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A STOCHASTIC EXAMINATION OF THE SIRI EPIDEMIC MODEL INCLUDING NONLINEAR RELAPSE AND THE ORNSTEIN–UHLENBECK PROCESS

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Abstract. In this study, we conduct an analysis of a stochastic SIRI model with nonlinear relapse that incorporates the Ornstein–Uhlenbeck process. Initially, we establish the existence of a unique global positive solution. Then, we present the sufficient conditions for both disease extinction and persistence. By constructing an appropriate Lyapunov function, we further demonstrate the existence of a stationary distribution for our model. Finally, we present some computer simulations that demonstrate the theoretical insights that we have obtained. The results of this study have the potential to enhance our understanding of epidemic models and contribute to the development of viable strategies for disease prevention and control.

Keywords: stochastic SIRI epidemic model; nonlinear relapse; Ornstein–Uhlenbeck process; stationary distribution; persistence in mean; extinction.

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1. INTRODUCTION

Many factors, such as changes in the environment, globalization, human behavior, and microbial evolution, contribute to the origin and resurgence of infectious diseases. Despite the recent focus on diseases like MERS, pandemic influenza, SARS, Zika, and Ebola, ancient diseases like plague, cholera, and yellow fever still pose problems and have even made a comeback. For epidemiologists, researchers, and public health officials, the application of mathematical modeling

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to the study of infectious diseases has proved immensely valuable. In order to explain the dynamics of disease transmission, project the trajectory of the diseases, and evaluate the potential effects of intervention methods, these models provide a systematic framework that can support these endeavors. In mathematical epidemiology, the SIR model, developed by Kermack and McKendrick [11], has been crucial. Susceptible "S", infectious "I", and recovered "R"are the three groups that this model divides the population into when considering the spread of disease. The phenomenon of latent infections, in which persons who have healed from an infection may reawaken or reactivate the pathogen, making them contagious again. This is especially crucial for infectious disorders like herpes and tuberculosis, which can cause latent infections in the host. Generally, the traditional SIR framework in epidemiology is not appropriate for modeling this type of behavior. It may be more acceptable to use models that take into consideration latent periods, reactivation, and potential re-entry into the pool of infectious individuals. Extensions to the basic SIR model, such as the SEIR (Susceptible-Exposed-Infectious-Recovered) model or models with additional compartments indicating delay, can be utilized to reflect these dynamics [5, 6, 8, 13].

2. PRELIMINARIES

2.1. Mathematical Model. In the context of alcohol consumption or other behaviors that might influence the reactivation of latent infections, an interdisciplinary approach that considers both epidemiological and behavioral aspects would be necessary. Modeling such phenomena requires a nuanced understanding of the specific disease, its biological characteristics, and the social and behavioral factors that may influence relapse or reactivation. Sanchez et al. [19] used a SIRI epidemic model with nonlinear relapse to give a full explanation of the patterns of drinking that happen because of social interactions that happen in places where people drink together. The following set of nonlinear differential equations describes the model.

$$\begin{cases} \dot{S} &= b - bS - \beta SI, \\ \dot{I} &= -(b+\delta)I + \beta SI + \lambda RI, \\ \dot{R} &= -bR + \delta I - \lambda RI. \end{cases} \tag{1}$$

In this instance, there are three groupings within the population: steady and random streaks "S", big or bothersome streaks "I", and short-term healing "R". The real touch rate is shown by the parameter β. This means that βIS shows the change from class S to class I caused by interactions between people from classes S and I that depend on how often they happen. b shows the adoption rate of weak people who are born or immigrate, and λ shows the return rate. This means that λIR shows the speed at which things change from R to I. This happens because R and I interact with each other in ways that depend on frequency. This nonlinear process assumes that people R, I, and S live in the same setting. The recovery rate is denoted by δ.

Regarding the system (1) stated above, we can easily show that the total population $\Upsilon = S+I+R$ verifies

$$d\Upsilon(s) = d\big[S(s)+I(s)+R(s)\big] = \big[b-b(S(s)+I(s)+R(s))\big]\mathrm{d}s = b(1-\Upsilon(s))\mathrm{d}s,$$

which implies by integration

$$1-\Upsilon(s) = (1-\Upsilon(0))e^{-bs}.$$

Then, if $\Upsilon(0) = S(0)+I(0)+R(0) = 1$, we obtain

$$S(s)+I(s)+R(s) = \Upsilon(s) = 1.$$

In this paper, we are going to define $\mathbb{R}^d_+ = \big\{(u_1,...,u_d)|u_k > 0,\, k = 1,...,d\big\}$. This means that the set

$$\mathscr{D}_1 = \Big\{\mathscr{X} = (u_1,u_2,u_3) \in \mathbb{R}^3_+;\; u_1+u_2+u_3 = 1\Big\}$$

is a positively invariant region of system (1). In order to investigate a two-dimensional system, the model (1) is reduced as follows:

$$\left\{\begin{array}{lcl} \dot{I} &=& -(b+\delta)I+\beta(1-I-R)I+\lambda RI, \\ \dot{R} &=& -bR+\delta I-\lambda RI. \end{array}\right. \tag{2}$$

For system (2), the basic repetition number is represented by

$$\mathscr{R}_\delta = \frac{\beta}{\delta+b}.$$

The dimensionless quantity $\mathscr{R}_\delta$ represents the number of I-individuals that are generated in a population that is predominantly composed of S-individuals sharing a common environment. It has been demonstrated by Sanchez et al. [19] that the dynamics of system (1) also depend on the initial population size and ratios

$$\mathscr{R}_1 = \frac{\lambda}{\beta} \times \left[1 - \mathscr{R}_\delta\right], \;\; \mathscr{R}_2 = \frac{\lambda}{\beta} \times \left[\frac{1}{1+\frac{b}{\beta}} - 2\sqrt{\frac{b}{\beta} - \frac{b}{\lambda}}\right].$$

- The I-class is established if $\mathscr{R}_\delta > 1$. This guarantees the continuation of a regular drinking class over time.
- The I-class is extinct if $\mathscr{R}_2 < \mathscr{R}_\delta < 1$ and $\mathscr{R}_1 < 1$, or $\mathscr{R}_\delta < \mathscr{R}_2 < 1$.
- If $\mathscr{R}_2 < \mathscr{R}_\delta < 1$ and $\mathscr{R}_1 > 1$, the starting size of the I-individuals shows if the I-class is set up or not.

