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DYNAMICAL ANALYSIS OF A FRACTIONAL ORDER SEIT EPIDEMIC MODEL FOR TB SPREAD UNDER INFLUENCE OF VACCINATION

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Abstract. In this work, we present a fractional *SEIT* mathematical model for Tuberculosis (TB) spread in a human population under the influence of vaccination and investigate the stability of the endemic and disease-free equilibrium points of the model. In addition, we also study the effect of vaccination on the basic reproduction number and the number of infected individuals. It was shown that the stability of disease-free and endemic equilibrium points depends on the basic reproduction number. The analysis reveals that as vaccination increases, both the basic reproduction number and the number of infected individuals decrease.

Keywords: fractional-order derivative; SEIT model; basic reproduction number; equilibrium.

2020 AMS Subject Classification: 49L20, 92D25.

1. INTRODUCTION

Epidemics of tuberculosis (TB) have had a significant effect on the human population. Despite several decades of study, the widespread availability of a vaccine, and a clear WHO drive to support a coordinated worldwide TB control plan, tuberculosis remains a major cause of infectious disease-related mortality [1, 2].

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Numerous researchers have used mathematical models in the form of nonlinear differential equations to study the spread of tuberculosis (TB) disease, see [3, 4, 5, 6, 7, 8, 9]. It has been demonstrated that mathematical modeling plays a significant role in assessing the efficacy of different control and preventative measures as well as in better understanding the dynamics of tuberculosis transmission.

One of the much-explored models of tuberculosis transmission in the form of non-linear differential equations is the $SEIT$ model [10, 11, 12]. In their model, the observed human population at time t $(N(t))$ is divided into fourth epidemiological sub-compartments, that are susceptible $S(t)$, exposed during the latent period $E(t)$, TB active (infected) $I(t)$, undergoing treatment $T(t)$. In general, a simple compartment diagram for $SEIT$ model is given in the Figure 1, where the parameter Λ denotes the birth rate, μ is the natural death rate, β is the contact rate betwen susceptible and infected, η is the rate of moving from exposed to infected, and γ is the rate of moving from infected to treated.

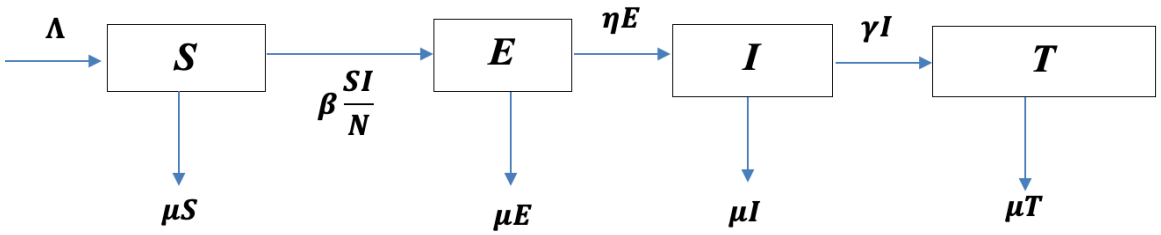

FIGURE 1. Simple Compartment diagram of $SEIT$ model

Based on Figure 1, various variations of the $SEIT$ model in the form of non-linear differential equations with usual derivatives have been developed as given in [10, 11, 12, 13].

In this paper, we modify these existing model $SEIT$ by replacing the usual derivative with a fractional derivative and providing vaccination to susceptible individuals. It is known that the use of fractional derivatives in epidemic models is currently widely explored, because fractional derivatives are trusted as a generalization of integer order derivatives, so modeling using fractional differential equations is a powerful method for studying the overall spread of the disease see [14, 15, 16]. The $SEIT$ model to be studied in this paper is given as follows:

$$
\begin{aligned}
\mathscr{D}^{(\sigma)}S &= \Lambda - (\mu+\varepsilon)S - (1-\varepsilon)\beta S\frac{I}{N},\\
\mathscr{D}^{(\sigma)}E &= (1-\varepsilon)\beta S\frac{I}{N} - (\mu+\eta)E,\\
\mathscr{D}^{(\sigma)}I &= \eta E - (\mu+\alpha+\gamma)I,\\
\mathscr{D}^{(\sigma)}T &= \gamma I + \varepsilon S - \mu R.
\end{aligned} \tag{1}
$$

