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THE INFLUENCE OF HUNTING COOPERATION, AND ANTI-PREDATOR BEHAVIOR ON AN ECO-EPIDEMIOLOGICAL MODEL WITH HARVEST

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Abstract: In this paper, an eco-epidemiological model consisting of a diseased predator with hunting cooperation and anti-predator property is formulated and studied as a three-dimensional system of ordinary differential equations. The solution's properties such as existence, uniqueness, and bounded are discussed. The conditions for the extinction of populations and the existence of equilibria are found, and the local and global stabilities are investigated. The possibility of the occurrence of local bifurcation was also studied. The conditions of occurrence of Hopf bifurcation are determined. To study the global dynamics and how changing parameters affect the system's asymptotic behavior, numerical simulation has been used.

Keywords: eco-epidemiological; hunting cooperation; anti-predator; harvest; Hopf- bifurcation.

2020 AMS Subject Classification: 92D40, 34D20, 37G10.

1. INTRODUCTION

The predation process assumes an essential part in advancing life evolution and maintaining ecological balance and biodiversity [1]. The interaction between prey and predator is an important topic of research in the study of ecological communities. This interaction varies in nature due to existence of infectious diseases that affect some or all species. Understanding the dynamics of prey-predator pathogens requires the use of mathematical modeling to formulate the model and

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then analyze the proposed model where one or more of the main populations become infected with an infection. The term eco-epidemiological model is used to describe models that incorporate diseases in ecological communities [2]. Anderson and May [3] introduced the first eco-epidemiological model including an infectious disease in prey. Later, Eco-epidemiological models incorporating a variety of biological factors were presented and investigated by several researcher; see [4-11]. It has been observed that the spread of diseases among a population is the main reason for the species extinction.

Prey-predator models have been used to explain a wide range of animal behavior, including the hunting and predation behaviors exhibited by predators and prey [12-14]. Straightforward of mathematical, ecological, and eco-epidemiological models, many studies have studied the implications of intra-species cooperation. For illustration, cooperative hunting can change how predator-prey models behave and may result in intricate patterns with several periodic cycles [15-22]. Therefore, studying and analyzing species interactions and population dynamics requires taking into the function of cooperation in ecological and eco-epidemiological systems. In recent years, various predator-prey mathematical models with hunting cooperation and the presence of disease have been proposed and studied [23-27].

Prey-predator interactions to avoid prey extinctions were studied, taking prey behavior in defending against predation pressure, called anti-predator behavior. Anti-predator behavior, which is a natural response of the prey, occurs when the prey is threatened, one at the expense of certain body parts [28-31].Most of the above studies use mathematical modeling to consider the impact of the hunting cooperation capability of the predators and the anti-predator capability of the prey on the prey-predator model dynamics separately.

In addition, one of the most important factors in population ecology is the effect of harvesting a natural population. Harvesting means a reduction of the population due to hunting or capturing individuals. It harms the harvested population size. Consequently, it is important to understand the effect of harvesting on the multi-species ecological systems dynamics, for example [2,5,32]. Furthermore, the effect of harvesting a particular species on its dynamic behavior is also studied. The existence of harvesting on some interacting species is beneficial from both ecological as well as economic. Since the pioneering work by Clark [33] that investigated the role of harvesting, few studies of the effect of harvesting on the dynamics of prey-predator systems in the existence of hunting cooperation and anti-predator properties have been done using different types of harvesting functions, for example, see the recent work [34] and the references therein.

The present research attempts to create and investigate an eco-epidemiological model of prey-predator that incorporates anti-predator, cooperative hunting, and harvesting. We present the mathematical formulation in the section that follows. In section 2, we examine the boundedness of the model and the properties of it's solution. In section3. looks into the model's equilibria and local stability (LS) analysis. Persistence is examined in Section 4. Section 5 examines the system's global stability (GS). Hopf-bifurcation (HB) analysis and the stability of limit cycles are covered in Sections 6 and 7. Local bifurcation (LB) (such as saddle-node(SNB), pitchfork(PB), and a transcritical bifurcation (TB)) near the equilibrium points (EPs) of system (1) are investigated. The numerical results are provided to support our theoretical analysis in Section 8.

2. MODEL FORMULATION

An eco-epidemiological system with a prey-predator interaction that involves an infectious disease in the predator population is offered and examined. There are two population classes within the predator population: the susceptible and the infected predator classes, illustrated by $S(t)$ and $I(t)$ appropriately. In comparison, $X(t)$ indicates the density of the prey population. The identified system is mathematically expressed by the following hypotheses.

- The disease is disseminated only within the predator population. In addition, the disease in predator limits their ability to hunt prey.
- The susceptible predator is expected to consume the prey depending on the Lotka-Volterra functional response. In the predator's absence, the prey population grows logistically.
- Predators exhibit cooperative hunting to capture and secure prey efficiently. The cooperation term can enhance the predator population attack rate $c_1 > 0$ to become $(c_1 + a_1 S)$, where $a_1 \geq 0$ represents the predator's cooperation in hunting,
- An external force imposes harvesting on the prey population.
- Prey has an anti-predator ability that decreases predation.

Therefore, the following nonlinear first-order differential equations system may describe the dynamics of the given eco-epidemiological system.

$$\begin{aligned}
\frac{dX}{dT} &= r\left(1 - \frac{X}{k}\right)X - (c_1 + a_1 S)XS - qEX, \\
\frac{dS}{dT} &= e_1(c_1 + a_1 S)XS - a_2 XS - \beta SI - d_1 S, \qquad (1) \\
\frac{dI}{dT} &= \beta SI - d_2 I,
\end{aligned}$$

where $X(0) = X_0 \geq 0,\ S(0) = S_0 \geq 0$, and $I(0) = I_0 \geq 0,$ the system's initial condition of the system (1), and all parameters are nonnegative and may be understood from Table 1.

Table 1: The description of the parameter.

Parameters	Description
r	The intrinsic growth in the prey population
k	Environmental carrying capacity
a_1	Hunting cooperation rate
c_1	The predation rate of prey
b_1	Hunting cooperative effort between predators.
e_1	The conversion rate of devouring prey by predator
a_2	The anti-predator rate
β	The infection rate
d_1	The mortality rate of susceptible predators in their native environment
d_2	The mortality rate of infected predator
E, q	Harvest rate

Therefore, the system (1) has the following domain

$$R_+^3 = \{(X, S, I) \in R^3, X \geq 0, S \geq 0, I \geq 0\}|$$

System (1) has a unique solution due to the continuity of its interaction functions along with the continuity of its partial derivatives. Consequently, these functions are Lipschitizian on R_+^3 . In the following theorem, the bound of the solution is established.

Theorem 1. All solutions to the system (1) are uniformly bounded.

Proof. Let $W_1 = X + S + I$, then $\frac{dW_1}{dT}$ can be written as

$$\frac{dW_1}{dT} \leq r\left(1 - \frac{X}{k}\right)X - qEX - d_1S - d_2I \leq \frac{rk}{4} - \mu W_1,$$

here $\mu = min\{qE, d_1, d_2\}$, therefore, for $T \to \infty$, we have $W_1 \leq L_1$, where $L_1 = \frac{rK}{4\mu}$.

Therefore, all the solutions are uniformly bounded.

3. Analysis of the Existence and Local Stability of Equilibrium Points

The system has a maximum of four nonnegative biologically potential EPs. The conditions for the existence of each one are specified in this section, and then their stability analysis is studied.

- The total extinction equilibrium point (TEEP), $\check{P}_0 = (0,0,0)$, exists without restriction.
- The axial equilibrium point (AEP), $\hat{P}_1 = \left(\hat{X}, 0,0\right)$, where $\hat{X} = \frac{k(r-qE)}{r}$, exists provided that

$$qE < r \tag{2}$$

Note that, condition (2) represents the survival condition of the prey species.

- The predator-free equilibrium point (PFEP), $\bar{P}_2 = (\bar{X}, \bar{S}, 0)$, where

$$\bar{X} = \frac{d_1}{e_1(c_1+a_1\bar{S})-a_2}, \tag{3}$$

while $\bar{S}$ is a positive root of the following 3^{rd}order polynomial equation:

$$B_3S^3 + B_2S^2 + B_1S + B_0 = 0, \tag{4}$$

where

$$B_3 = ka_1^2e_1 > 0.$$
$$B_2 = ka_1(2e_1c_1 - a_2) > 0.$$
$$B_1 = k[c_1(e_1c_1 - a_2) - e_1a_1(r - qE)].$$
$$B_0 = k[-(e_1c_1 - a_2)(r - qE) + rd_1].$$

The survival condition (2) with the use of the Descartes rule of signs, which specifies the number of positive roots under certain conditions, Eq. (4) has a unique positive root that is represented by $\bar{S}$ and hence $\bar{P}_2$ exists uniquely in the interior of positive quadrant XS −plane if

$$(e_1c_1 - a_2)(r - qE) > rd_1. \tag{5}$$

- The interior equilibrium point (IEP), $P_3^* = (X^*, S^*, I^*)$, where

$$\left.\begin{aligned} S^* &= \frac{d_2}{\beta}, \\ X^* &= \frac{k(r\beta^2 - q\beta^3E - \beta c_1d_2 - a_2d_2^2)}{r\beta^2}, \\ I^* &= \frac{K(r\beta^2 - q\beta^3E - \beta c_1d_2 - a_2d_2^2)[\beta(c_1e_1 - a_2) + a_1d_2e_1]}{r\beta^4} - d_1r\beta^3, \end{aligned}\right\} \tag{6}$$

exists uniquely in the interior of positive octant provided that the following conditions are held.

$$r\beta^2 > q\beta^3E + \beta c_1d_2 + a_2d_2^2. \tag{7}$$

$$\beta c_1e_1 + a_1d_2e_1 > \beta a_2. \tag{8}$$

The Jacobian matrix (JM) that is utilized to analyze the LS of the potential EPs stated earlier can be represented as:

$$J = (h_{ij})_{3\times3}. \tag{9}$$

where

$$h_{11} = \frac{-rX}{k} + r\left(1 - \frac{X}{k}\right) - qE - (c_1 + a_1 S)S;\ h_{12} = -(c_1 + 2a_1 S)X;\ h_{13} = 0.$$

$$h_{21} = [-a_2 + e_1(c_1 + a_1 S)]S;\ h_{22} = a_1 e_1 XS + [e_1(c_1 + a_1 S) - a_2]X - \beta I - d_1;$$

$$h_{23} = -\beta S;\ h_{31} = 0;\ h_{32} = \beta I;\ h_{33} = \beta S - d_2$$

Therefore, at $\check{P}_0 = (0,0,0)$, JM given by Eq. (10) converts to:

$$J(\check{P}_0) = \begin{bmatrix} r - qE & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}. \tag{10}$$