2.2. Stochastic Model. In the real world, ambient noise always has an impact on the transmission of infectious diseases. As a result, numerous authors have examined the epidemic model incorporating random perturbation and put forth a variety of random perturbation model types (see, for example, [1, 2, 6, 9, 12, 13, 21, 29]). Numerous scholars have proposed a type of SIRI epidemic model with random perturbation based on stochastic differential equations, such as [20], stochastic model under regime switching [7], and stochastic model with Lévy process [3].

In physics and biology, the Ornstein–Uhlenbeck process is used to model processes that have a tendency to return to a specific equilibrium state. It can be used to describe the mobility of particles subjected to friction as well as the behavior of biological systems with regulatory mechanisms. Currently, a number of random interference channels use the Ornstein–Uhlenbeck process (see [4, 15, 22, 28]). To determine the effect of ambient noise on transmission rate, we assume that β and λ follow the mean-reverting Ornstein–Uhlenbeck process in model (2) as follows [16, 17, 25, 26]:

$$d\beta(t) = -\alpha_1 \times \left(\beta(t) - \overline{\beta}\right)\mathrm{d}t + \xi_1 d\mathscr{W}_1(t), \quad d\lambda(t) = -\alpha_2 \times \left(\lambda(t) - \overline{\lambda}\right)\mathrm{d}t + \xi_2 d\mathscr{W}_2(t), \tag{3}$$

where $\overline{\beta}, \overline{\lambda}$ are measure the long-run mean levels of the infection rates β, λ; $\alpha_j (j = 1, 2)$ denote the speeds of reversion. $\mathscr{W}_j (j = 1, 2)$ are independent standard Brownian motion parameters

defined on a complete probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, and parameter $\xi_j (j = 1,2)$ represents the intensity of $\mathscr{W}_j$. In general, all parameters are considered to be nonnegative.

The following explicit form solution for the arithmetic Ornstein–Uhlenbeck process (3) can be obtained using the stochastic integral format.

$$\begin{cases} \beta(t) &= \overline{\beta} + \left(\beta_0 - \overline{\beta}\right) e^{-\alpha_1 t} + \xi_1 \int_0^t e^{-\alpha_1 (t-v)} d\mathscr{W}_1(v), \\ \lambda(t) &= \overline{\lambda} + \left(\lambda_0 - \overline{\lambda}\right) e^{-\alpha_2 t} + \xi_2 \int_0^t e^{-\alpha_2 (t-v)} d\mathscr{W}_2(v), \end{cases} \tag{4}$$

with initial values $\beta_0 = \beta(0)$ and $\lambda_0 = \lambda(0)$.

It is simple to determine that the term $\xi_i \int_0^t e^{-\alpha_i (t-v)} d\mathscr{W}_i(v)$ follows the normal distribution $\mathscr{N}\left(0, \frac{\xi_i^2}{2\alpha_i}(1 - e^{-2\alpha_i t})\right)$. When $t \longrightarrow \infty$, it is easy to demonstrate by calculating expectation and variance that

$$\mathbb{E}(\beta(t)) \longrightarrow \overline{\beta},\ \mathbb{V}(\beta(t)) \longrightarrow \frac{\alpha_1^2}{2\xi_1},\ \mathbb{E}(\lambda(t)) \longrightarrow \overline{\lambda},\ \mathbb{V}(\lambda(t)) \longrightarrow \frac{\alpha_2^2}{2\xi_2}.$$

Clearly, $\beta(t)$ follows the normal distribution $\mathscr{N}\left(\overline{\beta}, \frac{\alpha_1^2}{2\xi_1}\right)$ as t tends to infinity. Here is the definition of the probability density function:

$$\kappa_1(x) = \frac{\sqrt{\alpha_1}}{\sqrt{\pi}\xi_1} e^{-\frac{\alpha_1 (x-\overline{\beta})^2}{\xi_1^2}}. \tag{5}$$

To make the subsequent computations easier, it results in that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |\beta(v) - \overline{\beta}| \mathrm{d}v = \int_{-\infty}^{+\infty} |x - \overline{\beta}| \kappa_1(x) \mathrm{d}x = \frac{\xi_1}{\sqrt{\pi\alpha_1}}, \tag{6}$$

which shows β is ergodic and weakly converges to probability density function $\kappa_1(x)$.

In order to discuss the positivity of model (2) with Ornstein–Uhlenbeck process, we define $\beta^+(t) = \max\{0, \beta(t)\}$ and $\lambda^+(t) = \max\{0, \lambda(t)\}$. As a result, the following stochastic model with Ornstein–Uhlenbeck process is the subject of our investigation in this study.

$$\begin{cases} dI(t) &= \Big(-(b+\delta) I(t) + \beta^+(t)(1 - R(t) - I(t)) I(t) + \lambda^+(t) R(t) I(t) \Big) \mathrm{d}t, \\ dR(t) &= \Big(-bR(t) + \delta I(t) - \lambda^+(t) R(t) I(t) \Big) \mathrm{d}t, \\ d\beta(t) &= -\alpha_1 \times (\beta(t) - \overline{\beta}) \mathrm{d}t + \xi_1 d\mathscr{W}_1(t), \\ d\lambda(t) &= -\alpha_2 \times (\lambda(t) - \overline{\lambda}) \mathrm{d}t + \xi_2 d\mathscr{W}_2(t). \end{cases} \tag{7}$$

Additionally, $u_1 \wedge u_2$ is written as $\min\{u_1, u_2\}$; in a similar manner, $u_1 \vee u_2$ is written as $\max\{u_1, u_2\}$.

3. Existence and Uniqueness of a Global Solution

Within the scope of this section, we will discuss the existence of a global solution for model (7), as well as its uniqueness. Prior to beginning our investigation of the model, it is necessary for us to specify the following subset:

$$\mathscr{D}_2 = \Big\{(I,R,\beta,\lambda) \in \mathbb{R}^2_+ \times \mathbb{R}^2;\ I+R<1\Big\}.$$

Theorem 1. *If $(I(0),R(0),\beta(0),\lambda(0)) \in \mathscr{D}_2$, then $\mathbb{P}\big((I(t),R(t),\beta(t),\lambda(t)) \in \mathscr{D}_2\big) = 1$ for any $t \geq 0$. The assertion that the set $\mathscr{D}_2$ is almost certainly positively invariant by the system (7) is supported by this evidence.*

