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THE ROLE OF THE FEAR, HUNTING COOPERATION, AND ANTI-PREDATOR BEHAVIOR IN THE PREY PREDATOR MODEL HAVING DISEASE IN PREDATOR

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Abstract: This article aims to study the dynamical behavior of an eco-epidemiological model. A mathematical eco-epidemiological model consisting of a prey-predator model with disease in predator involving fear that induced due to the intensity of hunting cooperation, and anti-predator property is formulated and studied. The existence, uniqueness, and boundedness of the solution of the model are investigated. The persistence condition of the system has been established. The dynamic behavior of the system is analyzed, including the stability analysis of all possible equilibrium points is discussed. The local bifurcation analysis is carried out. Finally, numerical simulations are provided to check the validity of the theoretical results.

Keywords: eco-epidemiological; fear; anti-predator; hunting cooperation; bi-stability.

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1. INTRODUCTION

Eco-epidemiology merges two important fields in biomathematics, namely demographic

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systems, in which two populations interact either by competition or associate for mutual benefit, and models in which the spread of diseases is studied [1].

Eco-epidemiological systems, which describe the interactions between diseases and prey and predator in one population or both, have to be essential tools in the research of infectious disease management and transmission due to the high probability of the infection's disease transmission between the interacting species. Thus, in eco-epidemiology systems, several studies looked at ecological systems where the disease affects populations of prey, predators, or both [2-7]. In cases where the infectious disease is present within the prey population, the predators may consume vulnerable and sick creatures. Further studies on the role of illness and infection on the dynamics of prey and predators, including different ecological and biological factors, have been published recently; see [8–15] and the references therein.

Also, numerous data and field experiments on terrestrial vertebrates have shown that fear of predators causes significant variation in prey demographics. Fear of predator population also enhances the survival probability of the prey population, and reduces significantly reproduction [16-18].

The fear effect is a behavioral and stress-related physiological change in the prey population in the presence of a predator, as prey species are always wary of possible attack. Recent experimental findings have explored that fear of predators alone can change prey’s behavior. In the mathematical modeling approach, many authors investigated the impacts of hunting cooperation and fear effect in the predator-prey system [15, 19]. However, additional research concentrated on the eco-epidemiological systems related to Hunting cooperation and fear [15, 20-21]. Predators' cooperative hunting behavior is a significant biological occurrence in an ecosystem, which is commonly seen in carnivores [22]. There are numerous benefits to cooperative hunting for the evolution of predator populations. Cooperation, for instance, raises the likelihood of many kills, prey mass, and hunting success [23]. Additionally, it makes it easier for the population to obtain food more quickly, shortens the distance to pursue prey, raises the likelihood of catching large prey, and gives individuals in large packs hunting advantages by preventing other predators from stealing the carcass [22-23].

On the other hand, when the prey feels threatened, it naturally engages in anti-predator behavior, which can involve risking various bodily parts. Biological protections known as "anti-predator adaptations" are developed by evolution to help prey animals fight off predators in their constant struggle. Throughout the animal kingdom, adaptations have evolved for every phase of this fight. Prey-predator models with anti-predator characteristics have recently been presented and investigated by certain academics [24–25].

The current study aims to investigate the effect of hunting cooperation, anti-predator, and fear effects in a predator-prey model simultaneously. We consider that hunting cooperation among predators induces fear in the prey population and as a result birth rate of the prey population reduces. The formulation of the model with their basic properties such as boundedness and survival of the model is discussed in Section 2. The equilibrium analysis and stability of the model is discussed in Section 3. The model's persistence criteria is provided in section 4. Section 5, is concerned with global stability. Section 6, is investigates the local bifurcation (LB). Numerical simulations are performed in 7. Finally, the paper ends with a conclusion.

2. THE MODEL FORMULATION

A mathematical model that simulates the dynamics of the prey $x(t)$, susceptible predator $y(t)$, and infected predator $z(t)$; it is assumed that there is hunting cooperation between susceptible predator individuals that induces a fear behavior between the individuals of the prey. The prey species have an anti-predator capability to defend themselves. Accordingly, such a prey-predator model with disease in predator can be formulated mathematically using the following set of nonlinear first-order differential equations:

$$\begin{aligned} \frac{dx}{dt} &= \frac{rx}{1+py} - d_1 x - bx^2 - \frac{(c+my)xy}{a+x} \\ \frac{dy}{dt} &= h\frac{(c+my)xy}{a+x} - d_2 y - \delta yz - \eta xy \\ \frac{dz}{dt} &= \delta yz - (d_2 + d_3)z \end{aligned} \tag{1}$$

where $x(0) = x_0 \geq 0, y(0) = y_0 \geq 0$, and $z(0) = z_0 \geq 0$ represent IC of the model (1). The variables and other parameters are revealed in Table 1 and all the parameter values are regarded as nonnegative.

Table (1) Biological description of the system (1) parameter

Parameters	**Description**
$x(t)$	The density of the prey individuals at time t.
$y(t)$	The density of the susceptible predator individuals at time t.
$z(t)$	The density of the infected predator individuals at time t.
r	The prey's birth rate
p	The level of fear that reduces the growth of the prey
d_1	Natural death rate of the prey
b	The intraspecific competition
c	The attack rate of the susceptible predator on the prey
m	The level of predator's cooperation in hunting
a	The half-saturation constant
h	The conversion factor of prey biomass to the susceptible predator biomass
d_2	The predator natural death rate
δ	The infection rate
η	The anti-predator rate
d_3	Additional mortality rate of predators due to infection.

Now, rewritten system (1) as a Kolmogorov system in the form:

$$\begin{aligned} \frac{dx}{dt} &= x\left(\frac{r}{1+py} - d_1 - bx - \frac{(c+my)y}{a+x}\right) = xf_1 \\ \frac{dy}{dt} &= y\left(h\frac{(c+my)x}{a+x} - d_2 - \delta z - \eta x\right) = yf_2 \\ \frac{dz}{dt} &= z\big(\delta y - (d_2 + d_3)\big) = zf_3 \end{aligned} \tag{2}$$

System (2) with the given initial values satisfies the requirement of the fundamental theorem of the existence and uniqueness of the solution as the right-hand side functions are continuous and have continuous partial derivatives. Moreover, the solutions of system (2) is bounded, as shown in the next theorem 1. It is easy to confirm that the requirement for the survival of all species in the system (2) is given by:

$$d_1 < r \tag{3}$$

Theorem 1: All solutions of the system (2) with the initial condition $x(0) > 0,\ y(0) > 0$, and $z(0) > 0$ are positive for all the $t > 0$.

Proof: Let $\aleph = \{(x, y, z): x > 0,\ y > 0, z > 0\}$. Integrate the equations of system (1) using the initial conditions $x(0) > 0,\ y(0) > 0,\ z(0) > 0$, gives that:

$$x(t) = x(0)\exp\left\{\int_0^t \left(\frac{r}{1+py(s)} - d_1 - bx(s) - \frac{(c+my(s))y(s)}{a+x(s)}\right) ds\right\}.$$

$$y(t) = y(0)\exp\left\{\int_0^t \left(h\frac{(c+my(s))X(s)}{a+x(s)} - d_2 - \delta z(s) - \eta x(s)\right) ds\right\}.$$

$$z(t) = z(0)\, exp\left\{\int_0^t \left(\delta y - (d_2 + d_3)\right) ds\right\}.$$

Then, from the definition of the exponential function, any solution in $\aleph$ that begins with positive conditions remains in $\aleph$ for all $t > 0$.

Theorem 2: All system (2) solutions initiating in $\mathbb{R}_+^3$ are bounded.

Proof. From the prey equation in system (2) we get $x \le \frac{(r-d_1)}{b} = \mu_1$. That is biologically for x to be survival we should have $d_1 < r$. Now to show that each population size is bounded if and only if the total population size is bounded. It is sufficient to prove the total population size $M = x + y + z$ is bounded for all t.

Now, differentiating M for t along the solutions of system (1), we have

$$\frac{dM}{dt} = \frac{rx}{1+py} - d_1x - bx^2 - \frac{(c+my)xy}{a+x} + h\frac{(c+my)xy}{a+x} - d_2y - \delta yz$$
$$-\eta xy + \delta yz - (d_2 + d_3)z$$
$$\le rx - bx^2 - d_1x - d_2y - (d_2 + d_3)z.$$
$$\le \frac{r^2}{4b} - d_1x - d_2y - (d_2 + d_3)z.$$

Thus, we obtain that $\frac{dM}{dt} \le L - \mu_1 M$. where $\mu_2 = min\{d_1, d_2\}$, and $L = \frac{r^2}{4b}$.Therefore, solving the differential inequality gives, $M(t) \le \frac{L}{\mu_2} = \mu_3\ as\ t \to \infty$. Thus, every solution of system (1) is bounded in the region $\Lambda = \{(x, y, z) \in \mathbb{R}^3_+ : x(t) + y(t) + z(t) \le \mu_2\}$.

3. MODELS ANALYSIS

The equilibrium points (EPs) of the system (2) are found in this section. Then, their LS is examined using the linearization technique. System (2) has the following biologically feasible EPs.

- The extinction equilibrium point (EEP), $p_0 = (0,0,0)$ always exists.
- The axial equilibrium point (AEP), $p_1 = (\frac{r-d_1}{b}, 0,0)$ which exists under condition (3).
- The disease-free equilibrium point DFEP, $p_2 = (\bar{x}, \bar{y}, 0)$, where

$$\bar{y} = \frac{(a\eta - hc_1 + d_2)\bar{x} + \eta\bar{x}^2 + ad_2}{hm\bar{x}} \tag{4}$$

while $\bar{x}$ represents a positive root for the equation:

$$N_1x^5 + N_2x^4 + N_3x^3 + N_4x^2 + N_5x + N_6 = 0, \tag{5}$$

with

$$\begin{aligned}
N_1 &= \eta p(bh^2m + \eta^2), \\
N_2 &= bh^3m^2 + hm\eta^2 + abh^2m\eta p + 2a\eta^3p - bh^3mcp - 2h\eta^2cp \\
&\quad + h^2m\eta d_1p + bh^2md_2p + 3\eta^2d_2p, \\
N_3 &= -h^3m^2r + ahm\eta^2 - h^2m\eta c + h^3m^2d_1 + 2hm\eta d_2 + a^2\eta^3p \\
&\quad - 2ah\eta^2cp + h^2\eta c^2p + ah^2m\eta d_1p - h^3mcd_1p + abh^2md_2p \\
&\quad + 6a\eta^2d_2p - 4h\eta cd_2p + h^2md_1d_2p + 3\eta d_2^2p, \\
N_4 &= 2ahm\eta d_2 - h^2mcd_2 + hmd_2^2 + 3a^2\eta^2d_2p - 4ah\eta cd_2p + h^2c^2d_2p \\
&\quad + ah^2md_1d_2p + 6a\eta d_2^2p - 2hcd_2^2p + d_2^3p, \\
N_5 &= ad_2^2(hm + p(3a\eta - 2hc + 2d_2^2)), \\
N_6 &= a^2d_2^3p.
\end{aligned}$$

