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ASSESSING BLACK RUST SPREADING IN WHEAT PRODUCTION USING MATHEMATICAL MODEL

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Abstract. This paper presents and examines a model for a black rust (Puccinia graminis) epidemic in wheat. The spread of black rust can lead to significant yield losses, adversely affecting food security and agricultural economies. Furthermore, controlling the epidemic requires substantial resources, posing challenges for sustainable agricultural practices. The model used in this research comprises three compartments representing susceptible, infected, and recovered wheat. Initially, we will address the well-posedness of the proposed model, focusing on classical results related to existence, positivity, and boundedness. For the analysis of local stability, we identify two equilibria: the disease-free equilibrium and the endemic equilibrium. The local stability of each equilibrium is shown to depend on the basic reproduction number. Numerical simulations are conducted, demonstrating that the numerical results align well with the theoretical findings.

Keywords: mathematical model; basic reproduction number; local stability; black rust; wheat.

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1. INTRODUCTION

Black rust, also known as stem rust, is a devastating fungal disease caused by Puccinia graminis f. sp. tritici that primarily affects cereal crops such as wheat, barley, and rye. This pathogen has historically posed a significant threat to global food security due to its capacity to cause severe yield losses [1]. Beyond immediate yield loss, the weakened plants become more susceptible to other biotic and abiotic stresses, further compounding the adverse effects on crop health and productivity [10]. The disease manifests as reddish-brown pustules on the stems and leaves of infected plants, which eventually turn black, hence the name ”black rust” [2]. The life cycle of P. graminis involves both sexual and asexual reproduction, allowing for high genetic variability and adaptability to different environments and host resistance genes [3]. The presence of alternate hosts, such as barberry (Berberis spp.), plays a crucial role in the pathogen’s epidemiology by facilitating the sexual stage of its life cycle [4]. Recent outbreaks of black rust have been particularly alarming due to the emergence of new virulent strains, such as the Ug99 lineage, which can overcome previously resistant wheat varieties [5]. The spread of Ug99 and its variants across Africa, the Middle East, and into Asia underscores the urgent need for integrated disease management strategies that include breeding for durable resistance, monitoring pathogen populations, and implementing effective agronomic practices [6, 7].

Climate change and rising temperatures significantly influence the epidemiology of black rust. Higher temperatures and altered precipitation patterns can create favorable conditions for the growth and spread of P. graminis. Studies have shown that warmer climates can accelerate the development of the fungus, increase its reproductive cycles, and expand its geographical range [8]. Additionally, climate change may impact the effectiveness of resistance genes in crops, necessitating continuous monitoring and adaptation of breeding programs [9]. Understanding the complex interactions between climate change and black rust dynamics is essential for predicting future outbreaks and developing sustainable disease management strategies. Research into the biology and epidemiology of black rust, as well as the development of resistant crop varieties, is crucial for mitigating the impact of this disease on global agriculture. Management strategies must be multifaceted, incorporating resistant crop varieties, cultural practices, and chemical controls. Breeding programs aim to develop cereal varieties with enhanced resistance to black rust, while cultural practices such as crop rotation, removal of volunteer plants, and optimal planting densities help reduce the disease's spread [11]. Additionally, fungicides remain an important tool in managing black rust, especially in regions where the disease is endemic [12]. Given the critical role cereals play in global food security, continued research and innovation in managing black rust are imperative. Advances in genetic and molecular biology provide new opportunities to understand the mechanisms of resistance and susceptibility, paving the way for the development of more robust cereal varieties. By enhancing our understanding of Puccinia graminis and its interaction with temperature when it comes to cereal crops, we can better safeguard these essential plants and ensure sustainable food production for the future [7, 11].