Proof. Clearly, the coefficients of model (7) satisfy the local Lipschitz conditions, hence, for any starting point $(I(0),R(0),\beta(0),\lambda(0)) \in \mathscr{D}_2$, there exists a unique local solution $\mathscr{X}(t) = (I(t),R(t),\beta(t),\lambda(t))$ on $t \in [0,\tau_e)$, where τ_e denotes the explosion time. We only need to show that $\tau_e = \infty$ to show that this solution is global. Let $p_0 > 0$ such that $e^{\beta(0)}, e^{\lambda(0)}, I(0), R(0) \in [\frac{1}{p_0}, p_0]$. For each integer $p > p_0$ considering the stopping times

$$\tau_p = \inf\left\{t \in [0,\tau_e):\ \min\left\{e^{\beta(t)}, e^{\lambda(t)}, I(t), R(t)\right\} \leq \frac{1}{p} \ \text{ or } \ \max\left\{e^{\beta(t)}, e^{\lambda(t)}, I(t), R(t)\right\} \geq p\right\}.$$

We set out in this paper to $\inf\emptyset = \infty$. Evidently, τ_p is increasing as $p \longrightarrow \infty$. Set $\tau_\infty = \lim\limits_{p\to\infty} \tau_p$, next, we determine that $\tau_\infty \leq \tau_e$ a.s. If we demonstrate that $\tau_\infty = \infty$ a.s., so $\tau_e = \infty$ a.s., then $\big(I(t),R(t),\beta(t),\lambda(t)\big) \in \mathscr{D}_2$ a.s. for any $t \geq 0$.

Let us define a Lyapunov function $\varphi : \mathbb{R}^2_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}_+$,

$$\varphi\big(I,R,\beta,\lambda\big) = (I-1-\ln I) + (R-1-\ln R) + \Big[(1-R-I)-1-\ln(1-R-I)\Big] + \frac{\beta^2}{2} + \frac{\lambda^2}{2}.$$

Using Itô's formula, we obtain

$$\mathrm{d}\varphi = \mathscr{L}\varphi \mathrm{d}t + \xi_1 \beta \mathrm{d}\mathscr{W}_1(t) + \xi_2 \lambda \mathrm{d}\mathscr{W}_2(t), \tag{8}$$

where

$$\begin{aligned}\mathscr{L}\varphi \ = \ & -\frac{1}{I} \times \Big[-(b+\delta)I + \beta^+(1-R-I)I + \lambda^+ RI\Big] - \frac{1}{R} \times \Big[-bR + \delta I - \lambda^+ RI\Big] \\ & -\frac{1}{1-R-I}\Big[b(R+I) - \beta^+(1-R-I)I\Big] + \alpha_1\beta(\overline{\beta}-\beta) + \frac{\xi_1^2}{2} + \alpha_2\lambda(\overline{\lambda}-\lambda) + \frac{\xi_2^2}{2}.\end{aligned}$$

We easily obtain

$$\begin{aligned}\mathscr{L}\varphi &\leq 2b+\delta+\beta^{+}+\lambda^{+}+\frac{\xi_1^2}{2}+\frac{\xi_2^2}{2}+\alpha_1\beta(\overline{\beta}-\beta)+\alpha_2\lambda(\overline{\lambda}-\lambda)\\ &\leq 2b+\delta+|\beta|+|\lambda|+\frac{\xi_1^2}{2}+\frac{\xi_2^2}{2}-\alpha_1\beta^2+\alpha_1\overline{\beta}|\beta|-\alpha_2\lambda^2+\alpha_2\overline{\lambda}|\lambda|\\ &\leq 2b+\delta+\frac{\xi_1^2}{2}+\frac{\xi_2^2}{2}+\frac{(\alpha_1\overline{\beta}+1)^2}{4\alpha_1}+\frac{(\alpha_2\overline{\lambda}+1)^2}{4\alpha_2}:=\mathfrak{A},\end{aligned}$$

where $\mathfrak{A}$ is a positive constant independent of R, I and t. For the purpose of brevity, we will skip over the remainder of the demonstration that is identical to the proof of Theorem 2 in [13]. The proof of theorem is thus completed. □

4. Existence of a Stationary Distribution

This section pertains to the stationary distribution in the system (7), which has implications for the continued transmission of infectious diseases. We present Lemma 1 before beginning to establish that system (7) has a stationary distribution.

Lemma 1. [18] *For every starting value, $Z_0(0)=(I(0),R(0),\beta(0),\lambda(0))\in\mathscr{D}_2$, if there exists a bounded closed domain $\mathscr{H}_\eta\subset\mathscr{D}_2$ with regular boundary, and obeys*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{P}(v,Z_0(0),\mathscr{H}_\eta)dv>0\ a.s., \tag{9}$$

where $\mathbb{P}(v,Z_0(0),\mathscr{H}_\eta)$ denotes the transition probability of $Z_0(t)$. In this situation, the stochastic system (7) *has at least one stationary distribution.*

Define a critical value,

$$\widetilde{\mathscr{R}}_\beta=\frac{\overline{\beta}}{b+\delta+\frac{\xi_1}{\sqrt{\pi\alpha_1}}}. \tag{10}$$

Theorem 2. *Let $\big(I(0),R(0),\beta(0),\lambda(0)\big)\in\mathscr{D}_2$ be a starting departure point. If $\widetilde{\mathscr{R}}_\beta>1$, then the stochastic system* (7) *has at least one stationary distribution $\varpi(.)$ on $\mathscr{D}_2$.*

Proof. The definition of a C^2-function Ψ_1 will be given first, as stated below.

$$\Psi_1=-\ln(I)+\frac{\overline{\beta}}{b}R.$$

Using Itô's formula, we have

$$\begin{aligned}
\mathscr{L}\Psi_1 &= (b+\delta)-\beta^+(1-I-R)-\lambda^+R+\frac{\overline{\beta}}{b}\times(-bR+\delta I-\lambda^+RI)\\
&\leq b+\delta-\overline{\beta}(1-I-R)+|\overline{\beta}-\beta^+|(1-I-R)+\frac{\overline{\beta}}{b}\times(-bR+\delta I)\\
&\leq -\overline{\beta}+b+\delta+\overline{\beta}I+\frac{\delta\overline{\beta}}{b}I+|\beta-\overline{\beta}| \qquad (11)\\
&\leq -(\widetilde{\mathscr{R}}_\beta-1)\left(b+\delta+\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right)+I\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)+|\beta-\overline{\beta}|-\frac{\xi_1}{\sqrt{\pi\alpha_1}}.
\end{aligned}$$

Denote

$$\Psi_2=-\ln(R),\quad \Psi_3=-\ln(1-I-R),\quad \Psi_4=\frac{\beta^2}{2}+\frac{\lambda^2}{2}.$$

We have

$$\begin{aligned}
\mathscr{L}\Psi_2 &= b-\frac{\delta I}{R}+\lambda^+I,\\
\mathscr{L}\Psi_3 &= \frac{-b(R+I)}{1-(R+I)}+\beta^+I \;\leq\; \beta^+I-\frac{bI}{1-(R+I)},\\
\mathscr{L}\Psi_4 &= \alpha_1\beta(\overline{\beta}-\beta)+\alpha_2\lambda(\overline{\lambda}-\lambda)+\frac{\xi_1^2}{2}+\frac{\xi_2^2}{2}.
\end{aligned}$$

Let us define a function $\Psi:\mathbb{R}^2_+\times\mathbb{R}^2\longrightarrow\mathbb{R}$,

$$\Psi(I,R,\beta,\lambda) = \mathscr{A}\Psi_1+\Psi_2+\Psi_3+\Psi_4,$$

where the positive constant $\mathscr{A}>0$ is sufficiently large to be selected in an appropriate manner at a later time.

