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# THE EFFECT OF COMPETING PREDATORS IN AN ECOSYSTEM

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**Abstract:** The presence of nutrients is an important factor that affects the growth rate of organisms found in nature In this research, we presented a mathematical model in which we studied the effect of the concentration of nutrients on the growth rate of organisms with two predators competing to feed on organisms. We designed all the feeding processes in this system according to the Holling type -II and linear type functional response, we found five biologically plausible critical points. We studied for these five points local stability and also studied for the positive point global stability. In addition, we found the conditions for the local bifurcation of the positive point, finally, we studied the system numerically.

**Keywords:** concentration of nutrients; competition; stability; bifurcation; numerical analysis.
**2020 AMS Subject Classification:** 92D25.

## 1. INTRODUCTION

Competition is considered one of the types of interactions that can occur between species existing in nature, regardless of the differences between these species and the form of this competition. Where competition occurs between individuals of the same species that live in the same clan. Competition also occurs between creatures of different species that live in the same environment and consume similar resources. Competition can occur between plants and animals as

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well as humans. Many studies have dealt with competition between plants, see [10, 29, 30]. Many researchers are interested in studying competition between animals or other living organisms, whether they are of the same species or different species, see [1, 2, 3, 4, 6, 14, 15, 20, 23, 24, 25]. The concentration of nutrients found in nature has a significant impact on the growth rate of the organisms that feed on them, and thus also affects the presence and growth rate of predatory organisms that feed on these organisms [8, 16, 17, 18, 19]. Bhattacharyya [3] focused his research on a special study of an aquatic food chain, the presence of a constant rate of flow of input nutrients, the presence of organisms that feed on the nutrients, and the presence of two predators of the same species that feed on organisms, regardless of their age stages. In this research, the focus was on studying the presence and concentration rate of nutrients found in nature, the growth rate of organisms that feed on these nutrients, and the existence of competition between two predators of two different species competing for their food over these organisms. We noticed that when increasing the concentration of nutrients, the growth rate of organisms and predators increases, While when the concentration of nutrients decreases, it leads to a decrease in the growth rate of living organisms and thus to a decrease in the growth rate of predatory animals, which leads to their gradual extinction. As for competition between predators to obtain food, the greater the rate of competition between one predator, the more it leads to the extinction of the other. The effect of the rate of nutrient concentration as well as competition between predators on the dynamic behavior of the system was acceptable and clear analytically and numerically. The conditions for bifurcation of the system were found in the presence of nutrients, organisms, and predators together, and the results were clear.

## 2. MODEL ASSUMPTION

In this part, we formulated an ecological mathematical model in which we studied the effect of competition between predators. The model consists of naturally occurring nutrients whose concentration in the system at the time $t$ is $x(t)$. Organisms $y(t)$ at time $t$ grow by feeding on those nutrients. We also took into the system two predators whose total population density at time

$t$ are $z(t)$ and $w(t)$. We assumed predators $z(t)$ and $w(t)$ feeding on living organisms $y(t)$ and competing with each other to obtain their food.

The mathematical model can be represented by four of the differential equations as show in the following with parameters:

$$\frac{dx}{dt} = (x^0 - x)d_1 - \frac{\propto xy}{\beta_1(a+x)} = xf_1(x, y, z, w)$$

$$\frac{dy}{dt} = \frac{\propto xy}{\beta_1(a+x)} - d_2 y - n_1 yz - n_2 yw = yf_2(x, y, z, w)$$

$$\frac{dz}{dt} = n_3 yz - d_3 z - \gamma_1 zw = zf_3(x, y, z, w)$$

$$\frac{dw}{dt} = n_4 yw - d_4 w - \gamma_2 zw = wf_4(x, y, z, w) \qquad (1)$$

System (1) was analyzed by adopting the initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0$. where all parameters of model (1) are positive, these parameters have been described as follows: $x^0$ represents the rate of increase of nutrients and $d_1$ is the rate of decrease of these nutrients, $\propto$ represent maximum nutrition, $\beta$ is a constant of what nutrients are transformed into the organism, $a$ is the half saturation constant " Michaelis-Menten" which is the nutrient concentration at which the functional response of the organism is half maximal. $(d_i, i = 2,3,4)$ it represents the death of species $y, z$ and $w$ respectively. $n_1, n_2$ describes the rate at which predators attack $z, w$ an organism respectively, while $n_3, n_4$ It represents the rate of predation of organisms by predators $z$ and $w$. Finally, both $\gamma_1$ and $\gamma_2$ represent competition coefficients between predators.

## 3. BOUNDEDERY

**Theorem 1.** The solutions $x(t), y(t), z(t)$ with $w(t)$ of a system (1), which are start in $R_+^4$ will be uniformly bounded.

**Proof**: Let us assume that (x(t),y(t),z(t),w(t)) is a solution of system (1) provided that it is non-negative**.**

Let $M(t) = x(t) + y(t) + z(t) + w(t)$, we obtained $\frac{dM}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} + \frac{dw}{dt}$

$$\frac{dM}{dt} = x^0 d_1 - x d_1 - \frac{\propto xy}{\beta(a+x)} + \frac{\propto xy}{\beta(a+x)} - d_2 y - n_1 yz - n_2 yw$$

$$+ n_3 yz - d_3 z - \gamma_1 zw + n_4 yw - d_4 w - \gamma_2 zw$$

Hence, $\frac{dM}{dt} + mM(t) \leq x^0 d_1 = \delta$, where, $m = min\{d_1, d_2, d_3, d_4\}$,

Then,

$$M(t) \leq \delta - \delta e^{-mt} + M_0 e^{-mt},$$

where $M_0 = M(x(0), y(0), z(0), w(0))$.

Now , for $T \geq 0$ we will be obtained $0 \leq M(T) \leq \delta$

So, any solution of model (1) starting at $R_+^4$ will be within the following region:

$\vartheta = \{(x(t), y(t), z(t), w(t)) \in R_+^4 : M = x + y + z + w \leq \delta + \varepsilon,\ for\ any\ \varepsilon > 0\}$.

## 4. Equilibrium Points with an Analysis of Their Stability

In this part of the manuscript, we studied the existence of equilibrium points for a system (1) and analyzed their local stability, we found five equilibrium points which are:

1. The nutrient equilibrium point is $E_0 = (x^0, 0,0,0)$.
2. The predator's free equilibrium point is $E_1 = (\breve{x}, \breve{y}, 0,0)$, where $\breve{x} = \frac{a d_2 \beta}{\alpha - d_2 \beta}$ and

   $\breve{y} = \frac{d_1\left(x^0 \alpha - d_2 \beta (x^0 + a)\right)}{\alpha - d_2 \beta}$ exists if the following condition holds:

   $$\alpha > d_2 \beta + \frac{d_2 \beta a}{x^0} \qquad (2)$$

3. The equilibrium point without predator $w$ is $E_2 = (\bar{x}, \bar{y}, \bar{z}, 0)$, where $\bar{x} = \frac{-h_1 + \sqrt{h_1^2 + 4ax^0}}{2}$,

   $\bar{y} = \frac{d_3}{n_3}$ and $\bar{z} = \frac{\alpha \bar{x} - d_2 \beta (a + \bar{x})}{n_1 \beta (a + \bar{x})}$ exists if the following conditions hold:

   $$\alpha \bar{x} > d_2 \beta (a + \bar{x}) \qquad 3(a)$$

   $$\sqrt{h_1^2 + 4ax^0} > h_1 \qquad 3(b)$$

   $$h_1 > 0 \qquad 3(c)$$

   where $h_1 = \frac{a d_1 n_3 \beta + d_3 \alpha - d_1 n_3 x^0 \beta}{d_1 n_3 \beta}$ .

