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EFFECTS OF POPULATION DENSITY ON PREY-PREDATOR SYSTEM STABILITY

ARRIVAL RINCE PUTRI[1,*], NILA DESTIA NASRA[1], MAHDHIVAN SYAFWAN[1], MASAJI WATANABE[2]

[1]Department of Mathematics and Data Science, Universitas Andalas, Padang 25231, Indonesia

[2]Graduate School of Environmental, Life, Natural Science, and Technology, Okayama University, Japan

Abstract: The prey-predator model is explained by considering the population densities of both immature and mature prey and predators. The models were divided into two cases: the absence of the mature prey population density and the absence of the immature predator population density. These models were then analyzed around the eiop[quilibrium point, with the stability determined based on the eigenvalues of the Jacobian matrix. Stability analysis was also performed using Routh's criteria. Furthermore, numerical simulations confirmed the analytical results of the model. The dependence on parameters in each case was investigated for extreme values. Based on the analytical results and numerical simulations, it can be concluded that the equilibrium point is always stable whenever it exists. Thus, the populations of immature and mature prey, as well as immature and mature predators, will not undergo extinction.

Keywords: prey-predator system; stability; functional response; Routh's criterian; numerical simulations.

2020 AMS Subject Classification: 37N30, 65L07.

1. INTRODUCTION

Interaction between two species is a common occurrence in ecosystems. There are three main types of interactions: symbiosis, competition, and predation. Some species interact with others to sustain their lives. These interactions can benefit both species, benefit only one species, or involve

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*Corresponding author
E-mail address: arrival@sci.unand.ac.id

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competition between two species. Population dynamics refer to changes in population sizes in an area, influenced by abiotic and biotic factors. A key biotic factor influencing population dynamics is the interaction between prey and predators, known as predation. This prey-predator relationship is a fundamental process in ecosystems [4], [15].

The first mathematical model that describes the population dynamics between prey and predators is known as the Lotka-Volterra model [2], [3]. This model is often used to describe the dynamics of a system with two populations: the prey population and the predator population. It is therefore also known as the prey-predator model. The prey-predator model indicates that the prey population will increase if it is initially small. Conversely, the predator population will increase if the prey population is large and will decrease when the prey is exhausted. The increase in prey and predator populations depends on time [11], [13], [14].

The Lotka-Volterra model assumes that the growth of the prey population follows exponential growth, where the population grows indefinitely over time. This assumption is unrealistic because, in reality, a population cannot increase continuously without limit due to factors such as living space, food, and environmental conditions. These factors are collectively known as the carrying capacity of the environment. Therefore, the Lotka-Volterra model was modified to assume that prey growth follows logistic growth [7]. The logistic growth model accounts for the limited carrying capacity of the environment, preventing the population from growing exponentially.

The prey-predator model has also been reformulated to account for populations with age groups, such as immature and mature individuals. Wang et al. [16] introduced a prey-predator model that includes age groups for predators, specifically the immature predator population and the mature predator population. A study focused on a prey-predator model with age groups for both prey and predators, where the predator is a generalist type, has been discussed by [5]. This study showed that adding more predators led to a steady increase in biomass for mature prey. Another study considered the prey-predator model by taking into account growth rates that depend on population density in immature and mature prey groups [1]. This study analyzed stability conditions and the impact of mortality on predators. Ghosh et al. [8] discussed prey-predator models for several age groups, including immature and mature prey as well as immature and mature predators. They analyzed the stability of the model and the impact of mortality on each population group.

This study focuses on the model by Ghosh et al. [8]. The stability of the model will be analyzed by considering the population density in two cases: i) the absence of mature prey and ii) the absence of mature predators.

2. Mathematical Model

The mathematical model used in this study is a modification of the Lotka-Volterra model that accounts for age groups (immature and mature) in both prey and predator populations. The following assumptions leads to the model [1], [12]:

a. Each population, both prey and predators, is divided into two age groups: immature and mature.
b. The prey reproduction rate depends on the population density of mature prey, while the predator reproduction rate does not depend on the population density of mature predators.
c. The rate of transition from immature to mature prey is independent of the population density of mature prey, while the rate of transition from immature to mature predators is dependent on the population density of mature predators.
d. Only mature predators participate in predation and prefer to consume the mature prey population.