We define a non-negative C^2-function $\overline{\Psi}$ as follows

$$\overline{\Psi}(I,R,\beta,\lambda)=\Psi(I,R,\beta,\lambda)-\Psi(\check{\mathscr{X}}_0),$$

where $\Psi(\mathscr{X})$ is a continuous function, $\check{\mathscr{X}}_0$ is a minimum point $(\check{I}_0,\check{R}_0,\check{\beta}_0,\check{\lambda}_0)$ in the interior domain of $\mathscr{D}_2$. Using Itô's formula, we obtain

$$\begin{aligned}
\mathscr{L}\overline{\Psi} \leq\; & -\mathscr{A}(\widetilde{\mathscr{R}}_\beta-1)\left(b+\delta+\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right)+\mathscr{A}\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)I+\mathscr{A}\left(|\beta-\overline{\beta}|-\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right)\\
&+b-\frac{\delta I}{R}+\lambda^+I+\beta^+I-\frac{bI}{1-(R+I)}+\alpha_1\beta(\overline{\beta}-\beta)+\alpha_2\lambda(\overline{\lambda}-\lambda)+\frac{\xi_1^2+\xi_2^2}{2}
\end{aligned}$$

$$
\begin{aligned}
&\leq -\mathscr{A}(\widetilde{\mathscr{R}}_\beta - 1)\left(b+\delta+\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right)+\mathscr{A}\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)I+\mathscr{A}\left(|\beta-\overline{\beta}|-\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right)\\
&\quad +b+\frac{\xi_1^2+\xi_2^2}{2}-\frac{\delta I}{R}-\frac{bI}{1-(R+I)}+|\lambda|+|\beta|+\alpha_1|\beta|\overline{\beta}-\alpha_1\beta^2+\alpha_2|\lambda|\overline{\lambda}-\alpha_2\lambda^2\\
&\leq -\mathscr{A}(\widetilde{\mathscr{R}}_\beta - 1)\left(b+\delta+\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right)+\mathscr{A}\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)I+\mathscr{A}\left(|\beta-\overline{\beta}|-\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right)\\
&\quad +b+\frac{\xi_1^2+\xi_2^2}{2}-\frac{\delta I}{R}-\frac{bI}{1-(R+I)}+\frac{(1+\alpha_1\overline{\beta})^2}{2\alpha_1}+\frac{(1+\alpha_2\overline{\lambda})^2}{2\alpha_2}-\frac{\alpha_1\beta^2}{2}-\frac{\alpha_2\lambda^2}{2}\\
&= G\big(I,R,\beta,\lambda\big)+\mathscr{A}\left(|\beta-\overline{\beta}|-\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
G\big(I,R,\beta,\lambda\big) &= -\mathscr{A}(\widetilde{\mathscr{R}}_\beta - 1)\left(b+\delta+\frac{\xi_1}{\sqrt{\pi\alpha_1}}\right)+\mathscr{A}\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)I+b+\frac{\xi_1^2+\xi_2^2}{2}\\
&\quad +\frac{(1+\alpha_1\overline{\beta})^2}{2\alpha_1}+\frac{(1+\alpha_2\overline{\lambda})^2}{2\alpha_2}-\frac{\delta I}{R}-\frac{bI}{1-(I+R)}-\frac{\alpha_1\beta^2}{2}-\frac{\alpha_2\lambda^2}{2}.
\end{aligned}
$$

To begin, we select a sufficiently large constant $\mathscr{A}>0$ that

$$
-\mathscr{A}(\widetilde{\mathscr{R}}_\beta - 1)\times\left(b+\delta+\frac{\xi_1}{\sqrt{2\alpha_1}}\right)+\mathscr{B}<-2,
$$

where

$$
\mathscr{B}=b+\frac{\xi_1^2+\xi_2^2}{2}+\frac{(1+\alpha_1\overline{\beta})^2}{2\alpha_1}+\frac{(1+\alpha_2\overline{\lambda})^2}{2\alpha_2}.
$$

We construct the bounded closed set

$$
\mathscr{H}_\eta=\left\{(I,R,\beta,\lambda)\in\mathscr{D}_2:\ I\geq\eta, I+R\leq 1-\eta^2, R\geq\eta^2, |\beta|\leq\frac{1}{\eta}, |\lambda|\leq\frac{1}{\eta}\right\},
$$

the constant η is sufficiently small to meet the conditions that are listed below.

$$
-2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)\eta \leq -1, \tag{12}
$$

$$
-2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)-\frac{b}{\eta} \leq -1, \tag{13}
$$

$$
-2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)-\frac{\delta}{\eta} \leq -1, \tag{14}
$$

$$
-2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)-\frac{\alpha_1}{2\eta^2} \leq -1, \tag{15}
$$

$$-2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)-\frac{\alpha_2}{2\eta^2} \leq -1. \tag{16}$$

Just to make things more clear, we divide $\mathscr{D}_2\setminus\mathscr{H}_\eta$ into five domains.

$$\begin{aligned}
\mathscr{H}^c_{\eta,1} &= \{(I,R,\beta,\lambda)\in\mathscr{D}_2:\ I<\eta\},\\
\mathscr{H}^c_{\eta,2} &= \left\{(I,R,\beta,\lambda)\in\mathscr{D}_2:\ I\geq\eta, I+R>1-\eta^2\right\},\\
\mathscr{H}^c_{\eta,3} &= \left\{(I,R,\beta,\lambda)\in\mathscr{D}_2:\ I\geq\eta, R<\eta^2\right\},\\
\mathscr{H}^c_{\eta,4} &= \left\{(I,R,\beta,\lambda)\in\mathscr{D}_2:\ |\beta|>\frac{1}{\eta}\right\},\\
\mathscr{H}^c_{\eta,5} &= \left\{(I,R,\beta,\lambda)\in\mathscr{D}_2:\ |\lambda|>\frac{1}{\eta}\right\}.
\end{aligned}$$

Obviously, $\mathscr{D}_2\setminus\mathscr{H}_\eta=\cup_{i=1}^5\mathscr{H}^c_{\eta,i}$. We will demonstrate that $G\big(I,R,\beta,\lambda\big)\leq -1$ on $\mathscr{D}_2\setminus\mathscr{H}_\eta$, this is identical to displaying it on the five previous domains.