4. The equilibrium point without predator $z$ is $E_3 = (\dot{x}, \dot{y}, 0, \dot{w})$, where $\dot{x} = \frac{-h_2+\sqrt{h_2^2+4ax^0}}{2}$, $\dot{y} = \frac{d_4}{n_4}$ and $\dot{w} = \frac{\alpha\dot{x}-d_2\beta(a+\dot{x})}{n_2\beta(a+\bar{x})}$ exists if the following conditions hold:

$$\alpha\dot{x} > d_2\beta(a+\dot{x}) \quad 4(a)$$

$$\sqrt[2]{h_2^2 + 4ax^0} > h_2 \quad 4(b)$$

$$h_2 > 0 \quad 4(c)$$

Where $h_2 = \frac{ad_1n_4\beta+d_4\alpha-d_1n_4x^0\beta}{d_1n_3\beta}$

5. The positive equilibrium point $E_4 = (\ddot{x}, \ddot{y}, \ddot{z}, \ddot{w})$, where $\ddot{y} = \frac{d_1\beta(a+\ddot{x})(x^0-\ddot{x})}{\alpha\ddot{x}}$, $\ddot{z} = \frac{n_4\ddot{y}-d_4}{\gamma_2}, \ddot{w} = \frac{n_3\ddot{y}-d_3}{\gamma_1}$ while the positive solution of the following polynomial will be the value of $\ddot{x}$:

$\ddot{A}x^3 + \ddot{B}x^2 + \ddot{C}x + \ddot{D} = 0$

Where $\ddot{A} = d_1\beta^2(n_1n_4\gamma_1 + n_2n_3\gamma_2) > 0$

$\ddot{B} = \alpha^2\gamma_1\gamma_2 + d_1\beta(2a\beta + x^0)(n_1n_4\gamma_1 + n_2n_3\gamma_2) - \alpha\beta\gamma_1(d_2\gamma_2 + d_4n_1) - n_2d_3\gamma_2\beta$

$\ddot{C} = -a\alpha\beta(\gamma_1(d_2\gamma_2 + n_1d_4) + n_2d_3\gamma_2) - (n_1n_4\gamma_1 + n_2n_3\gamma_2)(ad_1\beta(x^0\beta - (a\beta + x^0))$

$\ddot{D} = -a^2\beta^2x^0d_1(n_1n_4\gamma_1 + n_2n_3\gamma_2) < 0$

So, the above equations will have a positive root according to the discard rule of sign. Let us call it $\ddot{x}$ if it fulfills the following conditions.

$$\alpha^2\gamma_1\gamma_2 + d_1\beta(2a\beta + x^0)(n_1n_4\gamma_1 + n_2n_3\gamma_2) < \alpha\beta\gamma_1(d_2\gamma_2 + d_4n_1) + n_2d_3\gamma_2\beta \quad 5(a)$$

$$x^0\beta > a\beta + x^0 \quad 5(b)$$

Or when $B$ and $C$ are negative. Then $E_4 = (\ddot{x}, \ddot{y}, \ddot{z}, \ddot{w})$ exist under conditions 5(a-b) and when the following conditions are met

$$\ddot{y} > max\left\{\frac{d_4}{n_4}, \frac{d_4}{n_4}\right\} \quad 5(c)$$

$$x^0 > \ddot{x} \quad 5(d)$$

If not, the positive equilibrium point for system (1) will not exist.

Now, the local behavior of the five equilibrium points that we found for system (1) can be known by calculating the Jacobian matrix for system (1) and finding the eigenvalues of the matrix at each of the equilibrium points. Jacobian matrix for system (1) can be written at $(x, y, z, w)$ as follows

$$J = \begin{bmatrix} -d_1 - \frac{a\alpha y}{\beta(a+x)^2} & \frac{-\alpha x}{\beta(a+x)} & 0 & 0 \\ \frac{a\alpha y}{\beta(a+x)^2} & \frac{\alpha x}{\beta(a+x)} - d_2 - n_1 z - n_2 w & -n_1 y & -n_2 y \\ 0 & n_3 z & -d_3 & -\gamma_1 z \\ 0 & n_4 w & -\gamma_2 w & -d_4 \end{bmatrix} \tag{6}$$

Stability at $E_0$:

At $\mathrm{E}_0 = (x^0, 0,0,0)$ can be written Jacobian matrix of system (1) in the following form

$$J(E_0) = \begin{bmatrix} -d_1 & \frac{-\alpha x^0}{\beta(a+x^0)} & 0 & 0 \\ 0 & \frac{\alpha x^0}{\beta(a+x^0)} - d_2 & 0 & 0 \\ 0 & 0 & -d_3 & 0 \\ 0 & 0 & 0 & -d_4 \end{bmatrix}$$

The eigenvalues of $J(E_0)$ are: $\lambda_{0x} = -d_1$, $\lambda_{0y} = \frac{\alpha x^0}{\beta(a+x^0)} - d_2, \lambda_{0z} = -d_3$ and $\lambda_{0w} = -d_2$

This means that if the condition (7) is met, point $E_0$ will be locally asymptotically stable and vice versa

$$\alpha < d_2\beta + \frac{d_2\beta a}{x^0} \tag{7}$$

While the point is unstable saddle point in the $\mathrm{R}_+^4$ with a locally unstable manifold of dimension one (i.e. $\dim\omega^{\mathrm{u}} = 1$) and with a locally stable manifold of dimension three (i.e. $\dim\omega^{\mathrm{s}} = 3$) if the condition (2) is met. Therefore stability at $E_0$ leads to the non-existence of $E_1$.

Stability at $E_1$:

At $\mathrm{E}_1 = (\check{x}, \check{y}, 0,0)$ can be written Jacobian matrix of system (1) as follows:

$$J(E_1) = \begin{bmatrix} -d_1 - \frac{a\alpha\check{y}}{\beta(a+\check{x})^2} & -d_2 & 0 & 0 \\ \frac{a\alpha\check{y}}{\beta(a+\check{x})^2} & 0 & -n_1\check{y} & -n_2\check{y} \\ 0 & 0 & -d_3 & 0 \\ 0 & 0 & 0 & -d_4 \end{bmatrix}$$

Where the eigenvalues of $J(E_1)$ satisfy the following relations:

$$\lambda_{1x} + \lambda_{1y} = -d_1 - \frac{a\alpha\check{y}}{\beta(a+\check{x})^2} < 0 \quad 8(a)$$

$$\lambda_{1x}.\lambda_{1y} = \frac{ad_2\alpha\check{y}}{\beta(a+\check{x})^2} > 0 \quad 8(b)$$

$$\lambda_{1z} + \lambda_{1w} = -d_3 - d_4 < 0 \quad 8(c)$$

$$\lambda_{1z}.\lambda_{1w} = d_3 d_4 > 0 \quad 8(d)$$

According to Eqs. 8(a-d), $E_1$ is locally asymptotical stable according to Routh–Hurwitz criterion

Stability at $E_2$:

