}\mathscr{I}_k \tag{16}$$

However, if $\varepsilon_i = 0$ for $i = 1,2$, the control attached to this case will be eliminated and removed.

By the bounds in $\mathscr{U}_{ad}$ of the controls, it is easy to obtain u_1^* and u_2^* in the form (11).

Finally, we have proofed the theorem. □

4. Numerical Simulation

In this section, we discuss the numerical simulations of the state system (1) and presenting the corresponding results found.

The numerical solutions of the optimality system comprising the equation of state (2) and the adjoint equation (13) are performed with **Python 3.12.3** software using the parameters shown in Table2, taking into consideration the initial conditions $\mathscr{S}_0 = 120.000$, $\mathscr{I}_0 = 25.000$, $\mathscr{E}_0 = 1.000$ and $\mathscr{R}_0 = 0$ individual (the choice of these values is approximate to reality) and considering the following weighting factors A = 20 and B = 25.

The u_1 and u_2 commands are updated and used to solve the state system and then the adjoint system.

This iterative process ends when the current state, adjoint and control values converge sufficiently.

Parameters	Baseline	Reference
$\mathscr{S}_0$	120.000	Assumed
$\mathscr{I}_0$	25.000	Assumed
$\mathscr{E}_0$	1.000	Assumed
$\mathscr{R}_0$	0	Assumed
Λ	100	Assumed
α	0.05	Assumed
β	0.05	Assumed
γ	0.02	Assumed
θ	0.05	Assumed
δ_1	0.1	Assumed
δ_2	0.1	Assumed
μ	0.01	Assumed

TABLE 3. The description of parameters used for the definition of discrete time system (1).

We just considered an academic data. Combining forward and backward difference approximations, we obtain the following algorithm.

4.1 ALGORITHM

This system consists of the state system, adjoint system, initial and final time conditions, and the characterization of controls. The optimality system is given by the following.

Step 1.

$$\mathscr{S}_0 = S_0, \quad \mathscr{I}_0 = I_0, \quad \mathscr{E}_0 = E_0, \quad \mathscr{R}_0 = R_0, \quad \lambda_{1,T} = \lambda_{4,T} = 0, \quad \lambda_{2,T} = A_T, \quad \lambda_{3,T} = B_T.$$

The optimal controls $u^*_{1k,0}$ and $u^*_{2k,0}$ are specified.

Step 2.

For $k = 0, 1, \ldots, T-1$, do:

$$\begin{aligned}
\mathscr{S}_{k+1} &= \Lambda - \alpha \mathscr{I}_k \mathscr{S}_k + (1-\beta-\mu)\mathscr{S}_k - \varepsilon_1 u_{1,k}\mathscr{S}_k,\\
\mathscr{I}_{k+1} &= \delta_1 \alpha \mathscr{I}_k \mathscr{S}_k + \delta_2 \theta \mathscr{E}_k + (1-\gamma-\mu)\mathscr{I}_k - \varepsilon_2 u_{2,k}\mathscr{I}_k,\\
\mathscr{E}_{k+1} &= \beta \mathscr{S}_k + \gamma \mathscr{I}_k + (1-\theta-\mu)\mathscr{E}_k,\\
\mathscr{R}_{k+1} &= (1-\delta_1)\alpha \mathscr{I}_k \mathscr{S}_k + (1-\delta_2)\theta \mathscr{E}_k + (1-\mu)\mathscr{R}_k + \varepsilon_1 u_{1,k}\mathscr{S}_k + \varepsilon_2 u_{2,k}\mathscr{I}_k.\\
\lambda_{1,T-k} &= A_k + \lambda_{1,k+1}(1-\beta-\mu-\alpha \mathscr{I}_k - \varepsilon_1 u_{1,K}) + \lambda_{2,k+1}\delta_1 \alpha \mathscr{I}_k + \lambda_{3,k+1}\beta\\
&\quad + \lambda_{4,k+1}((1-\delta_1)\alpha \mathscr{I}_k + \varepsilon_1 u_{1,k})\\
\lambda_{2,T-k} &= B_k - \lambda_{1,k+1}\alpha \mathscr{S}_k + \lambda_{2,k+1}(1-\gamma-\mu+\delta_1 \alpha \mathscr{S}_k - \varepsilon_2 u_{2,K}) + \lambda_{3,k+1}\gamma\\
&\quad + \lambda_{4,k+1}((1-\delta_1)\alpha \mathscr{S}_k + \varepsilon_2 u_{2,k})\\
\lambda_{3,T-k} &= \lambda_{2,k+1}\delta_2 \theta + \lambda_{3,k+1}(1-\theta-\mu) + \lambda_{4,k+1}(1-\delta_2)\theta\\
\lambda_{4,T-k} &= \lambda_{4,k+1}(1-\mu)
\end{aligned}$$