Current Situation and Economic Impact Black rust has significantly impacted wheat cultivation worldwide, especially in regions like Central and West Asia and North Africa (CWANA). Wheat is one of the most cultivated cereal crops globally and is highly susceptible to stem rust, the disease has caused severe economic damage in these regions, leading to substantial yield losses and financial burdens on farmers. The researchers provide a comprehensive overview of the disease's impact on both cultivated and wild barley, emphasizing the necessity for effective disease management strategies. The study underscores the urgency of implementing robust control measures to prevent the spread of the disease and protect crop yields [13]. The resistance of various wheat genotypes to stem rust has been studied extensively. Omrani and Roohparvar (2023) used GGE biplot analysis to reveal that stem rust can cause up to 100 % yield loss under epidemic conditions, underscoring the critical need for breeding resistant wheat varieties. The research highlights the importance of continuous monitoring and genetic analysis to mitigate the impact of this disease on wheat production. Their findings indicate that breeding programs should prioritize the development of resistant genotypes to ensure the sustainability of wheat production in rust-prone areas [14]. A biometrical analysis conducted of resistance to stem rust in winter wheat genotypes, focusing on the genetic variability and resistance levels in different wheat lines emphasizes the importance of selecting resistant varieties to combat the widespread damage caused by black rust. The researchers identified several genotypes with high levels of resistance, which could be utilized in breeding programs to enhance the resilience of wheat crops against stem rust [15]. Another study on the occurrence of Karnal bunt and black point disease of wheat in the northern part of Haryana highlights the complex nature of disease interactions in wheat and underscores the need for integrated disease management approaches [16].

Changing Climate and Disease Dynamics The prevalence and severity of black rust are expected to increase with climate change. A recentstudy explored the perceptions of disease and pest status in the Indian Himalayas under changing climatic conditions, reporting an increase in black rust occurrences with significant damage to hill crops [17]. The study highlights that climate change may exacerbate the prevalence and severity of black rust, making it a critical issue for future crop management strategies. The potential damage from black rust in this region is estimated to be around 30%, affecting food security and agricultural sustainability. The need for adaptive strategies to mitigate the impact of climate change on plant diseases, including the development of climate-resilient crop varieties and improved disease monitoring systems. In Western Europe, the resurgence of wheat stem rust infections has been noted, particularly with a large-scale outbreak in the UK. The authors attribute the increasing severity of outbreaks to climatic factors and emphasize the need for robust disease management practices. The study details how changing weather patterns, such as increased humidity and temperature fluctuations, create favorable conditions for the proliferation of stem rust [18]. The implementation of integrated disease management approaches, combining genetic resistance, cultural practices, and fungicide applications to effectively control the disease and prevent future outbreaks.

Advances in Disease Management A research reviewed emerging and reemerging plant pathogens affecting major food crops, including wheat highlights the threats posed by black rust to food security and discuss various strategies for disease control and management [19].This study underscores the need for comprehensive approaches to mitigate the impact of black rust on global food supplies. Investigating the role of seed priming with selected biocontrol agents in the eco-safe management of leaf rust disease in wheat found that biocontrol agents significantly reduce the incidence of rust diseases, including black rust, suggesting a potential strategy for sustainable disease management [20]. The research demonstrates that seed priming with biocontrol agents can enhance the natural defense mechanisms of plants, providing an environmentally friendly alternative to chemical fungicides. Lastly, a study on the physiological traits related to stem rust disease resistance in elite wheat lines. Their findings contribute to the understanding of how specific physiological traits can be harnessed to improve disease resistance in wheat [15].

FIG. 1. Pictures of Black rust affecting wheat [30].

The images provide a detailed visual representation of the pathogenic effects of Puccinia graminis, on wheat (Triticum aestivum). The infected wheat stalks exhibit prominent dark, elongated pustules along the stems, leaves, and grains, which are indicative of the disease. These pustules contain large quantities of urediniospores, which rupture the plant tissue, leading to significant damage. The affected wheat plants display considerable deterioration, with reduced vigor and compromised grain quality and yield. These images underscore the critical importance of implementing robust disease management strategies to mitigate the impact of black rust on wheat production [21, 22, 23, 24, 25, 26].