Since the signs of N_1 and N_6 coincide then the 5^{th} order polynomial either has no positive roots where all the other coefficients have the same signs as N_1 and N_6 or there are multiple positive roots. Also, for the positivity of $\bar{y}$, the resulting positive roots should satisfy the following condition:

$$(a\eta - hc + d_2)\bar{x} + \eta\bar{x}^2 + ad_2 > 0. \tag{6}$$

- The interior equilibrium point, (IEP) that is represented by $p_3 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$,

  where

$$\bar{\bar{y}} = \frac{d_2+d_3}{\delta}$$
$$\bar{\bar{z}} = \frac{(-\eta\bar{\bar{x}}^2-(a\eta+d_2-hc)\bar{\bar{x}})\delta+hm(d_2+d_3)\bar{\bar{x}}-ad_2}{(a+\bar{\bar{x}})\delta^2}. \tag{7}$$

while $\bar{\bar{x}}$ represents the positive root of the following equation

$$K_1x^2 + K_2x + K_3 = 0 \tag{8}$$

where

$$K_1 = b\delta^2(\delta + p(d_2 + d_3)),$$
$$K_2 = \delta^2((ab + d_1)\delta - r\delta + p(d_2 + d_3)(ab + d_1)),$$
$$K_3 = -ar\delta^3 + a\delta^3 d_1 + \delta^2 c(d_2 + d_3) + m\delta(d_2^2 + d_3^2) + 2m\delta d_2 d_3$$
$$+p((d_2 + d_3)(a\delta^2 d_1 + 3md_2d_3) + (d_2^2 + d_3^2)(m + \delta c) + 2\delta c d_2 d_3).$$

So by "Descartes' rule of sign", equation (8) has a unique positive root, and hence, system (2) has a unique IEP in $\mathbb{R}^3_+$ if

$$K_3 < 0. \tag{9a}$$
$$[\eta\bar{\bar{x}}^2 + (a\eta + d_2)\bar{\bar{x}}]\delta + ad_2 < hm(d_2 + d_3)\bar{\bar{x}} + h\delta c\bar{\bar{x}}. \tag{9b}$$

LS of the EPs of the system (2) is determined by determining the eigenvalues of the Jacobian matrix (JM). Now, the JM of system (2)

$$J = [J_{ij}]_{3\times3}, \tag{10}$$

where

$$J_{11} = x\left(-b + \frac{y(my+c)}{(a+x)^2}\right) + \frac{r}{1+yp_1} - d_1 - bx - \frac{y(my+c)}{a+x},$$
$$J_{12} = x\left(-\frac{my}{a+x} - \frac{my+c}{a+x} - \frac{rp_1}{(1+yp_1)^2}\right),\ J_{13} = 0,$$
$$J_{21} = y\left(-\eta - \frac{hx(my+c)}{(a+x)^2} + \frac{h(my+c)}{a+x}\right),$$
$$J_{22} = \frac{hmxy}{a+x} + \frac{hx(my+c)}{a+x} - \delta z - x\rho - d_2,$$
$$J_{23} = -\delta y,\ J_{31} = 0, J_{32} = \delta z,\ J_{33} = \delta y - d_2 - d_3.$$

The JM at p_0 becomes:

$$J(p_0) = \begin{pmatrix} r - d_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & -(d_2 + d_3) \end{pmatrix} \tag{11a}$$

So, the eigenvalues of $J(p_0)$ are $\lambda_{01} = r - d_1, \lambda_{02} = -d_2$ and $\lambda_{03} = -(d_2 + d_3)$. Hence, the EEP is locally asymptotically stable (LAS) and unstable if the following conditions hold respectively.

$$r < d_1. \tag{11b}$$

$$r > d_1. \tag{11c}$$

The JM of the system (2) at the $p_1 = (\hat{x}, 0,0)$ becomes

$$J(p_1) = \begin{pmatrix} -r + d_1 & -rp\hat{x} - \frac{c\hat{x}}{a+\hat{x}} & 0 \\ 0 & \frac{hc\hat{x}}{a+\hat{x}} - d_2 - \eta\hat{x} & 0 \\ 0 & 0 & -(d_2 + d_3) \end{pmatrix} \tag{12}$$

Hence, the eigenvalues are determined by $\lambda_{11} = -r + d_1$, $\lambda_{12} = \frac{hc\hat{x}}{a+\hat{x}} - d_2 - \eta\hat{x}$, and $\lambda_{13} = -(d_2 + d_3)$, then AEP is LAS provided that

$$\begin{gathered} \frac{hc\hat{x}}{a+\hat{x}} < d_2 + \eta\hat{x} \\ d_1 < r \end{gathered} \tag{13}$$

The JM of the system (2) at $p_2 = (\bar{x}, \bar{y}, 0)$ becomes:

$$J(p_2) = \begin{pmatrix} \bar{x}\left(-b + \frac{\bar{y}(m\bar{y}+c)}{(a+\bar{x})^2}\right) & \bar{x}\left(-\frac{m\bar{y}}{a+\bar{x}} - \frac{m\bar{y}+c}{a+\bar{x}} - \frac{rp_1}{(1+\bar{y}p)^2}\right) & 0 \\ \bar{y}\left(-\eta - \frac{h\bar{x}(m\bar{y}+c)}{(a+\bar{x})^2} + \frac{h(m\bar{y}+c)}{a+\bar{x}}\right) & \frac{hm\bar{x}\bar{y}}{a+\bar{x}} & -\eta\bar{y} \\ 0 & 0 & \delta\bar{y} - (d_2 + d_3) \end{pmatrix}. \tag{14a}$$

One of the eigenvalues is $\lambda_{23} = \delta\bar{y} - (d_2 + d_3)$ and the other two eigenvalues are given by:

$$\lambda_{21} = \frac{T}{2} + \frac{1}{2}\sqrt{T^2 - 4D};\ \lambda_{22} = \frac{T}{2} - \frac{1}{2}\sqrt{T^2 - 4D}, \tag{14b}$$

where

$$T = \bar{x}\left(-b + \frac{\bar{y}(m\bar{y}+c)}{(a+\bar{x})^2}\right) + \frac{hm\bar{x}\bar{y}}{a+\bar{x}},$$

$$D = \left(-b + \frac{\bar{y}(m\bar{y}+c)}{(a+\bar{x})^2}\right)\frac{hm\bar{x}^2\bar{y}}{a+\bar{x}} + \bar{x}\bar{y}\left(\frac{m\bar{y}}{a+\bar{x}} + \frac{m\bar{y}+c}{a+\bar{x}} + \frac{rp}{(1+\bar{y}p)^2}\right)\left(-\eta - \frac{h\bar{x}(m\bar{y}+c)}{(a+\bar{x})^2} + \frac{h(m\bar{y}+c)}{a+\bar{x}}\right).$$

Therefore, the other eigenvalues have negative real parts, and then DFEP is LAS if and only if the following condition holds:

$$\begin{gathered} \left(-b + \frac{\bar{y}(m\bar{y}+c)}{(a+\bar{x})^2}\right)\frac{hm\bar{x}^2\bar{y}}{a+\bar{x}} + \bar{x}\bar{y}\left(\frac{m\bar{y}}{a+\bar{x}} + \frac{m\bar{y}+c}{a+\bar{x}} + \frac{rp}{(1+\bar{y}p)^2}\right)\left(-\rho - \frac{h\bar{x}(m\bar{y}+c)}{(a+\bar{x})^2} + \frac{h(m\bar{y}+c)}{a+\bar{x}}\right) > 0 \\ \bar{x}\left(-b + \frac{\bar{y}(m\bar{y}+c)}{(a+\bar{x})^2}\right) + \frac{hm\bar{x}\bar{y}}{a+\bar{x}} < 0 \\ \delta\bar{y} < (d_2 + d_3) \end{gathered} \tag{15}$$

Finally, the LS conditions for IEP, which are represented by $p_3 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$, are showed in the next theorem.

Theorem 3. The IEP of the model (2) is LAS if

$$\begin{gathered} \bar{\bar{x}}\left(-b + \frac{\bar{\bar{y}}(c+m\bar{\bar{y}})}{(a+\bar{\bar{x}})^2}\right) + \frac{hm\bar{\bar{x}}\bar{\bar{y}}}{a+\bar{\bar{x}}} < 0 \\ m_{11}m_{22} - m_{12}m_{21} > 0 \\ -(m_{11}+m_{22})(m_{11}m_{22} - m_{12}m_{21}) + m_{22}m_{23}m_{32} > 0 \end{gathered} \tag{16}$$

Proof. The JM of the system (2) at $p_3 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$, is given by

$$J(p_3) = \left(m_{ij}\right)_{3\times 3}, \tag{17}$$

where

$$m_{11} = \bar{\bar{x}}\left(-b + \frac{\bar{\bar{y}}(c+m\bar{\bar{y}})}{(a+\bar{\bar{x}})^2}\right), m_{12} = -\bar{\bar{x}}\left(\frac{m\bar{\bar{y}}}{a+\bar{\bar{x}}} + \frac{c+m\bar{\bar{y}}}{a+\bar{\bar{x}}} + \frac{rp_1}{(1+p\bar{\bar{y}})^2}\right), m_{13} = 0$$

$$m_{21} = \bar{\bar{y}}\left(-\eta - \frac{h\bar{\bar{x}}(c+m\bar{\bar{y}})}{(a+\bar{\bar{x}})^2} + \frac{h(c+m\bar{\bar{y}})}{a+\bar{\bar{x}}}\right), m_{22} = \frac{hm\bar{\bar{x}}\bar{\bar{y}}}{a+\bar{\bar{x}}}, m_{23} = -\delta\bar{\bar{y}}$$

$$m_{31} = 0, m_{32} = \lambda\bar{\bar{z}},\ m_{33} = 0$$

Then the characteristic equation of $J(p_3)$ can be written as:

$$\lambda_3^3 + B_1\lambda_3^2 + B_2\lambda_3 + B_3 = 0, \tag{18}$$

where

$$B_1 = -(m_{11} + m_{22}),$$

$$B_2 = (m_{11}m_{22} - m_{12}m_{21}) - m_{23}m_{32}$$

$$B_3 = m_{11}m_{23}m_{32}$$

with

$$\Delta = B_1B_2 - B_3 = -(m_{11}+m_{22})(m_{11}m_{22} - m_{12}m_{21}) + m_{22}m_{23}m_{32}.$$

The LS of the p_3 depends on values of B_1, B_2, and B_3 using the “Routh–Hurwitz criterion” [26], the sign of the real part of the equations can be easily determined. The equation (18) has all negative real roots if B_1, B_3, and Δ are positive, under sufficient conditions (16).

