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ANALYSIS OF THE FRACTIONAL-ORDER EPIDEMIC MODEL OF SMOKING TOBACCO WITH THE IMPACT OF NICOTINE GUM

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Abstract. In this work, we present a fractional dynamic model to explain the spread of the smoking habit among individuals when parameters for nicotine gum consumption are given to active smokers. In line to the epidemy theory, it was shown that the stability of the smoking-endemic equilibrium as well as the smoking-free equilibrium is influenced by the basic reproduction number. A numerical example is provided to show that the results are valid. The results show that providing nicotine gum to smokers who are currently in the habit can decrease the number of active smokers and raise the number of secondhand smokers.

Keywords: Caputo fractional-order derivative; *PSQ* model; nicotine gum; basic reproduction number; equilibrium.

2020 AMS Subject Classification: 26A33, 34C60, 34D23.

1. INTRODUCTION

The use of tobacco is emerging as a significant dependency that requires global regulation, as evidenced by the rising mortality rate among consumers on a global scale. Numerous individuals began the practice of smoking in their youth without a complete understanding of the risks associated with tobacco use. Smoking poses risks that extend beyond the direct impacts

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on smokers and also affect individuals who do not engage in smoking because they are also exposed to cigarette smoke. Those individuals who are not active smokers but are affected by the effects of active smoking are commonly referred to as secondhand smokers. For example, if one inhales the cigarette smoke emitted by active smokers, then he is a secondhand smoker.

The World Health Organization (WHO) estimates that at least 8 million deaths are caused by cigarette smoke and 1.2 million of these cases occur in seconhand smokers. The exposure to secondhand smoke is a subject that garners substantial attention from us due to its widely recognized negative impacts on human health [1]. Hence, it is imperative to deter smoking behaviors to mitigate the diverse array of risks that may emerge.

Numerous scientific inquiries have been conducted in efforts to mitigate the habit of smoking. Mathematical modeling serves as a rigorous scientific approach aimed at enhancing comprehension regarding the dynamics associated with the transmission of smoking habits, as well as assessing the efficacy of different strategies for control and prevention. Various scholarly investigations have been carried out on the utilization of mathematical models in examining patterns related to tobacco consumption, as referenced in [2, 3, 4, 5, 6]. These models, called compartmental models, are divided into compartments filled with the population. The movement of data from one compartment to another depends on the type and speed of the data. These ideas' fundamental tenet is that people will start living healthy lifestyles in communities. Even while healthy people can get diseases, those affected can recover and healthily return to society [3].

One of the renowned models for analyzing the propagation of the habit of smoking is the *PSQ* compartment model, which is represented by a nonlinear differential equation [7]. In the framework of the *PSQ* model, the population under observation (N) is categorized into three distinct epidemiological compartments identified as the subpopulation of secondhand smoking (P), the subpopulation of active smoking (S), and the subpopulation stopped smoking (Q). Consequently, the overall population size at a given time t can be expressed as $N(t) = P(t) + S(t) + Q(t)$.

Based on the compartment diagram in reference [7], the dynamics of PSQ model for smoking habit spread in human population are governed by the system (1) of coupled nonlinear differential equation [7],

$$\begin{cases} \dot{P}(t) = \alpha - \lambda P(t)S(t) - (\nu + \mu + \sigma)P(t) \\ \dot{S}(t) = \lambda P(t)S(t) + \delta S(t)Q(t) - (\mu + \kappa + \zeta)S(t) \\ \dot{Q}(t) = (\zeta + \rho)S(t) - \delta S(t)Q(t) + (\eta + \mu + \xi)Q(t), \end{cases} \tag{1}$$

with the initial conditions $P(0) = P_0 \geq 0, S(0) = S_0 \geq 0, Q(0) = Q_0 \geq 0$, where the involved various parameters are given in Table 1.

TABLE 1. Parameter occuring in the model (1).

Parameter	**Biological meaning**
α	inflow rate of individuals who have a risk of smoking class
μ	natural per capita death rate
ν	the death rate of secondhand smoker
κ	the death rate of smoker by smoking tobacco
η	the death rate of individuals who quit smoking
σ	the exit rate of secondhand smoker to the health people
ξ	the exit rate of people who have stopped smoking to the healthy population
λ	infection rate from P to S
ζ	exit rate from S to Q
δ	infection rate from Q to S

In the present era, numerous epidemiological models have been developed using fractional-order differential equations, and have been extensively deliberated by a multitude of researchers, see [2, 8, 9, 10, 11, 12, 13, 14]. Fractional-order derivatives are acknowledged as extensions of integer order derivatives, thus the utilization of fractional differential equations for modeling represents a robust approach in examining the comprehensive propagation of diseases.