$$u_{1,k} = \frac{\lambda_{1,k+1} - \lambda_{4,k+1}}{C_k}\mathscr{S}_k$$
$$u_{2,k} = \frac{\lambda_{2,k+1} - \lambda_{4,k+1}}{D_k}\mathscr{I}_k$$

end for.

Step 3.

For $k = 0, 1, \ldots, T$, write $S_k^* = S_k, \quad I_k^* = I_k, \quad E_k^* = E_k, \quad R_k^* = R_k, \quad u_{1k}^* = u_{1,k}, \quad u_{2k}^* = u_{2,k}$
end for.

In this formulation, there were initial conditions for the state variables and terminal conditions for the adjoints functions.

That is, the optimality system is a two-point boundary value problem with separated boundary conditions at time steps $k = 0$ and $k = T$. We solve the optimality system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the controls at the first iteration and then before the next iteration we update the controls by using the characterization.

We continue until convergence of successive iterates is achieved.

4.2 Discussion

In this section, we study and analyse numerically the effects of optimal control strategies such as the awareness program of education in detains centers and financial support with follow-up after releasing of detains.

4.2.1 Strategy A

First of all, we observe that when we don't activate any control on the dynamic system we have (Figure2), we obtain a decrease in susceptible individuals but in favor of infected people, i.e. all prisoners become recidivists, and this is the problem we want to remedy. Control with Awareness Program. Given the priority to the awareness program of education in detains centers, we suggest an optimal strategy for this purpose. Hence, we activate the optimal control variable u_1 which represents the awareness program of education in detains centers.

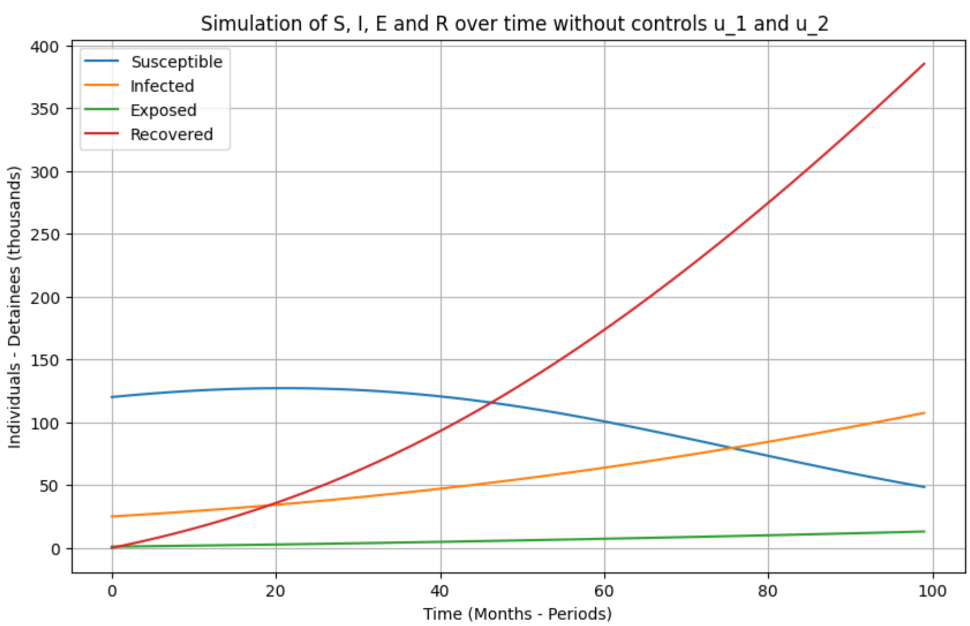

FIGURE 2. Without controls u_1 and u_2

Figure3 shows the evolution of the Susceptible individuals in detains centers with and without control u_1, which the impact of the proposed awareness program of education in detains centers is proven positively to decreasing the number the detains for the second time.