4. PERMANENCE

An ecological system is said to be ‘permanent’ when all the species in the system survive in the long run irrespective of any initial population size. This means the solution has no omega limit set in the boundary planes of the state space Λ mathematically. Now according to the system (2) if the predator individuals disappear then the following subsystem is obtained

$$\begin{aligned}\frac{dx}{dt} &= x[\frac{r}{1+py} - d_1 - bx - \frac{(c+my)y}{a+x}] = xf_{11} \\ \frac{dy}{dt} &= [h\frac{(c+my)x}{a+x} - d_2 - \eta x] = yf_{22}\end{aligned} \tag{19}$$

Define the Dulac functions as $\Phi_1 = \frac{1}{xy}$, it is obtained that:

$$\nabla = \frac{\partial}{\partial x}(\Phi_1 x f_{11}) + \frac{\partial}{\partial y}(\Phi_1 y f_{22}) = -\frac{b}{y} + \frac{(c+my)+hm(a+x)}{(a+x)^2}.$$

Therefore, according to the "Dulac-Bendixon criterion" [27], the subsystem (19) has no closed curve in the $\mathbb{R}_+^2$ of the xy −plane if one of the following conditions is met.

$$\begin{gathered}\frac{(c+my)+hm(a+x)}{(a+x)^2} < \frac{b}{y} \\ OR \\ \frac{b}{y} < \frac{(c+my)+hm(a+x)}{(a+x)^2}\end{gathered} \tag{20}$$

Since $\nabla \neq 0$ that does not change the sign in the $Int.\mathbb{R}_+^2$ of the $xy -$ plane under condition (20). So the "Dulac-Bendixon criterion" system (19) has no periodic solution lying entirely in the interior of the $xy -$ plane. Hence, using the "Poincare Bendixon theorem" [28], the unique EP in $\mathbb{R}_+^2$ of the $xy -$ plane, is globally asymptotically stable (GAS) whenever it exists provided that (20) holds. Then the system (1) has no periodic dynamics in the boundary xy −plane.

Theorem 4. System (2) is permanent if and only if the following condition is met

$$\frac{hc\hat{x}}{a+\hat{x}} > d_2 + \eta\hat{x}. \tag{21a}$$

$$\delta\bar{y} > (d_2 + d_3). \tag{21b}$$

Proof. Consider the function $L(x,y,z) = x^{\tau_1}y^{\tau_2}z^{\tau_3}$ where τ_1, τ_1 and $\tau_1$are positive constants. Clearly, $L(x,y,z)$ is a C^1 nonnegative function in the interior of Λ. Hence,

$$\pi(x,y,z) = \frac{L'(x,y,z)}{L(x,y,z)} = \tau_1 f_1 + \tau_2 f_2 + \tau_3 f_3,$$

where the functions $f_i, i = 1,2,3,$ are given in system (2). Accordingly, we have

$$\begin{gathered}\pi(x,y,z) = \tau_1\left[\frac{r}{1+py} - d_1 - bx - \frac{(c+my)y}{a+x}\right] + \tau_2[h_1\frac{(c+my)x}{a+x} - d_2 - \delta z - \eta x] \\ +\tau_3[\delta y - (d_2 + d_3)]\end{gathered} \tag{22}$$

Now, since there are no periodic attractors in the boundary planes, then the only possible $\omega-$ limit sets of system (2) are the EPs denoted by p_0, p_1, p_2 and p_3. Moreover, the direct calculation gives that

$$\pi(p_0) = \tau_1(r - d_1) - \tau_2 d_2 - \tau_3(d_2 + d_3)$$
$$\pi(p_1) = \tau_2\left(\frac{hc\hat{x}}{a+\hat{x}} - d_2 - \eta\hat{x}\right) - \tau_3(d_2 + d_3)$$
$$\pi(p_2) = \tau_3\big(\delta\bar{y} - (d_2 + d_3)\big)$$

Clearly, by choosing the constant $\tau_1 > 0$ sufficiently large for positive constants τ_2 and τ_3, it is obtained that $\pi(p_0) > 0$, However, $\pi(p_1)$ and $\pi(p_2)$ are positive under the conditions (21a) and (21b) with a suitable choice of positive constants τ_2 and τ_3, respectively. Therefore, due to the average Lyapunov method [29], model (2) is uniformly persistent.

5. GLOBAL STABILITY

Concerned with the GS property of the equilibria of model (2), we have the following results, which proved depending on using a suitable Lyapunov function (LF).

Theorem 5. The EP given by p_0 is GAS whenever it's LAS.

Proof. The GAS of the EEP is shown with the help of the LF. Let us consider

$$v_o = \gamma_1 x + \gamma_2 y + \gamma_3 z.$$

Direct computation shows that $v_o: \mathbb{R}^3_+ \to \mathbb{R}$ is a continuously differentiable function such that $v_o(0,0,0) = 0$, and $v_o(x,y,z) > 0,\ \forall (x,y,z) \neq (0,0,0)$. Further,

$$\frac{dv_0}{dt} = \gamma_1\left[\frac{rx}{1+py} - d_1 x - bx^2 - \frac{(c+my)xy}{a+x}\right] + \gamma_2\left[h\frac{(c+my)xy}{a+x} - d_2 y - \delta yz - \eta xy\right]$$
$$+\gamma_3[\delta yz - (d_2 + d_3)z\,].$$

Choosing the positive constants as $\gamma_1 = h, \gamma_2 = 1$ and $\gamma_3 = 1$, it is obtained that:

$$\frac{dv_0}{dt} \leq h(r - d_1)x - d_2 y - (d_2 + d_3)z.$$

Therefore, the function $\frac{dv_0}{dt}$ is negative definite under the condition (11b). Thus p_0 is GAS.

Theorem 6. The EP represented by $p_1(\hat{x}, 0,0)$ is GAS provided that the following condition holds:

$$hrp\hat{x} + \frac{h(c+m\mu_2)\hat{x}}{a} < d_2. \tag{23}$$

Proof. The GAS of the AEP is shown with the help of the following LF. Let us consider

$$v_1 = \gamma_4\left[x - \hat{x} - \hat{x}\ln\left(\frac{x}{\hat{x}}\right)\right] + \gamma_5 y + \gamma_6 z.$$

It is obvious that $v_1 : \mathbb{R}^3_+ \to \mathbb{R}$ is a continuously differentiable function such that $v_1(\hat{x}, 0,0) = 0$ and $v_1(x, y, z) > 0, \forall(x, y, z) \neq (\hat{x}, 0,0)$. Further,

$$\begin{aligned}\frac{dv_1}{dt} = &-\frac{\gamma_4 rpxy}{1+py} + \frac{\gamma_4 rp\hat{x}y}{1+py} - \gamma_4 b(x-\hat{x})^2 - \frac{\gamma_4(c+my)xy}{a+x} + \frac{\gamma_4(c+my)\hat{x}y}{a+x} \\ &+\gamma_5 h\frac{(c+my)xy}{a+x} - \gamma_5 d_2 y - \gamma_5\delta yz - \gamma_5\eta xy \\ &+\gamma_6\delta yz - \gamma_6(d_2+d_3)z\,].\end{aligned}$$

Choosing the arbitrary positive value $\gamma_4 = h, \gamma_5 = 1$ and $\gamma_6 = 1$, it is obtained that:

$$\frac{dv_1}{dt} \leq -hb(x-\hat{x})^2 - \left(d_2 - hrp\hat{x} - \frac{h(c+m\mu_2)\hat{x}}{a}\right)y - (d_2+d_3)z,$$

where μ_3 is given by Theorem 1.

Therefore, the function $\frac{dv_1}{dt}$ is negative definite under the condition (23). Thus p_1 is GAS.

Remark: Since system (2) has either multiple planar EPs or zero EPs, then the EP $p_2(\bar{x}, \bar{y}, 0)$ cannot be GS in the $\mathbb{R}^3_+$.

Theorem 7. Suppose the (IEP), $p_3 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$ is (LAS) in the $\mathbb{R}^3_+$. Then p_3 is GAS provided that the following conditions hold:

$$\begin{aligned}&\frac{hm\mu_1}{B} < \frac{S_1}{2} \\ &\frac{(c+m\bar{\bar{y}})\bar{\bar{y}}}{aB^*} < b\end{aligned}, \qquad (24)$$

where μ_1 and μ_3 is given in Theorem (1).

Proof. The GAS of the IEP is shown with the help of the following LF. Let us consider

$$v_2 = \gamma_7\left[x - \bar{\bar{x}} - \bar{\bar{x}}\ln\left(\frac{x}{\bar{\bar{x}}}\right)\right] + \gamma_8\left[y - \bar{\bar{y}} - \bar{\bar{y}}\ln\left(\frac{y}{\bar{\bar{y}}}\right)\right] + \gamma_9\left[z - \bar{\bar{z}} - \bar{\bar{z}}\ln\left(\frac{z}{\bar{\bar{z}}}\right)\right].$$

Clearly, $v_2 : \mathbb{R}^3_+ \to \mathbb{R}$ is a continuously differentiable function such that $v_2(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}) = 0$ and $v_2(x, y, z) > 0, \forall(x, y, z) \neq (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$, Further,

$$\begin{aligned}\frac{dv_2}{dt} = &-\left[\frac{\gamma_7 rp_1}{AA^*} + \frac{\gamma_7(c+m(y+\bar{\bar{y}}))}{B} - \frac{\gamma_8 ah(c+m\bar{\bar{y}})}{BB^*} + \gamma_8\eta\right](x-\bar{\bar{x}})(y-\bar{\bar{y}}) \\ &-\gamma_7\left[b - \frac{(c+m\bar{\bar{y}})\bar{\bar{y}}}{BB^*}\right](x-\bar{\bar{x}})^2 + \frac{\gamma_8 hmx}{B}(y-\bar{\bar{y}})^2 - \gamma_8\delta(y-\bar{\bar{y}})(z-\bar{\bar{z}}) \\ &+\gamma_9\delta(y-\bar{\bar{y}})(z-\bar{\bar{z}}).\end{aligned}$$

Choosing the positive constants as $\gamma_7 = \frac{ah}{B^*}, \gamma_8 = 1$ and $\gamma_9 = 1$, it is obtained that:

$$\frac{dv_2}{dt} \leq -\left[\frac{ahrp}{B^*AA^*} + \frac{ahmy}{BB^*} + \eta\right]\left[\frac{(x-\bar{\bar{x}})^2}{2} + \frac{(y-\bar{\bar{y}})^2}{2}\right] - \frac{ah}{B^*}\left[b - \frac{(c+m\bar{\bar{y}})\bar{\bar{y}}}{BB^*}\right](x-\bar{\bar{x}})^2 + \frac{hmx}{B}(y-\bar{\bar{y}})^2.$$

$$\frac{dv_2}{dt} \leq -\left[\frac{ah}{B^*}\left(b - \frac{(c+m\bar{\bar{y}})\bar{\bar{y}}}{BB^*}\right) + \frac{S_1}{2}\right](x-\bar{\bar{x}})^2 - \left[\frac{S_1}{2} - \frac{hmx}{B}\right](y-\bar{\bar{y}})^2,$$

where

$$S_1 = \frac{ahrp}{B^*AA^*} + \frac{ahmy}{BB^*} + \eta\,, A = 1 + py,\ A^* = 1 + p\bar{\bar{x}}, B = a + x \text{ and } B^* = a + \bar{\bar{x}}.$$

Therefore, the function $\frac{dv_2}{dt}$ is negative semi-definite under the condition (24). Therefore, the IEP is a stable point. By using "LaSalle's invariance principle" [27], it's attracting. Hence, P_3 is a GAS.