At $E_2 = (\bar{x}, \bar{y}, \bar{z}, 0)$ can be written Jacobian matrix of system (1) as follows:

$$J(E_2) = \begin{bmatrix} -d_1 - \frac{a\alpha\bar{y}}{\beta(a+\bar{x})^2} & \frac{-\alpha\bar{x}}{\beta(a+\bar{x})} & 0 & 0 \\ \frac{a\alpha y}{\beta(a+x)^2} & 0 & -n_1\bar{y} & -n_2\bar{y} \\ 0 & n_3\bar{z} & -d_3 & \gamma_1\bar{z} \\ 0 & 0 & 0 & -d_4 \end{bmatrix}$$

One of the eigenvalues of $J(E_2)$ is $\lambda_{2w} = -d_4$ and the other three eigenvalues can be given by the following quadratic equation: $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$

Where $\dot{A}_1 = d_1 + d_3 + \frac{a\alpha\bar{y}}{\beta(a+\bar{x})^2} > 0$ 9(a)

$$\dot{A}_2 = d_3\left(d_1 + \frac{a\alpha\bar{y}}{\beta(a+\bar{x})^2}\right) + n_1 n_3 \bar{y}\bar{z} + \frac{a\alpha^2\bar{x}\bar{y}}{\beta^2(a+\bar{x})^3} > 0 \quad 9(b)$$

$$\dot{A}_3 = n_1 n_3 \bar{y}\bar{z}\left(d_1 + \frac{a\alpha\bar{y}}{\beta(a+\bar{x})^2}\right) + \frac{d_3 a\alpha^2\bar{x}\bar{y}}{\beta^2(a+\bar{x})^3} > 0 \quad 9(c)$$

By Routh-Hurwitz criterion for dimension three, all the eigenvalues of $J(E_2)$ have roots with negative real parts if and only if $\mathrm{A_i}(\mathrm{i} = 1,3) > 0$ and $\Delta > 0$, where

$$\mathit{\Delta} = A_1 A_2 - A_3$$
$$= \left(d_1 + \frac{a\alpha\bar{y}}{\beta(a+\bar{x})^2}\right)\left[d_3\left(d_1 + \frac{a\alpha\bar{y}}{\beta(a+\bar{x})^2}\right) + \frac{a\alpha^2\bar{x}\bar{y}}{\beta^2(a+\bar{x})^3} + {d_3}^2\right] + d_3 n_1 n_3 \bar{y}\bar{z} > 0 \quad 9(d)$$

This is certain and clear from the Eqs. 9(a-d) the conditions of the Routh-Hurwitz are met, so $E_2$ is a locally asymptotical stable point wherever it is located.

Stability at $E_3$:

At $E_3 = (\dot{x}, \dot{y}, 0, \dot{w})$ can be written the Jacobian matrix of system (1) as follows:

$$J(E_3) = \begin{bmatrix} -d_1 - \frac{a\alpha\dot{y}}{\beta(a+\dot{x})^2} & \frac{-\alpha\dot{x}}{\beta(a+\dot{x})} & 0 & 0 \\ \frac{a\alpha\dot{y}}{\beta(a+\dot{x})^2} & 0 & -n_1\dot{y} & -n_2\dot{y} \\ 0 & 0 & -d_3 & 0 \\ 0 & n_4\dot{w} & -\gamma_2\dot{w} & -d_4 \end{bmatrix}$$

One of the eigenvalues of $J(E_3)$ is $\dot{\lambda}_{2z} = -d_3$ and the other three eigenvalues can be given by the following quadratic equation: $\dot{\lambda}^3 + B_1\dot{\lambda}^2 + B_2\dot{\lambda} + B_3 = 0$

Where $\dot{B}_1 = d_1 + d_4 + \frac{a\alpha\dot{y}}{\beta(a+\dot{x})^2} > 0$ 10(a)

$$\dot{B}_2 = d_4\left(d_1 + \frac{a\alpha\dot{y}}{\beta(a+\dot{x})^2}\right) + n_2n_4\dot{y}\dot{w} + \frac{a\alpha^2\dot{x}\dot{y}}{\beta^2(a+\dot{x})^3} > 0 \quad 10(b)$$

$$\dot{B}_3 = n_2n_4\dot{y}\dot{w}\left(d_1 + \frac{a\alpha\dot{y}}{\beta(a+\dot{x})^2}\right) + \frac{d_4a\alpha^2\dot{x}\dot{y}}{\beta^2(a+\dot{x})^3} > 0 \quad 10(c)$$

By Routh-Hurwitz criterion for dimension three, all the eigenvalues of $J(E_3)$ have roots with negative real parts if and only if $B_j(j = 1,3) > 0$ and $\dot{\Delta} > 0$, where

$$\dot{\Delta} = \dot{B}_1\dot{B}_2 - \dot{B}_3$$
$$= \left(d_1 + \frac{a\alpha\dot{y}}{\beta(a+\dot{x})^2}\right)\left[d_4\left(d_1 + \frac{a\alpha\dot{y}}{\beta(a+\dot{x})^2}\right) + \frac{a\alpha^2\dot{x}\dot{y}}{\beta^2(a+\dot{x})^3} + {d_4}^2\right] + d_4n_2n_4\dot{y}\dot{w} > 0 \quad 10(d)$$

This is certain and clear from the Eqs. 10(a-d) the conditions of the Routh-Hurwitz are met, so $E_3$ is a locally asymptotical stable point wherever it is located.

Stability at $E_4$:

Finally, at the positive equilibrium point $E_4 = (\ddot{x}, \ddot{y}, \ddot{z}, \ddot{w})$ can be written Jacobian matrix of system (1) as follows:

$$J(E_4) = \begin{bmatrix} -\left(d_1 + \frac{a\alpha\ddot{y}}{\beta(a+\ddot{x})^2}\right) & \frac{-\alpha\ddot{x}}{\beta(a+\ddot{x})} & 0 & 0 \\ \frac{a\alpha\ddot{y}}{\beta(a+\ddot{x})^2} & \frac{\alpha\ddot{x}}{\beta(a+\ddot{x})} - d_2 - n_1\ddot{z} - n_2\ddot{w} & -n_1\ddot{y} & -n_2\ddot{y} \\ 0 & n_3\ddot{z} & -d_3 & -\gamma_1\ddot{z} \\ 0 & n_4\ddot{w} & -\gamma_2\ddot{w} & -d_4 \end{bmatrix} \quad (11)$$

The characteristic equation of $J(E_4)$ can be written as follows:

$$\ddot{\lambda}^4 + \ddot{A}_1\ddot{\lambda}^3 + \ddot{A}_2\ddot{\lambda}^2 + \ddot{A}_3\ddot{\lambda} + \ddot{A}_4 = 0 \quad (12)$$

Where, $\ddot{A}_1 = -(m_1 + m_2)$, 12(a)

$$\ddot{A}_2 = a_{11}m_2 + m_3 + m_4 + m_5 + m_6, \tag{12(b)}$$

$$\ddot{A}_3 = a_{12}a_{21}m_2 - a_{11}m_6 - a_{11}m_4 - a_{22}m_6 - a_{11}m_5 + a_{42}m_7 + a_{32}m_8, \tag{12(c)}$$

$$\ddot{A}_4 = m_3m_6 - a_{11}a_{42}m_7 - a_{11}\, a_{32}m_8,$$

With

$$m_1 = a_{11} + a_{22},\ m_2 = a_{33} + a_{44} < 0,\ m_3 = a_{11}a_{22} - a_{12}a_{21},$$

$$m_4 = a_{22}a_{33} - a_{23}a_{32}\,, m_5 = a_{22}a_{44} - a_{24}a_{42},\ m_6 = a_{33}a_{44} - a_{34}a_{43} > 0,$$

$$m_7 = a_{42}(a_{24}a_{33} - a_{23}a_{34}),\ m_8 = a_{32}(a_{23}a_{44} - a_{24}a_{43}).$$