Infectious disease models were first used to understand the temporal dynamics of an epidemic and then to apply a therapeutic or control strategy against infectious diseases. Mathematical models are being used more and more frequently in medicine and even in biology, in a growing number of fields of application and increasingly varied areas of application. Formalising complex biological phenomena can be used to evaluate hypotheses and provide elements of understanding or prediction.

A mathematical model uses equations to represent a simplified view of reality. In particular, modelling in epidemiology is at the crossroads of epidemiology, medicine, biology and mathematics. The primary motivation at the outset was to study human-to-human contagion. Depending on the pathology being studied and its complexity, a greater or lesser number of variables and factors are used in communicable diseases; the central paradigm is human-to-human contagion.

Nowadays, realistic modeling is an increasing requirement, which leads to greater complexity in models to address ever more complex questions. This evolution stems from the sophisticated nature of the questions addressed and the increase in both quantity and precision of the data collected. This paper aims to provide a novel model that accurately replicates the spread of black rust in a wheat field. The objective of this model is to comprehend the intricacies of this contagious illness in order to formulate effective measures to eliminate or, at the absolute least, minimise its transmission.

2. FORMULATION OF THE MATHEMATICAL MODEL, POSITIVITY AND BOUNDEDNESS

2.1. Mathematical model. In this paper, we consider a system of WB_RR with three compartments to describe the spread of Black Rust in wheat field.

In defining an epidemic, the time interval considered is generally a few weeks to a few months. During this period, demographic variations in the density of the susceptible wheat plants are negligible. For the purposes of studying black rust (Puccinia graminis) spread, we assume that the size of the wheat host population remains constant. This implies that plant mortality is offset by new plant growth, suggesting that the growth rate is equal to the mortality rate.

Assumptions and Influences

Constant Host Population When examining black rust dynamics, we assume a stable population of wheat plants in the field. Mortality due to disease and other factors is balanced by the growth of new plants, maintaining a steady population size.

Infectivity and Population Density The total population density of wheat plants significantly influences the infectivity of black rust. Higher plant density can lead to a more rapid spread of the fungus, as spores have a greater chance of encountering susceptible hosts. Conversely, the infectivity rate of black rust does not affect the overall population density of wheat plants directly. This is similar to how measles spreads in human populations where the number of susceptible individuals affects the spread, but the spread does not directly alter the total population.

Epidemic Time Frame During the few weeks to few months that define an epidemic period, variations in the number of susceptible wheat plants are minimal. This period allows for a focused study on the factors contributing to the spread and control of black rust without the confounding effects of significant demographic changes.

The compartement W: The number of Wheat infected with the Puccinia graminis.

The coupartement B_R: The number of Wheat that have survived the infection and have developed resistance, or, are no longer capable of spreading the Puccinia graminis.

The compartment R: The number of Wheat that have survived the infection and have developed resistance, or are no lager capable of spreading the disease black rust.

$$
\begin{cases}
\frac{dW(t)}{dt} = \mu N - \beta \frac{W(t)B_R(t)}{N} - \mu W(t), \\
\frac{dB_R(t)}{dt} = \beta \frac{W(t)B_R(t)}{N} - (\gamma + \mu)B_R(t), \\
\frac{dR(t)}{dt} = \gamma B_R(t) - \mu B_R(t) - \mu R(t),
\end{cases}
\tag{2.1}
$$

Where

$N = W + B_R + R$ is the total size of the population. The arrival of newborns in W class at the rate μN. The birth rate is equal to the death rate, so the population size N is constant.

β: Transmission rate, which might depend on factors such as the density of the wheat and environmental conditions like humidity and temperature, which affect spore viability and dispersal.

μ: Rate of natural mortality .

γ: Recovery rate, the rate at which infectious wheat become recovered or die, thus removing them from the pool of infectious individuals.