Consequently, the eigenvalues of $J(\check{P}_0)$ are given by the following:

$$\lambda_{01} = r - qE,\ \lambda_{02} = -d_1 < 0, \lambda_{03} = -d_2 < 0. \tag{11}$$

So if the following condition is fulfilled, $\check{P}_0$ becomes local asymptotic stability (LAS) if the next condition holds

$$r < qE. \tag{12}$$

At $\hat{P}_1 = (\hat{X}, 0,0)$, the JM given by (9) becomes.

$$J(\hat{P}_1) = \begin{bmatrix} -r + qE & -\frac{k(r-qE)c_1}{r} & 0 \\ 0 & -d_1 + \frac{k(r-qE)(c_1 e_1 - a_2)}{r} & 0 \\ 0 & 0 & -d_2 \end{bmatrix}. \tag{13}$$

Consequently, the eigenvalues of $J(\hat{P}_1)$ provided by the following:

$$\lambda_{11} = -r + qE,\ \lambda_{12} = -d_1 + \frac{k(r-qE)(c_1 e_1 - a_2)}{r}, \lambda_{13} = -d_2. \tag{14}$$

Therefore, by using the existence condition (2) of $\hat{P}_1$, the following condition guarantees the LAS of P_1.

$$k(r - qE)(c_1 e_1 - a_2) < rd_1. \tag{15}$$

For the EP, $\bar{P}_2 = (\bar{X}, \bar{S}, 0)$, the JM given by (9) turns into

$$J(\bar{P}_2) = \begin{bmatrix} -\frac{r\bar{X}}{k} & -(c_1 + 2a_1\bar{S})\bar{X} & 0 \\ (-a_2 + e_1(c_1 + a_1\bar{S}))\bar{S} & a_1 e_1 \bar{X}\bar{S} & -\beta\bar{S} \\ 0 & 0 & \beta\bar{S} - d_2 \end{bmatrix}. \tag{16}$$

Thus, The characteristic equation of the $J(\bar{P}_2)$ can be written as:

$$[\lambda_2^2 - T_1\lambda_2 + D_1](\beta\bar{S} - d_2 - \lambda_2) = 0, \tag{17}$$

where

$$D_1 = \left(-\frac{r\bar{X}}{k}\right) a_1 e_1 \bar{X}\bar{S} + (c_1 + 2a_1\bar{S})[-a_2\bar{S} + e_1(c_1 + a_1\bar{S})]\bar{X}\bar{S}.$$

$$T_1 = -\frac{r\bar{X}}{k} + a_1 e_1 \bar{X}\bar{S}.$$

The roots of this equation are as follows:

$$\lambda_{21} = \frac{T_1}{2} + \frac{1}{2}\sqrt{{T_1}^2 - 4D_1};\ \lambda_{22} = \frac{T_1}{2} - \frac{1}{2}\sqrt{{T_1}^2 - 4D_1},\ \lambda_{23} = \beta\bar{S} - d_2. \tag{18}$$

The eigenvalues $\lambda_{21}, \lambda_{22}$ and λ_{23} have negative real-parts, indicating that $\bar{P}_2$ is LAS under the following conditions:

$$\beta\bar{S} < d_2. \tag{19}$$

$$a_1 e_1 \bar{S} < \frac{r}{k}. \tag{20}$$

$$\left(\frac{r}{k}\right) a_1 e_1 \bar{X} + a_2\bar{S}(c_1 + 2a_1\bar{S}) < (c_1 + 2a_1\bar{S})(c_1 + a_1\bar{S})e_1 \tag{21}$$

Lastly, the evaluation of the Jacobin matrix at the IEP, P_3^* is provided by:

$$J(P_3^*) = \begin{bmatrix} -\frac{rX^*}{K} & -(c_1 + 2a_1S^*)X^* & 0 \\ -a_2S^* + e_1(c_1 + a_1S^*)S^* & a_1e_1X^*S^* & -\beta S^* \\ 0 & \beta I^* & 0 \end{bmatrix} = [h_{ij}]. \tag{22}$$

Then the characteristic equation of $J(P_3^*)$ becomes

$$\lambda_3^3 + A_1\lambda_3^2 + A_2\lambda_3 + A_3 = 0, \tag{23}$$

where

$$A_1 = -(h_{11} + h_{22}) = \frac{rX^*}{K} - a_1e_1X^*S^*$$

$$\begin{aligned} A_2 &= (h_{11}h_{22} - h_{12}h_{21}) - h_{23}h_{32} \\ &= \left[-\frac{rX^*}{K}a_1e_1 - a_2(c_1 + 2a_1S^*) + e_1(c_1 + 2a_1S^*)(c_1 + a_1S^*)\right]X^*S^* + \beta^2S^*I^* \end{aligned}$$

$$A_3 = h_{11}h_{23}h_{32} = \beta^2S^*I^*\frac{rX^*}{K} > 0$$

with

$$\Delta = A_1A_2 - A_3 = -(h_{11} + h_{22})[h_{11}h_{22} - h_{12}h_{21}] + h_{22}h_{23}h_{32}.$$

Based on the "Routh-Hurwitz criterion", the equilibrium point is LAS, with three eigenvalues having negative real parts if $A_1 > 0$, $A_3 > 0$, and $\Delta = A_1A_2 - A_3 > 0$. The following theorem concerns the LS of the IEP.

Theorem 2. The IEP of the system (1) is LAS if the following condition is satisfied.

$$a_1e_1X^*S^* < \frac{rX^*}{K}. \tag{24}$$

$$\frac{rX^*}{K}a_1e_1 + a_2(c_1 + 2a_1S^*) < e_1(c_1 + 2a_1S^*)(c_1 + a_1S^*). \tag{25}$$

$$\begin{gathered}\beta^2 a_1e_1X^*S^{*2}I^* < \left(\frac{rX^*}{K} - a_1e_1X^*S^*\right) \\ \left[-\frac{rX^*}{K}a_1e_1 - a_2(c_1 + 2a_1S^*) + e_1(c_1 + 2a_1S^*)(c_1 + a_1S^*)\right]X^*S^* > 0\end{gathered} \tag{26}$$

Proof: According to the "Routh–Hurwitz criterion", the roots of the $J(h_{ij})$ have negative real parts provided that $A_1 > 0$, $A_3 > 0$, and $\Delta > 0$. Direct computation shows that conditions (24)-(26) guarantee the satisfaction of "Routh–Hurwitz criterion" requirements.

4. PERSISTENCE

An eco-epidemiological model's persistence and extinction properties are examined in this section. The goal is to look into how hunting cooperation and anti-predator behavior affect the persistence and extinction of system species. It is necessary to comprehend the dynamics at the system's boundary levels to identify the conditions that guarantee continuation.

Now the following subsystem is obtained

$$\begin{aligned}\frac{dX}{dt} &= r\left(1 - \frac{X}{k}\right)X - (c_1 + a_1S)XS - qEX = \ell_1(X,S), \\ \frac{dS}{dt} &= e_1(c_1 + a_1S)XS - a_2XS - d_1S = \ell_2(X,S).\end{aligned} \tag{27}$$

Now, to investigate the existence of periodic dynamics in the $Int.\mathbb{R}^2_+$ of $XS-$plane, define the Dulac function as $\mathcal{L}_1(X,S) = \frac{1}{XS}$ that satisfies $\mathcal{L}_1(X,S) > 0$ and C^1 function. Hence, it is obtained that

$$\mathcal{L}_1\ell_1 = \frac{1}{S}\left[r\left(1 - \frac{X}{k}\right) - (c_1 + a_1S)S - qE\right], \text{and } \mathcal{L}_1\ell_2 = \frac{1}{X}[\,e_1(c_1 + a_1S)X - a_2X - d_1].$$

Thus, it is obtained that

$$\Delta(x,y) = \frac{\partial(\mathcal{L}_1\ell_1)}{\partial X} + \frac{\partial(\mathcal{L}_1\ell_2)}{\partial S} = -\frac{r}{kS} + e_1a_1.$$

It's clear that Δ has the same sign and does not equal zero under the following conditions (28). Therefore, due to "Dulac-Bendixon criterion", a subsystem (27) does not have periodic dynamics in $XS-$plane provided that:

$$\begin{gathered}e_1a_1 > \frac{r}{kS} \\ OR \\ e_1a_1 < \frac{r}{kS}\end{gathered} \tag{28}$$

Hence, according to the "Bendixson–Dulac theorem," there are no periodic dynamics in the interior of the positive quadrant of the $XS-$ plane. As a result, the "Poincare-Bendixon theorem" asserts that whenever the border XS −plane is L.A.S, the unique EP in $int.\mathbb{R}_+^2$ is G.A.S

Theorem 3. Assume that the condition (28) are met then system (1) is uniformly persistent if

$$\begin{gathered} e_1 c_1 \hat{X} > a_2 \hat{X} + d_1 \\ \beta \bar{S} > d_2 \end{gathered} \tag{29}$$

Proof. Define $\mathcal{H}(X,S,I) = X^{\tau_1} S^{\tau_2} I^{\tau_3}$, where $\tau_1, \tau_2,$ and τ_3 are positive constants. It is clear that $\mathcal{H}(X,S,I) > 0$ for each $(X,S,I) \in Int\ \mathbb{R}_+^3$, and $\mathcal{H}(X,S,I) = 0\ if\ X,S,$ or I approaches zero. Consequently, it is obtained that

$$\Phi(X,S,I) = \frac{\mathcal{H}'(X,S,I)}{\mathcal{H}(X,S,I)} = \tau_1 \left[r\left(1 - \frac{X}{k}\right) - (c_1 + a_1 S)S - qE \right] \\ + \tau_2 [e_1(c_1 + a_1 S)X - a_2 X - d_1] + \tau_3 [\beta S - d_2]$$

Now, due to "average Lyapunov function" the proof will follows if and only if $\Phi(P_i) > 0$ for every boundary points P_i.

Now,

$$\begin{aligned} \Phi(\check{P}_0) &= \tau_1(r - qE) - \tau_2 d_1 - \tau_3 d_2 \\ \Phi(\hat{P}_1) &= \tau_2(e_1 c_1 \hat{X} - a_2 \hat{X} - d_1) - \tau_3 d_2 \\ \Phi(\bar{P}_2) &= \tau_3(\beta \bar{S} - d_2) \end{aligned}$$

Then the first expression is positive as the positive constants $\tau_1, \tau_2,$ and τ_3 are arbitrary constants and we are always can choose that τ_1 is sufficiently larger than τ_2 and τ_3. Hence, the requirements of the Lyapunov average method are met provided that the conditions (29) hold, which means the system (1) is uniformly persistent.

