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A MODIFIED METHOD FOR SOLVING DELAY FUZZY VARIABLE-ORDER FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. This work focuses on providing an efficient well known method which is so called Adomain decomposition method for solving non-linear delay fuzzy fractional variable-order partial differential equations. The fractional order derivative will be in the Caputo sense. According to this method, the solution can be simply computed as a components of a convergent power series. Several numerical examples are presented to illustrate that the proposed method is precise and accurate and can be enforced to other non-linear problems.

Keywords: delay partial differential equations; Adomian decomposition method; fuzzy set theory; fractional calculus.

2020 AMS Subject Classification: 35R60.

1. Introduction

Numerous scientific and engineering fields are recognized to be well-represented by the non-linear partial differential equations. Numerous studies, such as those in [6, 23], have been conducted to get numerical solutions for these kinds of problems when specific computational

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difficulties arise. Typically, rounding error results in accuracy loss, although the Adomian decomposition Method (ADM) only requires a small number of calculations. [1, 2, 4, 11, 18] Over the years, many researchers have studied the delay differential equations since delays are ubiquitous in real-world systems. [4] Fractional delay differential equations, or FDDES, are a relatively new subject, nonetheless.Though modeling some processes and systems in life science and engineering with these kinds of equations appears reasonable, the scientific community has only recently focused on them. [12, 15] We refer to [4, 17] for the existence of solutions for (FDDES). Taking into account the fractional derivative in the Riemann-Liouville sense.The e Lakshmikantham [23] provided the sufficient conditions for the availability of solutions to single term nonlinear delay fractional differential equations. Liao et al[4]. explore the possibility of positive solutions to a certain type of fractional differential equations with a single term delay. Sufficient requirements to ensure the solution's uniqueness are stated for the same class of equations later in [7] Delay differential equations (DDEs) or time-delay non-linear systems play a significant part in a variety of real life and physical phenomenon problems including time delays [22, 7, 11]. Few researchers have studied the FPDEs [5, 18] . However, the VFDPDEs have not been investigated yet. This paper investigates the implementation of ADM to solve a specific class of FVFDDEs. [2, 35, 34, 36, 29, 31].

Adomian proposed a novel approach to solve a few functional equations at the start of 1980 [25].The advantage of the Adomian decomposition approach is that it can handle a large class of differential and integral equations, both linear and nonlinear, with ease and convergence to the exact solution. The solution is broken down into a sequence of easily computed parts that quickly converge to the exact solution In.Refs. [1, 21, 26, 27, 20]. , The Adomian decomposition technique's theoretical treatment of convergence has been examined.

2. PRELIMINARIES

this section provides an overview of the fundamental concepts of fractional calculus that are related to present paper.

Definition 2.1. *The Riemann-Liouville variable order fractional integral operator of order* $m-1<\gamma(\varsigma,\nu)\leq m,\ \nu>0$ *of* $w(\varsigma,\nu)$*, is represented by the equation that follows* [35].

$$I_{\nu}^{\gamma(\varsigma,\nu)}w(\varsigma,\nu) = \frac{1}{\Gamma(\gamma(\varsigma,\nu))}\int_{0}^{\nu}(\nu-\rho)^{\gamma(\varsigma,\nu)-1}w(\varsigma,\nu)\,d\rho,\ \gamma>0,\ \varsigma>0$$

where $\nu > 0$.

Consequently, the following property is satisfied by the variable-order fractional integration:

$$I_{\nu}^{\gamma(\varsigma,\nu)}\nu^{\delta} = \begin{cases} \frac{\Gamma(\delta+1)}{\Gamma(\delta+\gamma(\varsigma,\nu)+1)}\nu^{(\delta+\gamma(\varsigma,\nu))} & ,\ \delta>-1 \\ 0 & ,\ otherwise \end{cases}$$

Definition 2.2. *The variable-order fractional derivative operator of* $w(\varsigma,\nu)$ *is provided as follows in the Caputo experience:*

$$\begin{aligned} {}_{0}^{C}D_{\nu}^{\gamma(\varsigma,\nu)}w(\varsigma,\nu) &= I_{\tau}^{n-\gamma(\varsigma,\nu)}D_{\nu}^{n}w(\varsigma,\nu) \\ &= \frac{1}{\Gamma(n-\gamma(\varsigma,\nu))}\int_{0}^{\nu}(\nu-\rho)^{n-\gamma(\varsigma,\nu)-1}\frac{\partial^{n}w(\varsigma,\rho)}{\partial\rho^{n}}\,d\rho \end{aligned}$$

For $m-1<\gamma(\varsigma,\nu)\le m,\quad \nu>0$ *and* $m\in Z^{+}$, *therefore, the following relation is utilized:*

$${}_{0}^{C}D_{\nu}^{\gamma(\varsigma,\nu)}\nu^{m} = \begin{cases} \frac{\Gamma(n+1)}{\Gamma n-\gamma(\varsigma,\nu)+1}\nu^{n-\gamma(\varsigma,\nu)} & ,\ m\le n\in N_{0} \\ 0 & ,\ otherwise \end{cases}$$

3. Fuzzy Set Theory

In order to understand fuzzy set theory, it is necessary to understand some of the fundamental ideas that are covered in this part. Presented here are a few key definitions pertaining to fuzzy numbers and fuzzy valued functions28 [28, 29, 30, 31, 32, 33]. The notation for the set of all real numbers is denoted by $\mathbb{R}$

A mapping $w:\mathbb{R}\to[0,1]$ with the subsequent characteristics is called a fuzzy number:

(1) w is upper semi-continuous.
(2) w is fuzzy convex, i.e $w(\mu t+(1-\mu)s)\ge min\{w(t),w(s)\}\ for\ all\ t,s\in\mathbb{R},\mu\in[0,1]$.
(3) w is normal,i.e.,$\exists\ t_0\in\mathbb{R}\ for\ which\ w(t_0)=1$.
(4) $suppw = t\in\mathbb{R}|w(t)>0$. is the support of the w, and its closer cl(suppw) is compact.