Obtaining the equations of the system from the division 2.1 by N and by posing $\mathscr{W} = \frac{W}{N}$, $\mathscr{B}_R = \frac{B_R}{N}$ and $\mathscr{R} = \frac{R}{N}$, the following system

$$\left\{ \begin{array}{l} \frac{d\mathscr{W}(t)}{dt} = \mu - \beta \mathscr{W}(t)\mathscr{B}_R(t) - \mu \mathscr{W}(t), \\ \frac{d\mathscr{B}_R(t)}{dt} = \beta \mathscr{W}(t)\mathscr{B}_R(t) - (\gamma + \mu)\mathscr{B}_R(t), \end{array} \right. \tag{2.2}$$

With initial condition $(\mathscr{W}(0), \mathscr{B}_R(0))$

2.2. Positivity and boundedness of solutions.

Given that our research focuses on population dynamics, we shall demonstrate that all variables in the model are both positive and bounded. Initially, we will make the assumption that all the parameters in our model possess a positive value.

Theorem 1. *The solution $(\mathscr{W}(t), \mathscr{B}_R(t))$ of the model (2.2) are positive for all $t > 0$ with positive initial condition $(\mathscr{W}(0), \mathscr{B}_R(0))$ in $\mathbf{R}^2$.*

Proof. We have

$$\left\{ \begin{array}{l} \left.\frac{d\mathscr{W}(t)}{dt}\right|_{\mathscr{W}(t)=0} = \mu \geqslant 0 \\ \left.\frac{d\mathscr{B}_R(t)}{dt}\right|_{\mathscr{B}_R(t)=0} = 0 \end{array} \right.$$

Lemma 2 in [14] ensures that any solutions originating from $\mathbb{R}^2_+$ with positive beginning conditions will be positive. ■

Remark 2. *The set $M = \{(\mathscr{W}, \mathscr{B}_R)/\mathscr{W} \geq 0; \mathscr{B}_R \geq 0; \mathscr{W} + \mathscr{B}_R \leq 1\}$ is positively invariant.*

3. BASIC REPRODUCTION NUMBER (B.R.N) AND STABILITY ANALYSIS

3.1. Basic Reproduction Number (B.R.N).

The basic reproduction number: In the context of agriculture, car the basic reproduction number, R_0, could be adapted to describe the spread of plant disease (Black rust in our case) or pest within a crop population. It would represent the average number of new cases (i.e., infected plants or newly colonized plants by pests) that one diseased or infested plant would produce in a population of entirely susceptible plants, assuming no control measures are in place to prevent the spread.

This concept is crucial for understanding and managing agricultural diseases and pests to prevent Widespread damage to crops.

Mathematically, the basic reproduction number is defined as the spectral radius of the nextgeneration matrix FV^{-1} [28], expressed as $R_0 = \rho\left(FV^{-1}\right)$. Here, F is the non-negative matrix representing the new infection terms, and V is the matrix associated with the transition of infections.

In our model,

$$R_0 = \frac{\beta}{\gamma+\mu}.$$

In fact, this quantity is the product of the contact rate multiplied by the average period of infection adjusted by deaths $\frac{1}{\gamma+\mu}$.

3.2. Stability analysis.

In this subsection, we show the conditions on which we have local stability.

Theorem 3. *Let* $(\mathscr{W}(t), \mathscr{B}_R(t))$ *the solution of 2.2 in M. If* $R_0 \leq 1$ *or* $\mathscr{W}(0) = 0$*, then any solution starting in M converges to disease-free equilibrium* $(1;0)$*. If* $R_0 > 1$*, then any solution with* $\mathscr{B}_R(0) > 0$ *converges towards endemic equilibrium* $(\mathscr{W}^*, \mathscr{B}_R^*) = \left(\frac{1}{R_0}, \frac{\gamma(R_0-1)}{\beta}\right)$.