And

$$\ddot{\Delta} = \ddot{A}_1\ddot{A}_2\ddot{A}_3 - \ddot{A}_3^2 - \ddot{A}_1^2\ddot{A}_4 \tag{12(d)}$$

$$= -(m_1 + m_2)(a_{11}m_2 + m_3 + m_4 + m_5 + m_6)(a_{12}a_{21}m_2 - a_{11}(m_6 + m_4 + m_5) - a_{22}m_6 + a_{42}m_7 + a_{32}m_8) - \left(a_{12}a_{21}m_2 - a_{11}(m_6 + m_4 + m_5) - a_{22}m_6 + a_{42}m_7 + a_{32}m_8)\right)^2 - \left((m_1 + m_2)\right)^2(m_3m_6 - a_{11}a_{42}m_7 - a_{11}\, a_{32}m_8)$$

Thus, it can be proven that the positive equilibrium point is locally asymptotically stable according to the following theorem

**Theorem 2.** The positive equilibrium point $E_4 = (\ddot{x}, \ddot{y}, \ddot{z}, \ddot{w})$ of system (1) is locally asymptotically stable in the $Int.\,R_+^4$ under the following conditions:

$$\frac{\alpha\ddot{x}}{\beta(a+\ddot{x})} < d_2 + n_1\ddot{z} + n_2\ddot{w}, \tag{13(a)}$$

$$\frac{n_2d_3}{n_3d_4} < n_1\ddot{z} < \frac{n_2n_4d_3\ddot{w}}{\gamma_1}, \tag{13(b)}$$

$$\text{and} \quad \ddot{\Delta} > 0 \tag{13(c)}$$

**Proof:** By Routh-Hurwitz criterion for dimension four, all the eigenvalues of $J(E_4)$ have roots with negative real parts, if and only if $\ddot{A}_{\mathrm{i}}(\mathrm{i} = 1{,}3{,}4) > 0$ and $\ddot{\Delta} > 0$. Now, straightforward computations and elements of $J(E_4)$ due to the coefficients of equation (12), we get that $\ddot{A}_1 > 0$ under condition 13(a), so $\ddot{A}_{\mathrm{i}}(\mathrm{i} = 3{,}4) > 0$ under conditions 13(a-b), also the positive terms are greater than the negative terms for equation 12(d) under condition 13(c). Thus, all the eigenvalues of $J(E_4)$ contain negative real parts. As a result $E_4$ is locally asymptotically stable in the $Int.R_+^4$ and thus the proof ends.

**Theorem 3.** Assume that $E_4$ of model (1) is locally asymptotically stable in the $IntR_+^4$, and if The following terms are met:

$$\frac{an_4}{\beta(a+x)(a+\ddot{x})} - a < \ddot{x} < \frac{x^0 d_1}{\alpha\ddot{y}} \tag{14}$$

Then $E_4$ is globally asymptotically stable in the following region:

$\varphi = \{(x,y,z,w): x > \ddot{x}, y > \ddot{y}, z > \ddot{z}, w > \ddot{w}\}$

**Proof:**

Consider the following:

$$U(x,y,z,w) = c_1\left[x - \ddot{x} - \ddot{x}ln\frac{x}{\ddot{x}}\right] + c_2\left[y - \ddot{y} - \ddot{y}ln\frac{y}{\ddot{y}}\right] + c_3\left[z - \ddot{z} - \ddot{z}ln\frac{z}{\ddot{z}}\right]$$
$$+c_4\left[w - \ddot{w} - \ddot{w}ln\frac{w}{\ddot{w}}\right]$$

Clearly $U: R_+^4 \to R$ is $C^1$ . Now

$$\frac{dU}{dt} = c_1\frac{(x-\ddot{x})}{x}\frac{dx}{dt} + c_2\frac{(y-\ddot{y})}{y}\frac{dy}{dt} + c_3\frac{(z-\ddot{z})}{z}\frac{dz}{dt} + c_4\frac{(w-\ddot{w})}{w}\frac{dw}{dt}$$
$$= -c_1\left[\frac{x^0 d_1 - \alpha\ddot{x}\ddot{y}}{x\ddot{x}}\right](x-\ddot{x})^2 - \left(c_1(a\alpha + \alpha\ddot{x}) - c_2\frac{a\alpha}{\beta(a+x)(a+\ddot{x})}\right)(x-\ddot{x})(y-\ddot{y})$$
$$-[c_2n_1 - c_3n_3](y-\ddot{y})(z-\ddot{z}) - [c_2n_2 - c_4n_4](y-\ddot{y})(w-\ddot{w})$$
$$-[c_3\gamma_1 + c_4\gamma_2](w-\ddot{w})(z-\ddot{z})$$

By choosing the positive constant as:

$c_1 = 1,\ c_2 = n_4,\ c_3 = \frac{n_1n_4}{n_3},\ c_4 = n_2$

Then we obtain:

$$\frac{dU}{dt} = -\left[\frac{x^0 d_1 - \alpha\ddot{x}\ddot{y}}{x\ddot{x}}\right](x-\ddot{x})^2 - \left[\alpha(a+\ddot{x}) - \frac{a\alpha n_4}{\beta(a+x)(a+\ddot{x})}\right](x-\ddot{x})(y-\ddot{y})$$
$$-\left[\frac{n_1n_4\gamma_1}{n_3} + n_2\gamma_2\right](w-\ddot{w})(z-\ddot{z})$$

Clearly, $\frac{dU}{dt} < 0$ under condition (14).

Hence, $U$ is strictly a Lyapunov function. So, $E_4$ is a globally asymptoticaly stable in the $\varphi$.

## 5. BIFURCATION

In this part of the manuscript, we studied the type of local bifurcation of the positive point, as it is the most important point among the points we found. In order to know the type of local bifurcation of this point, we used Sotomayor's theorem. Most researchers studied some different types of bifurcation by using Sotomayor's' theory [5, 7, 9, 11, 12, 13, 21, 22, 25, 27, 28, 31], such as transcortical, saddle nodes and pitchfork bifurcation Further, can be reformulated model (1) as follows:

$$\frac{dN}{dt} = F(N) \text{ with } N = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \text{ and } F = \begin{bmatrix} xf_1(x,y,z,w) \\ yf_2(x,y,z,w) \\ zf_3(x,y,z,w) \\ wf_4(x,y,z,w) \end{bmatrix}.$$