- **Case 1**. Whenever $(I,R,\beta,\lambda)\in\mathscr{H}^c_{\eta,1}$, according to (12), we have

$$G(I,R,\beta,\lambda)\leq -2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)\eta\leq -1.$$

- **Case 2**. Whenever $(I,R,\beta,\lambda)\in\mathscr{H}^c_{\eta,2}$, in light of (13), we have

$$G(I,R,\beta,\lambda)\leq -2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)-\frac{b}{\eta}\leq -1.$$

- **Case 3**. Whenever $(I,R,\beta,\lambda)\in\mathscr{H}^c_{\eta,3}$, in light of (14), we get

$$G(I,R,\beta,\lambda)\leq -2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)-\frac{\delta}{\eta}\leq -1.$$

- **Case 4**. Whenever $(I,R,\beta,\lambda)\in\mathscr{H}^c_{\eta,4}$, according to (15), we get

$$G(I,R,\beta,\lambda)\leq -2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)-\frac{\alpha_1}{2\eta^2}\leq -1.$$

- **Case 5**. Whenever $(I,R,\beta,\lambda)\in\mathscr{H}^c_{\eta,5}$, according to (16), we have

$$G(I,R,\beta,\lambda)\leq -2+\mathscr{A}\times\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)-\frac{\alpha_2}{2\eta^2}\leq -1.$$

When all five cases are considered together, it becomes clear that

$$G(I,R,\beta,\lambda)\leq -1 \text{ for all } (I,R,\beta,\lambda)\in\mathscr{D}_2\setminus\mathscr{H}_\eta.$$

Furthermore, there exists a positive constant Λ which satisfies $G\big(I,R,\beta,\lambda\big) \leq \Lambda$. Here, denote $Z_0(t) = \big(I(t),R(t),\beta(t),\lambda(t)\big)$. As a result, we can obtain

$$\begin{aligned} 0 &\leq \frac{1}{t}\mathbb{E}\Big[\overline{\Psi}(Z_0(t))\Big] = \frac{1}{t}\mathbb{E}\Big[\overline{\Psi}(Z_0(0))\Big] + \frac{1}{t}\int_0^t \mathbb{E}\Big[\mathscr{L}\overline{\Psi}(Z_0(v))\Big]\mathrm{d}v \\ &\leq \frac{1}{t}\mathbb{E}\Big[\overline{\Psi}(Z_0(0))\Big] + \frac{1}{t}\int_0^t \mathbb{E}\Big[G(Z_0(v))\Big]\mathrm{d}v \\ &\quad +\mathscr{A}\left\{\mathbb{E}\left[\frac{1}{t}\int_0^t |\beta(v)-\overline{\beta}|\mathrm{d}v\right] - \frac{\xi_1}{\sqrt{\pi\alpha_1}}\right\}. \end{aligned} \tag{17}$$

Taking the inferior limit on both sides of (17) and combining it with (6), we have

$$\begin{aligned} 0 &\leq \liminf_{t\to\infty}\left(\frac{1}{t}\mathbb{E}\Big[\overline{\Psi}(Z_0(0))\Big]\right) + \liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \mathbb{E}\Big[G(Z_0(v))\Big]\mathrm{d}v\right) \\ &\quad +\mathscr{A}\left\{\liminf_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t |\beta(v)-\overline{\beta}|\mathrm{d}v\right] - \frac{\xi_1}{\sqrt{\pi\alpha_1}}\right\} \\ &= \liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \mathbb{E}\Big[G(Z_0(v))\Big]\mathbb{1}_{\{Z_0(v)\in\mathscr{H}_\eta\}}\mathrm{d}v\right) \\ &\quad +\liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \mathbb{E}\Big[G(Z_0(v))\Big]\mathbb{1}_{\{Z_0(v)\in\mathscr{D}_2\setminus\mathscr{H}_\eta\}}\mathrm{d}v\right) \\ &\leq \Lambda\times\liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \mathbb{P}\big\{Z_0(v)\in\mathscr{H}_\eta\big\}\mathrm{d}v\right) - \liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \mathbb{P}\big\{Z_0(v)\in\mathscr{D}_2\setminus\mathscr{H}_\eta\big\}\mathrm{d}v\right) \\ &\leq -1+(1+\Lambda)\times\liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \mathbb{P}\big\{Z_0(v)\in\mathscr{H}_\eta\big\}\mathrm{d}v\right). \end{aligned}$$

This indicates that

$$\liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \mathbb{P}\big\{Z_0(v)\in\mathscr{H}_\eta\big\}\mathrm{d}v\right) \geq \frac{1}{1+\Lambda},$$

then

$$\liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \mathbb{P}\big\{v,Z_0(0),\mathscr{H}_\eta\big\}\mathrm{d}v\right) \geq \frac{1}{1+\Lambda} > 0,\ \forall Z_0(0)\in\mathscr{D}_2\ a.s. \tag{18}$$

The inequality (18) and the invariance of $\mathscr{D}_2$ suggest the existence of an invariance probability measure for model on $\mathscr{D}_2$. The existence of the invariant probability measure also makes it easy to derive the positive recurrence of model (7). Therefore, the system (7) has a stationary distribution $\varpi(.)$. □

5. Disease Extinction

In this section, we will discuss the required condition for the disease to become extinct. Prior to delving into this matter, we will first define

$$\widetilde{\mathscr{R}}_E = \frac{\frac{\xi_1}{2\sqrt{\alpha_1\pi}}e^{-\frac{\overline{\beta}^2\alpha_1}{\xi_1^2}} + \overline{\beta}\Phi\left(\frac{\overline{\beta}\sqrt{\alpha_1}}{\xi_1}\right) + \frac{\xi_2}{2\sqrt{\alpha_2\pi}}e^{-\frac{\overline{\lambda}^2\alpha_2}{\xi_2^2}} + \overline{\lambda}\Phi\left(\frac{\overline{\lambda}\sqrt{\alpha_2}}{\xi_2}\right)}{b+\delta},$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \mathrm{d}v$.

Theorem 3. *With a starting value of* $(I(0),R(0),\beta(0),\lambda(0)) \in \mathscr{D}_2$, *let* $(I(t),R(t),\beta(t),\lambda(t))$ *be the solution of the system* (7). *If* $\widetilde{\mathscr{R}}_E < 1$, *then*

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{\ln I(t)}{t} < 0\right) = 1.$$

To put it another way, the disease almost surely goes away exponentially.