where ε is percentage of susceptible population vaccinated, $\mathscr{D}^{(\sigma)}$ is the Caputo fractional derivative operator of oder σ with $0 < \sigma < 1$. The development of the model (1) is subject to assumptions that all the newborns are in the susceptible. After one unit time, a susceptible individual can be infected through contacting with the infectious individuals and enter the latent class or is still in the susceptible class or dies. A latent individual may become infectious and enter the infectious class or still stay in the latent class or die. An infectious individual may be treated and enter the treated/recovery class or stay in the infectious class or die. For a treated individual, he (or she) may recover by effective treatment. The treated individuals are assumed to have recovered or died, so the T and R compartments can be combined. Figure 2 shows the relationship between the four variables of their $SEIT$ model.

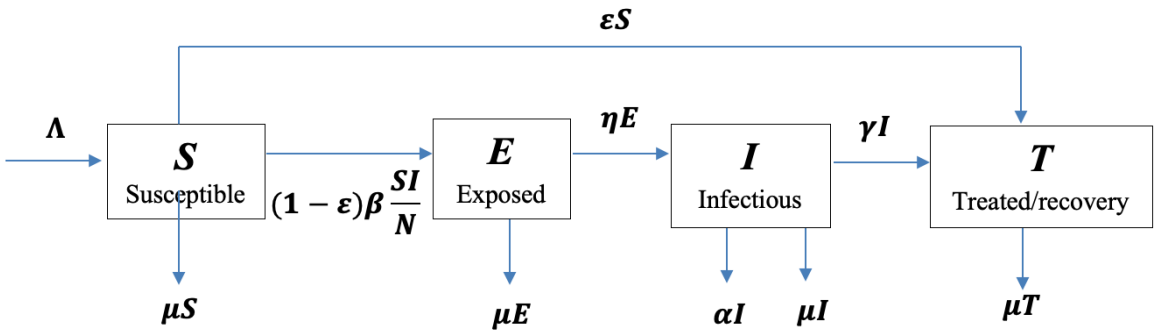

FIGURE 2. Compartment diagram of $SEIT$ model

We examine the stability of the endemic and disease-free equilibriums of the model (1) as a new $SEIT$ model. In addition, we also study the effect of vaccination on the basic reproduction number and the number of infected individuals. As far as the authors know, there is still no

solution to this problem. As a result, the findings of this study represent both a novel and a fresh advancement in the field of fractional-order epidemic dynamics.

The paper is organized as follows: Section 2 presents some mathematical concepts needed to analyze the stability of fractional order dynamical systems. The main result of this article is presented in the section 3. Section 4 concludes the paper.

2. PREREQUISITES

The requirements and fundamental ideas used throughout this paper are covered in this part. The Caputo fractional derivative of order σ with $\sigma \in (j-1, j)$, $j \in \mathbb{N}$ for the integrable vector function $\mathbf{x} : [0,\infty) \to \mathbb{R}^n$, is defined by

$$\mathscr{D}^{(\sigma)}\mathbf{x}(t) = \frac{1}{\Gamma(j-\sigma)} \int_0^t (t-\tau)^{j-\sigma-1} \mathscr{D}^{(j)}\mathbf{x}(\tau) d\tau \tag{2}$$

where $\Gamma(.)$ is the Euler Gamma function [17], and $\mathscr{D}^{(j)}\mathbf{x}(.)$ is the usual j th derivative of function $\mathbf{x}(.)$.

Let us consider the fractional-order nonlinear system involving Caputo derivative

$$\mathscr{D}^{(\sigma)}\mathbf{x}(t) = \mathbf{g}(t, \mathbf{x}(t)) \tag{3}$$

with suitable initial conditions $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector of the system (3), $\mathbf{g} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$. Specifically, the system (3) can be written as

$$\mathscr{D}^{(\sigma)}\mathbf{x}(t) = A\mathbf{x}(t)), \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$ if $\mathbf{g}$ is linear. The point $\mathbf{x}^*$ is said the equilibrium point of the system (3) if $\mathbf{f}(t, \mathbf{x}^*) = \mathbf{0}$.