Based on the assumptions, the prey-predator model consisting of groups of immature and mature prey as well as immature and mature predators is formulated in the following system of differential equations [8]:

$$
\begin{aligned}
\frac{dx_1}{dt} &= r_1 x_2(1 - c_1 x_2) - b_1 x_1 - \mu_1 x_1,\\
\frac{dx_2}{dt} &= b_1 x_1 - \gamma x_2^2 - \frac{\alpha x_2 y_2}{h + x_2} - \mu_2 x_2,\\
\frac{dy_1}{dt} &= r_2 y_2 - b_2 y_1(1 - c_2 y_1) - m_1 y_1,\\
\frac{dy_2}{dt} &= b_2 y_1(1 - c_2 y_1) + \frac{\beta \alpha x_2 y_2}{h + x_2} - m_2 y_2,
\end{aligned} \tag{1}
$$

where $x_1(t)$ and $x_2(t)$ are the populations of immature prey and mature prey at time *t*, respectively; $y_1(t)$ and $y_2(t)$ are populations of immature predator and mature predators at time *t*, respectively; $r_1 x_2(1 - c_1 x_2)$ is the reproduction rate of prey, which depends on the population density of mature prey x_2, with r_1 being the maximum per capita growth rate and c_1 representing the per capita birth rate that reduces as the population density of mature prey increases; b_1 is the transition rate from immature prey to mature prey; μ_1 and μ_2 are the natural mortality rates of immature and mature prey, respectively; γx_2^2 represents the crowding effect among mature prey, where γ is the strength of competition between mature prey; $\frac{\alpha x_2}{h+x_2}$ is the Holling type II functional response for predation, where α is the attack rate, h is the half-saturation constant, and β is the conversion coefficient (efficiency of converting consumed prey into predator offspring); r_2 is the reproductive rate of mature predator; $b_2y_1(1-c_2y_1)$ is the transition rate from immature predators to mature predators, where b_2 is the maximum per capita growth rate and c_2 represents the per capita birth rate that reduces as the population density of mature predators increases; m_1 and m_2 are the natural mortality ratesimmature and matures predators, respectively [9].

3. Main Results

3.1. The Prey-Predator System with Absence of Mature Prey Population

By providing the asumptions $c_1 = 0$ and $c_2 \neq 0$, the system of equations (1) simplifies as follows:

$$\begin{aligned}
\frac{dx_1}{dt} &= r_1x_2 - b_1x_1 - \mu_1x_1,\\
\frac{dx_2}{dt} &= b_1x_1 - \gamma x_2^2 - \frac{\alpha x_2y_2}{h+x_2} - \mu_2x_2,\\
\frac{dy_1}{dt} &= r_2y_2 - b_2y_1(1-c_2y_1) - m_1y_1,\\
\frac{dy_2}{dt} &= b_2y_1(1-c_2y_1) + \frac{\beta\alpha x_2y_2}{h+x_2} - m_2y_2.
\end{aligned} \tag{2}$$

This system of differential equations now reflects the assumption that the reproduction rate of the prey is not limited by the carrying capacity (as $c_1 = 0$), while the predator reproduction rate still depends on the population density of mature predators (as $c_2 \neq 0$).There are four equilibrium points of the system of equation (2):

1. Extinction of All Species $S^0 = (0,0,0,0)$.
2. Extinction of Prey $\hat{S} = (0,0,\hat{y}_1,\hat{y}_2)$, where
$$\hat{y}_1 = \frac{m_2b_2 + m_1m_2 - r_2b_2}{b_2c_2(m_2-r_2)},$$
$$\hat{y}_2 = \frac{b_2\hat{y}_1(1-c_2\hat{y}_1)}{m_2}.$$
3. Extinction of Predator $\bar{S} = (\bar{x}_1,\bar{x}_2,0,0)$, where
$$\bar{x}_1 = \frac{r_1\bar{x}_2}{b_1+\mu_1},$$
$$\bar{x}_2 = \frac{1}{\gamma}\left(\frac{b_1r_1}{b_1+\mu_1} - \mu_2\right).$$
4. Existence of All Species $S^* = (x_1^*,x_2^*,y_1^*,y_2^*)$, where
$$x_1^* = \frac{r_1x_2^*}{b_1+\mu_1},$$