Inspired by the present investigation, we adjusted model (1) in this study by replacing fractional order derivatives for the first-order derivative and assigning the consumption of nicotine gum within a rate ρ to the active smoker. It is commonly recognized that smokers who chew

nicotine gum can lessen their reliance on smoking [15]. Based on the compartment diagram in Figure 1, a novel model is established as follows:

$$
\begin{cases}
\mathscr{D}^{(\gamma)}P(t) = \alpha - \lambda P(t)S(t) - (\nu+\mu+\sigma)P(t) \\
\mathscr{D}^{(\gamma)}S(t) = \lambda P(t)S(t) + \delta S(t)Q(t) - (\mu+\kappa+\zeta+\rho)S(t) \\
\mathscr{D}^{(\gamma)}Q(t) = (\zeta+\rho)S(t) - \delta S(t)Q(t) + (\eta+\mu+\xi)Q(t),
\end{cases} \tag{2}
$$

where $\rho \in [0,1)$ and $\mathscr{D}^{(\gamma)}$ is the fractional derivative operator of order γ as defined in [16, 17, 18], where $0 < \gamma < 1$.

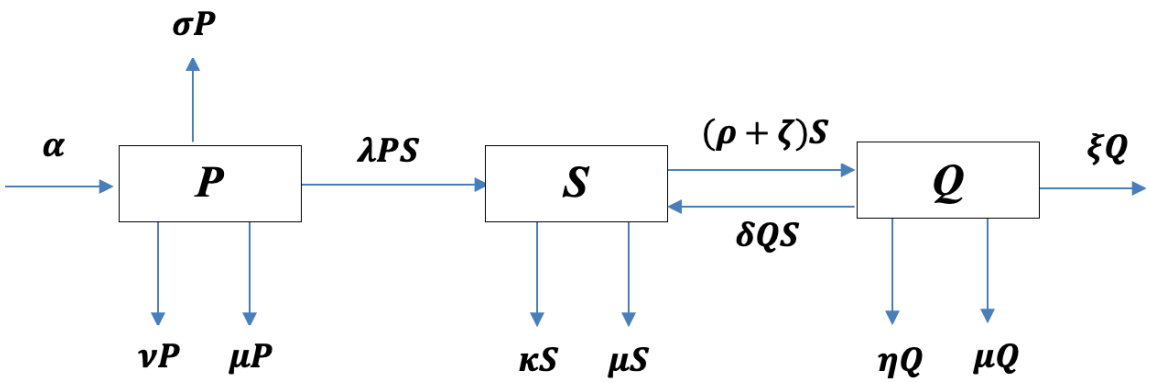

FIGURE 1. Compartment diagram for PSQ model

In this study, we examined the impact of providing nicotine gum on the reduction of the population that smokes actively by examining the stability of the equilibrium points of the model (2). As far as the authors are aware, this issue has not yet been fixed. As a result, the research results offer a fresh development in the area of fractional-order epidemic dynamics.

2. PRELIMINARIES

In this segment, we revisit certain definitions and characteristics related to fractional calculus and the stability of the equilibrium points.

Let $\mathscr{B} : [0,\infty) \to \mathbb{R}^n$ is an integrable vector function and $\gamma \in (m-1, m)$, $k \in \mathbb{N}$. The Caputo fractional-order derivative of order γ is defined by

$$
\mathscr{D}^{(\gamma)}\mathscr{B}(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^t \frac{\mathscr{B}^{(m)}(\tau)}{(t-\tau)^{1-m+\gamma}} d\tau \tag{3}
$$

where $\Gamma(.)$ is the Euler Gamma function [16]. One can readily ascertain that when K is a constant, then $\mathscr{D}^{(\gamma)}K = 0$.