6. LOCAL BIFURCATION

The influence of varying the parameter values on the dynamic of the system (2) is investigated in this section. Now, to compute the second derivative of the JM system (2) is rewritten in the vector form as follows:

$$\frac{dX}{dt} = F(X), \text{with } X = (x, y, z)^T \text{and } F = (xf_1, yf_2, zf_3)^T$$

Let $W = (\omega_1, \omega_2, \omega_3)$ be any nonzero vector and ϑ is any parameter, Hence the second directional derivatives for system (2) can be written as:

$$D^2F(W, W) = [\rho_{ij}]_{3\times1}, \tag{25}$$

where

$$\rho_{11} = 2\left(-b + \frac{a(my+c)y}{(a+x)^3}\right)\omega_1{}^2 + \left(\frac{-a(2my+c)}{(a+x)^2} - \frac{rp}{(1+py)^2}\right)\omega_1\omega_2 + x\left(\frac{-m}{a+x} + \frac{rp^2}{(1+py)^3}\right)\omega_2{}^2,$$

$$\rho_{21} = -\frac{2ahy(my+c)}{(a+x)^3}\omega_1{}^2 - 2\left(\frac{ah(c+2my)}{(a+x)^2} - \eta\right)\omega_1\omega_2 - \frac{2hmx}{a+x}\omega_2{}^2 - 2\delta\omega_2\omega_3,$$

$$\rho_{31} = 2\delta\omega_2\omega_3.$$

So, the third directional derivative for (1) is

$$D^3F(W, W, W) = [b_{ij}]_{3\times1}, \tag{26}$$

where

$$b_{11} = 6[-\frac{a(my+c)y}{(a+x)^4}{\omega_1}^3 + \frac{a(2my+c)}{(a+x)^3}{\omega_1}^2\omega_2 + (-\frac{am}{(a+x)^2} + \frac{rp^2}{(1+py)^3})\,\omega_1{\omega_2}^2 - \frac{rp^3x}{(1+py)^4}{\omega_2}^3],$$

$$b_{21} = 6ah\left[-\frac{(my+c)y}{(a+x)^4}{\omega_1}^3 - \left(\frac{(2my+c)}{(a+x)^3}\right){\omega_1}^2\omega_2 + \frac{m}{(a+x)^2}\omega_1{\omega_2}^2\right],$$

$$b_{31} = 0.$$

Now, the following theorems investigate the possibility of occurrence of (LB) in the system (2).

Theorem 8. The system (2) at the EEP undergoes a transcritical bifurcation (TB) when the parameter r passes through the value $r^* = d_1$

Proof. The JM of the system (2) at (p_0, r^*) is

$$J_0 = J_{(p_0,r^*)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & -(d_2+d_3) \end{bmatrix}.$$

Therefore, the eigenvalues of J_0 are given by $\lambda_{01}^* = 0, \lambda_{02}^* = -d_2,\ \lambda_{03}^* = -(d_2+d_3)$. So, the EEP is a non-hyperbolic point.

Let $V_0 = (\mathrm{v}_{01}, \mathrm{v}_{02}, \mathrm{v}_{03})^T$ be the eigenvectors corresponding to $\lambda_{01}^* = 0$.Thus $J_1V_0 = 0$, gives that $V_0 = (\nu_{01}, 0, 0)^T$, with ($\mathrm{v}_{01} \neq 0$).

Now, let $U_0 = (\kappa_{01}, \kappa_{02}, \kappa_{03})^T$ represents the eigenvector of ${J_0}^T$with the eigenvalue $\lambda_{01}^* = 0$, then ${J_0}^TU_0 = 0$ gives $U_0 = (\kappa_{01}, 0, 0)^{\mathrm{T}}$, with ($\kappa_{01} \neq 0$). Since $\frac{\partial F}{\partial r} = F_r = (\frac{x}{1+py}, 0, 0)^T$.

Hence we obtain that $F_r(p_0, r^*) = (0, 0, 0)^T$.

Therefore,${U_0}^TF_r(p_0, r^*) = 0$. Hence system (2) has no saddle-node bifurcation (SNB).

Now, we have ${U_0}^T[DF_r(p_0, r^*)V_0] = \kappa_{01}\nu_{01} \neq 0.$

Moreover,

$${U_0}^T\ [D^2F(p_0, r^*)(V_0, V_0)] = -2b\,{\nu_{01}}^2\kappa_{01} \neq 0.$$

Clearly ${U_0}^T\ [D^2F(p_0, r^*)(V_0, V_0)] \neq 0$, by Sotomayor theorem [27], system (2) undergoes a TB at the p_0.

Theorem 9. The system (2) undergoes a TB at the AEP when the parameter d_2 passes through the value $d_2^* = \frac{hc\,\hat{x}}{a+\hat{x}} - \eta\,\hat{x}$ if the next condition is met:

$$\left[\left(\eta - \frac{ahc}{(a+\hat{x})^2}\right)\alpha_1 - \frac{hm\hat{x}}{a+\hat{x}}\right] \neq 0, \tag{27}$$

where α_1 will be defined in the proof. Otherwise, pitchfork bifurcation (PB) takes place.

Proof. The JM of the system (2) at (p_1, d_2^*) can be represented by:

$$J_1 = J_{(p_1,d_2^*)} = \begin{bmatrix} -r + d_1 & -rp\hat{x} - \frac{c\hat{x}}{a+\hat{x}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(d_2 + d_3) \end{bmatrix} = [a_{ij}].$$

Therefore, the eigenvalues of J_1 are given by $\lambda_{11}^* = -r + d_1$, $\lambda_{12}^* = 0$, and $\lambda_{13}^* = -(d_2 + d_3)$. So, the AEP is a non-hyperbolic point.

Let $V_1 = (\nu_{11}, \nu_{12}, \nu_{13})^T$ be the eigenvectors corresponding to $\lambda_{12}^* = 0$. Thus $J_1 V_1 = 0$ gives that $V_1 = (\alpha_1 \nu_{12}, \nu_{12}, 0)^T$, $(\nu_{12} \neq 0)$ where $\alpha_1 = \frac{-a_{12}}{a_{11}}$.

Now, let $U_1 = (\kappa_{11}, \kappa_{12}, \kappa_{13})^T$ represents the eigenvector of $J_1{}^T$with the eigenvalue $\lambda_{12}^* = 0$, then $J_1{}^T \ U_1 = 0$ gives $U_1 = (0, \kappa_{12}, 0)^T$, $(\kappa_{12} \neq 0)$.

Since $F_{d_2} = (0, -y, -z)^T$. Hence we obtain that $F_{d_2}(p_1,\ d_2^*) = (0, 0, 0)^T$

Therefore, $U_1{}^T F_{d_2}(p_1,\ d_2^*) = 0$. Hence system (2) has no SNB.

Now, we have $U_1{}^T \left[DF_{d_2}(p_1,\ d_2^*) V_1 \right] = -\kappa_{12}\, \nu_{12} \neq 0$. Moreover, using condition (27) leads to:

$$U_1{}^T \ [D^2 F(p_1, d_2^*)(V_1, V_1)] = 2\left[\left(\eta - \frac{ahc}{(a+\hat{x})^2} \right) \alpha_1 - \frac{hm\hat{x}}{a+\hat{x}} \right] \kappa_{12} \nu_{12}{}^2 \neq 0.$$

Hence, the system (2) near p_1 with $d_2 = d_2{}^*$ possesses a TB. However, violating condition (27) leads to

$$U_1{}^T D^3 F(p_1, d_2^*)(V_1, V_1, V_1) = 6ah\alpha_1 \left[-\left(\frac{c}{(a+\hat{x})^3} \right) \alpha_1 + \frac{m}{(a+\hat{x})^2} \right] \nu_{12}{}^3 \kappa_{12} \neq 0.$$

Hence, system (2) undergoes a PB.

Theorem 10. Assume that conditions (15a)-(15b) hold, then the system (2) undergoes a TB at the DFEP when the parameter δ passes through the value $\delta^* = \frac{d_2+d_3}{\bar{y}}$.

Proof: The JM of the system (2) at (p_2, δ^*)

$$J_2 = J_{(p_2,\delta^*)} = \begin{bmatrix} \bar{x}\left(-b + \frac{(c+m\bar{y})\bar{y}}{(a+\bar{x})^2} \right) & \frac{-rp\bar{x}}{(1+p\bar{y})^2} - \frac{(c+2m\bar{y})\bar{x}}{a+\bar{x}} & 0 \\ \frac{ah_1\, \bar{y}(c+m\bar{y})}{a+\bar{x}} - \eta\bar{y} & \frac{h_1 m\bar{x}\, \bar{y}}{a+\bar{x}} & -(d_2 + d_3) \\ 0 & 0 & 0 \end{bmatrix} = [b_{ij}]$$

Therefore, J_2has two eigenvalues having negative real part under conditions (15a)-(15b). While the third is zero. Thus it is a non-hyperbolic point.

Let $V_2 = (\nu_{21}, \nu_{22}, \nu_{23})^T$ be the eigenvectors corresponding to $\lambda_{23}^* = 0$. Thus $J_2 V_2 = 0$ gives that $V_2 = (\alpha_2 \nu_{23}, \alpha_3 \nu_{23}, \nu_{23})^T$, where $\alpha_2 = \frac{b_{11} b_{23}}{b_{11} b_{22} - b_{12} b_{21}}$ and $\alpha_3 = \frac{-b_{12} b_{23}}{b_{11} b_{22} - b_{12} b_{21}}$ and, $(\nu_{33} \neq 0)$.

Now, let $U_2 = (\kappa_{21}, \kappa_{22}, \kappa_{23})^T$ represents the eigenvector of ${J_2}^T$ with the eigenvalue $\lambda_{23}^* = 0$, then ${J_2}^T\ U_2 = 0$ gives $U_2 = (0, 0, \kappa_{23})^T$, $(\kappa_{23} \neq 0)$.

Since $F_\delta(X, \delta) = (0, -yz, yz)^T$. Hence we obtain that $F_\delta(p_2,\ \delta^*) = (0, 0, 0)^T$

Therefore, ${U_2}^T F_\delta(p_2,\ \delta^*) = 0$. Hence system (2) has no SNB.

Now, we have ${U_2}^T [DF_\delta(p_2,\ \delta^*) V_2] = \overline{y} \kappa_{23} v_{23} \neq 0$.