Now, (6) it gives us the Jacobian matrix at any point, then for non-zero vector $A = (a_1, a_2, a_3, a_4)^T$:

$$DF = \begin{bmatrix} -\left(d_1 + \frac{a\alpha y}{\beta(a+x)^2}\right) a_1 - \frac{\alpha x}{\beta(a+x)} a_2 \\ \frac{a\alpha y}{\beta(a+x)^2} a_1 + \left(\frac{\alpha x}{\beta(a+x)} - d_2 - n_1 z - n_2 w\right) a_2 - n_1 y a_3 - n_2 y a_4 \\ n_3 z a_2 - d_3 a_3 - \gamma_1 z a_4 \\ n_4 w a_2 - \gamma_2 w a_3 - d_4 a_4 \end{bmatrix}$$

and,

$$D^2F = \begin{bmatrix} \frac{2a\alpha y}{\beta(a+x)^3} a_1^2 - \frac{2a\alpha}{\beta(a+x)^2} a_1 a_2 \\ \frac{-2a\alpha y}{\beta(a+x)^3} a_1^2 + \frac{2a\alpha}{\beta(a+x)^2} a_1 a_2 \\ n_3 a_2 a_3 - \gamma_1 a_3 a_4 \\ n_4 a_2 a_4 - \gamma_2 a_3 a_4 \end{bmatrix}$$

$$D^3F = \begin{bmatrix} \frac{-6a\alpha y}{\beta(a+x)^4} a_1^3 + \frac{6a\alpha}{\beta(a+x)^3} a_1^2 a_2 \\ \frac{6a\alpha y}{\beta(a+x)^3} a_1^3 - \frac{6a\alpha}{\beta(a+x)^3} a_1^2 a_2 \\ 0 \\ 0 \end{bmatrix}$$

**Theorem 4.** If $d_3 = d_3^*$ where

$$d_3^* = \frac{(a_{11}+a_{22}+a_{44})A_2A_3 - A_3^2 - A_1^2A_4}{A_2A_3}$$

And if condition (15) is met , system (1) have a saddle-noade bifurcation at $E_4$

$$\left(B^{[4]}\right)^T\left[D^2F_d(E_4, d_3^*)A^{[4]}\right] \neq 0 \tag{15}$$

**Proof:** $J(E_4)$, given by (11) at $d_3 = d_3^*$ can be inscribed as:

$$J^*(E_4, d_3^*) = \begin{bmatrix} -\left(d_1 + \frac{a\alpha\ddot{y}}{\beta(a+\ddot{x})^2}\right) & \frac{-\alpha\ddot{x}}{\beta(a+\ddot{x})} & 0 & 0 \\ \frac{a\alpha\ddot{y}}{\beta(a+\ddot{x})^2} & \frac{\alpha\ddot{x}}{\beta(a+\ddot{x})} - d_2 - n_1\ddot{z} - n_2\ddot{w} & -n_1\ddot{y} & -n_2\ddot{y} \\ 0 & n_3\ddot{z} & -d_3^* & -\gamma_1\ddot{z} \\ 0 & n_4\ddot{w} & -\gamma_2\ddot{w} & -d_4 \end{bmatrix}$$

The calculation tells that $(J^*(E_4, d_3^*)$ has zero eigenvalue, say $\lambda_{4z} = 0$.

Now, let $A^{[4]} = \left(a_1^{[4]}, a_2^{[4]}, a_3^{[4]}, a_4^{[4]}\right)^T$ the eigenvector matching to $\lambda_{4z} = 0$ , accordingly $(J^*(E_4) - \lambda_{4z}F)A^{[4]} = 0$ gives: $a_1^{[4]} = \frac{\alpha\ddot{x}(a+\ddot{x})}{k_1}a_3^{[4]}, a_2^{[4]} = \frac{k_1\beta\ddot{y}(a+\ddot{x})(n_1\ddot{z}k_4+n_2\ddot{w}k_3)}{k_4\ddot{z}(a\alpha^2\ddot{x}\ddot{y}+k_1k_2)}a_3^{[4]}, a_4^{[4]} = \frac{k_3\ddot{w}}{k_4\ddot{z}}a_3^{[4]}$, where $a_3^{[4]}$ be any nonzero real number.

Let $B^{[4]} = \left(b_1^{[4]}, b_2^{[4]}, b_3^{[4]}, b_4^{[4]}\right)^T$ the eigenvector associated to $\lambda_{4z} = 0$ of the $\left(J^*(E_4, d_3^*)\right)^T$.

$$J^{*T}(E_4, d_3^*) = \begin{bmatrix} -\left(d_1 + \frac{a\alpha\ddot{y}}{\beta(a+\ddot{x})^2}\right) & \frac{a\alpha\ddot{y}}{\beta(a+\ddot{x})^2} & 0 & 0 \\ \frac{-\alpha\ddot{x}}{\beta(a+\ddot{x})} & \frac{\alpha\ddot{x}}{\beta(a+\ddot{x})} - d_2 - n_1\ddot{z} - n_2\ddot{w} & n_3\ddot{z} & n_4\ddot{w} \\ 0 & -n_1\ddot{y} & -d_3^* & -\gamma_2\ddot{w} \\ 0 & -n_2\ddot{y} & -\gamma_1\ddot{z} & -d_4 \end{bmatrix}$$

Then $((J^*(E_4))^T - \lambda_{4z}I)B^{[4]} = 0$ gives : $b_1^{[4]} = \frac{a\alpha\ddot{y}}{k_1}b_2^{[4]}$ , $b_2^{[4]} = \frac{k_1\beta(a+\ddot{x})(n_3k_6\ddot{z}+n_4k_5\ddot{w})}{k_6(a\alpha^2\ddot{x}\ddot{y}-k_1k_2)}b_3^{[4]}$, $b_4^{[4]} = \frac{k_5}{k_6}b_3^{[4]}$ where $b_3^{[4]}$ any nonzero real number.

We will calculate the following to see whether the bifurcation of saddle nodes type satisfies the all conditions or not

$$\frac{\partial F}{\partial s} = F_{d_3}(N, d_3) = \left(\frac{\partial f_1}{\partial d_3}, \frac{\partial f_2}{\partial d_3}, \frac{\partial f_3}{\partial d_3}, \frac{\partial f_3}{\partial d_3}\right)^T = (0,0,1,0)^T$$

So $F_{d_3}(E_4, d_3^*) = (0,0,1,0)^T$ and $\left(B^{[4]}\right)^T F_{d_3^*} = \left(b_1^{[3]}, b_2^{[3]}, b_3^{[4]}, b_4^{[4]}\right)^T (0,0,1,0)^T = b_3^{[4]} \neq 0.$

Thus, the first condition for the bifurcation of the saddle nodes has been verified, while the conditions for pitchfork and transcortical bifurcation were not met

Here,

$$D^2 F_d(E_4, d_3^*) A^{[4]} = \begin{bmatrix} \frac{2a\alpha\ddot{y}}{\beta(a+\ddot{x})^3}\left(a_1^{[4]}\right)^2 - \frac{2a\alpha}{\beta(a+\ddot{x})^2} a_1^{[4]} a_2^{[4]} \\ \frac{-2a\alpha\ddot{y}}{\beta(a+\ddot{x})^3}\left(a_1^{[4]}\right)^2 + \frac{2a\alpha}{\beta(a+\ddot{x})^2} a_1^{[4]} a_2^{[4]} \\ n_3 a_2^{[4]} a_3^{[4]} - \gamma_1 a_3^{[4]} a_4^{[4]} \\ n_4 a_2^{[4]} a_4^{[4]} - \gamma_2 a_3^{[4]} a_4^{[4]} \end{bmatrix}$$