Let E represent the entire set of fuzzy numbers on $\mathbb{R}$. The $\beta-level$ set of a fuzzy number $w\in E\,0\le\beta\le1$, denoted by $[w]_{\beta}$, is defined as:

$$[w]_\beta = \begin{cases} t \in \mathbb{R} | w(t) \geq \beta \quad if \ \ 0 < \beta \leq 1. \\ cl(suppw) \qquad if \quad \beta = 0 \end{cases}$$

Definition 3.1. *In parametric form, a fuzzy number w is a pair* $(\underline{w}, \overline{w})$ *of functions* $\underline{w}(\beta), \overline{w}(\beta), 0 < \beta \leq 1$*, which meet the following criteria:*

(1) $\underline{w}(\beta)$*is a left continuous, bounded, non-decreasing function in* $(0,1]$*, while it is right continuous at 0.*

(2) $\overline{w}(\beta)$ *is a right continuous at 0 and a bounded non-increasing left continuous function in* $(0,1]$.

(3) $\underline{w}(\beta) \leq \overline{w}(\beta), 0 < \beta \leq 1$. *a crisp number* β *is expressed as follows:* $\underline{w}(\beta) = \overline{w}(\beta) = \beta, 0 < \beta \leq 1$.

Definition 3.2. *Let* $h : \mathbb{R} \to \mathbb{Q}$ *be a function with fuzzy values. If for a fixed, arbitrary* $s_0 \in \mathbb{R}$ *and* $\varepsilon > 0, \beta > 0$ *such that:*

$$|s - s_0| < \beta \Rightarrow d(h(s), h(s_0)) < \varepsilon$$

H is said to be continuous.

Definition 3.3. *A function h(t) is called the reference function of fuzzy number if and only if:*

(1) $h(-t) = h(t) \ (even \ function)$.

(2) $h(0) = 1$.

(3) $h(t) isdecreasingon [0, +\infty)$.

Definition 3.4. *Assume that* $\widetilde{h}(q)$ *is for any* $q \in [a,b]$ *and LR-fuzzy number. L for (left) and R for (right). It implies that for all* $q \in [a,b]$*:*

$$\mu_{\widetilde{h}(q)}(l) = \begin{cases} L\left(\frac{\Phi(q) - l}{\lambda(q)}\right) & if \ l \leq \Phi(q) \\ R\left(\frac{l - \Phi(q)}{p(q)}\right) & if \ l > \Phi(q) \end{cases}$$

Or symbolically:

$\widetilde{h}(q) = (\Phi(q), \lambda(q), p(q))_{LR}$

On the other hand, λ *and p are positive differentiable mappings from* $[0,\infty) to \ [0,1]$*, L and R are continuous strictly decreasing mappings from [a,b] to* R^+*.so that the fuzzy figure* $h\widetilde{(q)}$*. The* *degree of membership of $\Phi(q)$ to $h\widetilde{(q)}$ is equal to 1, and the spreads on the left and right are L(0)=1=R(0), $\lambda(q)$, and $p(q)$.*

The following equations describe the β-curves if the reference functions are fixed for all q and only their parameters depend on q: where the mapping from [a,b] to R is differentiable (Φ).

$$\left.\begin{aligned} h_\beta^-(q) &= \Phi(q) - L^-1(\beta)\lambda(q)). \\ h_1(q) &= \Phi(q). \\ h_\beta^+(q) &= \Phi(q) + R^-1(\beta)p(q) \end{aligned}\right\}$$

4. Description of the Method

Consider the FVFPDDES:

$$ {}_0^C D_\nu^{\delta(\varsigma,\nu)} \widetilde{w}(\varsigma,\nu) - \eta \frac{\partial^2 \widetilde{w}(\varsigma,\nu)}{\partial \varsigma^2} = F(\nu, \widetilde{w}(\varsigma,\nu), \widetilde{w}(\varsigma,\nu-k)) \qquad 0 \le \varsigma \le 1,\ 0 < \nu < \infty \tag{4.1}$$

Using the initial values

$$\widetilde{w}(\varsigma,0) = \widetilde{g}_0(\varsigma) \qquad \frac{\partial \widetilde{w}(\varsigma,\nu)}{\partial \nu} = \widetilde{g}_1(\varsigma) \tag{4.2}$$

where k is the delay term, g_0 *and* g_1 are fuzzy functions.