Our attention will be directed towards the broad fractional-order dynamic system

$$\mathscr{D}^{(\gamma)}\mathscr{B}(t) = \mathbf{h}(t, \mathscr{B}(t)) \tag{4}$$

with suitable initial conditions $\mathscr{B}(t_0) = \mathscr{B}_0$, where $\mathscr{B}(t)$ is the state at time t, $\mathbf{h} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathscr{D}^{(\gamma)}$ is the Caputo fractional derivative of order γ. It is important to note that the system (4) could potentially demonstrate non-linear behavior, or conversely. If $\mathbf{h}$ is linear, the system (4) can be written as

$$\mathscr{D}^{(\gamma)}\mathscr{B}(t) = \mathscr{A}\mathscr{B}, \tag{5}$$

where $\mathscr{A}$ is a n by n matrix.

The stability of the equilibrium points, or how the solution (4) behave when $t \to \infty$, is a crucial aspect of the system (4) [19]. If $\mathbf{h}(t, \mathscr{B}^*) = \mathbf{0}$ for a $\mathscr{B}^* \in \mathbb{R}^n$ then the point $\mathscr{B}^*$ is said the equilibrium point of the system (4). The equilibrium point, it should be noted, represents a constant solution within the dynamic system (4).

Definition 2.1. [16, 19] *Let $\mathscr{B}^*$ is an equilibrium point of the fractional-order system (19).*

(1). *$\mathscr{B}^*$ is said to be stable if for $\varepsilon > 0$, there exists a $\vartheta_\varepsilon > 0$ such that $\|\mathscr{B}(t_0) - \mathscr{B}^*\| < \vartheta_\varepsilon$ implies $\|\mathscr{B}(t) - \mathscr{B}^*\| < \varepsilon$ for $t \geq t_0$.*

(2). *$\mathscr{B}^*$ is said to be asymtotically stable if it is stable and $\lim_{t\to\infty}\mathscr{B}(t) = \mathscr{B}^*$.*

Theorem 2.2. [16, 19] *The equilibrium point $\mathscr{B}^*$ of the fractional-order linear system (5) with $\gamma \in (0,1)$ is asymptotically stable if*

$$|\arg(X_i)| > \frac{1}{2}\gamma\pi, \tag{6}$$

where X_i, $i = 1, 2, \cdots, n$ are eigenvalues of the matrix $\mathscr{A}$.

Theorem 2.3. [16, 19] *The equilibrium point $\mathscr{B}^*$ of the the fractional-order nonlinear system (4) with $\gamma \in (0,1)$ is asymptotically stable if*

$$|\arg(X)| > \frac{1}{2}\gamma\pi, \tag{7}$$

for all roots X of the characteristic equation

$$|J_{\mathscr{B}^*} - XI| = 0 \tag{8}$$

where $J_{\mathscr{B}^*}$ is the Jacobian matrix of system (4) around the equilibrium $\mathscr{B}^*$ and I is the identity matrix of suitable size.

3. Stability Analysis

We assume that $N(t)$ is constant. For the sake of computation, we treat $P(t), S(t)$, and $Q(t)$ as proportions of $N(t)$, where $P(t) + Q(t) + S(t) = 1$. It is simple to demonstrate that solution of the model under consideration is bounded in in the region given by

$$\mathscr{W} = \left\{ (P,S,Q) \in \mathbb{R}^3_+ \ : \ 0 \leq N(t) \leq \frac{\alpha}{\mu} \right\}.$$

In epidemiology, it is commonly recognized that the basic reproduction number $\mathscr{R}_0$ determines the dynamic behavior of model (4). The basic reproduction number of an infectious disease is the average number of secondary cases generated by a single primary case in a fully susceptible population [20]. Using the next generation technique, the model (2)'s basic reproduction number is provided by

$$\mathscr{R}_0 = \frac{\lambda \alpha}{(\mu + \nu + \sigma)(\mu + \kappa + \zeta + \rho)}. \tag{9}$$

We need to solve the following equations in order to determine the model (2)'s equilibrium point:

$$\mathscr{D}^{(\gamma)} P(t) = \mathscr{D}^{(\gamma)} S(t) = \mathscr{D}^{(\gamma)} Q(t) = 0. \tag{10}$$