5. Global Stability Analysis

The following theorems show that the GS of all EPs is investigated in this part using the Lyapunov method.

Theorem 4. The TEEP, $\check{P}_0$ of the system (1) is GAS in R_+^3, assuming that condition (12) is met.

Proof. Let the following function

$$\omega_0 = \zeta_1 X + \zeta_2 S + \zeta_3 I,$$

where ω_0 is C^1 function, which is a positive definite real-valued function, and $\zeta_i; i = 1,2,3$ are positive constants to be determined. Then we have

$$\frac{d\omega_0}{dT} = \zeta_1 r X - \zeta_1 \frac{rX^2}{k} - \zeta_1(c_1 + a_1 S)XS - \zeta_1 qEX + \zeta_2 e_1(c_1 + a_1 S)XS - \zeta_2 a_2 XS - \zeta_2 \beta SI \\ - \zeta_2 d_1 S + \zeta_3 \beta SI - \zeta_3 d_2 I$$

So, by selecting $\zeta_1 = e_1,\ \zeta_2 = \zeta_3 = 1$, we get that.

$$\frac{d\omega_0}{dT} \leq -e_1(qE-r)X - d_1S - d_2I$$

Therefore, $\frac{d\omega_0}{dT}$ is negative definite due to the above given condition (12). Hence, the TEEP is a GAS.

Theorem 5. The AEP, $\hat{P}_1$ of the system (1) is GAS in R_+^3, assuming that the following condition is met

$$e_1\hat{X}(c_1 + a_1L_1) < d_1, \tag{30}$$

where L_1 is the upper bound given in Theorem 1.

Proof. Let the following function

$$\omega_1 = \zeta_4\left(X - \hat{X} - \hat{X}\ln\frac{X}{\hat{X}}\right) + \zeta_5 S + \zeta_6 I,$$

where ω_1 is C^1 function, which is a positive definite real-valued function and $\zeta_i; i = 4,5,6$ are positive constants to be determined. Then we have.

$$\frac{d\omega_1}{dT} = -\zeta_4\frac{r}{k}\left(X-\hat{X}\right)^2 - \zeta_4(c_1+a_1S)XS + \zeta_4(c_1+a_1S)\hat{X}S + \zeta_5e_1(c_1+a_1S)XS - \zeta_5a_2XS$$
$$- \zeta_5\beta SI - \zeta_5d_1S + \zeta_6\beta SI - \zeta_6d_2I$$

So, by selecting $\zeta_4 = e_1$, $\zeta_5 = \zeta_6 = 1$ we get that

$$\frac{d\omega_1}{dT} \leq -\frac{re_1}{k}\left(X-\hat{X}\right)^2 - \left[d_1 - e_1\hat{X}(c_1+a_1S)\right]S - d_2I$$

Therefore, $\frac{d\omega_1}{dT}$ is negative definite due to the above-given condition (30). Hence, the AEP is a GAS.

Theorem 6. The PFEP, $\bar{P}_2$ of the system (1) is GAS in R_+^3, assuming that the following conditions is met

$$\frac{(a_1\bar{S}+a_2)}{2} > e_1a_1\bar{X}$$
$$d_2 > \beta\bar{S} \tag{31}$$

Proof. Let the following function

$$\omega_2 = \zeta_7\left(X - \bar{X} - \bar{X}\ln\frac{X}{\bar{X}}\right) + \zeta_8\left(S - \bar{S} - \bar{S}\ln\frac{S}{\bar{S}}\right) + \zeta_9 I,$$

where ω_2 is C^1 function, which is a non-negative definite and real-valued function and $\zeta_i; i = 7,8,9$ are positive constants then we have

$$\frac{d\omega_2}{dT} = -\frac{\zeta_7 r}{k}(X-\bar{X})^2 - \zeta_7c_1(X-\bar{X})(S-\bar{S}) - \zeta_7a_1(S+\bar{S})(X-\bar{X})(S-\bar{S})$$
$$+ \zeta_8e_1c_1(X-\bar{X})(S-\bar{S}) + \zeta_8e_1a_1\bar{X}(S-\bar{S})^2 + \zeta_8e_1a_1S(X-\bar{X})(S-\bar{S})$$
$$- \zeta_8a_2(X-\bar{X})(S-\bar{S}) - \zeta_8\beta SI + \zeta_8\beta\bar{S}I + \zeta_9\beta SI - \zeta_9d_2I$$

So, by selecting, $\zeta_7 = e_1,\ \zeta_8 = \zeta_9 = 1$, we get that:

$$\frac{d\omega_2}{dT} \leq -\frac{re_1}{k}(X-\bar{X})^2 - \frac{(a_1\bar{S}+a_2)}{2}(X-\bar{X})^2 - \frac{(a_1\bar{S}+a_2)}{2}(S-\bar{S})^2$$
$$+e_1a_1\bar{X}(S-\bar{S})^2 - (d_2-\beta\bar{S})I.$$

Thus

$$\frac{d\omega_2}{dT} \leq -\left[\frac{re_1}{k}+\frac{(a_1\bar{S}+a_2)}{2}\right](X-\bar{X})^2 - \left[\frac{(a_1\bar{S}+a_2)}{2} - e_1a_1\bar{X}\right](S-\bar{S})^2 - (d_2-\beta\bar{S})I.$$

Therefore, $\frac{d\omega_1}{dT}$ is negative definite due to the above-given conditions (31). Hence, the PFEP is a GAS.

Theorem 7. The IEP, P_3^* of the system (1) is GAS in R_+^3, assuming that the following conditions is met

$$\frac{(e_1a_1S^*+a_2)}{2} > e_1a_1X^* \tag{32}$$

Proof. Let the following function

$$\omega_3 = e_1\left(X - X^* - X^*\ln\frac{X}{X^*}\right) + \left(S - S^* - S^*\ln\frac{S}{S^*}\right) + \left(I - I^* - I^*\ln\frac{I}{I^*}\right),$$

where ω_3 is C^1 function, which is a non-negative, definite and real-valued function, then we have

$$\frac{d\omega_3}{dT} = \left[-\frac{e_1r}{k}(X-X^*)^2 - (e_1a_1S^*+a_2)(X-X^*)(S-S^*)\right] + e_1a_1X^*(S-S^*)^2$$

Further simplification leads to the following.

$$\frac{d\omega_3}{dT} \leq -\frac{e_1r}{k}(X-X^*)^2 - \frac{(e_1a_1S^*+a_2)}{2}(X-X^*)^2 - \frac{(e_1a_1S^*+a_2)}{2}(S-S^*)^2 + e_1a_1X^*(S-S^*)^2$$

$$\leq -\left[\frac{e_1r}{k}+\frac{(e_1a_1S^*+a_2)}{2}\right](X-X^*)^2 - \left[\frac{(e_1a_1S^*+a_2)}{2} - e_1a_1X^*\right](S-S^*)^2$$

Therefore, the IEP, P_3^* is a stable point under the condition (32). Now since the only invariant set that satisfies $\frac{d\omega_3}{dt}$ac$= 0$ is given by P_3^* then cording to "LaSalle's invariance principle", it's attracting. Hence, P_3^* is a GAS.

6. Bifurcation Analysis

This section examines how varying a control parameter causes a qualitative change in the system's (1) dynamic behavior (local bifurcation). Since the EPs non-hyperbolic property is necessary but insufficient for the occurrence of bifurcation, the parameter selected changes the EP from hyperbolic to non-hyperbolic. This influence is examined with the use of the Sotomayor theorem [35].

In order to simplify the notations, recast system (1) in vector form as follows

$$\frac{dX}{dT} = G(X),\ X = (X,S,I)^T \text{ and } \quad G = (Xg_1, Sg_2, Ig_3)^T. \tag{33}$$

Then, using the JM of the system (1) at the point (X,S,I), that is simple to confirm for any vector $N = (n_1, n_2, n_3)^T$, we have that

$$D^2G(X)(N,N) = [m_{ij}]_{3\times 1}, \tag{34}$$

where

$$m_{11} = -2\left[\frac{rn_1^2 + k(c_1 + 2a_1S)n_1n_2 + ka_1Xn_2^2}{k}\right]$$

$$m_{21} = 2\left[\left(-a_2 + e_1(c_1 + 2a_1S)\right)n_1n_2 + e_1a_1Xn_2^2 - \beta n_2n_3\right]$$

$$m_{31} = 2\beta n_2n_3$$

Furthermore, we have also

$$D^3G(X)(N,N,N) = [s_{ij}]_{3\times 1}, \tag{35}$$

where

$$s_{11} = -6a_1n_1n_2^2$$
$$s_{21} = 6e_1a_1n_1n_2^2$$
$$s_{31} = 0$$

The LB that occurs at the EPs, $\check{P}_0, \hat{P}_1, \bar{P}_2$, and P_3^* is examined in the corresponding theorems that follow.

Theorem 8. The system (1) at the TEEP, $\check{P}_0$ undergoes a TB at $r = qE = r^*$.

Proof. When $r = qE = r^*$, it is obvious that JM given by (10) becomes

$$J_0^* = J^*(\check{P}_0, r^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$

So $\lambda_{01}^* = 0$, $\lambda_{02}^* = -d_1$ and $\lambda_{03}^* = -d_2$ are the eigenvalues for J_0^*. As a result, $\check{P}_0$ is a non-hyperbolic point, which is a prerequisite for LB.

Let $N_1 = (n_{11}, n_{12}, n_{13})^T$ be J_0^* 's eigenvector that corresponds to $\lambda_{01}^* = 0$, then basic computation results in that $N_1 = (n_{11}, 0,0)^T$, where n_{11} represents any nonzero real number.

Let $\Psi_1 = (\psi_{11}, \psi_{12}, \psi_{13})^T$ be J_0^{*T} 's eigenvector that corresponds to $\lambda_{01}^* = 0$. Then, it is obtained $\Psi_1 = (\psi_{11}, 0,0)^T$, where ψ_{11} is any nonzero real number.