Problem eq(4.1)-(4.2) can be written in terms of upper and lower functions as:

$$ {}_0^C D_\nu^{\delta(\varsigma,\nu)} \overline{w}(\varsigma,\nu) - \eta \frac{\partial^2 \overline{w}(\varsigma,\nu)}{\partial \varsigma^2} = F(\nu, \overline{w}(\varsigma,\nu), \overline{w}(\varsigma,\nu-k)) \qquad 0 \le \varsigma \le 1,\ 0 < \nu < \infty \tag{4.3}$$

Considering the initial values

$$\overline{w}(\varsigma,0) = \overline{g_0}(\varsigma) \qquad \frac{\partial \overline{w}(\varsigma,\nu)}{\partial \nu} = \overline{g_1}(\varsigma) \tag{4.4}$$

And

$$ {}_0^C D_\nu^{\delta(\varsigma,\nu)} \underline{w}(\varsigma,\nu) - \eta \frac{\partial^2 \underline{w}(\varsigma,\nu)}{\partial \varsigma^2} = F(\nu, \underline{w}(\varsigma,\nu), \underline{w}(\varsigma,\nu-k)) \qquad 0 \le \varsigma \le 1,\ 0 < \nu < \infty \tag{4.5}$$

Considering the initial values

$$\underline{w}(\varsigma,0) = \underline{g_0}(\varsigma) \qquad \frac{\partial \underline{w}(\varsigma,\nu)}{\partial \nu} = \underline{g_1}(\varsigma) \tag{4.6}$$

Where ${}_0^C D_\nu^{\delta(\varsigma,\nu)}$ represents the variable fractional derivative of order $\delta(\varsigma,\nu)$, ν, $m-1 < \delta(\varsigma,\nu) \le m$ $m \in N^+$, and F is the partial differential operator, $\overline{w}$ and $\underline{w}$ are upper and lower solutions.

Now, equations (4.3) and (4.5) becomes:

(4.7) $${}_{0}^{C}D_{\nu}^{\delta(\varsigma,\nu)}\overline{w}(\varsigma,\nu) = F(\nu,\overline{w}(\varsigma,\nu),\overline{w}(\varsigma,\nu-k)) + \eta\frac{\partial^2\overline{w}(\varsigma,\nu)}{\partial\varsigma^2} \qquad 0 \le \varsigma \le 1,\ 0 < \nu < \infty$$

And

(4.8) $${}_{0}^{C}D_{\nu}^{\delta(\varsigma,\nu)}\underline{w}(\varsigma,\nu) = F(\nu,\underline{w}(\varsigma,\nu),\underline{w}(\varsigma,\nu-k)) + \eta\frac{\partial^2\underline{w}(\varsigma,\nu)}{\partial\varsigma^2} \qquad 0 \le \varsigma \le 1,\ 0 < \nu \le \infty$$

Apply the ADM to equations (4.5), we get:

(4.9) $$\mathscr{L}_{\nu}\overline{w}(\varsigma,\nu) = N\overline{w}(\varsigma,\nu)$$

Where $\mathscr{L}_{\nu} = {}_{0}^{C}D_{\nu}^{\delta(\varsigma,\nu)}$ and its inverse is $\mathscr{L}_{\nu}^{-1} = {}_{0}^{C}I_{\nu}^{\delta(\varsigma,\nu)}$ $N\overline{w}(\varsigma,\nu) = F(\nu,\overline{w}(\varsigma,\nu),\overline{w}(\varsigma,\nu-k)) + \eta\frac{\partial^2\overline{w}(\varsigma,\nu)}{\partial\varsigma^2}$, Applying $\mathscr{L}_{\nu}^{-1}$ to both sides of (4.7) we get:

(4.10) $$\mathscr{L}_{\nu}^{-1}\mathscr{L}_{\nu}\overline{w}(\varsigma,\nu) = \mathscr{L}_{\nu}^{-1}N\overline{w}(\varsigma,\nu), \qquad {}_{0}^{C}I_{\nu}^{\delta(\varsigma,\nu)\overline{w}(\varsigma,\nu)=\overline{w}(\varsigma,\nu)-\overline{w}(\varsigma,0)=\mathscr{L}_{\nu}^{-1}N\overline{w}(\varsigma,\nu)}$$

then

(4.11) $$\overline{w}(\varsigma,\nu) = \overline{w}(\varsigma,0) + \mathscr{L}_{\nu}^{-1}N\overline{w}(\varsigma,\nu)$$

following that, substitute $\overline{w}(\varsigma,\nu) = \sum_{n=0}^{\infty}\overline{w}_n(\varsigma,\nu)$ and $N\overline{w} = \sum_{m=0}^{\infty}A_m$ and the initial condition in (4.4)

(4.12) $$\sum_{m=0}^{\infty}\overline{w}_m(\varsigma,\nu) = \overline{g_0}(\varsigma) + \mathscr{L}_{\nu}^{-1}\sum_{m=0}^{\infty}\overline{A_m}$$

The recursive procedure is thus:

(4.13) $$\overline{w}_0(\varsigma,0) = \overline{g_0}(\varsigma), \qquad \overline{w}_{m+1}(\varsigma,\nu) = \mathscr{L}_{\nu}^{-1}A_m \quad m = 0,1,2,3,..$$