Moreover, ${U_2}^T [D^2 F(p_2,\ \delta^*)(V_2, V_2)] = 2\ \alpha_3 \delta^* \kappa_{23} v_{23}^2 \neq 0$, Hence, system (2) at the DFEP, p_2 with $\delta = \delta^*$ possesses a TB.

Theorem 11. As the parameter b^* passes through the value $b = b^* = \frac{(c + m\bar{\bar{y}})\bar{\bar{y}}}{(a + \bar{\bar{x}})^2}$, the system (2) undergoes a SNB near the IEP.

Proof. The free coefficient of the characteristic equation given by Eq. (18) is $B_3 = 0$ when $b = b^*$. Hence the characteristic equation has zero root (eigenvalue).

Hence, the JM of the system (2) around the point p_3 and $b = b^*$, can be written as

$$J_3 = J(\,p_3, b^*) = \begin{bmatrix} 0 & m_{12} & 0 \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & 0 \end{bmatrix}.$$

Now, let $V_3 = (\nu_{31}, \nu_{32}, \nu_{33})^T$ be the eigenvectors corresponding to $\lambda^* = 0$. Then $J_3 V_3 = 0$ gives $V_3 = (\nu_{31}, 0, \alpha_4 \nu_{31})^T$,where $\alpha_4 = \frac{-m_{21}}{m_{23}}$ and $(\nu_{31} \neq 0)$.

Now, let $U_3 = (\kappa_{31}, \kappa_{32}, \kappa_{33})^T$ represents the eigenvector J^T with the eigenvalue $\lambda^* = 0$ of ${J_3}^T$ then ${J_3}^T\ U_3 = 0$ gives $U_3 = (\kappa_{31}, 0, \alpha_5 \kappa_{31})^T$, where $\alpha_5 = \frac{-m_{12}}{m_{32}}$ and $(\kappa_{31} \neq 0)$.

Since $F_b(X, b) = (-x^2, 0, 0)^T$, Hence we obtain that $F_b(p_3,\ b^*) = (-\bar{\bar{x}}^2, 0, 0)^T$.

Clearly, ${U_3}^T F_b(p_3,\ b^*) = -\bar{\bar{x}}^2 \kappa_{31} \neq 0$, and hence the system (2) satisfies the first condition of an SNB in the sense of Sotomayor theorem.

Now, since ${U_3}^T [D^2 F(p_3, b^*)(U_3, U_3)] = \left(-b^* + \frac{a(m\bar{\bar{y}} + c)\bar{\bar{y}}}{(a + \bar{\bar{x}})^3}\right) \kappa_{31} \nu_{31}^2,$

$$= \frac{(c + m\bar{\bar{y}})\bar{\bar{y}}}{(a + \bar{\bar{x}})^2}\left(-1 + \frac{a}{(a + \bar{\bar{x}})}\right) \kappa_{31} \nu_{31}^2 \neq 0$$

Hence the system (2) undergoes an SNB near the IEP.

7. NUMERICAL SIMULATIONS

In this section, the effect of parameters on the model dynamics and the validation of the obtained results are numerically verified. For this purpose, the simulations were performed to investigate the behaviors of the system (2) using the Runge–Kutta fourth-order method with MATLAB software. The numerical observations of systems (2) will reinforce the analytical findings and provide some more insights into the dynamical properties of these systems. The sensitivity analysis of the system (2) is also included using the following estimated set of parameter values.

$$r = 2, p = 1, b = 0.3, d_1 = 0.001, c = 1.5, a = 2 , h = 0.75, m = 0.2,$$
$$\delta = 0.25, d_2 = 0.01, \eta = 0.01, d_3 = 0.15; \quad (28)$$

The numerical solution of the system (2) is determined and represented in the form of a phase portrait and their time series as shown in Fig. 1 using the data set (28) and starting from different initial points.

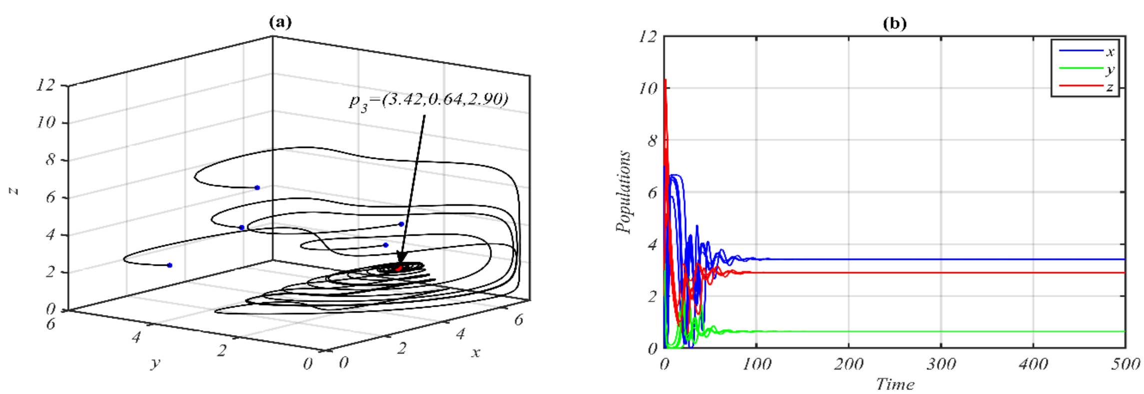
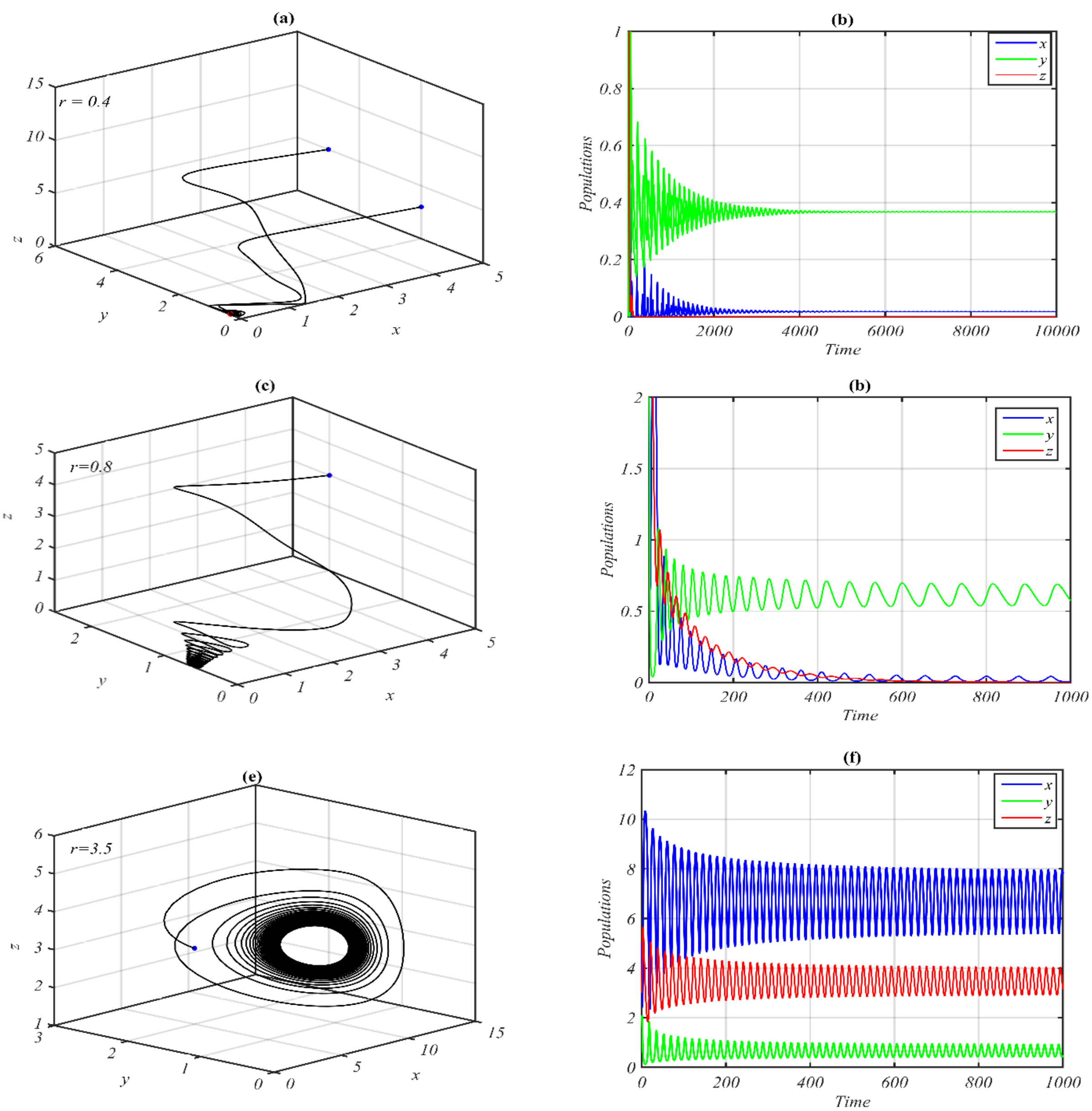


Figure 1. The trajectory of the model (2) utilizing data set (28). (a)Approach asymptotically to IEP, $p_3 = (3.42,0.64,2.90)$ for the given in (28) and different initial points.(b) Trajectories of populations versus time.

Clearly, Fig. 1 shows the asymptotic approach of the solutions, which started from different initial points to an IEP, (3.42, 0.64, 2.90), for the data given by (28). This confirms our obtained result regarding the existence of GAS in the system (2) provided that certain conditions hold.

Now, the effect of varying the value of r on the dynamic of the system (2) is discussed, and the obtained results are presented at selected values in Figure 2. It is obtained that, for r belongs to [0.01,0.54],[0.55,0.86], and $r \geq 3.44$ the system's (2) approaches to p_2, $2D$ period attractor, and $3D$ periodic attractor, respectively, see Fig. 2 for the selected values. Otherwise when $r \in$ [0.87, 3.44] the solution approaches p_3 as in Fig. 1.

Moreover, it is observed that for $r < d_1$the solution of system (2) approaches p_0 as in Fig. 2g and 2h, however for $d_1 \leq r$ the solution approaches p_3 as in Fig. 1.