Because $\frac{\partial G}{\partial r} = G_r = (X - \frac{X^2}{k}, 0,0)^T$, we get that $G_r(\check{P}_0, r^*) = (0,0,0)^T$, which produces

$$\Psi_1^T[G_r(\check{P}_0, r^*)] = 0.$$

As a result, the "Sotomayor theorem" rules out the SNB at $\check{P}_0$. In addition, we have

$$\Psi_1^T[DG_r(\check{P}_0, r^*)N_1] = n_{11}\psi_{11} \neq 0,$$

where DG_r represents the derivative of G_r w.r.t. X. So, by using Eq. (34) at $(\check{P}_0, r^*)$ with N_1, we get that

$$\Psi_1^T[D^2G(\check{P}_0, r^*)(N_1, N_1)] = \frac{-2r^*}{k} n_{11}^2\psi_{11} \neq 0.$$

After that, a TB take place; nevertheless, the "Sotomayor theorem" state that a PB cannot occur around $\check{P}_0$ with $r = r^*$.

Theorem 9. The system (1) at the AEP, $\hat{P}_1$undergoes a TB at $e_1 = \frac{d_1 r + k a_2(r - qE)}{k c_1 (r - qE)} = e_1{}^*$ if the following condition is met

$$\frac{kc_1}{r}(a_2 - e_1{}^* c_1) \neq e_1{}^* a_1 \hat{X}. \tag{36}$$

Otherwise, PB takes place.

Proof. The JM that is given by (13) for $e_1 = e_1{}^*$ becomes:

$$J_1^* = J^*(\hat{P}_1, e_1{}^*) = \begin{bmatrix} -r + qE & -\frac{k(r-qE)c_1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}.$$

So $\lambda_{11}^* = -r + qE$, $\lambda_{12}^* = 0$ and $\lambda_{13}^* = -d_2$ are the eigenvalues for J_1^*. As a result, $\hat{P}_1$ is a non-hyperbolic point and is a prerequisite for LB.

Let $N_2 = (n_{21}, n_{22}, n_{23})^T$ be J_1^* 's eigenvector that corresponds to $\lambda_{12}^* = 0$, then basic computation results in that $N_2 = \left(\frac{-kc_1}{r} n_{22}, n_{22}, 0\right)^T$, where n_{22} represents any nonzero real number.

Let $\Psi_2 = (\psi_{21}, \psi_{22}, \psi_{23})^T$ be J_1^{*T} 's eigenvector that corresponds to $\lambda_{12}^* = 0$. Then, direct computation shows that $\Psi_2 = (0, \psi_{22}, 0)^T$, where ψ_{22} is any nonzero real number.

Because$\frac{\partial G}{\partial e_1} = G_{e_1} = (0, (c_1 + a_1 S)XS, 0)^T$, we get that $G_{e_1}(\hat{P}_1, e_1{}^*) = (0,0,0)^T$,which produces

$$\Psi_2^T[G_{e_1}(\hat{P}_1, e_1{}^*)] = 0.$$

As a result, the "Sotomayor theorem" rules out the SNB at $\hat{P}_1$. In addition, we have

$$\Psi_2^T\left[DG_{e_1}(\hat{P}_1, {e_1}^*)N_2\right] = -c_1\hat{X}n_{22}\psi_{22} \neq 0,$$

So, by using Eq. (34) at $(P_1, {e_1}^*)$ with N_2 we get that by using condition (34) that:

$$\Psi_2^T\left[D^2G\left(\hat{P}_1, {e_1}^*\right)(N_2, N_2)\right] = 2\left[\frac{-kc_1}{r}(-a_2 + {e_1}^*c_1) + {e_1}^*a_1\hat{X}\right]{n_{22}}^2\psi_{22} \neq 0.$$

After that, a TB takes place.

Assume that condition (36) is not satisfied, then by using Eq. (35) at $(\hat{P}_1, {e_1}^*)$ with N_2 we get that

$$\Psi_2^T\left[D^3G\left(\hat{P}_1, {e_1}^*\right)(N_2, N_2, N_2)\right] = -6{e_1}^*a_1\frac{kc_1}{r}{n_{22}}^3\psi_{22} \neq 0$$

Therefore, a PB occurs around $\hat{P}_1$ with $e_1 = {e_1}^*$.

Theorem 10. The system (1) at the PFEP, $\bar{P}_2$ undergoes a TB at $d_2 = \beta\bar{S} = {d_2}^*$.

Proof. we get the Jacobin matrix for $d_2 = \beta\bar{S} = {d_2}^*$, as

$$J_2^* = J^*(\bar{P}_2, {d_2}^*) = \begin{bmatrix} -\dfrac{r\bar{X}}{k} & -(c_1 + 2a_1\bar{S})\bar{X} & 0 \\ [-a_2 + (c_1 + a_1\bar{S})e_1]\bar{S} & a_1e_1\bar{X}\bar{S} & -\beta\bar{S} \\ 0 & 0 & 0 \end{bmatrix} = \left[b_{ij}\right]_{3\times3}$$

So, the eigenvalues λ_{21} and λ_{22}, which are given by Eq. (18), and $\lambda_{23}^* = 0$. As a result, $\bar{P}_2$ is a non-hyperbolic point, which is a prerequisite for LB to occur.

Let $N_3 = (n_{31}, n_{32}, n_{33})^T$ be J_2^* 's eigenvector that corresponds to $\lambda_{23}^* = 0$, then basic computation results in that $N_3 = (\gamma_1 n_{33}, \gamma_2 n_{33}, n_{33})^T$, where v_{33} represents any nonzero real number and $\gamma_1 = \frac{b_{12}b_{23}}{b_{11}b_{22} - b_{21}b_{12}}$, $\gamma_2 = -\frac{b_{11}b_{23}}{b_{11}b_{22} - b_{21}b_{12}}$.

Let $\Psi_3 = (\psi_{31}, \psi_{32}, \psi_{33})^T$ be J_2^{*T} 's eigenvector that corresponds to $\lambda_{23}^* = 0$. Then, it results that $\Psi_3 = (0, 0, \psi_{33})^T$, where ψ_{33} is any nonzero real number.

Because $\frac{\partial G}{\partial d_2} = G_{d_2} = (0,0,-I)^T$, we get that $G_{d_2}(\bar{P}_2, {d_2}^*) = (0,0,0)^T$, which produces

$$\Psi_3^T\left[G_{d_2}(\bar{P}_2, {d_2}^*)\right] = 0.$$

As a result, the "Sotomayor theorem" rules out the SNB at $\bar{P}_2$. In addition, we have

$$\Psi_3^T\left[DG_{d_2}(\bar{P}_2, {d_2}^*)N_3\right] = -n_{33}\psi_{33} \neq 0,$$

Therefore, by using Eq. (34) at $(\bar{P}_2, {d_2}^*)$ with N_3 we get that

$$\Psi_2^T[D^2G(\bar{P}_2, {d_2}^*)(N_3, N_3)] = 2\beta\gamma_2 n_{33}^2\psi_{33} \neq 0.$$

Therefore, a TB occurs around $\bar{P}_2$ when the parameter $d_2 = {d_2}^*$.

Finally, because the determinant of $J(P_3^*)$, which is given by A_3 in Eq. (24), is always positive, then there is no possibility for the IEP to be a non-hyperbolic point and hence there is no possibility for LB to occur.

7. HOPF BIFURCATION

In this section, HB occurs if an EP of a system loses its stability, meanwhile, a pair of complex conjugate eigenvalues of the linearization around the EP crosses the imaginary axis in a complex plan [36-38]. The following theorem gives to conditions under which this type of bifurcation.

Theorem 11. If the following conditions hold

$$e_1(c_1 + a_1 S^*) > \rho_1 \tag{37}$$

$$A_3'(a_2^*) > A_2'(a_2^*)\sqrt{A_2(a_2^*)} + 2\big(A_1(a_2^*)\big)^2 \tag{38}$$

then as the parameter a_2 passes through the positive value $a_2 = \frac{h_{22}h_{23}h_{32} - h_{11}h_{22}(h_{11}+h_{22})}{S^* h_{12}(h_{11}+h_{22})} + e_1(c_1 + a_1 S^*) = a_2^*$, where $h_{ij}; i,j = 1,2,3$ represent the JM elements that are given in Eq.(22), while $A_i; i = 1,2,3$ are the coefficients of the characteristic Eq.(23), the system (1) possesses an HB at the IEP.

Proof. System (1) will undergo an HB at $a_2 = a_2{}^* = e_1(c_1 + a_1 S^*) - \rho_1$, where $\rho_1 = \frac{h_{11}h_{22}(h_{11}+h_{22}) - h_{22}h_{23}h_{32}}{S^* h_{12}(h_{11}+h_{22})}$, then the Jacobian matrix at the EP has a simple pair of complex eigenvalues, say $\lambda_{1,2} = \delta_1(a_2) \pm i\delta_2(a_2)$, such that they become purely imaginary at $a_2 = a_2{}^*$. Moreover, $\left.\frac{d\delta_1(a_2)}{d}\right|_{a_2 = a_2{}^*} \neq 0$ should be held. Hence, substituting $\lambda = \delta_1(a_2) \pm i\delta_2(a_2)$, in Eq.(23), then calculating the derivative w.r.t. the bifurcation parameter a_2 we get

$$\left.\begin{aligned}\Theta(a_2)\delta_1'(a_2) - \Phi(a_2)\delta_2'(a_2) &= -\theta(a_2)\\ \Phi(a_2)\delta_1'(a_2) + \Theta(a_2)\delta_2'(a_2) &= -\Gamma(a_2)\end{aligned}\right\} \tag{39}$$

where

$$\theta(a_2) = A_1'(a_2)[\delta_1(a_2)]^2 - A_1'(a_2)[\delta_2(a_2)]^2 + A_2'(a_2)\delta_1(a_2) + A_3'(a_2).$$
$$\Theta(a_2) = 3[\delta_1(a_2)]^2 + 2A_1(a_2)\delta_1(a_2) - 3[\delta_2(a_2)]^2 + A_2(a_2).$$
$$\Gamma(a_2) = 2A_1'(a_2)\delta_1(a_2)\delta_2(a_2) + A_2'(a_2)\delta_2(a_2).$$
$$\Phi(a_2) = 6\delta_1(a_2)\delta_2(a_2) + 2A_1(a_2)\delta_2(a_2).$$

Solving the liner system (39) then it gives that

$$\delta_1'(a_2) = -\frac{\theta(a_2)\Theta(a_2) + \Gamma(a_2)\Phi(a_2)}{[\Theta(a_2)]^2 + [\Phi(a_2)]^2}, \delta_2'(a_2) = -\frac{\Gamma(a_2)\Theta(a_2) - \theta(a_2)\Phi(a_2)}{[\Theta(a_2)]^2 + [\Phi(a_2)]^2}.$$

Notices that $\delta_1(a_2^*) = 0$ and $\delta_2(a_2^*) = \sqrt{A_2(a_2^*)}$, then at $a_2 = a_2^*$ the coefficients of system (39) are written as

$\Theta(a_2^*) = -2A_2(a_2^*)$,
$\Phi(a_2^*) = 2A_1(a_2^*)\sqrt{A_2(a_2^*)}$,
$\theta(a_2^*) = A_2'(a_2^*)[A_1(a_2^*) - \sqrt{A_2(a_2^*)}]$,
$\Gamma(a_2^*) = 2A_1(a_2^*)\sqrt{A_2(\omega_4^*)}$.