Using the Adomain formula [1], one could obtain the Adomain polynomial A_m where $N\overline{w}(\varsigma,\nu) = F(\nu,\overline{w}(\varsigma,\nu),\overline{w}(\varsigma,\nu-k)) + \eta\frac{\partial^2\overline{w}(\varsigma,\nu)}{\partial\varsigma^2}$ then the Adomian polynomials are:

$$A_0 = N(\overline{w}_0(\varsigma,\nu)) = \overline{w}_0(\varsigma,\nu)\frac{\partial}{\partial\varsigma}\overline{w}_0(\varsigma,\nu)$$

$$\overline{w}_1(\varsigma,\nu) = \mathscr{L}_{\nu}^{-1}A_0 = \mathscr{L}_{\nu}^{-1}(N(\overline{w}_0(\varsigma,\nu))) = \mathscr{L}_{\nu}^{-1}\left(\overline{w}_0(\varsigma,\nu)\frac{\partial}{\partial\varsigma}\overline{w}_0(\varsigma,\nu)\right)$$

$$A_1 = \frac{d}{d\psi}[N(\overline{w}_1(\varsigma,\nu) + \overline{w}_1(\varsigma,\nu)\psi]$$

$$\overline{w}_2(\varsigma,\nu) = \mathscr{L}_\nu^{-1}A_1 = \mathscr{L}_\nu^{-1}(\frac{d}{d\psi}N(\overline{w}_1(\varsigma,\nu)+\overline{w}_1(\varsigma,\nu)\,\psi)$$

$$A_2 = \frac{d^2}{d\psi^2}[N(\overline{w}_0(\varsigma,\nu)+\overline{w}_1(\varsigma,\nu)\,\psi+\overline{w}_2(\varsigma,\nu)\,\psi^2]$$

$$\overline{w}_3(\varsigma,\nu) = \mathscr{L}_\nu^{-1}A_2 = \mathscr{L}_\nu^{-1}(\frac{d^2}{d\psi^2}\overline{w}_0(\varsigma,\nu)+\overline{w}_1(\varsigma,\nu)\,\psi+\overline{w}_2(\varsigma,\nu)\,\psi^2)$$

$$\vdots$$

Hence, the N-terms that give an approximate solution is:

$\Phi_N(T) = \sum_{m=0}^{M-1}\overline{w}_n(\varsigma,\nu) \quad M \geq 1$

Where exact solution is $\overline{w}(\varsigma,\nu) = \lim_{M\to\infty}\Phi_M(t)$.

Apply the ADM to equation (4.8), we get:

$$\mathscr{L}_\nu \underline{w}(\varsigma,\nu) = N\underline{w}(\varsigma,\nu) \tag{4.14}$$

Where $\mathscr{L}_\nu = {}_0^C D_\nu^{\delta(\varsigma,\nu)}$ and its inverse is

$\mathscr{L}_\nu^{-1} = {}_0^C I_\nu^{\delta(\varsigma,\nu)}$ $N\underline{w}(\varsigma,\nu) = F(\nu,\underline{w}(\varsigma,\nu),\underline{w}(\varsigma,\nu-k)) + \eta\frac{\partial^2\underline{w}(\varsigma,\nu)}{\partial\varsigma^2}$

Applying $\mathscr{L}_\nu^{-1}$ to both sides of (4.8), we obtain:

$\mathscr{L}_\nu^{-1}\mathscr{L}_\nu\underline{w}(\varsigma,\nu) = \mathscr{L}_\nu^{-1}N\underline{w}(\varsigma,\nu)$

${}_0^C I_\nu^{\delta(\varsigma,\nu)}$ $\underline{w}(\varsigma,\nu) = \underline{w}(\varsigma,\nu) - \underline{w}(\varsigma,0) = \mathscr{L}_\nu^{-1}N\underline{w}(\varsigma,\nu)$.

Then

$$\underline{w}(\varsigma,\nu) = \underline{w}(\varsigma,0) + \mathscr{L}_\nu^{-1}N\underline{w}(\varsigma,\nu)) \tag{4.15}$$

following that, substitute $\underline{w}(\varsigma,\nu) = \sum_{m=0}^{\infty}\underline{w}_m(\varsigma,\nu)$ and $N\underline{w} = \sum_{m=0}^{\infty}A_m$ and the initial condition in (4.6)

$$\sum_{m=0}^{\infty}\underline{w}_m(\varsigma,\nu) = \underline{g_0}(\varsigma) + \mathscr{L}_\nu^{-1}\sum_{m=0}^{\infty}\underline{A_m} \tag{4.16}$$

The recursive technique is therefore:

$$\underline{w}_0(\varsigma,0) = \underline{g_0}(\varsigma)$$

$$\underline{w}_{m+1}\varsigma,\nu = \mathscr{L}_\tau^{-1}A_m \quad m = 0,1,2,3,..$$

Using the Adomian formula [1], we may get the Adomian polynomials A_m, where $N\underline{w}(\varsigma,\nu) = F(\nu,\underline{w}(\varsigma,\nu),\underline{w}(\varsigma,\nu-k))+\eta\frac{\partial^2 w(\varsigma,\nu)}{\partial\varsigma^2}$ then the Adomian polynomials are:

$$A_0 = N\left(\underline{w}_0\left(\varsigma,\nu\right)\right) = \underline{w}_0\left(\varsigma,\nu\right)\frac{\partial}{\partial\varsigma}\underline{w}_0\left(\varsigma,\nu\right)$$

$$\underline{w}_1\left(\varsigma,\nu\right) = \mathscr{L}_\nu^{-1}A_0 = \mathscr{L}_\nu^{-1}\left(N\left(\underline{w}_0\left(\varsigma,\nu\right)\right)\right) = \mathscr{L}_\nu^{-1}\left(\underline{w}_0\left(\varsigma,\nu\right)\frac{\partial}{\partial\varsigma}\underline{w}_0\left(\varsigma,\nu\right)\right)$$

$$A_1 = \frac{d}{d\Psi}[N(\underline{w}_1\left(\varsigma,\nu\right)+\underline{w}_1\left(\varsigma,\nu\right)\Psi]$$