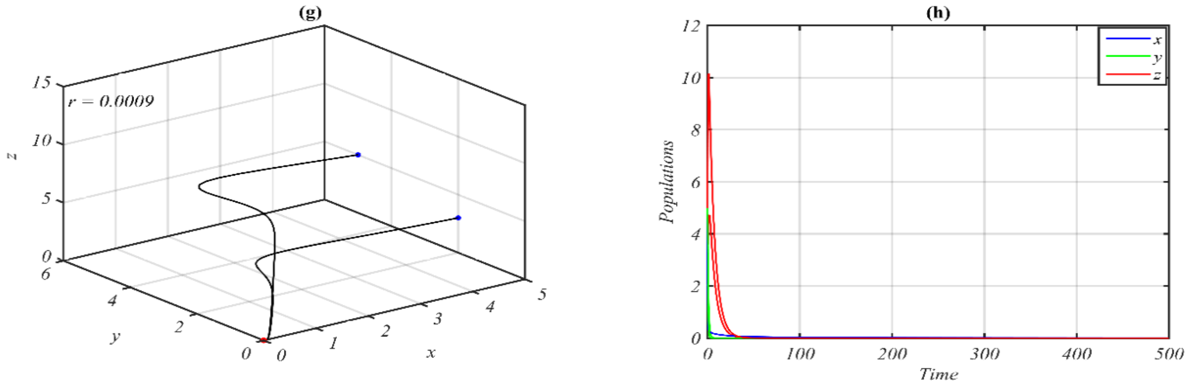
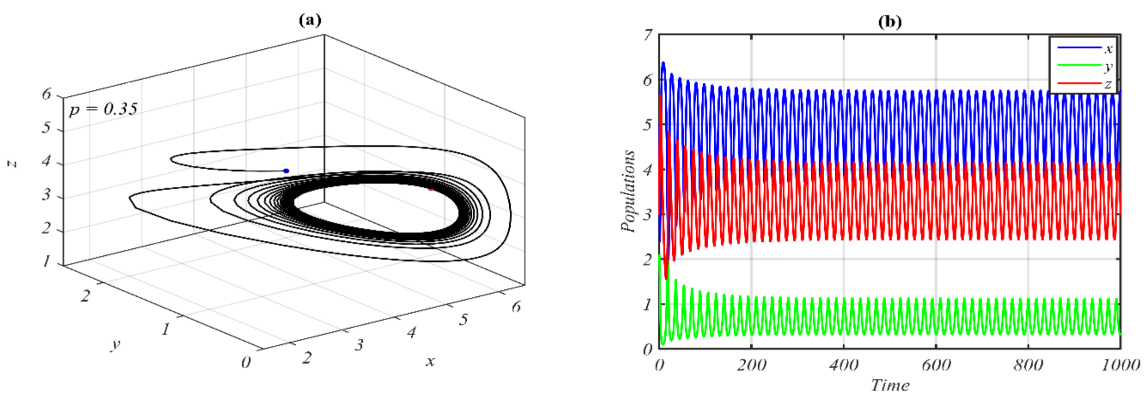


Figure 2. The trajectory of system (2) utilizing data set (28) with different values of r. (a) Approach to $p_2 = (0.01, 0.63, 0)$ when $r = 0.4$ (b) Time series when $r = 0.4$. (c) Periodic dynamics in $xy-$plane when $r = 0.8$ (d) Time series when $r = 0.8$. (e) Asymptotic stable limit cycle when $r = 3.5$. (f)Time series when $r = 3.5$. (g) Approach to p_0 when $r = 0.0009$. (h) Time series when $r = 0.0009$.

The effect of varying p is studied numerically on the dynamic of the system (2), and it is observed that for p belongs to$[4.30, 4.43]$ and $p \leq 0.38$ the system's solution converges asymptotically to $3D$ period attractor, while if $p \geq 4.44$ then, and $2D$ period attractor. See Fig.3 for the selected values. However for $p \in [0.39, 4.30]$ the solution approaches p_3 as in Fig. 1.



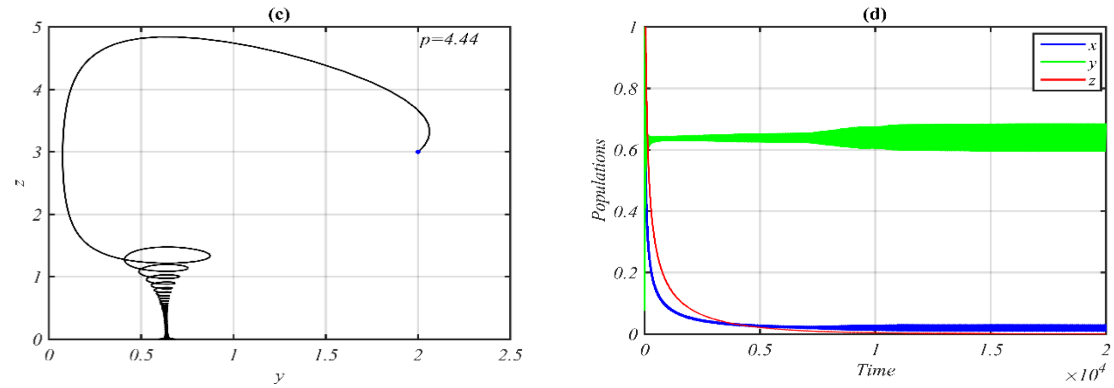


Figure 3. The trajectory of system (2) utilizing data set (28) with different values of p. (a) Asymptotic stable limit cycle when $p = 0.35$ (b) Time series when $p = 0.35$. (c) Periodic dynamics in $xy -$ plane when $p = 4.44$ (d) Time series when $p = 4.44$.

For the parameter b in the range $b \leq 0.16$ (similarly when $\delta \leq 0.15$) with the parameters sets as in (28), it's observed that the system's solution converges to $3D$ period attractor, as in Fig.(4), otherwise it's still persistent at p_3, as in Fig. (1).

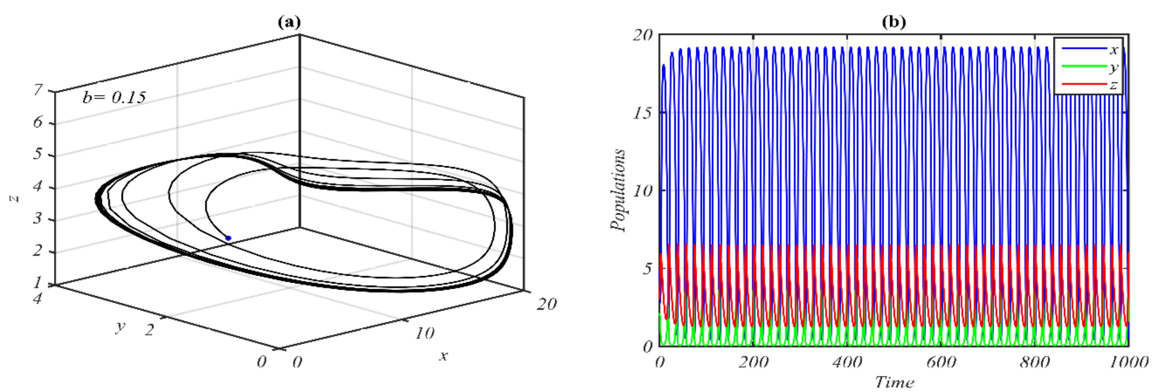


Figure 4. The trajectory of the system (2) utilizing data set (28) with different values of b.(a) Periodic dynamics in $\mathcal{R}_+^3$ when $b = 0.15$. (b) Time series when $b = 0.15$.

Now, for the parameter c in the ranges $c \leq 0.13$, $c \in [0.13, 0.78]$, and $c \geq 4.9$ the system's solution converges asymptotically p_1, $3D$ period attractor, and a bi-stable behavior between IEP and $3D$ periodic attractor respectively, see Fig. 5. However for $c \in [0.79, 4.08]$ the solution approaches p_3 as in Fig. 1.

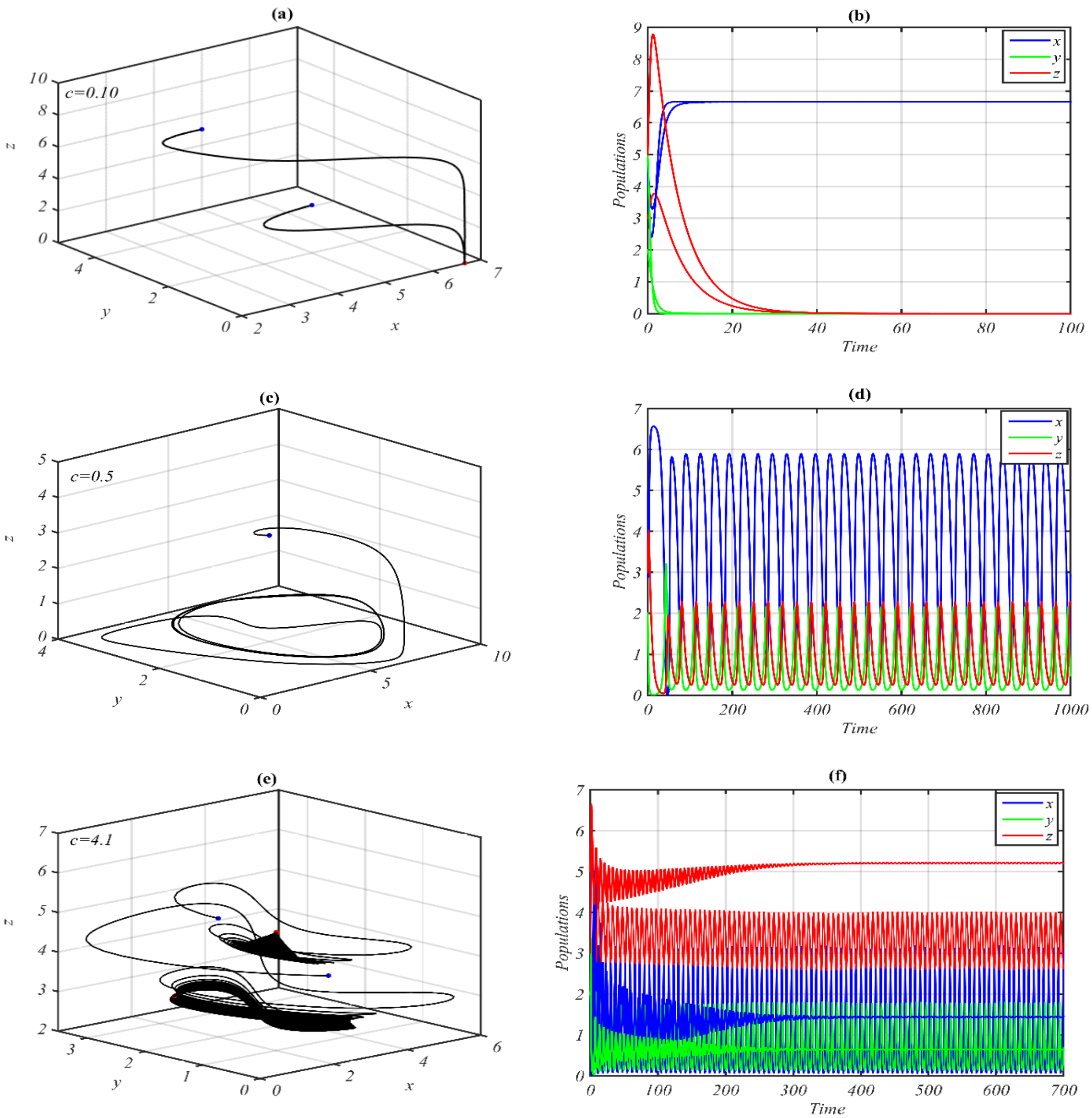


Figure 5. The trajectories of the system (2) utilizing data set (28) with different values of c (a) Approach to $p_1 = (6.66,0,0)$ when $c = 0.10$. (b) Time series when $c = 0.10$. (c) Periodic dynamics in $\mathcal{R}_+^3$ when $c = 0.5$ (d) Time series when $c = 0.5$. (e) bi- stable between $p_3 = (1.44,0.64,5.20)$ and $3D$periodic when $c = 4.1$. (f) Time series when $c = 4.1$.