Theorem 2.1. [18] *If* $|\arg(\lambda_j)| > \dfrac{\sigma\pi}{2}$ *for each eigenvalues* λ_j, $j = 1, 2, \cdots, n$ *of the matrix* A, *then the fractional-order linear system (4) with* $\sigma \in (0,1)$ *is asymptotically stable*

Theorem 2.2. [18] *Let* $\mathbf{x} = \mathbf{x}^*$ *is an equilibrium of the the fractional-order system (3) with* $\sigma \in (0,1)$. *The equilibrium point* $\mathbf{x} = \mathbf{x}^*$ *is asymptotically stable if*

$$|\arg(\lambda)| > \frac{\sigma\pi}{2}, \tag{5}$$

for all roots λ of the equation

$$|J_{\mathbf{x}^*} - \lambda I| = 0 \tag{6}$$

where $J_{\mathbf{x}^*}$ is the Jacobian matrix of system (3) at the equilibrium $\mathbf{x}^*$.

3. Mathematical Analysis of the System

3.1. Existence of Equilibria and Basic Reproduction Number.

By setting

$$\mathscr{D}^{(\sigma)}S = \mathscr{D}^{(\sigma)}E = \mathscr{D}^{(\sigma)}I = \mathscr{D}^{(\sigma)}T = 0, \tag{7}$$

the model system (1) gives two equilibria: the TB free-equilibrium, denoted as $\mathscr{E}_0$, and the TB endemic equilibrium, denoted as $\mathscr{E}_e$. By setting $I = 0$, one gets the TB free-equilibrium $\mathscr{E}_0 = \left(\frac{\Lambda}{\mu+\varepsilon},\ 0,\ 0,\ \frac{\Lambda\varepsilon}{\mu(\mu+\varepsilon)}\right)$. This TB free-equilibrium is used to determine the basic reproduction number.

It is well known that the basic reproduction number, denoted as $\Re_0$, is a measure of the potential for disease spread in a population. The basic reproduction number is the total number of secondary infections that a single primary infection can cause in a fully susceptible population. The basic reproduction number can be calculated using the Next-Generation Matrix technique [19]. The matrix $\mathscr{F}$ dan $\mathscr{V}$ for our model are as follows:

$$\mathscr{F} = \begin{pmatrix} (1-\varepsilon)\beta S\dfrac{I}{N} \\ 0 \end{pmatrix}, \mathscr{V} = \begin{pmatrix} (\mu+\eta)E \\ -\eta E + (\mu+\alpha+\gamma)I \end{pmatrix},$$

The Jacobian of the above matrices evaluated at the TB-free equilibrium point are given by

$$F = \begin{bmatrix} 0 & \frac{(1-\varepsilon)\beta\Lambda}{(\mu+\varepsilon)N} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \mu+\eta & 0 \\ -\eta & \mu+\alpha+\gamma \end{bmatrix}.$$

The spectral radius of the matrix FV^{-1} gives the basic reproduction number $\Re_0$ and it is given by

$$\Re_0 = \frac{\eta(1-\varepsilon)\beta\Lambda}{(\mu+\eta)(\mu+\alpha+\gamma)(\mu+\varepsilon)N}. \tag{8}$$

Next, by setting $I_e > 0$, one gets the TB endemic equilibrium $\mathscr{E}_e = (S_e,\ E_e,\ I_e,\ T_e)$, where

$$S_{\mathrm{e}} = \frac{\Lambda}{\Re_0(\mu+\varepsilon)},\ E_{\mathrm{e}} = \frac{\Lambda}{\mu+\eta}\left(1-\frac{1}{\Re_0}\right),\ I_{\mathrm{e}} = \frac{\Lambda\eta}{(\mu+\eta)(\mu+\alpha+\gamma)}\left(1-\frac{1}{\Re_0}\right),$$
$$T_{\mathrm{e}} = \frac{\gamma(\mu+\varepsilon)N}{\mu(1-\varepsilon)\beta}(\Re_0-1)+\frac{\varepsilon\Lambda}{\mu\Re_0(\mu+\varepsilon)}.$$