Hence,

$$\left(B^{[4]}\right)^T \left[D^2 F_d(E_4, d_3^*) A^{[4]}\right] = \left(b_1^{[4]}, b_2^{[4]}, b_3^{[4]}, b_4^{[4]}\right)^T \begin{bmatrix} \frac{2a\alpha\ddot{y}}{\beta(a+\ddot{x})^3}\left(a_1^{[4]}\right)^2 - \frac{2a\alpha}{\beta(a+\ddot{x})^2} a_1^{[4]} a_2^{[4]} \\ \frac{-2a\alpha\ddot{y}}{\beta(a+\ddot{x})^3}\left(a_1^{[4]}\right)^2 + \frac{2a\alpha}{\beta(a+\ddot{x})^2} a_1^{[4]} a_2^{[4]} \\ n_3 a_2^{[4]} a_3^{[4]} - \gamma_1 a_3^{[4]} a_4^{[4]} \\ n_4 a_2^{[4]} a_4^{[4]} - \gamma_2 a_3^{[4]} a_4^{[4]} \end{bmatrix}$$

$$= b_1^{[4]} \left[\frac{2a\alpha\ddot{y}\left(a_1^{[4]}\right)^2 - 2a\alpha(a+\ddot{x}) a_1^{[4]} a_2^{[4]}}{\beta(a+\ddot{x})^3}\right] + b_2^{[3]} \left[\frac{-2a\alpha\ddot{y}\left(a_1^{[4]}\right)^2 + 2a\alpha(a+\ddot{x}) a_1^{[4]} a_2^{[4]}}{\beta(a+\ddot{x})^3}\right]$$

$$+ b_3^{[4]} \left[a_3^{[4]} \left(n_3 a_2^{[4]} - \gamma_1 a_4^{[4]}\right)\right] + b_4^{[4]} \left[a_4^{[4]} \left(n_4 a_2^{[4]} - \gamma_2 a_3^{[4]}\right)\right]$$

It is clear that condition (15) guarantees the fulfillment of one of the conditions for the bifurcation of the saddle nodes. Thus, model (1) has saddle node bifurcation at $E_4$ with $d_3 = d_3^*$.

## 6. NUMERICAL ANALYSIS

In this part, we calculated the time series for system (1) by analyzing the system numerically by MATLAB and with appropriate data, as follows:

$x_0 = 0.69$ ; $d_1 = 0.25$; $\alpha = 0.35$; $\beta = 0.4$; $a = 0.2$; $d_2 = 0.4$; $n_1 = 0.7$; $n_2 = 0.6$; $n_3 = 0.61$; $d_3 = 0.12$; $\gamma_1 = 0.012$; $n_4 = 0.51$; $d_4 = 0.1$; $\gamma_2 = 0.02$ (16)

Now, after we took different ratios for the parameter ($x^0$) with parameters as in Eq.(16), we have seen an acceptable result on the dynamical behavior of model (1) as in fig. I (a-d)

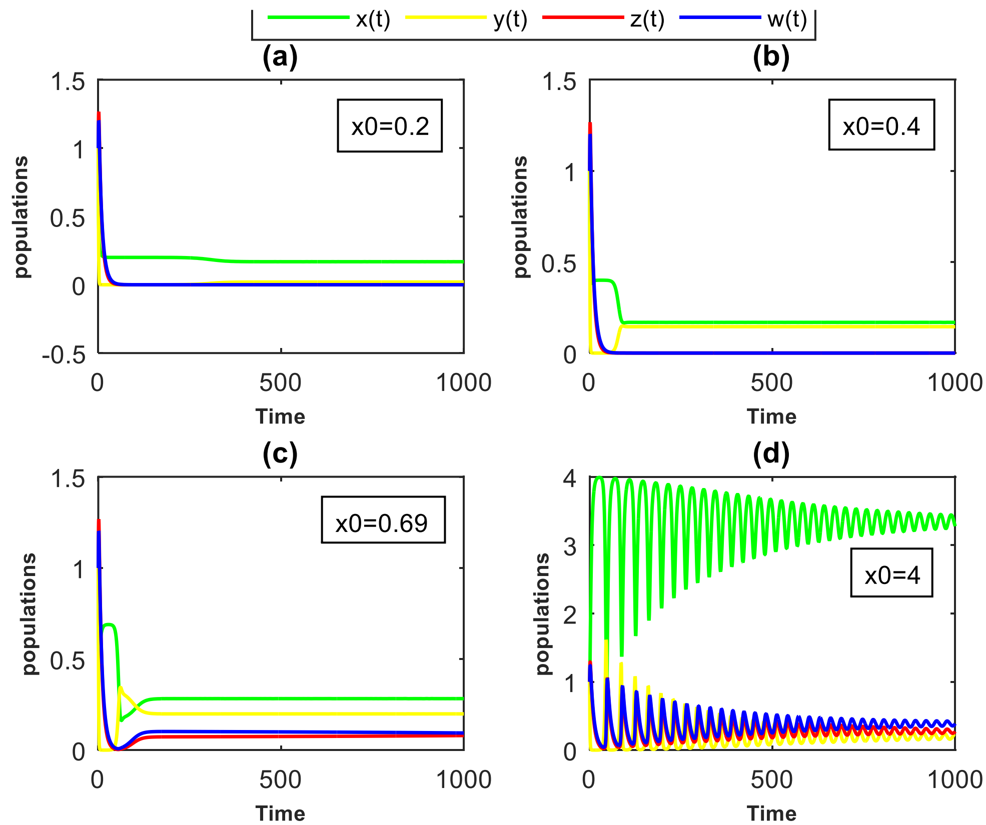


**fig. I:** Time series of model (1) with data as in Eq. (16) with varying of $x^0$ as shown in a,b,c and d respectively.

For fig. I, clearly that system (1) is stable at $E_0$ and $E_1$ for $x^0 < 0.21$ and for $0.21 \leq x^0 < 0.53$ as in the fig. I(c-d) respectively. While system (1) has been stable at $E_4$ for $0.53 \leq x^0 < 0.84$ as in the fig. I(c), finally the system oscillatory around $E_4$ for $x^0 \geq 0.84$ as in the fig. I(d).

Until we can study the effect of nutrient deficiencies $d_1$ with parameters as in Eq. (16), we have seen an acceptable result on the dynamical behavior of model (1) as in fig. II (a-d).