4. NUMERICAL SIMULATION

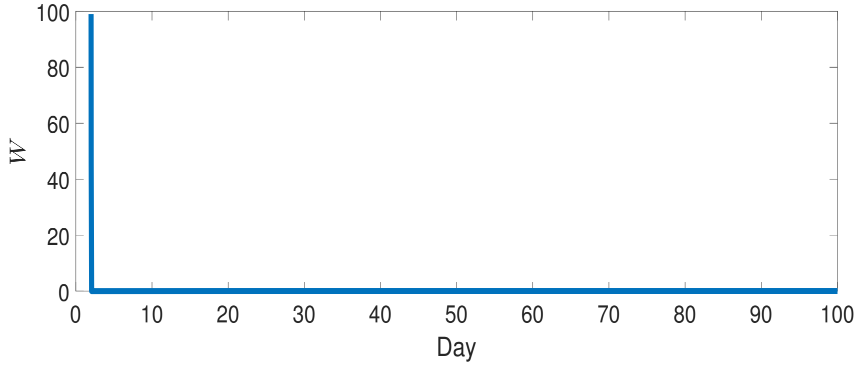

FIG. 2. Simulation of W when $R_0 = 3.5 > 1$

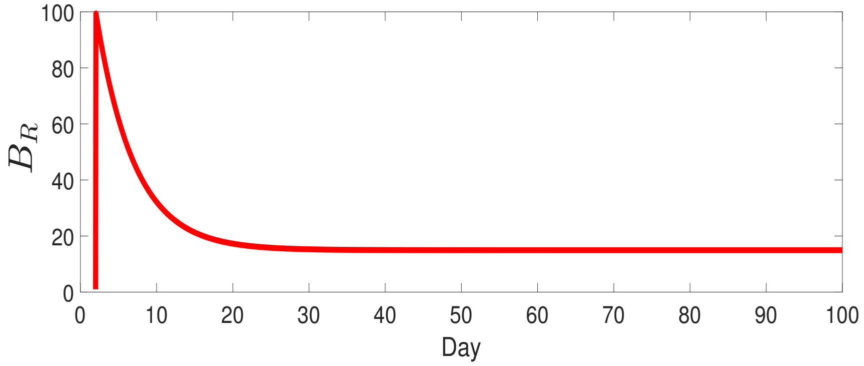

FIG. 3. Simulation of B when $R_0 = 3.5 > 1$, the disease is likely to spread rapidly, potentially leading to an outbreak that could affect a significant portion of the crop.