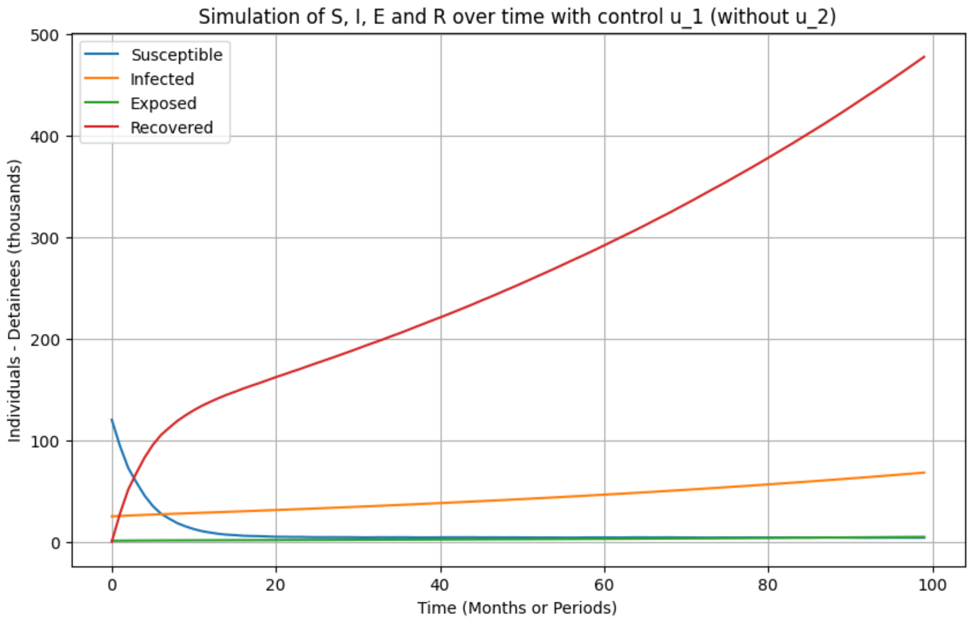

FIGURE 3. With control u_1 and without u_2

4.2.2 STRATEGY B

In this strategy, we have considered a financial support with follow-up after releasing of detains program. When the number of recidivists detains is so significant, this is why, it's obligatory to seek for some strategies such as the financial support with follow-up after releasing of detains in order to reduce the number of Recidivists (Infected Detains) at time they were arrested and simultaneously with the susceptible detains for the first time. Therefore, we propose an optimal strategy by using the optimal control u_2 in the beginning, a policy wich we have applied to achieve better results.

We can observe (Figure4) a significant decrease in the number of recidivists, but on the other hand we note that the number of susceptible individuals is very remarkable due to the activation of just the control u_2 and no acitivation of the u_1 control.

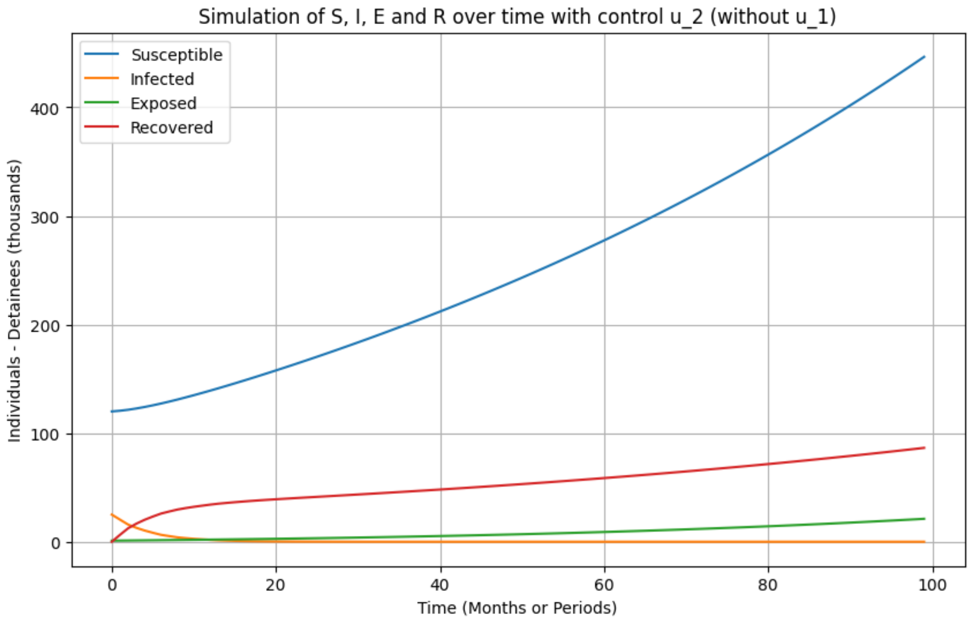

FIGURE 4. With control u_2 and without u_1

Finally, the Figure5 represents the activation of the two controls u_1 and u_2, wich reflect a significant reduction in the number of Susceptibles and Infected (Recidivists) individuals, wich demonstrating the expected results.