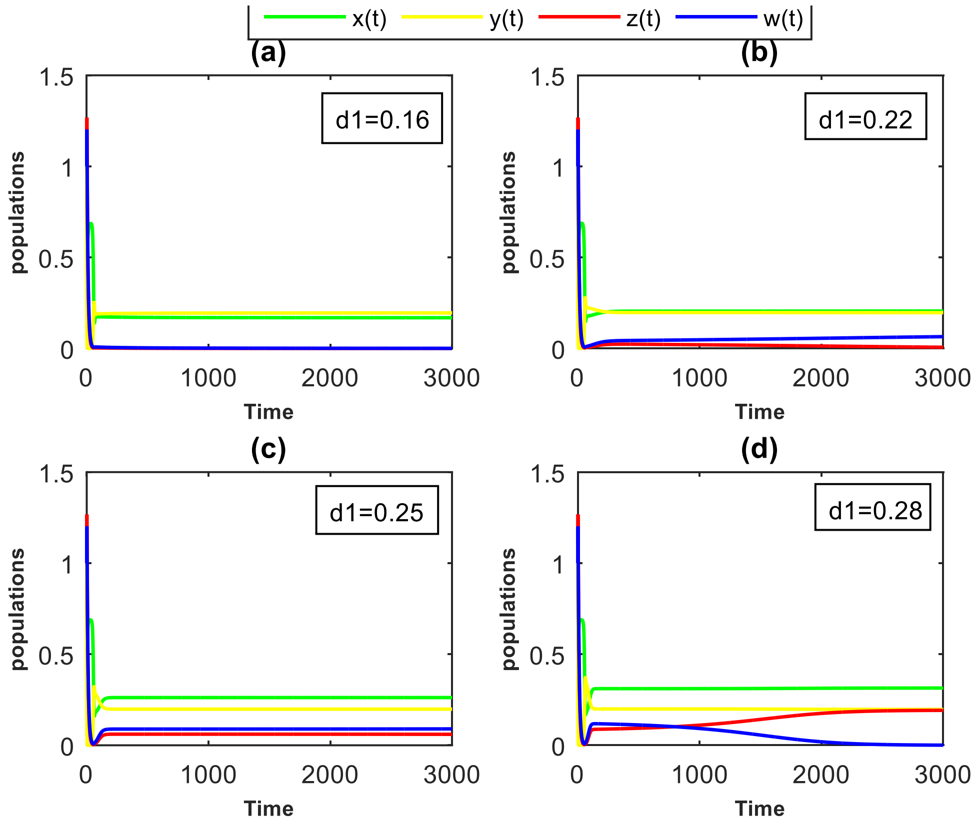


**fig. II:** Time series of model (1) with data as in Eq. (16) with varying of $d_1$ as shown in a,b,c and d respectively.

For fig. II, clearly that system (1) is stable at $E_1$ for $d_1 \leq 0.16$ as shown in the fig. II(a) while for $0.16 < d_1 < 0.23$ the system is stable at $E_2$ as shown in the fig. II(b) . Finally, the system is stable at $E_4$ and $E_3$ for $0.23 \leq d_1 < 0.26$ and $d_1 \geq 0.26$ as shown in the fig. II(c-d) respectively.

Until we can study the effect of maximum nutrition $\propto$ with parameters as in Eq. (16), we have seen an acceptable result on the dynamical behavior of model (1) as in fig. III (a-c)

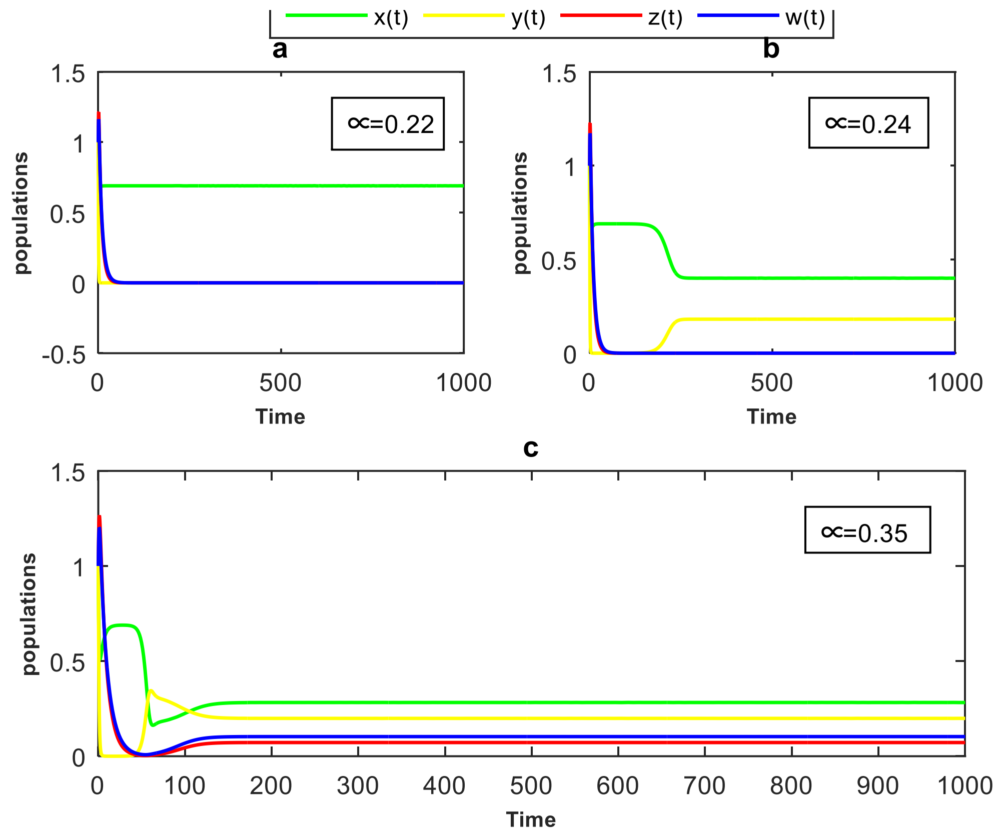


**fig. III:** Time series of model (1) with data as in Eq. (16) with varying of $\alpha$ as shown in a,b and c respectively.

Noticeably, $\propto$ when it decreases then the system is stable at $E_0$ for $\alpha \leq 0.22$ as shown in the fig. III(a), while the system has stable at $E_1$ for $0.22 < \alpha \leq 0.26$ as shown in the fig. III(b), finally for $0.26 < \alpha$ the system has stable at $E_4$ as shown in the fig. III(c).

With certain values of the coefficient of competition $\gamma_1$ of the first predator $z$ with parameters as in Eq. (16), we have seen an acceptable result on the dynamical behavior of model (1) as in fig. IV (a-c)

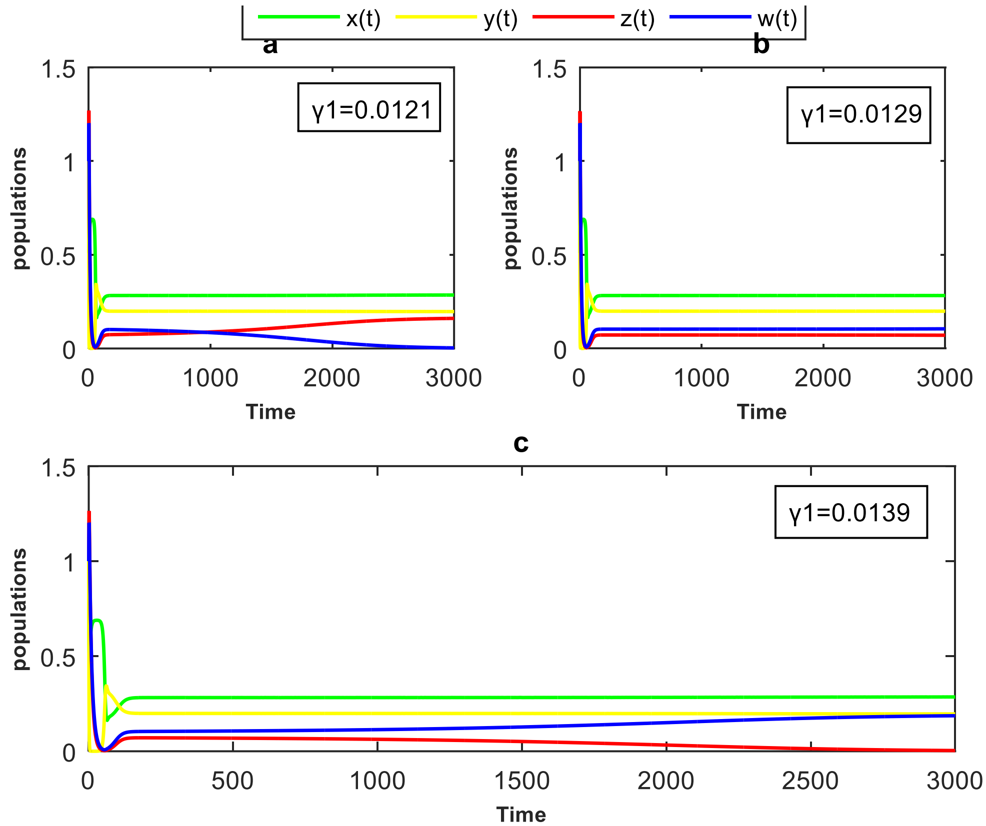


**fig. IV:** Time series of model (1) with data as in Eq. (16) with varying of $\gamma_1$as shown in a,b and c respectively.