$$\underline{w}_2\left(\varsigma,\nu\right) = \mathscr{L}_\nu^{-1}A_1 = \mathscr{L}_\nu^{-1}\frac{d}{d\Psi}[N(\underline{w}_1\left(\varsigma,\nu\right)+\underline{w}_1\left(\varsigma,\nu\right)\Psi$$

$$A_2 = \frac{d^2}{d\Psi^2}[N(\underline{w}_0\left(\varsigma,\nu\right)+\underline{w}_1\left(\varsigma,\nu\right)\Psi+\underline{w}_2\left(\varsigma,\nu\right)\Psi^2]$$

$$\underline{w}_3\left(\varsigma,\nu\right) = \mathscr{L}_\nu^{-1}A_2 = \mathscr{L}_\nu^{-1}\frac{d^2}{d\Psi^2}[N(\underline{w}_0\left(\varsigma,\nu\right)+\underline{w}_1\left(\varsigma,\nu\right)\Psi+\underline{w}_2\left(\varsigma,\nu\right)\Psi^2]$$

$$\vdots$$

Consequently, the following finite series provides the N-terms that approximate the solution:

$$\phi_M\left(T\right) = \sum_{m=0}^{M-1}\underline{w}_m\left(\varsigma,\nu\right) \quad M \geq 1$$

The exact solution is provided by $\underline{w}(\varsigma,\nu) = \lim_{M\to\infty}\phi_M(t)$

5. Illustrative Examples

In this part, the non-linear delay fuzzy variable order partial fractional differential equations will be solved using the ADM, and the outcomes produced by this method will be compared to the analytical solution.

Example 5.1. *The FVFDDES (4.1) with $\eta = 1$ and k=0.1 and subject to:*

$$\widetilde{w}\left(\varsigma,0\right) = 10\varsigma^2(1-\varsigma)^2 \qquad \frac{\partial}{\partial\nu}\widetilde{w}\left(\varsigma,0\right) = 0 \tag{5.1}$$

where

$$\begin{aligned} &f\left(\nu,\widetilde{w}\left(\varsigma,\nu\right),\widetilde{w}\left(\varsigma,\nu-k\right)\right) \\ &= 10\varsigma^2\left(1-\varsigma\right)^2\frac{\nu^2-\delta(\varsigma,\nu)}{\Gamma(3-\delta(\varsigma,\nu))}-20(6\varsigma^2-6\varsigma+1)(\nu^2+1)-10(\nu-0.1)^2\varsigma^2(1-\varsigma)^2 \end{aligned} \tag{5.2}$$

There is an exact solution for this particular problem as follows.

$w(\varsigma,\nu) = 10\varsigma^2(1-\nu)^2\left(\nu^2+1\right)$ *and* $\delta(\varsigma,\nu) = \frac{9}{5} - 0.005\cos(\varsigma\nu)\sin(\varsigma)$

First to derived a fuzzy for initial condition, Now we choose the left and right reference functions

Let $L(\beta) = e^{-\beta^2}$, $R(\beta) = e^{-|\beta|}$, *and let* $\lambda(\varsigma) = \frac{10\varsigma^2(1-\varsigma)^2}{7}$, $p(\varsigma) = \frac{10\varsigma^2(1-\varsigma)^2}{3}$.

We apply definition (3.4), we get:

$\underline{w}(\varsigma,0) = \left(1+\frac{1}{7}\sqrt{-\ln\beta}\right)10\varsigma^2(1-\varsigma)^2$ *and* $\overline{w}(\varsigma,0) = \left(1-\frac{\ln\beta}{3}\right)10\varsigma^2(1-\varsigma)^2$

Now equations (4.1) and (4.3) becomes:

$$ {}_0^C D_\nu^{\delta(\varsigma,\nu)}\underline{w}(\varsigma,\nu) = F(\nu,\underline{w}(\varsigma,\nu),\underline{w}(\varsigma\nu-0.1)) + \frac{\partial^2\underline{w}(\varsigma,\nu)}{\partial\varsigma^2} \tag{5.3}$$

With initial condition

$$\underline{w}(\varsigma,0) = \left(1+\frac{1}{7}\sqrt{-\ln\beta}\right)10\varsigma^2(1-\varsigma)^2 \tag{5.4}$$

and

$$ {}_0^C D_\rho^{\delta(\varsigma,\nu)}\overline{w}(\varsigma,\nu) = F(\nu,\overline{w}(\varsigma,\nu),\overline{w}(\varsigma,\nu-0.1)) + \frac{\partial^2\overline{w}(\varsigma,\nu)}{\partial\varsigma^2} \tag{5.5}$$

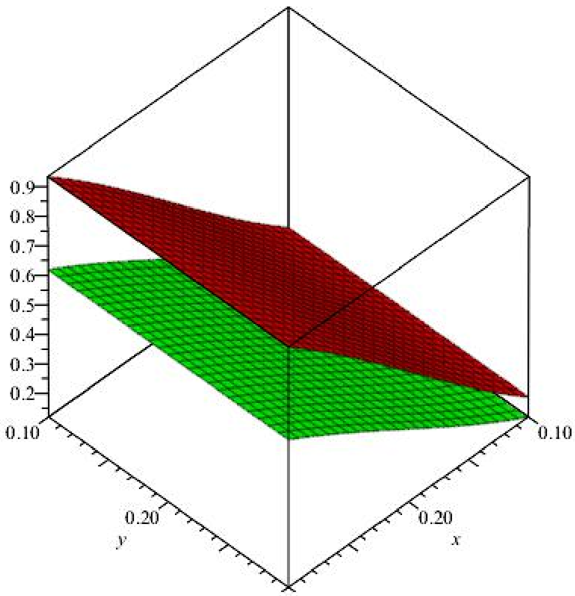
With initial condition

$$\overline{w}(\varsigma,0) = \left(1-\frac{\ln\beta}{3}\right)10\varsigma^2(1-\varsigma)^2 \tag{5.6}$$