Now, as h varies in the ranges $h \leq 0.06$ and $0.06 < h < 0.1$, with the parameters set as in (28), it is noted that the system's solution converges asymptotically to p_1, and $3D$ period attractor, respectively, as in Fig. (6). However for $h \geq 0.1$ the solution approaches p_3 as in Fig. 1.

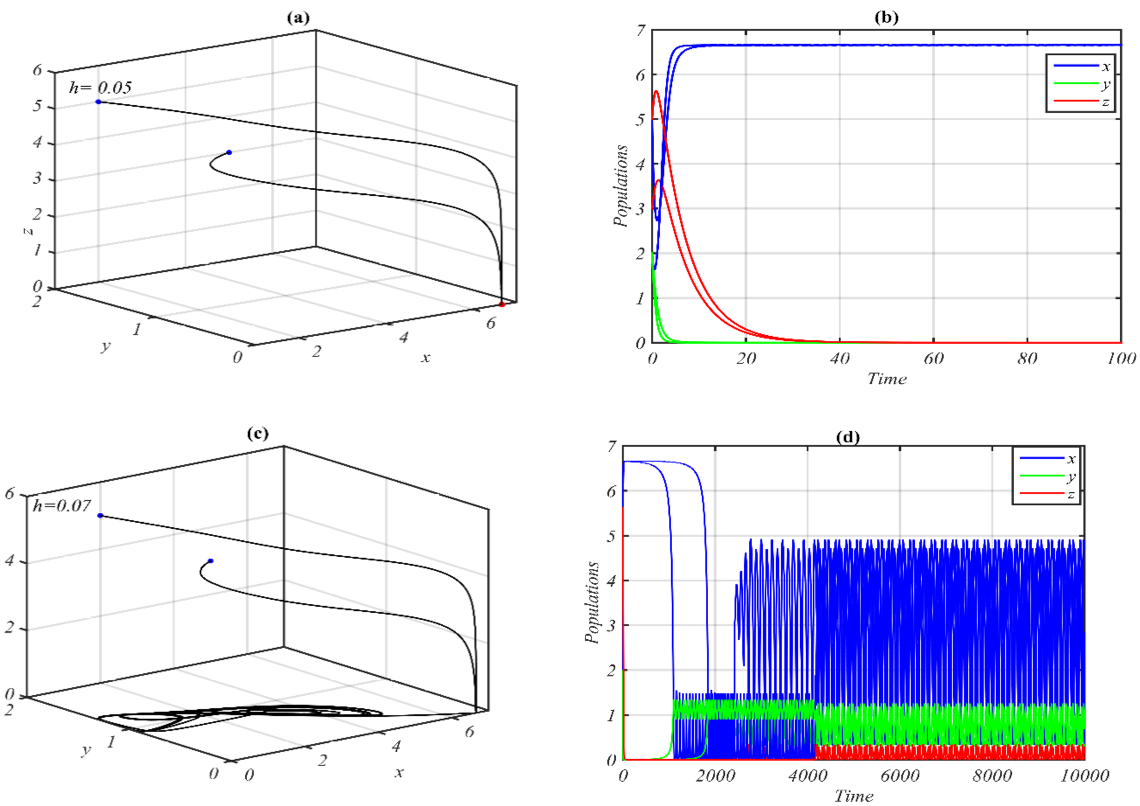


Figure 6. The trajectories of the system (2) utilizing data set (28) with different values of h (a) Approach to $p_1 = (6.66,0,0)$ when $h = 0.05$. (b) Time series when $h = 0.05$. (c) Periodic dynamics in $\mathcal{R}_+^3$ when $h = 0.07$. (d) Time series when $h = 0.07$.

Now for the ranges $a < 0.78$ and $a \in [0.79,0.84]$, the system's solution approaches asymptotically to $3D$ period attractor, and a bi-stable behavior between IEP and $3D$ periodic, respectively, see for the selected values Fig. 7. However for $a \geq 0.85$ the solution approaches p_3 as in Fig. 1.

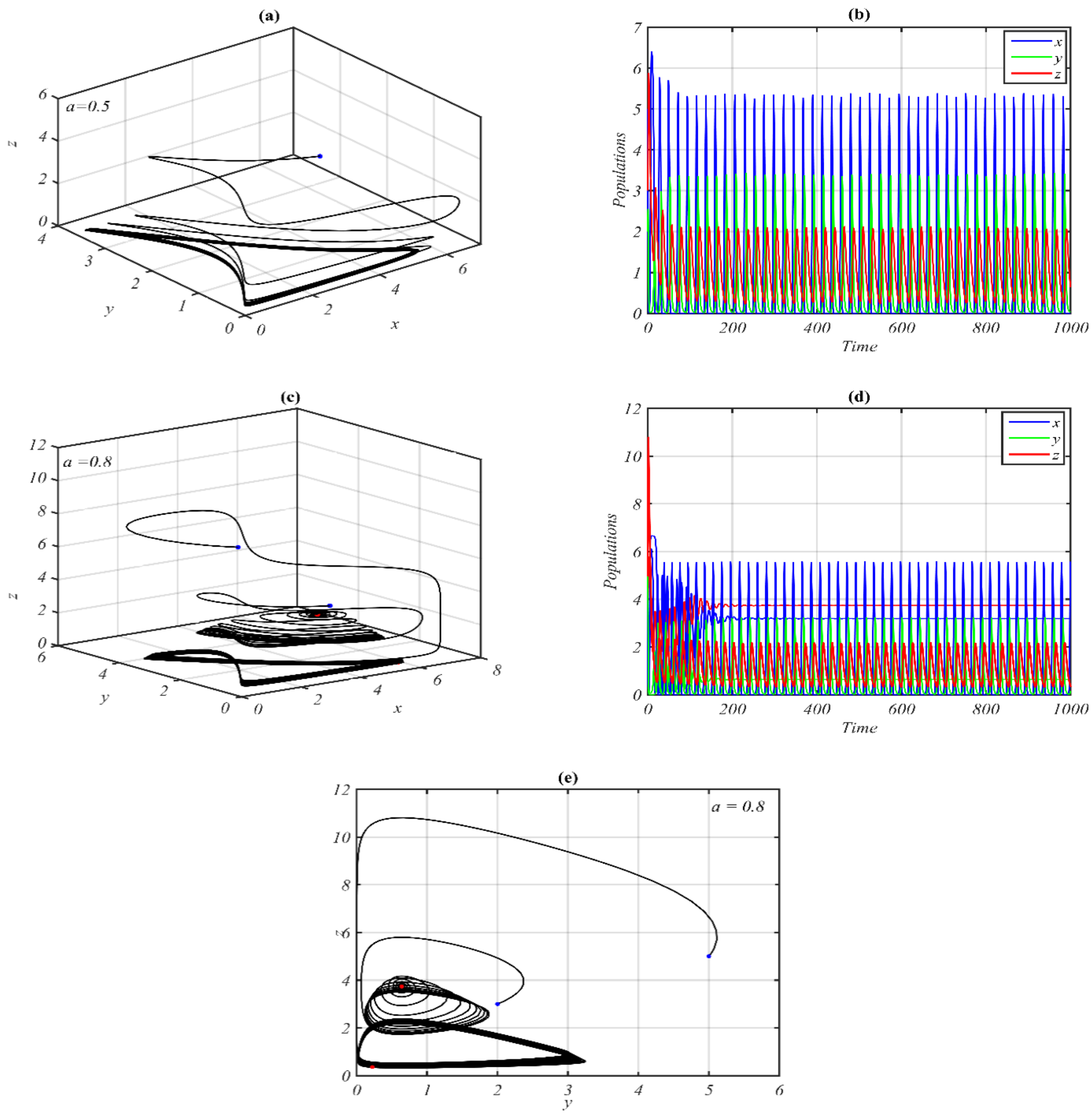


Figure 7. The trajectories of the system (2) utilizing data set (28) with different values of a (a) Periodic dynamics in $\mathcal{R}_+^3$ when $a = 0.5$. (b)Time series when $a = 0.5$. (c) bi- stable between $p_3 = (3.19,0.64,3.73)$ and $3D$ periodic when $a = 0.8$. (d) Time series when $a = 0.8$. (e) Projection of the trajectory on the $yz-$plane for $a = 0.8$.

Moreover, the system's solution approaches asymptotically to $3D$ period attractor and p_1, respectively, when $\eta \in [0.06,0.12]$, and $\eta \geq 0.13$, while the rest of parameters as given by (28), see for the selected values Fig. 8. However for $\eta \leq 0.06$ the solution approaches $p_3$as in Fig. 1.

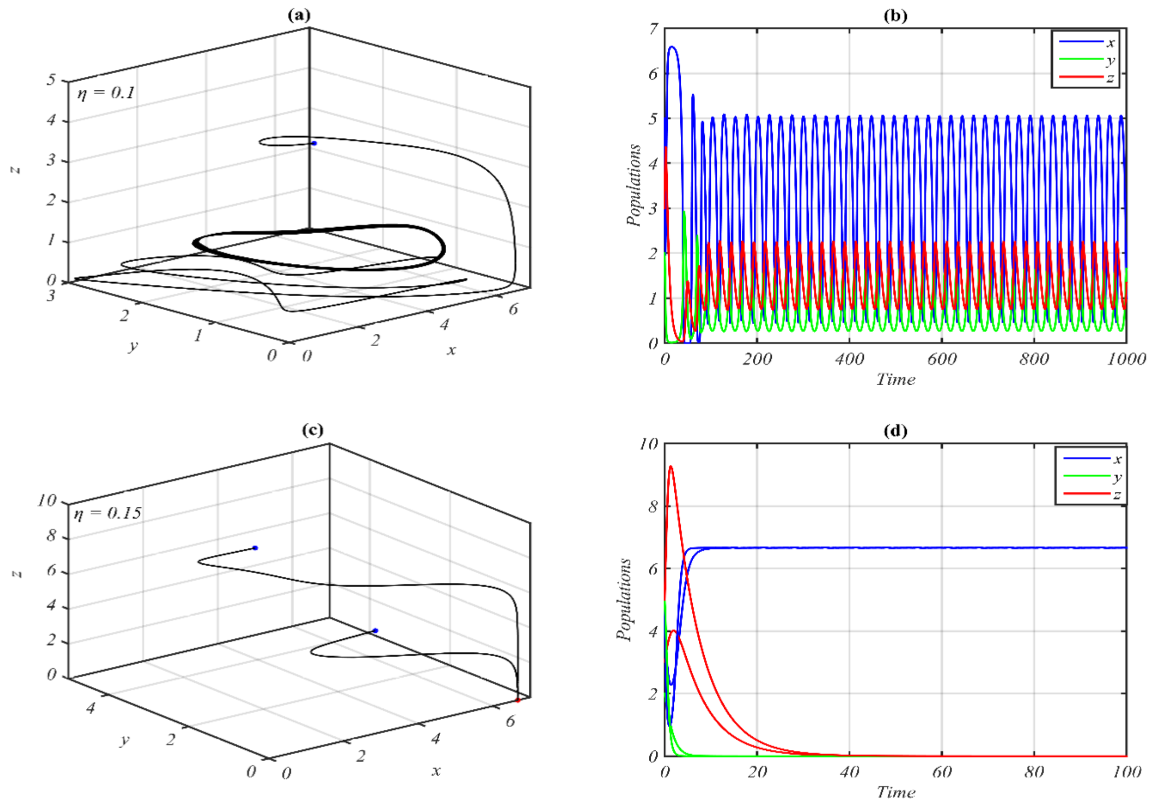


Figure 8. The trajectories of the system (2) utilizing data set (28) with different values of η (a) Periodic dynamics in $\mathcal{R}_+^3$ when $\eta = 0.1$. (b) Time series when $\eta = 0.1$. (c) Approach to $p_1 = (6.66,0,0)$ when $\eta = 0.15$. (d) Time series when $\eta = 0.15$.