There are two equilibrium points for the model (2): the smoking-free equilibrium point, and the smoking-endemic equilibrium point. The smoking-free equilibrium points, denoted by $\mathscr{E}_0 = (P^0, S^0, Q^0)$, are steady-state solutions of a mathematical model indicating that there is no active smoking; in this case $S = 0$. Thus, setting $S = 0$, one finds the smoking free-equilibrium of the fractional-order model (2) as follows:

$$\mathscr{E}_0 = \left(\frac{\alpha}{\mu + \nu + \sigma}, 0, 0 \right).$$

Smoking-endemic equilibrium, denoted by $\mathscr{E}^* = (P^*, S^*, Q^*)$, is a steady-state solution when smoking persists in the population; in this case $S > 0$. Thus, in model (2), if $S > 0$, one has

$$P = \frac{\alpha}{\nu + \mu + \sigma + \lambda S}, \tag{11}$$

$$Q = \frac{(\zeta + \rho)S}{\eta + \mu + \xi + \delta S}, \tag{12}$$

$$S = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \tag{13}$$

where

$$A = -\lambda\delta(\mu + \kappa), \tag{14}$$

$$\begin{aligned} B =& \delta(\mu + \kappa + \zeta + \rho)(\nu + \mu + \sigma)(\mathscr{R}_0 - 1) + \\ & \delta(\zeta + \rho)(\nu + \mu + \sigma) - \lambda(\eta + \mu + \xi)(\mu + \kappa + \zeta + \rho), \end{aligned} \tag{15}$$

$$C = (\nu + \mu + \sigma)(\mu + \kappa + \zeta + \rho)(\eta + \mu + \xi)(\mathscr{R}_0 - 1). \tag{16}$$

If $\mathscr{R}_0 > 1$ then

$$S^* = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \tag{17}$$

due to $A < 0$. In this case, P^* and Q^* can be found by substituting equation (17) into equation (11) and (12), respectively. If $\mathscr{R}_0 < 1$, then $-B > 0$ and $C < 0$, and thus either S is negative real number or complex number. In this case, S does not exist and hence P and Q. If $\mathscr{R}_0 = 1$, then $B = \delta(\zeta + \rho)(\nu + \mu + \sigma) - \lambda(\eta + \mu + \xi)(\mu + \kappa + \zeta + \rho)$ and $C = 0$. Consequently,

$$S = \frac{-B - |B|}{2A} \tag{18}$$

which is negatif if $B > 0$ and zero if $B \leq 0$. Based on this discussion, we have

i. the smoking-endemic equilibrium point of model (2) exist if $\mathscr{R}_0 > 1$ or $\mathscr{R}_0 = 1$ and $\delta\zeta(\nu + \mu + \sigma) > \lambda(\eta + \mu + \xi)(\mu + \kappa + \zeta + \rho)$, and does not exist if $\mathscr{R}_0 < 1$,
ii. in the case the smoking-endemic equilibrium point exist, P^* is given by (11), S^* is given by (17) and Q^* is given by (12).

We'll examine whether these two equilibrium points are stable. First, the dynamical system (2)'s Jacobian matrix is

$$J = \begin{bmatrix} -(\nu+\mu+\sigma+\lambda S) & -\lambda P & 0 \\ \lambda S & \lambda P+\delta Q-(\mu+\kappa+\zeta+\rho) & \delta S \\ 0 & \zeta+\rho-\delta Q & -(\delta S+\eta+\mu+\xi) \end{bmatrix}. \tag{19}$$

At around the smoking-free equilibrium point $\mathscr{E}_0$, the Jacobian matrix is

$$J_{\mathscr{E}_0} = \begin{bmatrix} -(\nu+\mu+\sigma) & -\dfrac{\lambda\alpha}{\nu+\mu+\sigma} & 0 \\ 0 & (\mathscr{R}_0-1)(\mu+\kappa+\zeta+\rho) & 0 \\ 0 & \zeta+\rho & -(\eta+\mu+\xi) \end{bmatrix} \tag{20}$$

which have eigenvalues

$$\lambda_1 = -(\nu+\mu+\sigma),\ \lambda_2 = (\mathscr{R}_0-1)(\mu+\kappa+\zeta+\rho) \text{ and } \lambda_3 = -(\eta+\mu+\xi). \tag{21}$$

Based on (21) we see that λ_1 and λ_3 are negative, hence they satisfy $|\arg(\lambda_i)| > \dfrac{\gamma\pi}{2}$, for $i = 1,3$, whereas $|\arg(\lambda_2)| > \dfrac{\gamma\pi}{2}$ if $\mathscr{R}_0 < 1$. Certainly, $|\arg(\lambda_2)| < \dfrac{\gamma\pi}{2}$ if $\mathscr{R}_0 > 1$. Hence, based on the Theorem 2.3, the smoking-free equilibrium point $\mathscr{E}_0$ is asymptotically stable if $\mathscr{R}_0 < 1$ and becomes unstable if $\mathscr{R}_0 > 1$.