Therefore, it is obtained that

$$\theta(a_2^*)\Theta(a_2^*) + \Gamma(a_2^*)\Phi(a_2^*) = -2A_2(a_2^*)\left[A_3'(a_2^*) - A_2'(a_2^*)\sqrt{A_2(a_2^*)} - 2\left(A_1(a_2^*)\right)^2\right]$$

As a result, under condition (38), $\delta_1'(a_2^*) > 0$, and then the system (2) undergoes HB at $a_2 = a_2^*$.

8. Numerical Simulations

In this section, numerical simulations have been performed to validate our analytical findings of previous sections, we have used MATHLAB version 14 for our numerical simulation portion. Accordingly, System (1) with the following hypothetical fixed parameters Dataset is investigated.

$$r = 2.5, k = 20, c_1 = 0.5, a_1 = 0.1, E = 0.2, q = 0.1, e_1 = 0.6, a_2 = 0.1, \beta = 0.15,$$
$$d_1 = 0.05, d_2 = 0.15 \qquad (40)$$

It is obtained that, the trajectory of the system (1) utilizing the parameters set (40) is approached to the IEP, $P_3^* = (15.03,1.25.73)$ starting from different initial points, see Figure (1).

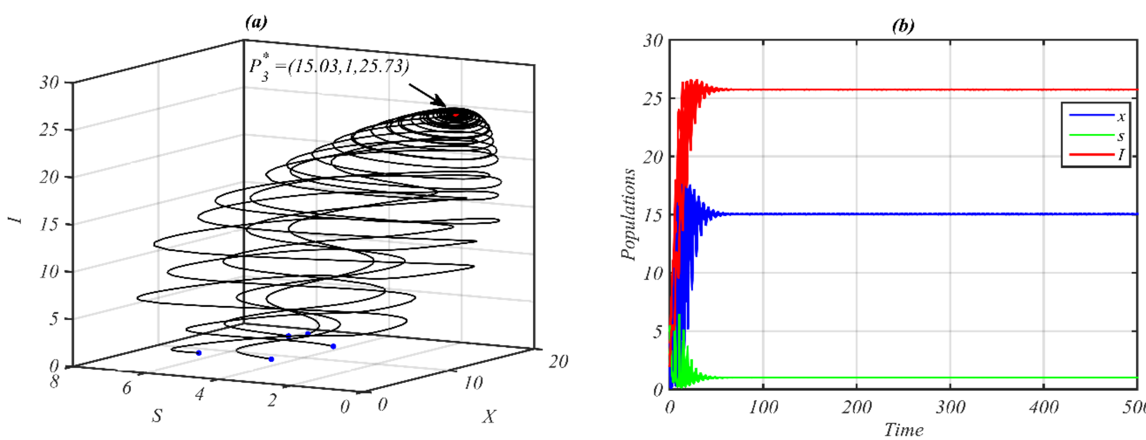

Figure 1. For Dataset (40), the solutions of system (1) approach to $P_3^* = (15.03,1,25.73)$ with multiple initial conditions. (a) 3D Phase portrait. (b) The populations against time.

Now, the impact of changing r value on the system's dynamic (1) is examined, and the findings are shown in Figure (2) for a selection of values. It is obtained that, for $r < 0.02$ the solution of system (1) still approaches asymptotically to the TEEP, and for $r \in [0.02,0.5]$ the system's solution converges asymptotically to $2D$ period attractor, while the system's solution converges asymptotically to $3D$ period attractor, $r \in [0.6,1.7]$ and $r \geq 3.23$. Otherwise the solution of system (2) approaches to the IEP for $r \in [1.71,3.22]$, as illustrated in Fig. (1).

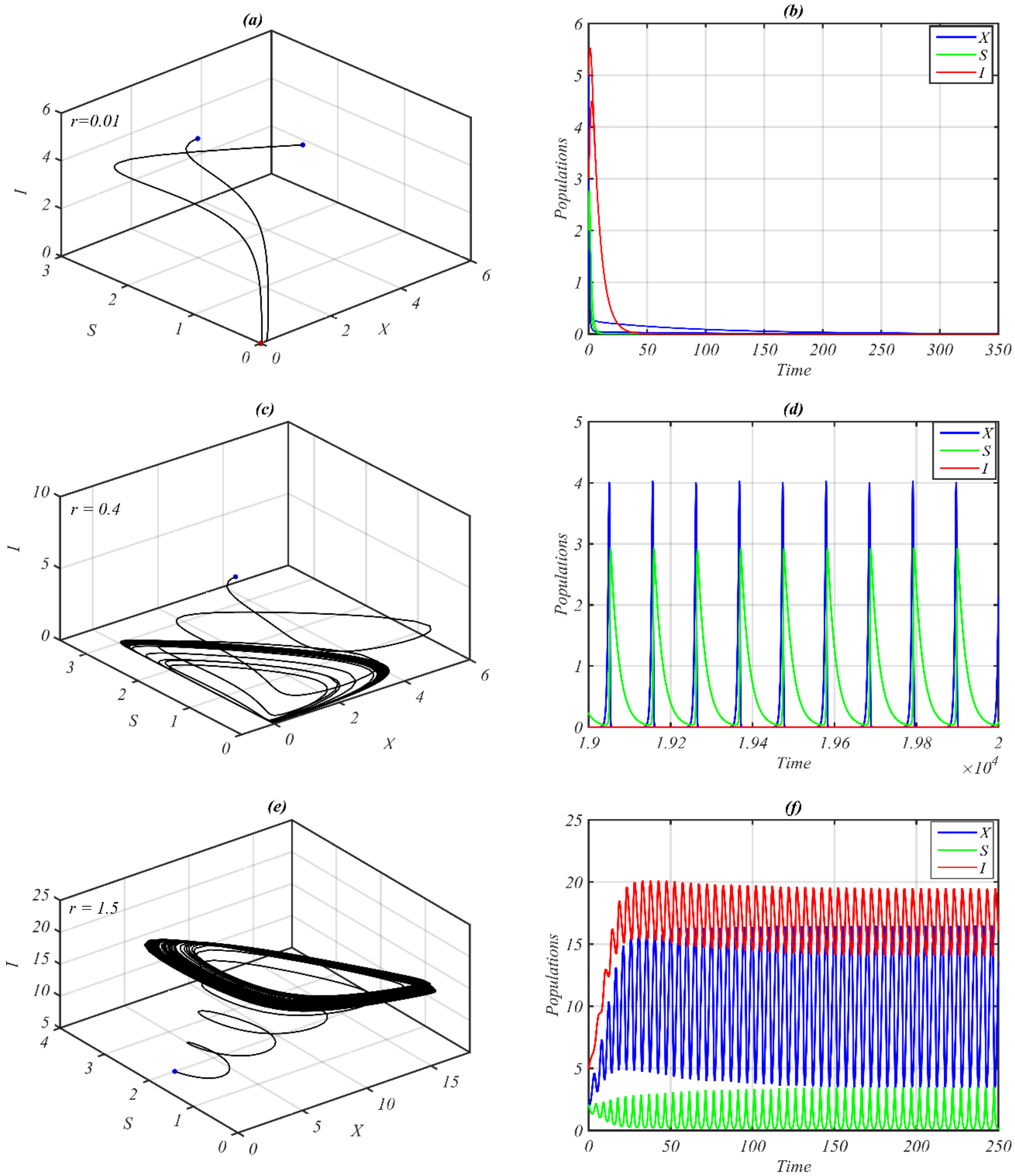

Figure 2. The trajectory of system (1) for parameters (40) with different values r. (a) Approach to $\check{P}_0 = (0,0,0)$ for $r = 0.01$. (b) Time series for $r = 0.01$. (c) (Periodic dynamics in $XS-$ plane for $r = 0.4$. (f) Time series for $r = 0.4$ (c). (e) Periodic dynamics in $\mathcal{R}^3_+$ for $r = 1.6$. (f) Time series for $r = 1.6$.

It is observed further that for $k \leq 0.25$ the solution of system (1) approaches to the AEP and the system's solution converges to $3D$ period attractor when $k \geq 26.55$, as illustrated in Fig. (3). While for $0.26 \leq k < 26.55$ the solution of system (1) approaches to IEP, as illustrated in Figure (1).