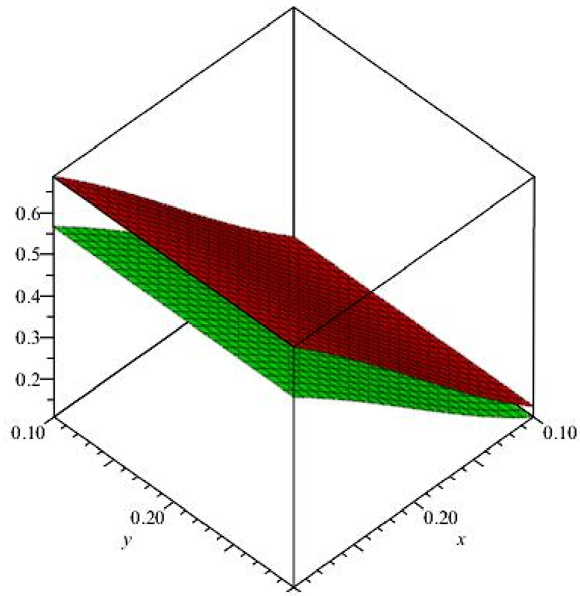
figures (1) show the ADM solution of example (5.1) for different values of $\delta(\varsigma,\nu)\, at\beta = 0.25\, 0.5, 0.75, 1$ respectively.

The absolute error levels at various values of ς are displayed in the following table (1), when $\nu = 0.1$ and $\beta = 1$.

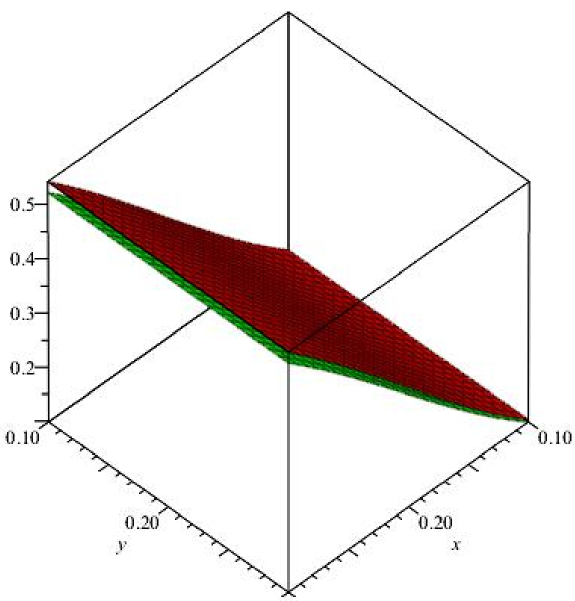
The absolute error levels at various values of ς are displayed in the following table (2), when $\nu = 0.01$ and $\beta = 1$.



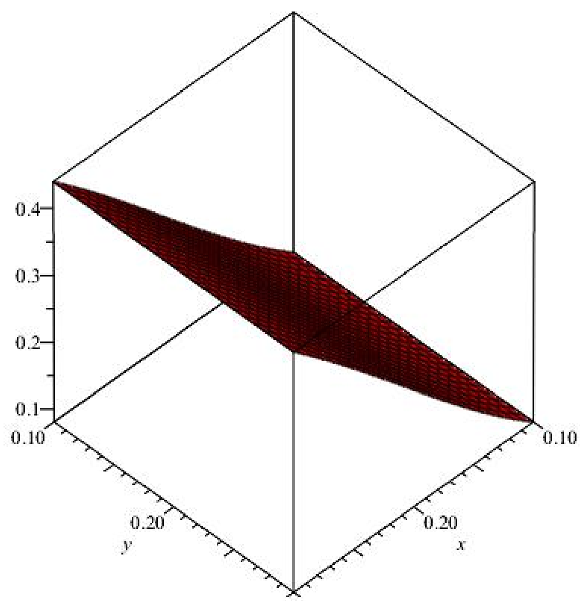
(A) at $\beta = 0.25$



(B) at $\beta = 0.5$



(C) at $\beta = 0.75$



(D) at $\beta = 1$

FIGURE 1. when $\delta(\varsigma, \nu) = \frac{9}{5} - 0.005\cos(\varsigma\nu)\sin(\varsigma)$

Red for Upper and Green for Lower

$(\varsigma\nu)$	Absolute error
$(0.01,0.1)$	9.800×10^{-6}
$(0.02,0.1)$	3.838×10^{-5}
$(0.03,0.1)$	8.447×10^{-5}
$(0.04,0.1)$	1.466×10^{-4}
$(0.05,0.1)$	2.233×10^{-4}
$(0.06,0.1)$	3.126×10^{-4}
$(0.07,0.1)$	4.127×10^{-4}
$(0.08,0.1)$	5.211×10^{-4}
$(0.09,0.1)$	6.355×10^{-4}
$(0.1,0.1)$	7.531×10^{-4}