$x_2^* = A,$

$$y_1^* = \frac{\beta\alpha m_1 x_2^* - m_1 m_2(h + x_2^*)}{b_2 c_2\big(r_2(h + x_2^*) + \beta\alpha x_2^* - m_2(h + x_2^*)\big)} + \frac{1}{c_2},$$

$y_2^* = \frac{b_2 y_1^*(1 - c_2 y_1^*) + m_1 y_1^*}{r_2}$, and

$A = A_3(x_2^*)^3 + A_2(x_2^*)^2 + A_1 x_2^* + A_0.$

3.2. The Prey-Predator System with Absence of Mature Prey Population

By providing assumptions $c_1 \neq 0$ and $c_2 = 0$, the system of equations (1) simplifies as follows:

$$\begin{aligned}
\frac{dx_1}{dt} &= r_1 x_2(1 - c_1 x_2) - b_1 x_1 - \mu_1 x_1,\\
\frac{dx_2}{dt} &= b_1 x_1 - \gamma x_2^2 - \frac{\alpha x_2 y_2}{h + x_2} - \mu_2 x_2,\\
\frac{dy_1}{dt} &= r_2 y_2 - b_2 y_1 - m_1 y_1,\\
\frac{dy_2}{dt} &= b_2 y_1 + \frac{\beta\alpha x_2 y_2}{h + x_2} - m_2 y_2.
\end{aligned} \tag{3}$$

There are three equilibrium of the system of equation (3).

1. Extinction of All Species $S^0 = (0,0,0,0)$.
2. Extinction of Predator $\bar{S} = (\bar{x}_1, \bar{x}_2, 0,0)$, where

$$\bar{x}_1 = \frac{r_1 \bar{x}_2(1 - c_1 \bar{x}_2)}{b_1 + \mu_1},$$

$$\bar{x}_2 = \frac{b_1 r_1 - b_1 \mu_2 - \mu_1 \mu_2}{b_1 r_1 c_1 + \gamma b_1 + \gamma \mu_1}.$$

3. Existence of All Species $S^* = (x_1^*, x_2^*, y_1^*, y_2^*)$, where

$$x_1^* = \frac{r_1 x_2^*(1 - c_1 x_2^*)}{b_1 + \mu_1},$$

$$x_2^* = \frac{hB}{\beta\alpha - B},$$

$$y_1^* = \frac{r_2 y_2^*}{b_2 + m_1},$$

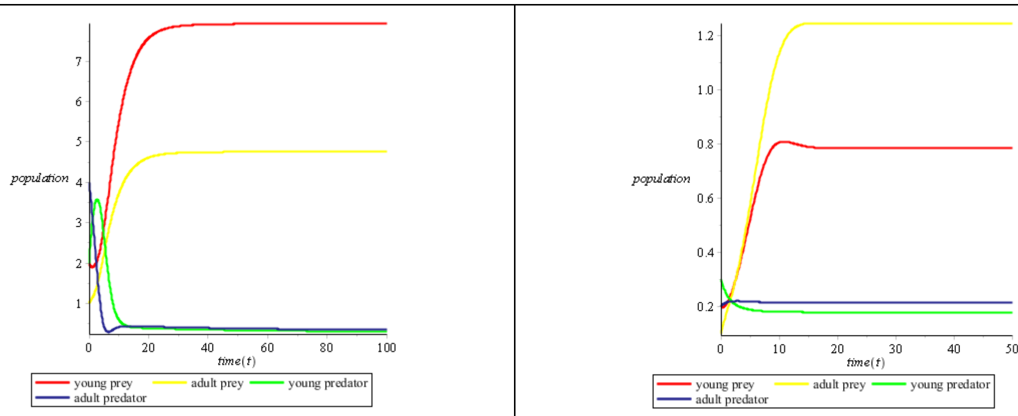
$y_2^* = \frac{h + x_2^*}{\alpha}\left(\frac{b_1 r_1(1 - c_1 x_2^*)}{b_1 + \mu_1} - \gamma x_2^* - \mu_2\right)$, and