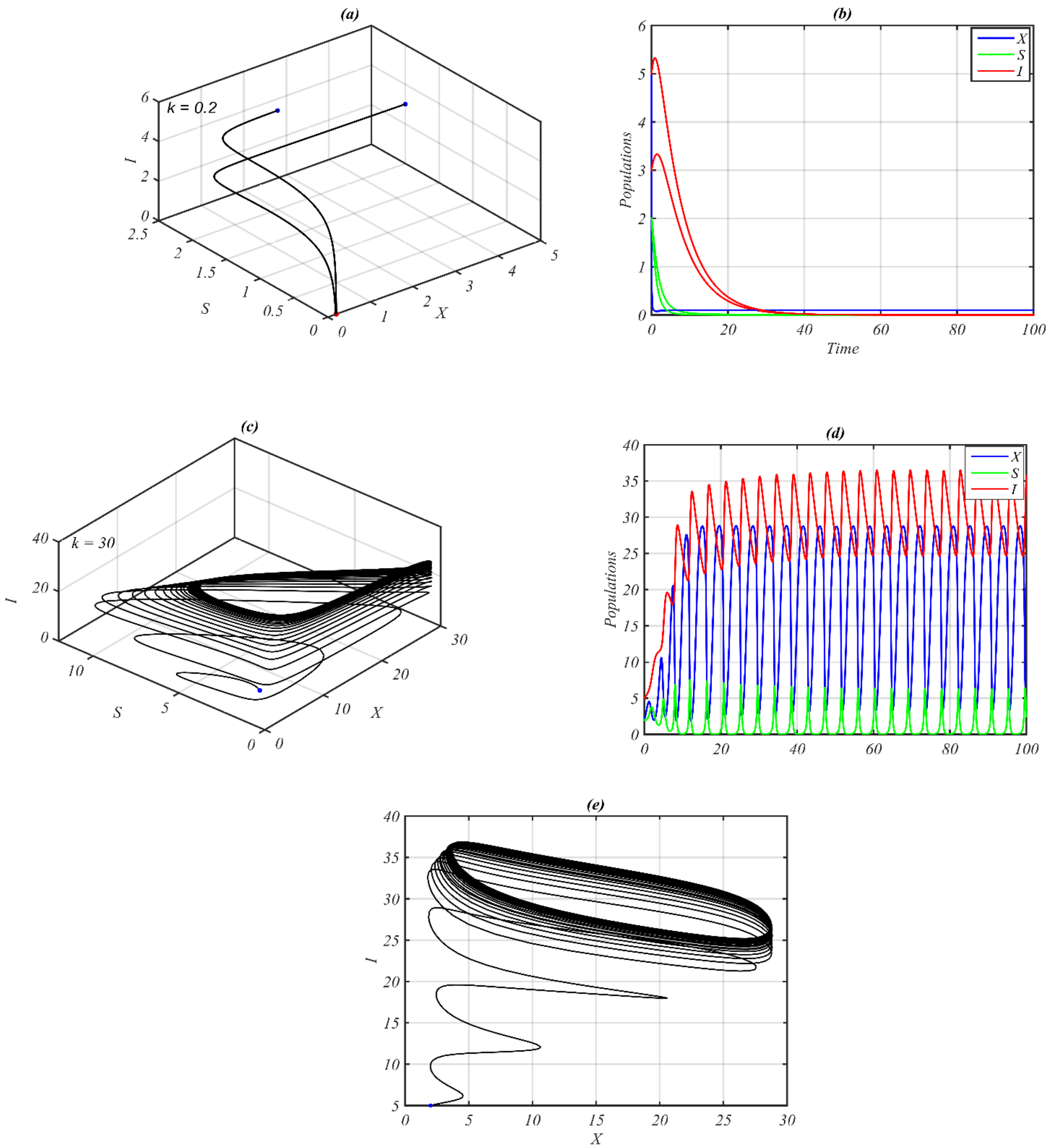

Figure 3. The trajectory of the system (1) for parameters set (40) with different values k. (a) Approach asymptotically to $\hat{P}_1 = (0.19,0,0)$ for $k = 0.2$. (b) Time series for $k = 0.2$. (c) Periodic dynamics in $\mathcal{R}_+^3$ for $k = 30$. (d) Time series for $k = 30$. (e) Projection on the $XI -$ plane for $k = 30$.

The influence of varying c_1 is studied numerically on the system's dynamic (1), and it is observed that for $c_1 \leq 0.1$ the system approaches to AEP. For $c_1 \in [0.2,0.44]$, the system approaches to a stable limit cycle. However, for $c_1 \geq 2.37$, the system's solution converges to $2D$ period attractor, as illustrated n in Figure (4). while $c_1 \in [0.45,2.36]$ the system approaches to the IEP, as illustrated in Figure (1).

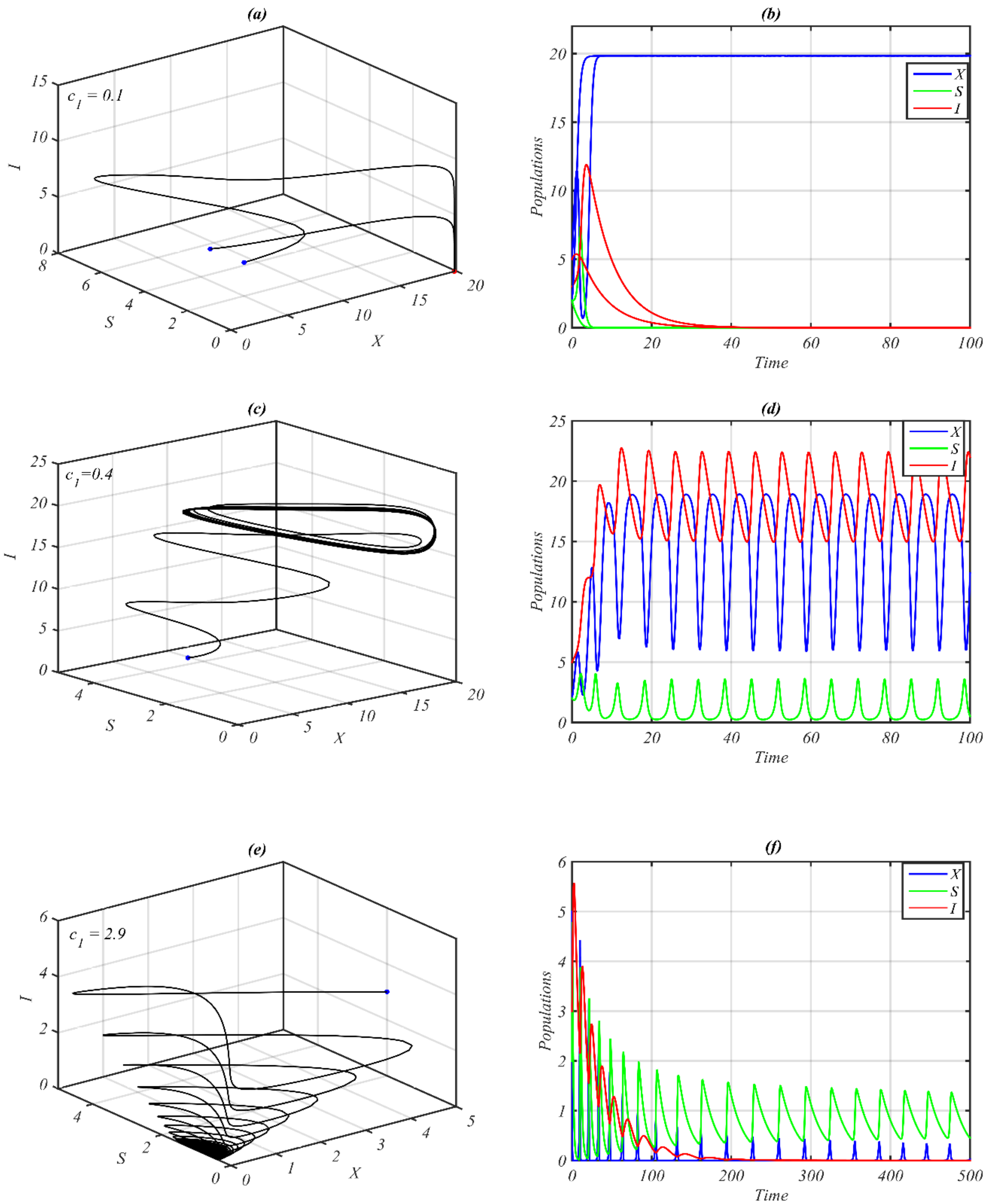

Figure 4. The trajectory of the system (1) for parameters set (40) with different values c_1. (a) Approach asymptotically to $\hat{P}_1 = (19.8,0,0)$ for $c_1 = 0.2$. (b) Time series for $c_1 = 0.2$. (c) Periodic dynamics in $\mathcal{R}_+^3$ for $c_1 = 0.4$. (d) Time series for $c_1 = 0.4$. (e) Periodic dynamics in $XS-$plane for $c_1 = 2.9$. (f) Time series for $c_1 = 2.9$.

Now, the influence of altering $a_1$is explored through Figure (5)

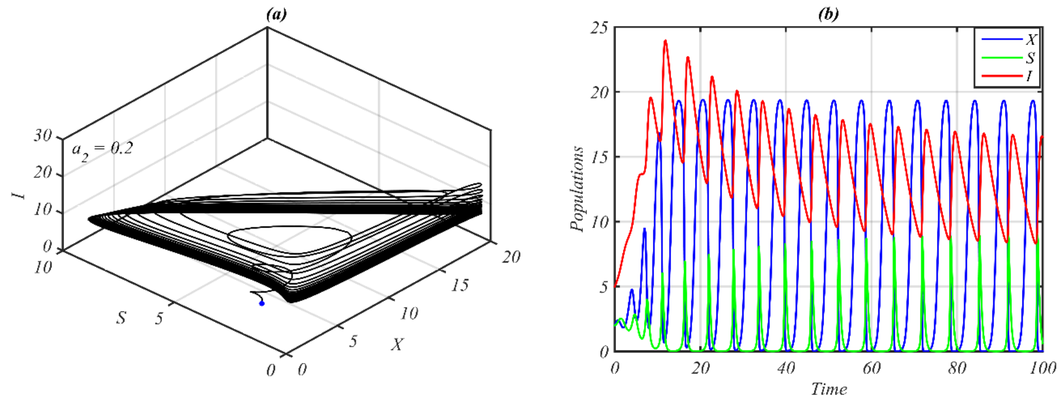

Figure 5. The trajectory of the system (1) for parameters set (40) with different values a_1. (a)Periodic dynamics in $\mathcal{R}_+^3$ for $a_1 = 0.2$. (d) Time series for $a_1 = 0.2$..

Clearly, increasing the value $a_1 \geq 0.14$ leads to periodic dynamics in $\mathcal{R}_+^3$, while decreasing $a_1 \leq 0.13$. it further leads to IEP as illustrated in Figure (1).

The influence of varying e_1 is numerically studied on the dynamic of the system (1), and it is observed that for $e_1 \leq 0.20$, the system approaches to AEP, for $e_1 \in [0.21,0.32]$ and $e_1 \geq 0.84$, the system approaches to a stable limit cycle as illustrated in Figure (6). while for $e_1 \in [0.33,0.83]$, the system approaches to a IEP, as illustrated in Figure (1).