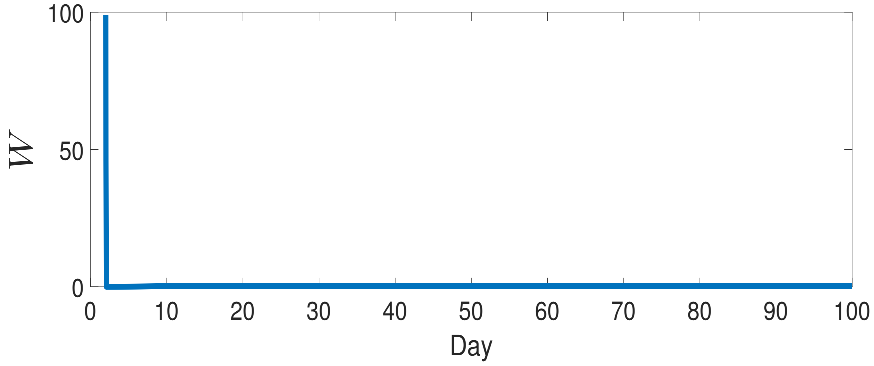

FIG. 4. Simulation of W when $R_0 = 0.95 < 1$

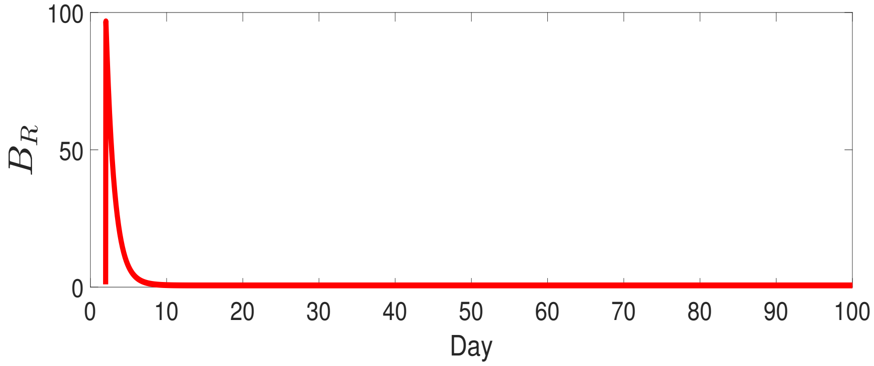

FIG. 5. Simulation of B when $R_0 = 0.95 < 1$, the infection will likely diminish over time as it fails to establish itself in enough hosts to maintain its presence in the coop population.

Simulation of B when $R_0 = 3.5 > 1$, suggest that the pathogen Puccinia graminis is likely to spread rapidly, potentially leading to a significant outbreak that could impact a considerable portion of the wheat crop. Such an outbreak poses substantial economic risks, as it threatens not only the yield and quality of the harvest but also increases the financial burden on farmers due to the heightened need for disease management and control measures. This situation could lead to significant financial losses and market instability. The broader agricultural sector may experience fluctuations in wheat prices and supply shortages, affecting both local and global markets. Therefore, implementing effective preventive and responsive strategies is crucial to mitigate these economic impacts and maintain the stability of wheat production.

Simulation of B when $R_0 = 0.95 < 1$, indicate that the spread of the pathogen Puccinia graminis is likely to be contained, reducing the likelihood of a significant outbreak. This lower transmission rate implies that the disease will not infect a large portion of the wheat crop, which has favorable economic implications. Farmers can expect more stable yields and reduced costs associated with disease management and control measures. Consequently, this stability helps to maintain market prices and ensures a steady supply of wheat, benefiting both local and global markets. Effective monitoring and early intervention strategies can further reinforce this positive outcome, safeguarding the agricultural economy from potential disruptions caused by black rust.

5. CONCLUSION

This paper presents and examines a model for a black rust epidemic in wheat, comprising three compartments: susceptible, infected, and recovered wheat. Our initial focus was on the well-posedness of the proposed model, addressing classical results related to existence, positivity, and boundedness. For local stability analysis, we identified two equilibria: the disease-free equilibrium and the endemic equilibrium, with the stability of each equilibrium shown to depend on the basic reproduction number. Numerical simulations were conducted, and the results were found to be in good agreement with the theoretical findings.

As a future direction, Given the significant threat posed by black rust to wheat production, ongoing research and development of resistant varieties remain crucial. Genetic studies and breeding programs should focus on incorporating multiple resistance genes to enhance durability and effectiveness against diverse rust strains. Additionally, integrated pest management practices that combine cultural, biological, and chemical controls can provide a holistic approach to managing black rust. Moreover, global collaboration and information sharing are essential to track and respond to the evolving dynamics of black rust, especially under changing climatic conditions. Countries and research institutions should work together to develop early warning systems, share best practices, and coordinate efforts to contain outbreaks swiftly. Black rust remains a significant threat to global wheat production, with substantial economic and agricultural impacts. Effective management strategies, including the development of resistant crop varieties and continuous monitoring, are essential to mitigate the effects of this disease. Additionally, understanding the implications of climate change on the dynamics of black rust will be crucial for future agricultural planning and food security. The integration of genetic resistance, cultural practices, and innovative disease management approaches will be key to safeguarding wheat crops and ensuring sustainable food production.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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