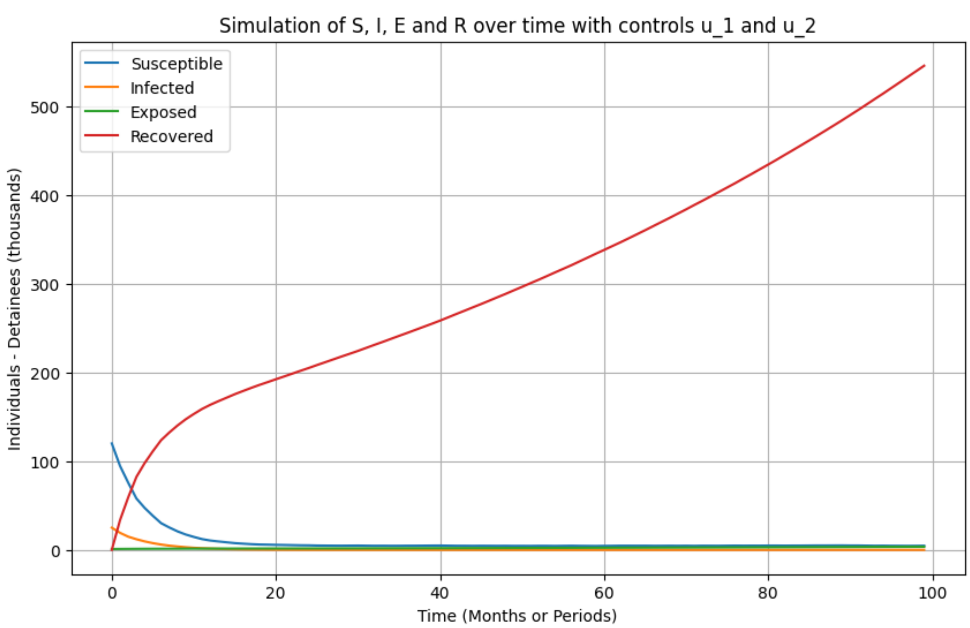

FIGURE 5. With controls u_1 and u_2

5. Conclusion

In this paper, we have defined a discrete-time model of the phenomenon of recidivism in Morocco with the aim of minimising the number of recidivists and individuals who are potentially recidivists after their release.
We also imposed two strategic controls which are respectively, an awareness raising program through education in detention centres and socio-economic support with post-release follow-up. We have succeeded in obtaining the expected characterisations of optimal controls after rigorous application of the results of control theory, and the effectiveness of the proposed strategies has been well demonstrated by numerical simulation of the theoretical results obtained.
And we wish to emphasise here that any other theoretical control will no longer be able to perform its main role of minimising the rising rate of recidivism in Morocco without a real political will to tackle this scourge, a problem with remarkable social and economic consequences, and above all the loss of potential among young prisoners and recidivists, who are the most influenced by this phenomenon.

Data Availability

We considered mainly academic data to approve this research.

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Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] WPB, The world prison brief, https://www.prisonstudies.org/world-prison-brief-data.

[2] Official National Gate of Morocco, M. Tamek: La nature complexe du phénomène de surpopulation carcérale nécessite des solutions pratiques dans le cadre d'un plan intégré, https://www.maroc.ma/fr/actualites/m-tamek-la-nature-complexe-du-phenomene-de-surpopulation-carcerale-necessite-des.

[3] Official National Gate of Morocco, M.Ramid: Le nombre des cas de récidive remet en question le système pénal, https://www.maroc.ma/fr/actualites/mramidle-nombre-des-cas-de-recidive-remet-en-question-le-systeme-penal.

[4] JFA Institute, The impact of COVID-19 on crime, arrests, and jail populations, 2021. https://safetyandjusticechallenge.org/wp-content/uploads/2021/07/The-Impact-of-COVID-19-on-Crime-Arrests-and-Jail-Populations-JFA-Institute.pdf.