Proof. Employing the Itô formula on $I \mapsto \ln I$, we can get

$$\begin{aligned} d\big[\ln I(t)\big] &= \Big[-(b+\delta)+\beta^+(1-R(t)-I(t))+\lambda^+R(t)\Big]\mathrm{d}t \\ &\leq \big[-(b+\delta)+\beta^+ +\lambda^+\big]\mathrm{d}t. \end{aligned} \tag{19}$$

Division by t after integrating this inequality (19) from 0 to t, we get

$$\frac{1}{t}\times \ln I(t) \leq \frac{1}{t}\times \ln I(0) + \frac{1}{t}\int_0^t \beta^+(v)\mathrm{d}v + \frac{1}{t}\int_0^t \lambda^+(v)\mathrm{d}v - (b+\delta). \tag{20}$$

$\beta(t)$ has the ergodic property and there is a unique stationary distribution with the density function $\kappa_1(x) = \frac{\sqrt{\alpha_1}}{\sqrt{\pi}\xi_1} e^{-\frac{\alpha_1(x-\overline{\beta})^2}{\xi_1^2}}$. Then the strong law of large numbers [11], we get

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t}\int_0^t \beta^+(v)\mathrm{d}v &= \int_0^\infty x\kappa_1(x)\mathrm{d}x \\ &= \int_0^\infty x\frac{\sqrt{\alpha_1}}{\sqrt{\pi}\xi_1}e^{-\frac{\alpha_1(x-\overline{\beta})^2}{\xi_1^2}}\mathrm{d}x \\ &= \frac{\xi_1}{2\sqrt{\alpha_1\pi}}e^{-\frac{\overline{\beta}^2\alpha_1}{\xi_1^2}} + \overline{\beta}\left[1-\Phi\left(-\frac{\overline{\beta}\sqrt{\alpha_1}}{\xi_1}\right)\right] \\ &= \frac{\xi_1}{2\sqrt{\alpha_1\pi}}e^{-\frac{\overline{\beta}^2\alpha_1}{\xi_1^2}} + \overline{\beta}\Phi\left(\frac{\overline{\beta}\sqrt{\alpha_1}}{\xi_1}\right). \end{aligned} \tag{21}$$

In a similar manner, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \lambda^+(v)\mathrm{d}v \;=\; \frac{\xi_2}{2\sqrt{\alpha_2\pi}}e^{-\frac{\overline{\lambda}^2\alpha_2}{\xi_2^2}}+\overline{\lambda}\Phi\left(\frac{\overline{\lambda}\sqrt{\alpha_2}}{\xi_2}\right). \tag{22}$$

In light of $\lim_{t\to\infty}\frac{1}{t}\times \ln I(0)=0$ combined with (20), (21) and (22); we obtain

$$\limsup_{t\to\infty}\left[\frac{\ln I(t)}{t}\right]\leq (b+\delta)\times\left(\widetilde{\mathscr{R}}_E-1\right)<0.$$

This concludes the proof. □

6. Persistence in Mean of the Disease

Extinction and persistence of diseases are the most interesting things to study in epidemic models. Section 5 talked about the first one. Following this section, we will demonstrate that the disease persists for a long time.

Theorem 4. *If $\widetilde{\mathscr{R}}_\beta>1$, then for any starting value $\left(I(0),R(0),\beta(0),\lambda(0)\right)\in\mathscr{D}_2$, the following inequality holds*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(v)dv \;\geq\; \frac{b\left(\frac{\xi_1}{\sqrt{\pi\alpha_1}}+b+\delta\right)}{\overline{\beta}\,(b+\delta)}\times\left(\widetilde{\mathscr{R}}_\beta-1\right)\ a.s.. \tag{23}$$

Proof. Through (11), we can get

$$\mathscr{L}\Psi_1 \;\leq\; -\overline{\beta}+b+\delta+\left(\overline{\beta}+\frac{\delta\overline{\beta}}{b}\right)\times I+|\beta-\overline{\beta}|.$$

Integrating both sides of the above inequality from 0 to t and dividing t, there is

$$\frac{\Psi_1(t)-\Psi_1(0)}{t} \;\leq\; (-\overline{\beta}+\delta+b)+\left(\overline{\beta}+\frac{\overline{\beta}\delta}{b}\right)\times\frac{1}{t}\int_0^t I(v)\mathrm{d}v+\frac{1}{t}\int_0^t|\beta(v)-\overline{\beta}|\mathrm{d}v.$$

Then

$$\left(\overline{\beta}+\frac{\overline{\beta}\delta}{b}\right)\times\frac{1}{t}\int_0^t I(v)\mathrm{d}v \;\geq\; \overline{\beta}-b-\delta-\frac{1}{t}\int_0^t|\beta(v)-\overline{\beta}|\mathrm{d}v-\frac{\Psi_1(0)}{t}. \tag{24}$$

Consequently, from (6) and (24) we easily get

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(v)\mathrm{d}v &\geq \frac{\overline{\beta}-b-\delta-\frac{\xi_1}{\sqrt{\pi\alpha_1}}}{\left(\overline{\beta}+\frac{\overline{\beta}\delta}{b}\right)}\\ &\geq \frac{b\left(\frac{\xi_1}{\sqrt{\pi\alpha_1}}+b+\delta\right)}{\overline{\beta}\,(b+\delta)}\times\left(\widetilde{\mathscr{R}}_\beta-1\right).\end{aligned}$$

Which concludes the proof. □

7. NUMERICAL EXAMPLES

In this part, various numerical simulations have to be carried out in order to show the analytical conclusions that were described before. Here, we apply the higher-order Milstein's method as stated in [10]. The discretized form is as follows:

$$\begin{cases}\beta_{p+1} &= \beta_p+\alpha_1\left(\overline{\beta}-\beta_p\right)\Delta t+\xi_1\rho_{1,p}\sqrt{\Delta t}+\frac{\xi_1^2}{2}(\rho_{1,p}^2-1)\Delta t,\\ \lambda_{p+1} &= \lambda_p+\alpha_2\left(\overline{\lambda}-\lambda_p\right)\Delta t+\xi_2\rho_{2,p}\sqrt{\Delta t}+\frac{\xi_2^2}{2}(\rho_{2,p}^2-1)\Delta t,\\ I_{p+1} &= I_p+\left[-(b+\delta)I_p+\beta_p^+(1-R_p-I_p)I_p+\lambda_p^+I_pR_p\right]\Delta t,\\ R_{p+1} &= R_p+\left[-bR_p+\delta I_p-\lambda_p^+I_pR_p\right]\Delta t,\end{cases}$$

where $\Delta t>0$ is time variation, $\rho_{1,p},\rho_{2,p}(p=1,2,...)$ are the Gaussian random variables which follows the standard normal distribution.