$$B = m_2 - \frac{b_2 r_2}{b_2 + m_1}.$$

3.3. Numerical Simulations

This section shows the results of a numerical simulation with several parameter values to confirm the analytical solutions obtained in the previous section. For models (2) and (3), specific parameter values were assumed, namely $r_1 = 1$, $r_2 = 0.5$, $b_1 = 0.5$, $b_2 = 0.2$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $m_1 = 0.4$, $m_2 = 0.2$, $\gamma = 0.15$, $c_1 = 0.5$, $c_2 = 0.5$, $\alpha = 0.3$, $\beta = 0.2$ and $h = 1$ [8].

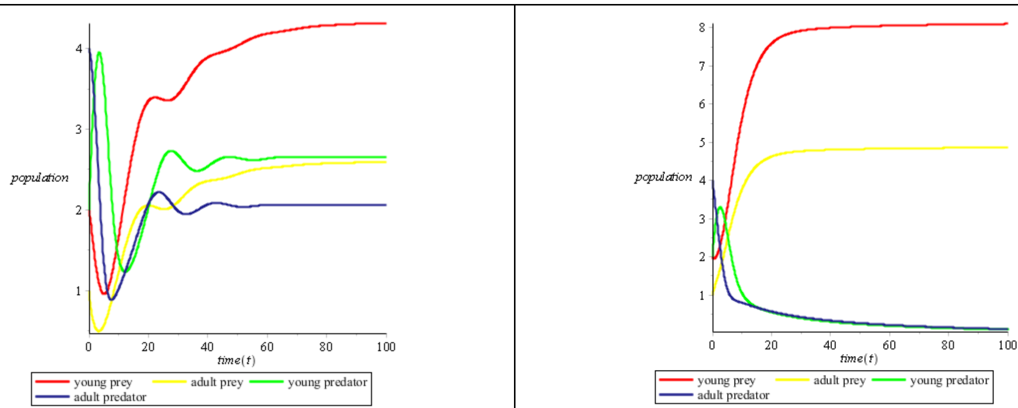
Using these parameter values, stability for each equilibrium point of system (2) that determined from eigenvalues are as follows: the equilibrium S^0 is unstable, the equilibrium point $\hat{S}$ is unstable, the equilibrium point $\bar{S}$ is unstable, and the equilibrium point S^* is asymptotically stable. Figure 1 (a) shows the graph of the solution of the model (2) for the initial values $x_1(0) = 2$, $x_2(0) = 1$, $y_1(0) = 2$, and $y_2(0) = 4$. The population of immature prey and mature prey increases until it approaches a stable population. Meanwhile, the immature predator population and the mature predator population decrease. Over time, the populations of both immature and mature prey, as well as immature and mature predators, approach the equilibrium point $S^* = (7.965, 4.779, 0.280, 0.3240)$. Stability for each equilibrium point of system (3) that determined from eigenvalues are as follows: the equilibrium S^0 is unstable, the equilibrium point $\hat{S}$ is unstable, the equilibrium point $\bar{S}$ is unstable, and the equilibrium point S^* is asymptotically stable. The graph of the solution of the model (3) for the initial values $x_1(0) = 0.2$, $x_2(0) = 0.1$, $y_1(0) = 0.3$, and $y_2(0) = 0.2$ is depicted in Figure 1 (b). The figure shows that the populations of immature prey and mature prey increases until approach a stable population. Meanwhile, when the immature predator population decrease, the mature predator population increases until it approach a stable population. When it reaches a certain time, the population of immature and mature prey as well as immature and mature predators approach the equilibrium point $S^* = (0.7825, 1.247, 0.1667, 0.2000)$.