Next, the Jacobian matrix of the model (2) around $\mathscr{E}^*$ is given by

$$J_{\mathscr{E}^*} = \begin{bmatrix} -(\nu+\mu+\sigma+\lambda S^*) & -\lambda P^* & 0 \\ \lambda S^* & 0 & \delta S^* \\ 0 & \zeta+\rho-\delta Q^* & -(\delta S+\eta+\mu+\xi) \end{bmatrix},$$

where P^* is given by (11), S^* is given by (17) and Q^* is given by (12). The characteristic polynomial of $J_{\mathscr{E}^*}$ is given by the following equation:

$$p(X) = X^3 + b_1X^2 + b_2X + b_3, \tag{22}$$

where

$$b_1 = \nu + \mu + \sigma + \lambda S^* + \eta + \mu + \xi \tag{23}$$

$$b_2 = (\delta\lambda S^* + \delta(\nu + \mu + \sigma) + \lambda(\eta + \mu + \xi) + \delta^2 Q^* - (\zeta + \rho)\delta + \lambda^2 P^*)S^* + (\nu + \mu + \sigma)(\eta + \mu + \xi) \tag{24}$$

$$b_3 = \Big(\delta^2(\nu + \mu + \sigma)Q^* + \lambda\delta^2 S^* Q^* - (\nu + \mu + \sigma)\delta(\zeta + \rho) - \lambda(\zeta + \rho)\delta S^* + \lambda^2\delta P^* S^* + \lambda^2(\eta + \mu + \xi)P^*\Big)S^* \tag{25}$$

Our aim in this analysis is to demonstrate that every root of the characteristic polynomial (22) possesses a negative real part, thus resulting $|\arg(X_i)| > \dfrac{\gamma\pi}{2}$ for $i = 1, 2, 3$. However, it is difficult to done because the complexity of b_1, b_2 and b_3. However, using the Routh–Hurwitz criterion, the real parts of roots of polynomial (22) are negative if $b_3 > 0$ and $b_1 b_2 - b_3 > 0$. Hence, the smoking-endemic equilibrium point $\mathscr{E}^*$ is asymptotically stable if $b_3 > 0$ and $b_1 b_2 - b_3 > 0$.

To demonstrate the veracity of the findings, let's look at the following numerical example. For the model (2), the values of parameters used are $\lambda = 0.6, \alpha = 0.17, \delta = 0.3, \nu = 0.02, \mu = 0.12, \xi = 0.02, \kappa = 0.04, \zeta = 0.03, \eta = 0.01, \xi = 0.1$, and the initial value $P(0) = 0.75, S(0) = 0.15, Q(0) = 0.1$. Using Matlab R2024a, graphs of the secondhand smoking subpopulation, the active smoking subpopulation, and the stopped smoking subpopulation without the effect of the nicotine gum for various order γ are given in Figure 2. One can see that with giving the nicotine gum, the $\mathscr{E}^*$ is asymptotic stable due to $\mathscr{R}_0 > 1$.

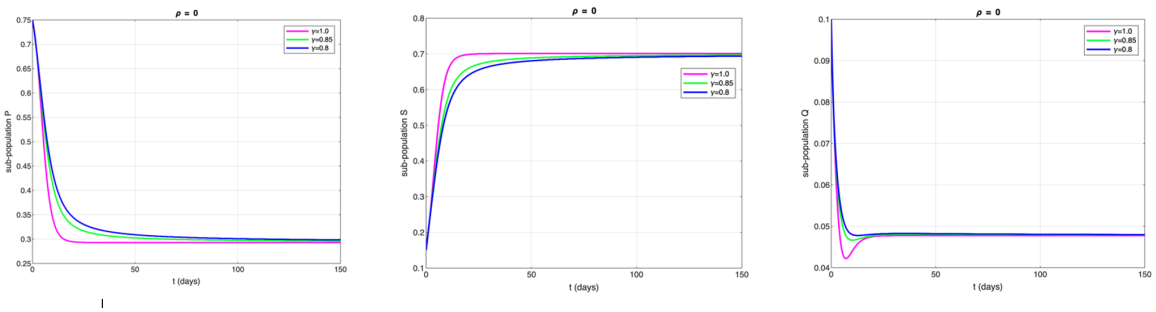

FIGURE 2. The curves of secondhand smoking, active smoking and stopped smoking for $\rho = 0$, where $\mathscr{E}^* = (0.2928, 0.7011, 0.0478)$ and $\mathscr{R}_0 = 3.3553$