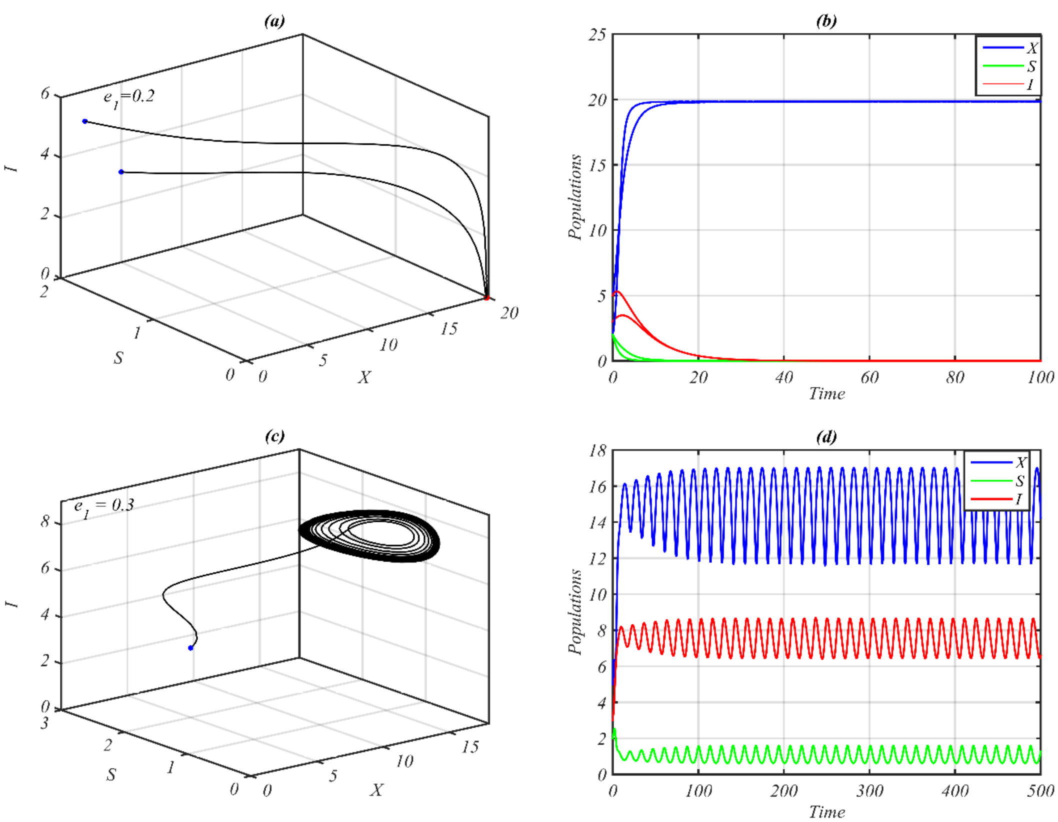

Figure 6. The trajectory of system (1) for dataset (40) with different values $e_1.2$. (a) Approach to $\hat{P}_1 = (0.19,0,0)$ for $e_1 = 0.2$. (b) Time series for $e_1 = 0.2$. (c) Periodic dynamics in $\mathcal{R}_+^3$ for $e_1 = 0.3$. (d) Time series for $e_1 = 0.2$.

The influence of a_2 on the dynamic of system (1) in studied numerically and the obtained results give the following. for $a_2 \leq 0.15$, the system approaches to the IEP, as illustrated in Figure (1). In figure (7), shows the HB of system (1) when $a_2 \in [0.16,0.29]$. However, for $a_2 \geq 0.3$, the system (1) approach approaches to AEP.

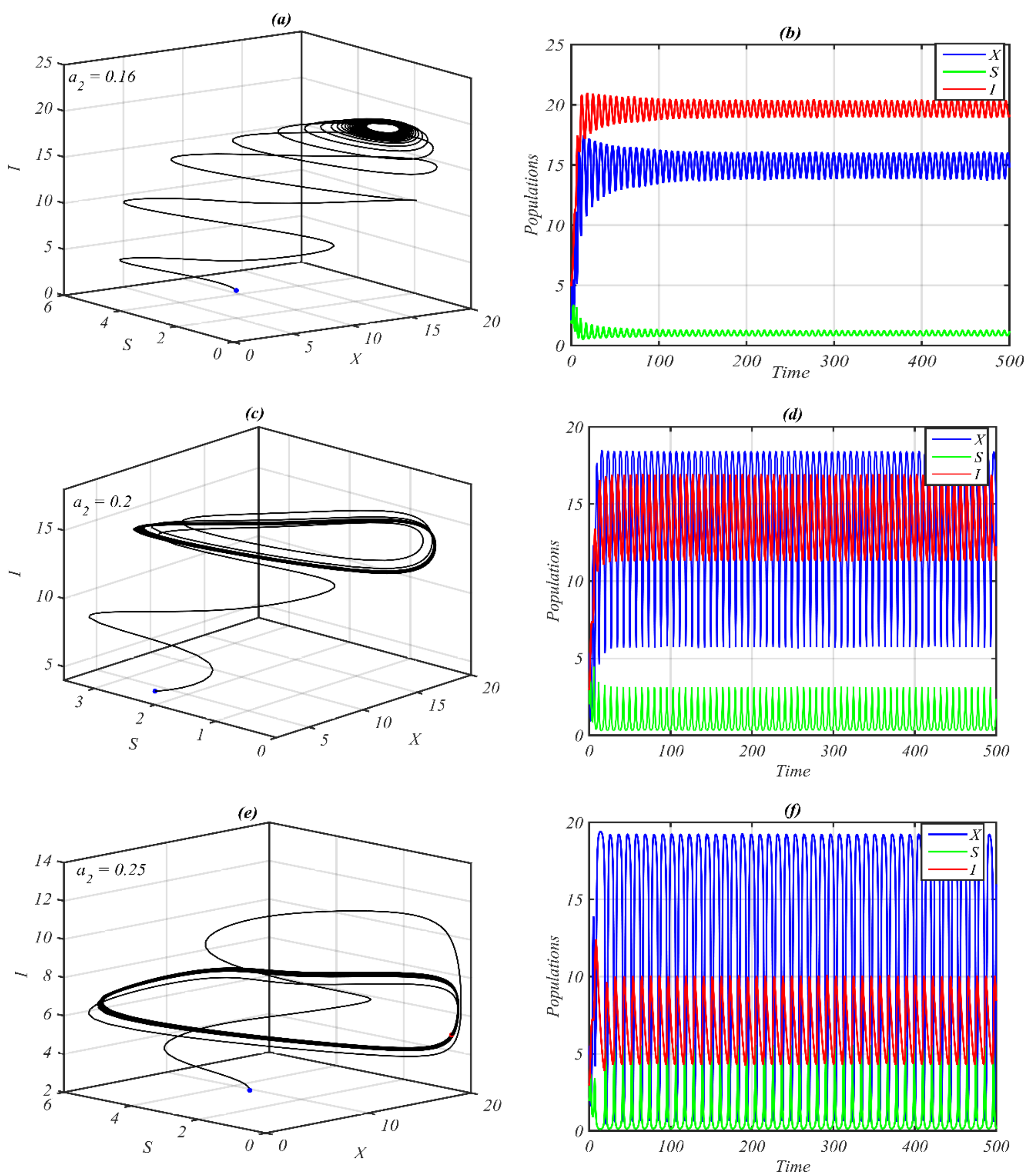

Figure 7. Dynamics of the trajectory showing the existence of limit cycle from the HB of system (1) (a) limit cycle behavior of solution for $a_2 = 0.16$. (b) Time series for $a_2 = 0.16$. (c) limit

cycle behavior for $a_2 = 0.2$. (d) Time series for $a_2 = 0.2$. (e) limit cycle behavior for $a_2 = 0.25$. (f) Time series for $a_2 = 0.25$.

The biological interpretation of the HB is that the prey with the predator, exhibits oscillatory behavior. Indeed, we observe that if increasing parameter a_2, we have periodic fluctuation of prey and predator species: Figure (7a)-(7f) show the existence of a limit cycle resulting from the HB. The effect of varying the parameters $q, E,$ and d_1 has a quantitative impact on the position of IEP. Finally, for $\beta \leq 0.09$ and $\beta \geq 0.23$ with the rest of the parameters as in (40), the trajectories of system (1) approach to a stable limit cycle, as illustrated in Figure (8). However, system (1) approaches the IEP otherwise, as illustrated in Figure (1). It is observed that the parameter d_2 has a similar influence on the dynamic of system (1) as that obtained for β.

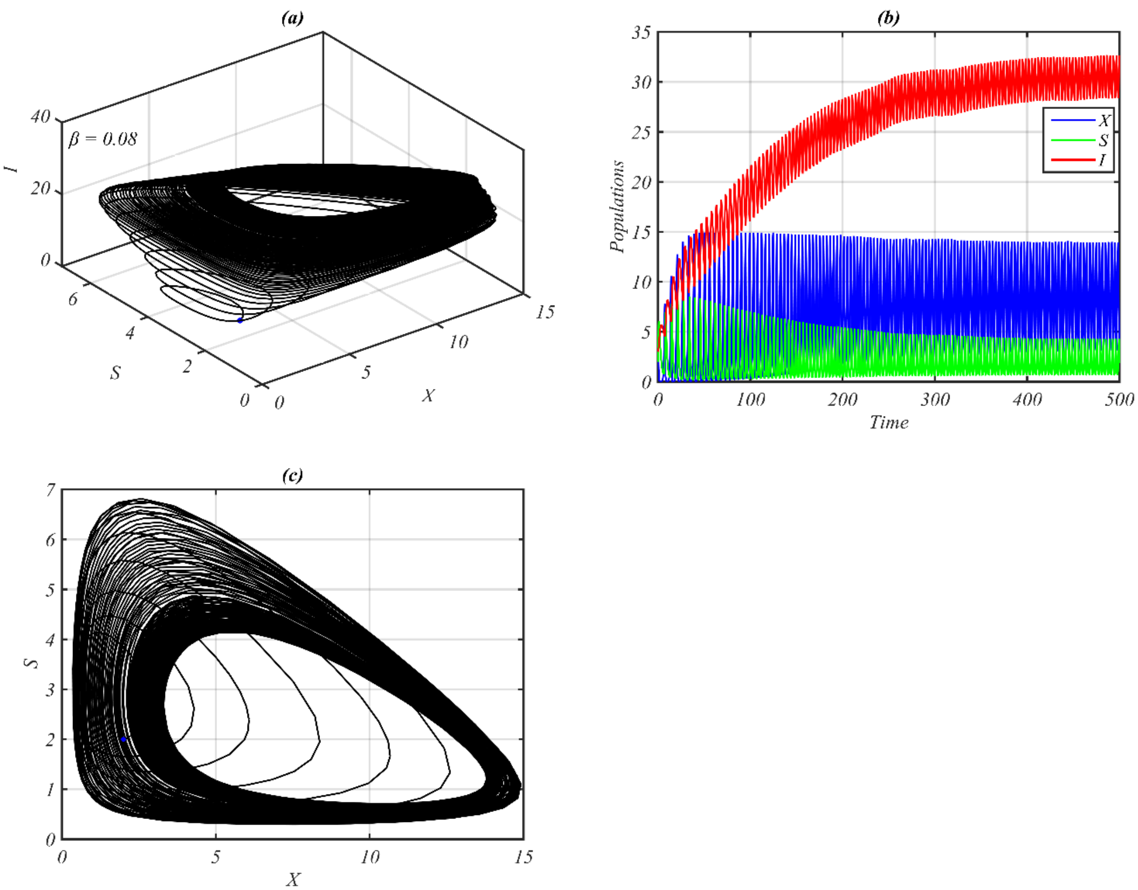

Figure 8. The trajectory of the system (1) for dataset (40) with different values c_1. (a). Periodic dynamics in $\mathcal{R}_+^3$ for $\beta = 0.08$. (b) Time series for $\beta = 0.08$. (c) Projection on the $XS-$ plane for $\beta = 0.08$.