It is observed that for $m \leq 0.59$ (similarly when $d_3 \leq 0.27$), system (2) approaches asymptotically to $3D$ period attractor, see for the selected values Fig.9. Otherwise, it approaches to p_3, as in Fig. (1)

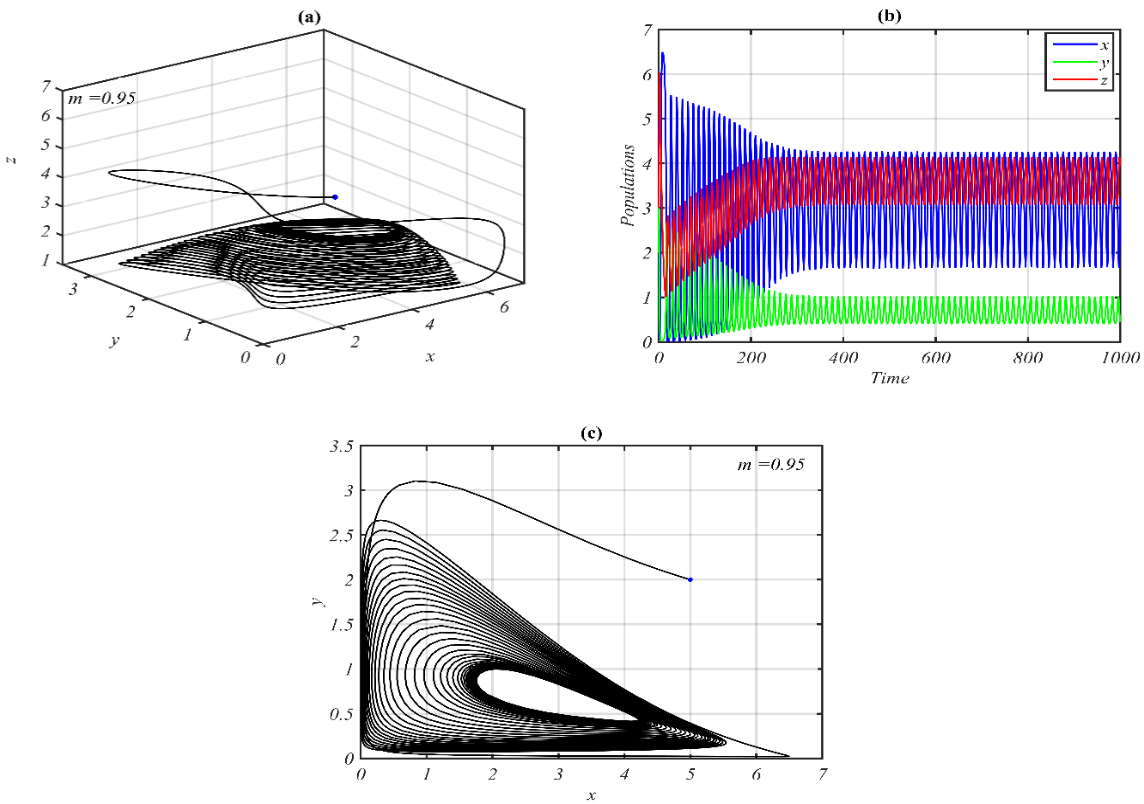


Figure 9. The trajectories of the system (2) utilizing data set (28) with different values of m. (a) Periodic dynamics in $\mathcal{R}_+^3$ when $m = 0.95$. (b) Time series when $m = 0.95$. (c) Projection of the trajectory on the $xy -$ plane when $m = 0.95$.

Finally, Fig. (10) demonstrates the influence of the parameter d_2 on the dynamics of the system (2), which is studied numerically. It is noted that the system (2) approaches asymptotically to $2D$ period attractor, p_2 and p_1 when the parameter d_2 belongs to the ranges, $d_2 \in [0.13, 0.45]$, $d_2 \in [0.46, 0.79]$, and $d_2 \geq 0.80$ while the rest of parameters as in (28). However for $d_2 \leq 0.12$ the solution approaches p_3 as in Fig. 1.

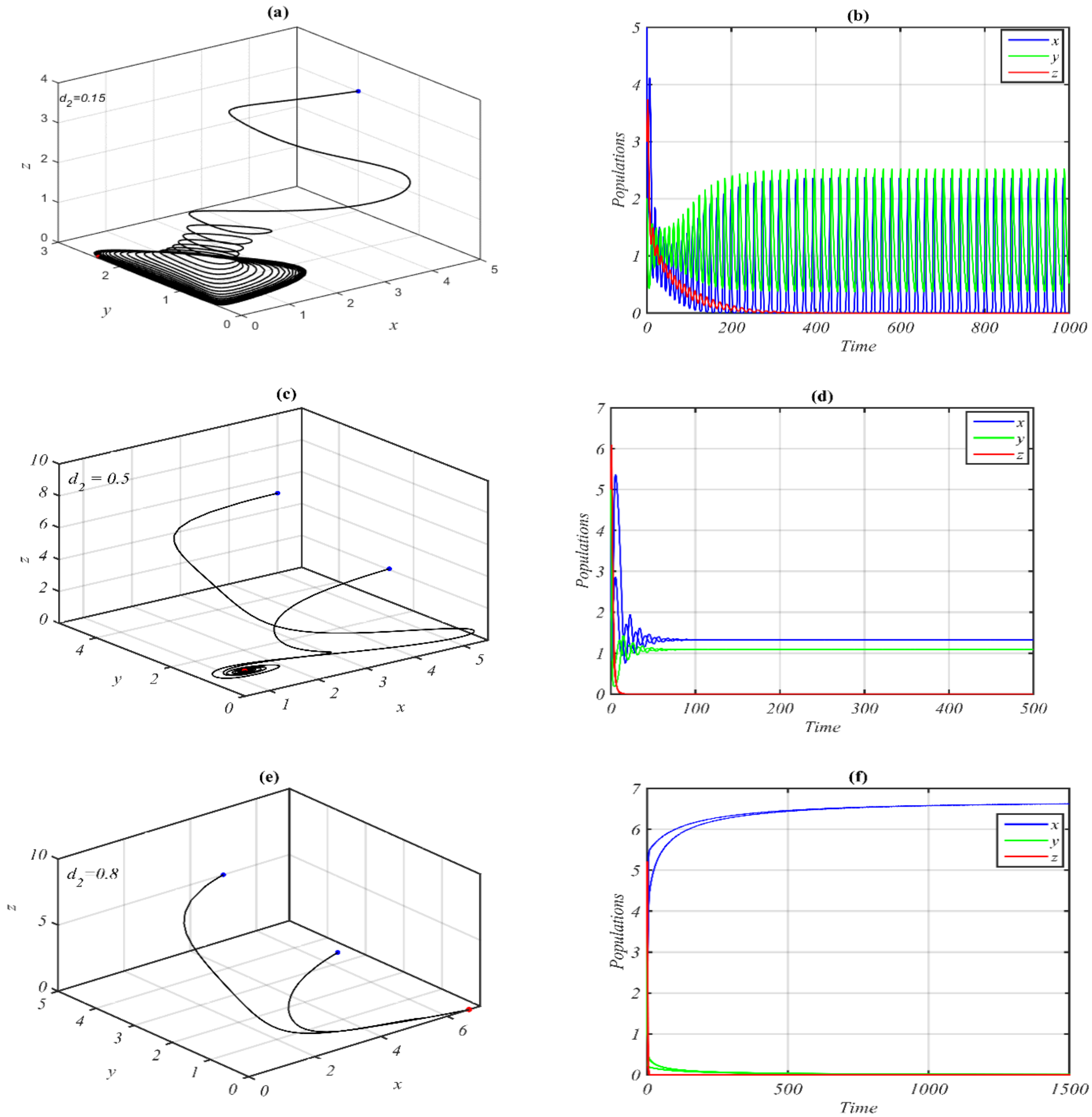


Figure 10. The trajectories of the system (2) utilizing data set (28) with different values of d_2. (a) Periodic dynamics in $xy-$plane when $d_2 = 0.25$. (b) Time series when d_2.. (c) Approach to $p_2 = (1.31,1.08,0)$ when $d_2 = 0.5$. (d) Time series when $d_2 = 0.5$. (e) Approach to $p_1 = (0.66,0,0)$ when $d_2 = 0.8$. (f)Time series when $d_2 = 0.8$.

8. CONCLUSION

Based on substantial and reliable biological assumptions, the eco-epidemiological model that consists of prey consumed by predators having infectious diseases is constructed. All the properties of the solutions are discussed. All potential EPs are calculated. Local stability analysis of EPs is performed. The persistence requirements of the proposed model are established. The GS analysis using the LF technique is performed whenever possible. The LB around the EPs is investigated using the Sotomayor theorem. After doing numerical simulations to confirm the theoretical finding and verify the control set of parameters, the following results are categorized using the parameters set of data given in (28).

The system has various types of attractors including the stable point, stable periodic, and bi-stable between stable point and periodic attractors. The parameters of intraspecific competition, infection rate, conversion factor, and half-saturation constant have a stabilizing role in the dynamic behavior of the system. However, the parameters of the prey's birth rate, the natural death rate of the prey, the level of predator's cooperation in hunting, and the additional mortality rate of predators due to infection play destabilizing role in the system's dynamics. Finally, all other parameters have extinction effects on the system.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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