TABLE 1. The absolute error of example (5.1) when $\nu = 0.1$

(ς,ν)	Absolute error
$(0.01,0.01)$	9.687×10^{-8}
$(0.02,0.01)$	3.531×10^{-7}
$(0.03,0.01)$	6.349×10^{-7}
$(0.04,0.01)$	6.534×10^{-7}
$(0.05,0.01)$	7.837×10^{-8}
$(0.06,0.01)$	2.275×10^{-6}
$(0.07,0.01)$	6.897×10^{-6}
$(0.08,0.01)$	1.516×10^{-5}
$(0.09,0.01)$	2.856×10^{-5}
$(0.1,0.01)$	4.883×10^{-5}

TABLE 2. The absolute error of example (5.1) when $\nu = 0.01$

Example 5.2. *The FVFDDES (4.1) with $\eta = 1$ and $k = 0.2$ and subject to:*

$$\widetilde{w}(\varsigma,0) = \frac{\partial}{\partial\nu}\widetilde{w}(\varsigma,0) = 5\varsigma(1-\varsigma) \tag{5.7}$$

where

$$f\left(\nu,\widetilde{w}\left(\varsigma,\nu\right),\widetilde{w}\left(\varsigma,\nu-k\right)\right)=5\varsigma\left(1-\varsigma\right)\frac{\nu^{1}-\delta(\varsigma,\nu)}{\Gamma(2-\delta(\varsigma,\nu))})-10\nu+5\varsigma(1-\varsigma)(\nu-0.2) \tag{5.8}$$

with an exact solution represented by:

$w\left(\varsigma,\nu\right)=5\varsigma(1-\varsigma)\left(\nu+1\right)$ *and* $\delta\left(\varsigma,\nu\right)=2-0.2\cos\left(\nu\right)\sin\left(\varsigma\right)$

first to derived a fuzzy for initial condition, Now we choose the left and right reference functions let $L\left(\beta\right)=e^{-\beta^{2}}$ $\quad R\left(\beta\right)=e^{-|\beta|}$ *and let*

$\lambda\left(\varsigma\right)=\frac{5\varsigma(1-\varsigma)}{7}$ *and* $p\left(\varsigma\right)=\frac{5\varsigma(1-\varsigma)}{3}$

We apply definition (3.5),we get:

$\underline{w}\left(\varsigma,0\right)=\left(1+\frac{1}{7}\sqrt{-\ln\beta}\right)5\varsigma(1-\varsigma)$ *and* $\overline{w}\left(\varsigma,0\right)=\left(1-\frac{\ln\beta}{3}\right)5\varsigma(1-\varsigma).$

Now equations (4.1)and (4.3) becomes:

$${}_{0}^{C}D_{\nu}^{\delta(\varsigma,\nu)}\underline{w}\left(\varsigma,\nu\right)=F\left(\nu,\underline{w}(\varsigma,\nu),\underline{w}\left(\varsigma,\nu-0.2\right)\right)+\frac{\partial^{2}\underline{w}\left(\varsigma,\nu\right)}{\partial\varsigma^{2}} \tag{5.9}$$

With initial condition

$$\underline{w}\left(\varsigma,0\right)=\left(1+\frac{1}{7}\sqrt{-\ln\beta}\right)5\varsigma(1-\varsigma) \tag{5.10}$$

and

$${}_{0}^{C}D_{\nu}^{\delta(\varsigma,\nu)}\overline{w}\left(\varsigma,\nu\right)=F\left(\nu,\overline{w}(\varsigma,\nu),\overline{w}\left(\varsigma,\nu-0.2\right)\right)+\frac{\partial^{2}\overline{w}\left(\varsigma,\nu\right)}{\partial\varsigma^{2}}. \tag{5.11}$$

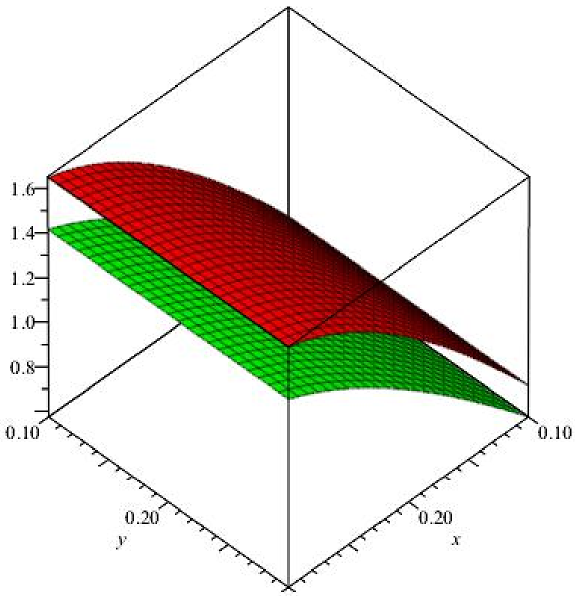
With initial condition

$$\overline{w}\left(\varsigma,0\right)=\left(1-\frac{\ln\beta}{3}\right)5\zeta\left(1-\varsigma\right) \tag{5.12}$$

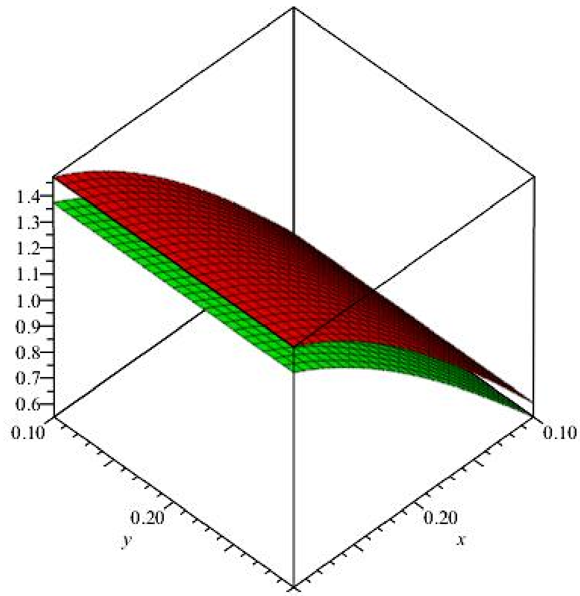
Figures (2) show the ADM solution of example (5.2) for different values of $\delta\left(\varsigma,\nu\right)$ at $\beta=0.25,0.5,0.75,1$ respectively.