In all examples, the values of $\beta_0=0.2,\lambda_0=0.5,I(0)=0.2$ and $R(0)=0.1$, are chosen to be the same.

Example 1. *To begin, we choose* $b=0.09,\delta=0.08,\overline{\beta}=0.3,\overline{\lambda}=0.2,\alpha_1=0.2,\alpha_2=0.2,\xi_1=0.02,\xi_2=0.02$. *Since* $\widetilde{\mathscr{R}}_\beta=1.5366>1$. *As a result, Theorem 2 indicates that the solution of model* (7) *has a stationary distribution, which is demonstrable through numerical simulation, as Figure 1 illustrates.*

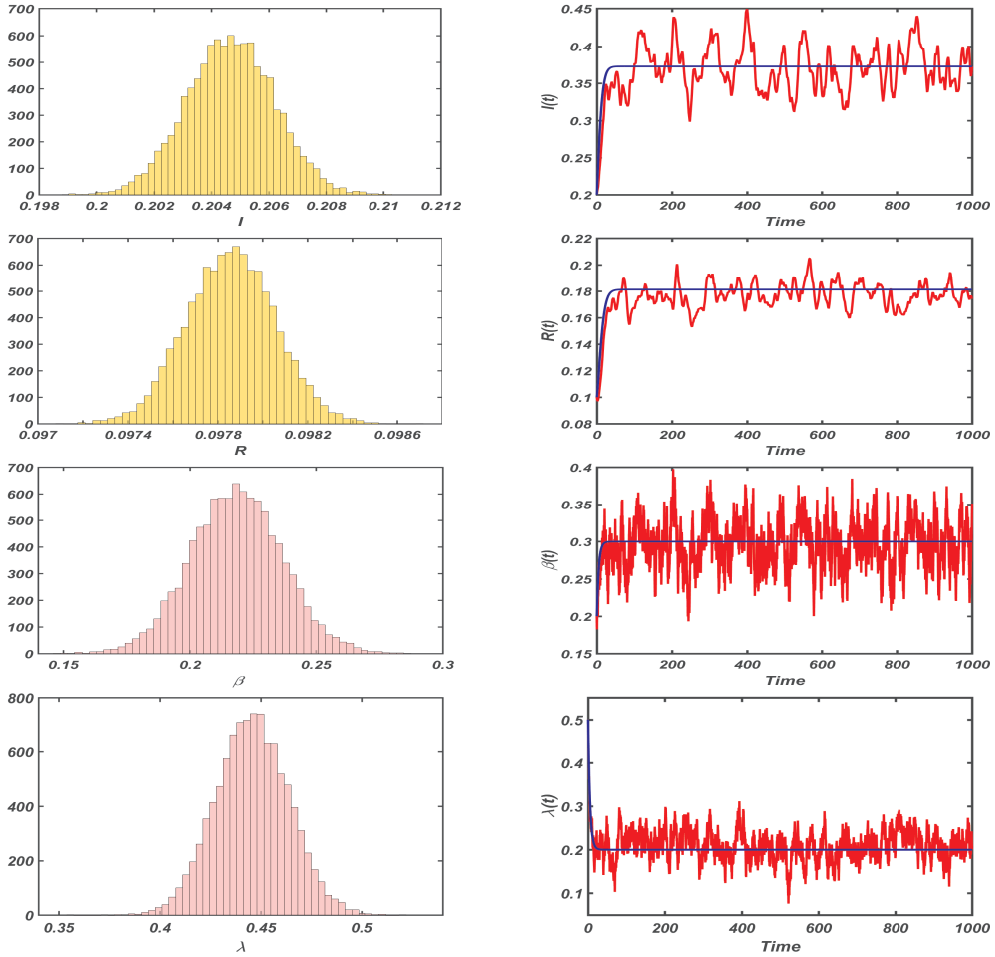

FIGURE 1. The stationary distribution of the stochastic system (7) is simulated when $\widetilde{\mathscr{R}}_\beta = 1.5366 > 1$. The column on the left represents the density of the stochastic system (7). On the right column, the stochastic solution of the system (7) is indicated by the red lines, and the deterministic solution of the system (2) is depicted by the blue lines.

Example 2. *We choose* $b = 0.09, \delta = 0.13, \overline{\beta} = 0.1, \overline{\lambda} = 0.1, \alpha_1 = 0.2, \alpha_2 = 0.2, \xi_1 = 0.05, \xi_2 = 0.05$. *Since* $\widetilde{\mathscr{R}}_E = 0.8652 < 1$. *According to Theorem 3, the stochastic model* (7) *predicts that the disease will inevitably vanish from the population. For more illustrations of this example, see Figure 2.*

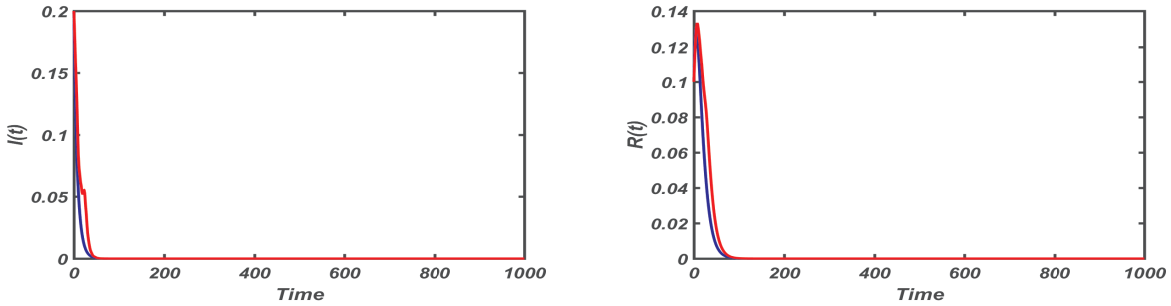

FIGURE 2. The red trajectories represent the solution to model (7), and the blue trajectories represent the corresponding deterministic system. When using the parameters from Example 2, track diagram examples of $I(t)$ and $R(t)$. By this point, $\widetilde{\mathscr{R}}_E = 0.8692 < 1$, the disease will be eliminated.

In numerical simulation, changing the essential parameters $\overline{\beta}, \xi_1, \alpha_1$ has a significant impact on the alteration of the curve of the system (7). To demonstrate the influence of simulation, we apply the control variables approach, which involves varying the values of a significant parameter together with a change in the curve. Figures 3-5 provide examples of this.