9. Conclusion

The effects of infected hunting cooperation, anti-predator, and harvest effect on the dynamics of the prey-predator eco-epidemiological system were studied in this work. The proposed mathematical model contains at most four EPs. The local and global stability analysis near EPs are studied. However, bifurcation analysis is used to understand the effects of varying the system parameters. Moreover, we have described the conditions of existence of the HB to analyze to what extent changes will influence the trajectories in the predation rate. We used a numerical simulation to confirm the analytical findings and understand the impact of parameters on the system dynamics (1).

It is observed that the system is very sensitive to changes in most of the system's (1) parameters so it has different types of attractors including point attractors and periodic attractors. Increasing the intrinsic growth in the prey, carrying capacity, hunting cooperation rate, conversion rate, infection rate, or mortality rate of infected predators above a vital value destabilizes the system and keeps its persistence. On the other hand, increasing the predation rate or the anti-predator rate above a vital point causes a loss of the persistence of the system. Decreasing the hunting cooperation rate or anti-predator rate below a vital point stabilizes the system. Finally, the harvesting rate and mortality rate of susceptible predators have a quantitative effect on the dynamic behavior of the system.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] S. Pal, N. Pal, S. Samanta, J. Chattopadhyay, Effect of hunting cooperation and fear in a predator-prey model, Ecol. Complex. 39 (2019), 100770. https://doi.org/10.1016/j.ecocom.2019.100770.

[2] M. Agarwal, R. Pathak, Persistence and optimal harvesting of prey-predator model with Holling type III functional response, Int. J. Eng. Sci. Technol. 4 (2012), 78-96.

[3] R.M. Anderson, R.M. May, Infectious diseases and population cycles of forest insects, Science. 210 (1980), 658–661. https://doi.org/10.1126/science.210.4470.658.

[4] N. Bairagi, P.K. Roy, J. Chattopadhyay, Role of infection on the stability of a predator–prey system with several response functions—A comparative study, J. Theor. Biol. 248 (2007), 10–25. https://doi.org/10.1016/j.jtbi.2007.05.005.

[5] R. Bhattacharyya, B. Mukhopadhyay, On an eco-epidemiological model with prey harvesting and predator switching: Local and global perspectives, Nonlinear Anal.: Real World Appl. 11 (2010), 3824–3833. https://doi.org/10.1016/j.nonrwa.2010.02.012.

[6] B.W. Kooi, G.A.K. van Voorn, K. pada Das, Stabilization and complex dynamics in a predator–prey model with predator suffering from an infectious disease, Ecol. Complex. 8 (2011), 113–122. https://doi.org/10.1016/j.ecocom.2010.11.002.

[7] X. Liu, Bifurcation of an eco-epidemiological model with a nonlinear incidence rate, Appl. Math. Comput. 218 (2011), 2300–2309. https://doi.org/10.1016/j.amc.2011.07.050.

[8] J. Chattopadhyay, O. Arino, A predator-prey model with disease in the prey, Nonlinear Anal.: Theory Methods Appl. 36 (1999), 747–766. https://doi.org/10.1016/s0362-546x(98)00126-6.

[9] H.A. Ibrahim, R.K. Naji, Chaos in Beddington–DeAngelis food chain model with fear, J. Phys.: Conf. Ser. 1591 (2020), 012082. https://doi.org/10.1088/1742-6596/1591/1/012082.

[10] W. Hussein, H.A. Satar, The dynamics of a prey-predator model with infectious disease in prey: Role of media coverage, Iraqi J. Sci. 62 (2021), 4930–4952. https://doi.org/10.24996/ijs.2021.62.12.31.

[11] W.M. Alwan, H.A. Satar, The effects of media coverage on the dynamics of disease in prey-predator model, Iraqi J. Sci. 62 (2021), 981–996. https://doi.org/10.24996/ijs.2021.62.3.28.

[12] Y. Yao, T. Song, Z. Li, Bifurcations of a predator–prey system with cooperative hunting and Holling III functional response, Nonlinear Dyn. 110 (2022), 915–932. https://doi.org/10.1007/s11071-022-07653-7.

[13] H.A. Adamu, Mathematical analysis of predator-prey model with two preys and one predator, Int. J. Eng. Appl. Sci. 5 (2018), 17-23.

[14] É. Diz-Pita, M.V. Otero-Espinar, Predator–prey models: a review of some recent advances, Mathematics. 9 (2021), 1783. https://doi.org/10.3390/math9151783.

[15] H.A. Satar, H.A. Ibrahim, D.K. Bahlool, On the dynamics of an eco-epidemiological system incorporating a vertically transmitted infectious disease, Iraqi J. Sci. 62 (2021), 1642–1658. https://doi.org/10.24996/ijs.2021.62.5.27.

[16] Y. Du, B. Niu, J. Wei, A predator-prey model with cooperative hunting in the predator and group defense in the prey, Discrete Contin. Dyn. Syst. Ser. B. 27 (2022), 5845–5881. https://doi.org/10.3934/dcdsb.2021298.

[17] S.R.J. Jang, A.M. Yousef, Effects of prey refuge and predator cooperation on a predator–prey system, J. Biol. Dyn. 17 (2023), 2242372. https://doi.org/10.1080/17513758.2023.2242372.

[18] M.T. Alves, F.M. Hilker, Hunting cooperation and Allee effects in predators, J. Theor. Biol. 419 (2017), 13–22. https://doi.org/10.1016/j.jtbi.2017.02.002.

[19] S. Pal, N. Pal, J. Chattopadhyay, Hunting cooperation in a discrete-time predator–prey system, Int. J. Bifurcation Chaos. 28 (2018), 1850083. https://doi.org/10.1142/s0218127418500839.

[20] S. Pal, N. Pal, S. Samanta, J. Chattopadhyay, Fear effect in prey and hunting cooperation among predators in a Leslie-Gower model, Math. Biosci. Eng. 16 (2019), 5146–5179. https://doi.org/10.3934/mbe.2019258.

[21] J. Zhang, W. Zhang, Dynamics of a predator–prey model with hunting cooperation and Allee effects in predators, Int. J. Bifurcation Chaos. 30 (2020), 2050199. https://doi.org/10.1142/s0218127420501990.

[22] S. Al-Momen, R.K. Naji, Effect of hunting cooperation and fear in a food chain model with intraspecific competition, Commun. Math. Biol. Neurosci. 2023 (2023), 119. https://doi.org/10.28919/cmbn/8246.

[23] J. Liu, B. Liu, P. Lv, T. Zhang, An eco-epidemiological model with fear effect and hunting cooperation, Chaos Solitons Fractals. 142 (2021), 110494. https://doi.org/10.1016/j.chaos.2020.110494.

[24] U. Ghosh, A.A. Thirthar, B. Mondal, et al. Effect of fear, treatment, and hunting cooperation on an eco-epidemiological model: Memory effect in terms of fractional derivative, Iran. J. Sci. Technol. Trans. Sci. 46 (2022), 1541–1554. https://doi.org/10.1007/s40995-022-01371-w.

[25] N.H. Fakhry, R.K. Naji, The dynamic of an eco-epidemiological model involving fear and hunting cooperation, Commun. Math. Biol. Neurosci. 2023(2023), 63. https://doi.org/10.28919/cmbn/7998.

[26] K.Q. Al-Jubouri, R.K. Naji, Delay in eco-epidemiological prey-predator model with predation fear and hunting cooperation, Commun. Math. Biol. Neurosci. 2023 (2023), 89. https://doi.org/10.28919/cmbn/8081.

[27] N.H. Fakhry, R.K. Naji, Fear and hunting cooperation's impact on the eco-epidemiological model's dynamics, Int. J. Anal. Appl. 22 (2024), 15. https://doi.org/10.28924/2291-8639-22-2024-15.

[28] D. Savitri, Dynamics analysis of anti-predator model on intermediate predator with ratio dependent functional responses, J. Phys.: Conf. Ser. 953 (2018), 012201. https://doi.org/10.1088/1742-6596/953/1/012201.

[29] P. Panja, S. Mondal, J. Chattopadhyay, Dynamical effects of anti-predator of adult prey in a pretator-prey model with ratio-dependent functional response, Asian J. Math. Phys. 1 (2017), 19-32.

[30] B. Tang, Y. Xiao, Bifurcation analysis of a predator–prey model with anti-predator behaviour, Chaos Solitons Fractals. 70 (2015), 58–68. https://doi.org/10.1016/j.chaos.2014.11.008.

[31] J. Harianto, T. Suparwati, A.L.P. Dewi, Local Stability Dynamics of Equilibrium Points in Predator-Prey Models with Anti-Predator Behavior, J. ILMU DASAR. 22 (2021), 153. https://doi.org/10.19184/jid.v22i2.23991.

[32] Z.K. Mahmood, H.A. Satar, The influence of fear on the dynamic of an eco-epidemiological system with predator subject to the weak Allee effect and harvesting, Commun. Math. Biol. Neurosci. 2022 (2022), 90. https://doi.org/10.28919/cmbn/7638.

[33] C.W. Clark, Mathematical bioeconomics: The optimal management of renewable resources, John Wiley & Sons, New York, 1976.

[34] R.K. Naji, Contribution of hunting cooperation and antipredator behavior to the dynamics of the harvested prey-predator system, Commun. Math. Biol. Neurosci. 2023 (2023), 99. https://doi.org/10.28919/cmbn/8164.

[35] L. Perko, Differential equations and dynamical systems, 3rd edition, Springer, New York, 2001.

[36] A.S. Deshpande, V. Daftardar-Gejji, Y.V. Sukale, On Hopf bifurcation in fractional dynamical systems, Chaos Solitons Fractals. 98 (2017), 189–198. https://doi.org/10.1016/j.chaos.2017.03.034.

[37] Y. Chen, J. Liu, Supercritical as well as subcritical Hopf bifurcation in nonlinear flutter systems, Appl. Math. Mech.-Engl. Ed. 29 (2008), 199–206. https://doi.org/10.1007/s10483-008-0207-x.

[38] A. Savadogo, B. Sangaré, H. Ouedraogo, A mathematical analysis of Hopf-bifurcation in a prey-predator model with nonlinear functional response, Adv. Differ. Equ. 2021 (2021), 275. https://doi.org/10.1186/s13662-021-03437-2.