Graphs of the secondhand smoking subpopulation, the active smoking subpopulation, and the stopped smoking subpopulation with the effect of giving the nicotine gum for various order γ are given in Figure 3 and Figure 4 . One can see that with giving the nicotine gum ($\rho = 0.15$ and $\rho = 0.3$), the $\mathscr{E}^*$ is asymptotic stable due to $\mathscr{R}_0 > 1$. The graphs and the smoking endemic equilibrium points show that giving the nicotine gum to active smoking can reduce the number of active smoking and increase the number of secondhand smoking.

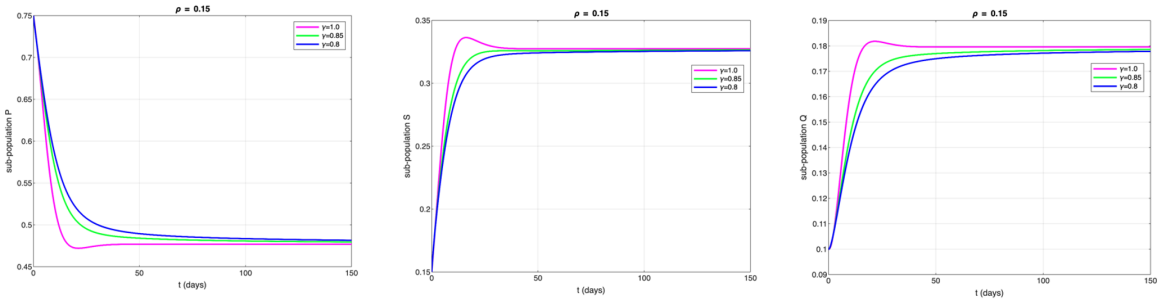

FIGURE 3. The curves of secondhand smoking, active smoking and stopped smoking for $\rho = 0.15$ where $\mathscr{E}^* = (0.4769, 0.3275, 0.1796)$ and $\mathscr{R}_0 = 1.875$

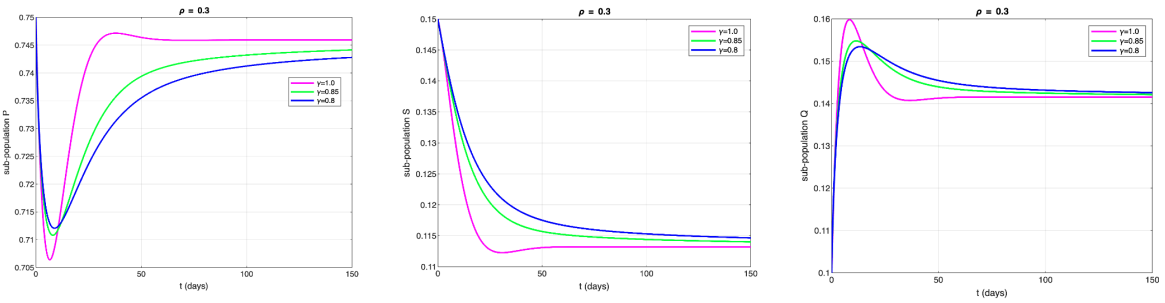

FIGURE 4. The curves of secondhand smoking, active smoking and stopped smoking for $\rho = 0.3$ where $\mathscr{E}^* = (0.7459, 0.1132, 0.1415)$ and $\mathscr{R}_0 = 1.3010$

4. CONCLUSION

The fractional *PSQ* model for the dynamics of the smoking tobacco pandemic with the impact of nicotine gum has been identified. A numerical example has been provided to demonstrate the outcome. According to the analysis, providing nicotine gum to smokers who are currently

in the habit can decrease the number of active smokers and raise the number of secondhand smokers. For this reason, the *PSQ* model provides sufficient information regarding the spread of smoking habits.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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