Clearly, as $\gamma_1$ when it decreases then the system is stable at $E_2$ for $\gamma_1 \leq 0.0124$ as shown in the fig. IV(a), while the system has stable at $E_4$ for $0.0124 < \gamma_1 < 0.0133$ as shown in the fig. IV(b), finally for $0.0133 \leq \gamma_1$ the system is stable at $E_3$ as shown in the fig. IV(c).

Finally, with certain values of the coefficient of competition $\gamma_2$ of the secnd predator $w$ with parameters as in Eq. (16), we have seen an acceptable result on the dynamical behavior of model (1) as in fig. V (a-c)

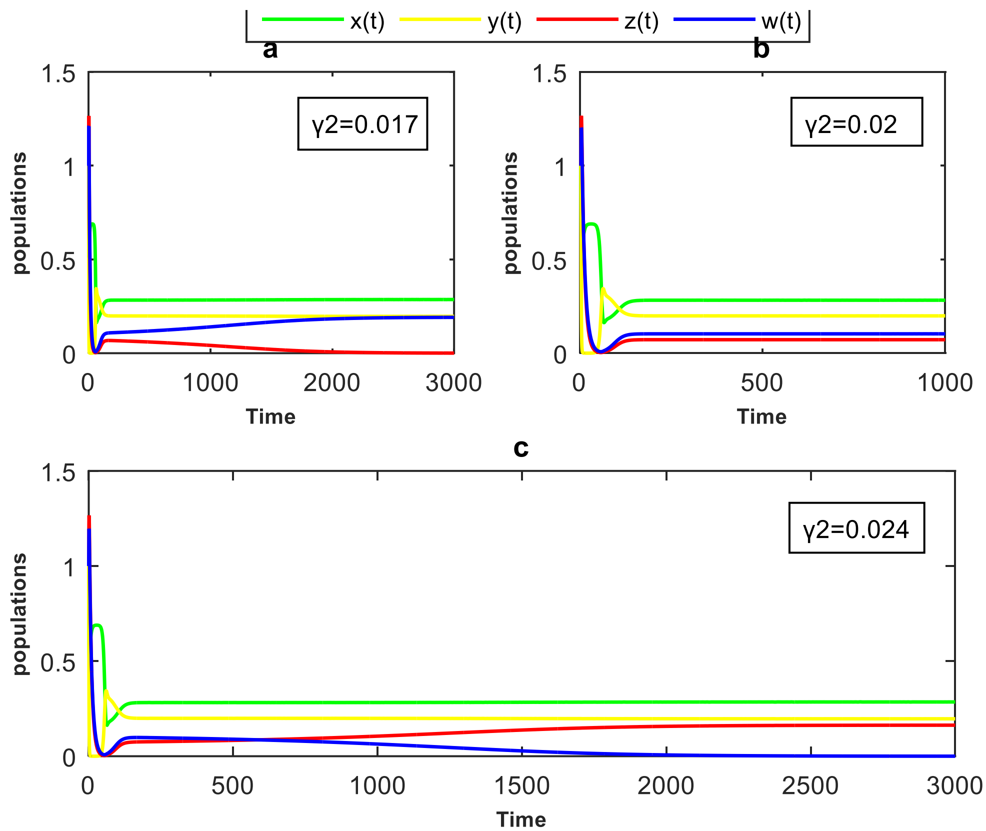


**fig. V:** Time series of model (1) with data as in Eq. (16) with varying of $\gamma_2$as shown in a,b and c respectively.

Clearly, as $\gamma_2$ when it decreases then the system is stable at $E_3$ for $\gamma_2 \leq 0.017$ as shown in the fig. V(a), while the system has stable at $E_4$ for $0.017 < \gamma_2 < 0.02$ as shown in the fig. V(b), finally for $0.02 \leq \gamma_2$ the system is stable at $E_3$ as shown in the fig. V(c).

## 7. DISSCUTION

In order to know the influence of increasing & decreasing the concentration of nutrients found in nature on the growth rate of organisms that feed on these nutrients, as well as studying the effect of competition between two predators competing to feed on those organisms for an ecosystem consisting of nutrients, organisms and two predators compete to feed on those organisms**.**We have studied the dynamical behavior of model (1) theoretically by finding local stability conditions of the five points, finding conditions for the global stability of the positive point, and also finding conditions for the bifurcation of the positive point.

After the numerical study that we studied in part (6) , we obtained acceptable results when any change occurred in some parameters, shown as follows:

Regarding the effect of changing the rate of concentration of nutrients in nature, with all parameters remaining the same in Eq. (16) we note the time series of model (1) approaches to $E_0$ and $E_1$ for $x^0 < 0.21$ and for $0.21 \leq x^0 < 0.53$ respectively. While the system has been stable at $E_4$ for $0.53 \leq x^0 < 0.84$, finally the system oscillatory around $E_4$ and still stable for $x^0 \geq 0.84$

The influence of the rate of decrease in nutrients $d_1$ on system dynamics with parameters as in (16), it is witnessed the system has the presence of nutrients only for $d_1 \leq 0.16$, while for $0.16 < d_1 < 0.23$ the system has the presence of nutrients with the presence of living organisms, while for $0.23 \leq d_1 < 0.26$, the system has the presence of nutrients with the presence of living organisms as well as the presence of predators. Finally, the system loses one of its predators and approaches to the $E_3$ for $d_1 \geq 0.26$.

For the influence of varying the maximum nutrition $\propto$ on system dynamics with parameters as in (16) it is witnessed the system has the presence of nutrients only for $\alpha \leq 0.22$, while for $0.22 < \alpha \leq 0.26$ the system has the presence of nutrients with the presence of living organisms, finally for $0.26 < \alpha$ the system has the presence of nutrients with the presence of living organisms as well as the presence of predators.

for the influence of varying of $\gamma_1$ on system dynamics with parameters as in (16) it is witnessed the system loses the presence of the second predator and approaches to $E_2$ for $\gamma_1 \leq 0.0124$, while for $0.0124 < \gamma_1 < 0.0133$, the system has the presence of nutrients with the presence of living organisms as well as the presence of predators, finally once again, the system loses one of the predators, but this time it loses the first predator, and the system approaches to $E_3$ for $0.0133 \leq \gamma_1$.

Finally, for the influence of varying $\gamma_2$ on system dynamics with parameters as in (16) it is witnessed the system loses the presence of the first predator and approaches to $E_3$ for $\gamma_2 \leq 0.017$, while for $0.017 < \gamma_2 < 0.02$, the system has the presence of nutrients with the presence of living organisms as well as the presence of predators, finally once again, the system loses one of the predators, but this time it loses the second predator, and the system approaches to $E_2$ for $0.02 \leq \gamma_2$.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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