The absolute error amounts at various values of ς are presented in the following table (3), when $\nu=0.01$ and $\beta=1$.

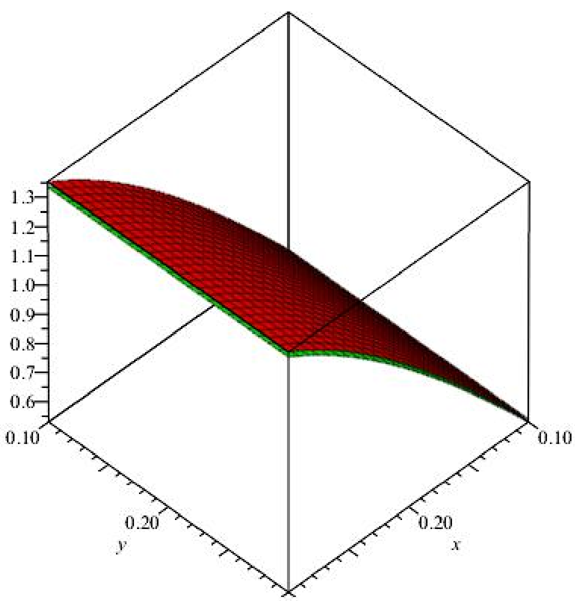
The absolute error amounts at various values of ς are presented in the following table (4), when $\nu=0.001$ and $\beta=1$.



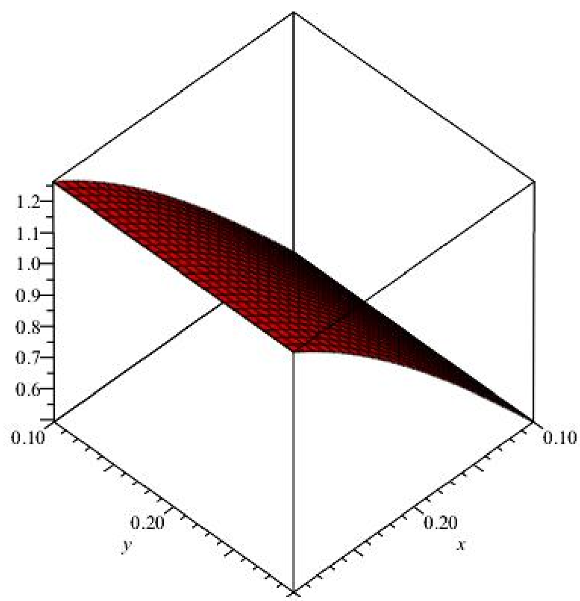
(A) at $\beta = 0.25$



(B) at $\beta = 0.5$



(C) at $\beta = 0.75$



(D) at $\beta = 1$

FIGURE 2. when $\delta(\varsigma, \nu) = 2 - 0.2\cos(\nu)\sin(\varsigma)$

Red for Upper and Green for Lower

(ς, ν)	Absolute error
$(0.01, 0.01)$	1.863×10^{-6}
$(0.02, 0.01)$	9.883×10^{-4}
$(0.03, 0.01)$	3.00×10^{-3}
$(0.04, 0.01)$	6.00×10^{-3}
$(0.05, 0.01)$	9.00×10^{-3}
$(0.06, 0.01)$	1.40×10^{-2}
$(0.07, 0.01)$	1.90×10^{-2}
$(0.08, 0.01)$	2.50×10^{-2}
$(0.09, 0.01)$	3.20×10^{-2}
$(0.1, 0.01)$	3.9×10^{-2}

TABLE 3. The absolute error of example (5.2) when $\nu = 0.01$

(ς, ν)	Absolute error
$(0.01, 0.001)$	4.474×10^{-4}
$(0.02, 0.001)$	2.00×10^{-3}
$(0.03, 0.001)$	4.00×10^{-3}
$(0.04, 0.001)$	7.00×10^{-3}
$(0.05, 0.001)$	1.20×10^{-2}
$(0.06, 0.001)$	1.70×10^{-2}
$(0.07, 0.001)$	2.20×10^{-2}
$(0.08, 0.001)$	2.90×10^{-2}
$(0.09, 0.001)$	3.60×10^{-2}
$(0.1, 0.001)$	4.3×10^{-2}

TABLE 4. The absolute error of example (5.2) when $\nu = 0.001$

6. CONCLUSIONS

The aim of this study is to investigate the solution of a certain class of FVFDDEs using (ADM). The proposed method is a widely recognized approach for solving non-linear problems without making linearization or dercretization. The approximate solution can be simply computed as a components of a convergent series. The results indicate that the (ADM) has an excellent agreement with the closed form of the FVFDDEs.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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