Example 3. *We choose* $b = 0.09, \delta = 0.1, \overline{\lambda} = 0.2, \alpha_1 = 0.2, \alpha_2 = 0.2, \xi_1 = 0.02, \xi_2 = 0.02$. *Here we choose four situations: the diseases will continue to exist for* $\overline{\beta} = 0.3, \overline{\beta} = 0.25, \overline{\beta} = 0.2$ *and will eventually become extinct for* $\overline{\beta} = 0.15$. *For more illustrations of this example, see Figure 3.*

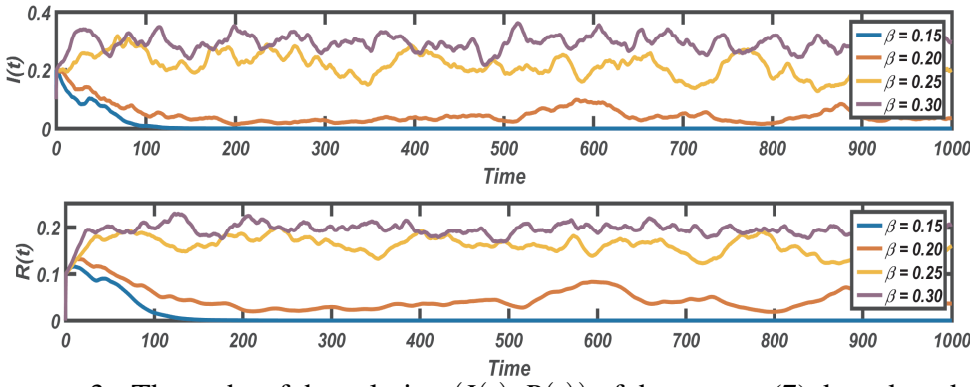

FIGURE 3. The paths of the solution $(I(t), R(t))$ of the system (7), based on data from Example 3.

Example 4. *We choose* $b = 0.09, \delta = 0.1, \overline{\beta} = 0.3, \overline{\lambda} = 0.2, \alpha_1 = 0.2, \alpha_2 = 0.2, \xi_2 = 0.02$. *Here we choose three situations:* $\xi_1 = 0.001, \xi_1 = 0.01$, *and* $\xi_1 = 0.1$. *It is evident from Figure 4 that the disease is more unstable as* ξ_1 *increases.*

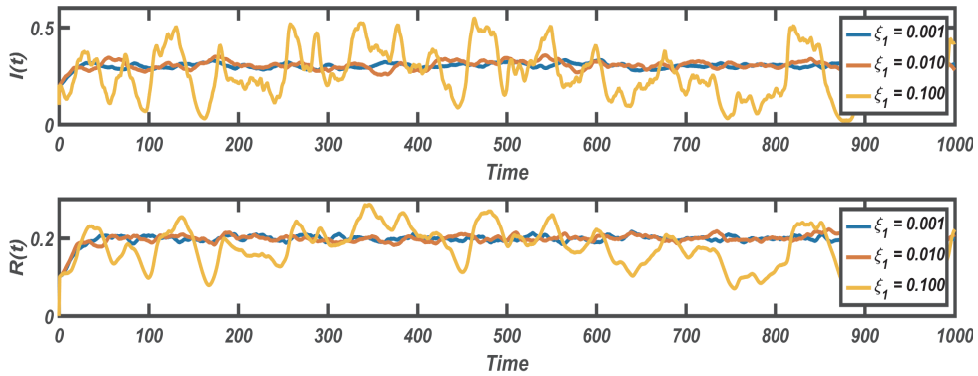

FIGURE 4. The paths of the solution $(I(t), R(t))$ of the system (7), based on data from Example 4.

Example 5. *We choose* $b = 0.09, \delta = 0.1, \overline{\beta} = 0.3, \overline{\lambda} = 0.2, \alpha_2 = 0.2, \xi_1 = 0.02, \xi_2 = 0.02$. *Here we choose three situations:* $\alpha_1 = 0.05, \alpha_1 = 0.5$, *and* $\alpha_1 = 1$. *Figure 5 shows that the disease gets more unstable as* α_1 *decreases.*

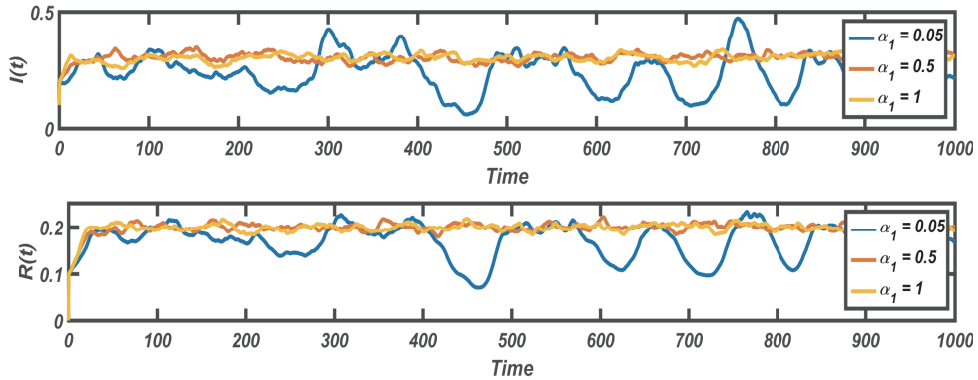

FIGURE 5. The paths of the solution $(I(t), R(t))$ of the system (7), based on data from Example 5.

8. CONCLUSION

This article looks at how a random SIRI epidemic model with nonlinear relapse works. It uses the mean-reverting Ornstein-Uhlenbeck process. First, we provide and demonstrate the

theoretical result that there is a unique global positive solution for the stochastic SIRI system (7). Then, it is decided that the epidemic disease will end or stay active based on two threshold values, $\widetilde{\mathscr{R}}_\beta$ and $\widetilde{\mathscr{R}}_E$, of the stochastic system (7). Based on theory, we show that there are ergodic stationary distributions for random SIRI systems (7). We find that if $\widetilde{\mathscr{R}}_E < 1$, the disease is almost surely exponential extinction (see Theorem 3). Nevertheless, based on Theorem 4, we showed that, if $\widetilde{\mathscr{R}}_\beta > 1$, the process I is persistent in mean. Importantly, we produce simulations with different parameter values in order to illustrate and verify our theoretical results.

These findings shed important light on how stochastic epidemic models function, which has applications in disease prevention and management. We can utilize them to formulate targeted strategies aimed at halting the transmission of infectious diseases. Moreover, additional stochastic epidemic models [24, 27] can benefit from the effective application of the theoretical techniques employed in this work. This project is presently in progress.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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