(a). System (2) (b). System (3)

Figure 1. Solution of the system

Furthermore, the following presents the results of numerical simulations based on models (2) and (3) with various values of the parameters α and β. These simulations are carried out to determine how changes in these parameters affect all existing populations. Additionally, parameter variation in model (2) is considered in the absence of the population of mature prey with specific values as shown in Figure 2. This is also applies to system (3) as shown in Figure 3.

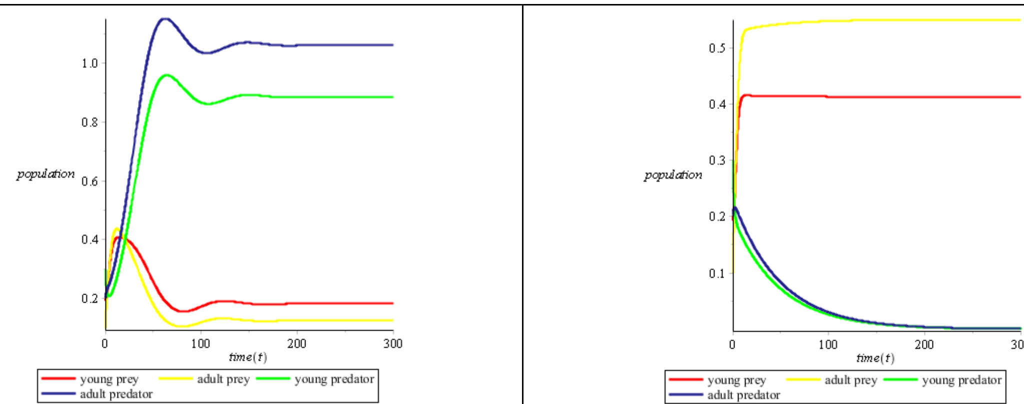


(a). $\alpha = 0.6, \beta = 0.5, c_2 = 0.4$ (b). $\alpha = 0.2, \beta = 0.1, c_2 = 0.4$

Figure 2. Solution of the system (2) with specific values of parameter

The parameters α and β affect the populations of immature and mature prey, as well as immature and mature predators. When the parameters α and β are increased, the populations remain stable after a certain time. Conversely, when the parameters α and β are decreased, the populations of immature and mature predators will experience extinction after a certain time. The parameter c_2 also affects each population. When c_2 is reduced, the populations of immature and mature prey also decrease. Meanwhile, the population of mature predators increases following the increase in the population of immature predators, which occurs due to the small value of the parameter c_2.

Additionally, the parameter c_1 affects the populations of immature and mature prey as well as the populations of immature and mature predators. Solutions of model (3) will approach the asymptotically stable equilibrium point faster if the value of the parameter c_1 is larger. This is due to the fact that a higher reproduction rate in mature prey stabilizes the immature prey population. Consequently, the mature predator population will be stable because the immature prey population will continue to grow into mature prey, ensuring that mature predators will always have mature prey to feed on. Furthermore, immature predators will also become stable due to the reproductive rate of mature predators. However, the parameters α and β also have a significant influence on all four populations. When these parameters are reduced, the populations will converge more closely.



(a). $\alpha = 0.6, \beta = 0.5, c_1 = 1$ (b). $\alpha = 0.2, \beta = 0.1, c_1 = 1$

Figure 3. Solution of the system (3) with specific values of parameter

4. Conclusions

The construction of the prey-predator model of system (1) consists of two cases: i) the mature prey population density is neglected in system (2), and ii) the mature predator population density is neglected in system (3). Systems (2) and (3) were analyzed using Routh's criteria to determine the roots of the polynomial equation that have negative real parts, without having to calculate the roots directly. Numerical simulations based on systems (2) and (3) were obtained using several parameter values. Using Routh's criteria, the equilibrium points in the two systems are only stable at the equilibrium point for the existence of all species. Thus, it can be interpreted that the prey and predator populations will not experience extinction.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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