[5] M. Meyer, A. Hassafy, G. Lewis, et al. Changes in crime rates during the COVID-19 pandemic, Stat. Public Policy 9 (2022), 97–109. https://doi.org/10.1080/2330443X.2022.2071369.

[6] R. Weisberg, Meanings and measures of recidivism, Southern California Law Rev. 87 (2014), 785.

[7] Office National des Oeuvres Universitaires, Sociales et Culturelles, Code penal. https://www.onousc.ma/storage/code_penal.pdf.

[8] UNODC, Criminal justice and reoffending. https://www.unodc.org/documents/justice-and-prison-reform/ReducingReoffending/OCPA_International_Organization.pdf.

[9] F.T. Cullen, C.L. Jonson, D.S. Nagin, Prisons do not reduce recidivism: the high cost of ignoring science, Prison J. 91 (2011), 48S-65S. https://doi.org/10.1177/0032885511415224.

[10] Third United Nations Congress on the Prevention of Crime and the Treatment of Offenders, Measure to combat recidivism. https://www.unodc.org/documents/congress//Previous_Congresses/3rd_Congress_1965/004_ACONF.26.4_Measures_to_Combat_Recidivism.pdf.

[11] A. Essounaini, S. Hilal, B. Khajji, et al. A new simple discrete-time model for the description of excessive alcohol consumption with n complications, Commun. Math. Biol. Neurosci., 2024 (2024), 2. https://doi.org/10.28919/cmbn/8124.

[12] A. Elqaddaoui, A. El Bhih, H. Laarabi, et al. A stochastic optimal control strategy for multi-strain COVID-19 spread, Commun. Math. Biol. Neurosci. 2023 (2023), 130. https://doi.org/10.28919/cmbn/8223.

[13] O. Balatif, A. Labzai, M. Rachik, A discrete mathematical modeling and optimal control of the electoral behavior with regard to a political party, Discr. Dyn. Nat. Soc. 2018 (2018), 9649014. https://doi.org/10.1155/2018/9649014.

[14] A. Labzai, O. Balatif, M. Rachik, Optimal control strategy for a discrete time smoking model with specific saturated incidence rate, Discr. Dyn. Nat. Soc. 2018 (2018), 5949303. https://doi.org/10.1155/2018/5949303.

[15] A. Kourrad, A. Alabkari, K. Adnaoui, et al. A spatiotemporal model with optimal control for the novel coronavirus epidemic in Wuhan, China, Commun. Math. Biol. Neurosci. 2021 (2021), 45. https://doi.org/10.28919/cmbn/5755.

[16] A. Kourrad, A. Alabkari, K. Adnaoui, F. Lahmidi, Y. Tabit, A. El Adraoui, A mathematical model and optimal control analysis for scholar drop out, Bol. Soc. Paran. Mat. 41 (2023), 1–11. https://doi.org/10.5269/bspm.62650.

[17] M. Rachik, H. Laarabi, O. El Kahlaoui, et al. Observer for perturbed linear continuous systems, Appl. Math. Sci. 4 (2010), 681–689.

[18] H. Laarabi, A. Abta, M. Rachik, et al. Stability analysis of a delayed rumor propagation model, Differ. Equ. Dyn. Syst. 24 (2016), 407–415. https://doi.org/10.1007/s12591-015-0251-0.

[19] W. Ding, R. Hendon, B. Cathey, et al. Discrete time optimal control applied to pest control problems, Involve, 7 (2014), 479–489. https://doi.org/10.2140/involve.2014.7.479.

[20] D.C. Zhang, B. Shi, Oscillation and global asymptotic stability in a discrete epidemic model, J. Math. Anal. Appl. 278 (2003), 194–202. https://doi.org/10.1016/S0022-247X(02)00717-5.

[21] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, et al. The mathematical theory of optimal processes, Wiley, 1962.

[22] V. Guibout, A. Bloch, A discrete maximum principle for solving optimal control problems, in: 2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No.04CH37601), IEEE, Nassau, Bahamas, 2004: pp. 1806-1811 Vol.2. https://doi.org/10.1109/CDC.2004.1430309.

[23] C.L. Hwang, L.T. Fan, A discrete version of pontryagin's maximum principle, Oper. Res. 15 (1967), 139–146. https://doi.org/10.1287/opre.15.1.139.