3.2. Stability Analysis. To conduct the stability analysis, we first obtain the system's Jacobian (1), as follows:

$$\mathscr{J} = \begin{bmatrix} -\mu-(1-\varepsilon)\beta\frac{I}{N}-\varepsilon & 0 & -(1-\varepsilon)\beta\frac{S}{N} & 0 \\ (1-\varepsilon)\beta\frac{I}{N} & -\mu-\eta & (1-\varepsilon)\beta\frac{S}{N} & 0 \\ 0 & \eta & -\mu-\alpha-\gamma & 0 \\ \varepsilon & 0 & \gamma & -\mu \end{bmatrix}.$$

The Jacobian at around the TB free-equilibrium is given by the following matrix:

$$\mathscr{J}_{\mathscr{E}_0} = \begin{bmatrix} -\mu-\varepsilon & 0 & -\frac{(1-\varepsilon)\beta\Lambda}{(\mu+\varepsilon)N} & 0 \\ 0 & -\mu-\eta & \frac{(1-\varepsilon)\beta\Lambda}{(\mu+\varepsilon)N} & 0 \\ 0 & \eta & -\mu-\alpha-\gamma & 0 \\ \varepsilon & 0 & \gamma & -\mu \end{bmatrix}. \tag{9}$$

Based on the Jacobian (9) one gets the following characteristic polynomial:

$$(-\mu-\lambda)(-\mu-\lambda)\left(\lambda^2+(a+b)\lambda+(1-\Re_0)ab\right) = 0, \tag{10}$$

where $a=\mu+\eta$ dan $b=\mu+\alpha+\gamma$. The Jacobian matrix $\mathscr{J}_{\mathscr{E}_0}$ possesses the four eigenvalues that given by $\lambda_1=-\mu-\delta, \lambda_2=-\mu$, and λ_3 and λ_4 are the roots of the equation:

$$\lambda^2+(a+b)\lambda+(1-\Re_0)ab=0 \tag{11}$$

It is obvious that eigenvalues λ_i, for $i=1,2$, are negative, thus they satisfy $|\arg(\lambda_i)|>\frac{\sigma\pi}{2}$, whereas $|\arg(\lambda_{3,4})|>\frac{\sigma\pi}{2}$ if $\Re_0<1$, and it implies $|\arg(\lambda_{3,4})|<\frac{\sigma\pi}{2}$ when $\Re_0>1$. Hence, based on the Theorem 2.2, $\mathscr{E}_0$ is asymptotically stable if $\Re_0<1$ and becomes unstable if $\Re_0>1$.

Furthermore, the Jacobian at around the TB endemic equilibrium $\mathscr{E}_{\mathrm{e}}$ is given by

$$\mathscr{J}_{\mathscr{E}_e} = \begin{bmatrix} -\mu-(1-\varepsilon)\beta\frac{I_e}{N}-\varepsilon & 0 & -(1-\varepsilon)\beta\frac{S_e}{N} & 0 \\ (1-\varepsilon)\beta\frac{I_e}{N} & -\mu-\eta & (1-\varepsilon)\beta\frac{S_e}{N} & 0 \\ 0 & \eta & -\mu-\alpha-\gamma & 0 \\ \varepsilon & 0 & \gamma & -\mu \end{bmatrix}. \tag{12}$$

Based on the Jacobian (12) one gets the following characteristic polynomial:

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{13}$$

where

$$\begin{aligned} a_1 &= 2\mu + (\mu+\varepsilon)(\Re_0-1) + \varepsilon + (\mu+\eta) + (\mu+\alpha+\gamma), \\ a_2 &= \mu\left(\mu + (\mu+\varepsilon)(\Re_0-1) + \varepsilon + (\mu+\eta) + (\mu+\alpha+\gamma)\right) \\ &\quad + \left(\mu + (\mu+\varepsilon)(\Re_0-1) + \varepsilon\right)(\mu+\eta)(\mu+\alpha+\gamma), \\ a_3 &= \mu\left(\mu + (\mu+\varepsilon)(\Re_0-1) + \varepsilon\right)\left((\mu+\eta) + (\mu+\alpha+\gamma)\right) \\ &\quad + (\mu+\eta)(\mu+\alpha+\gamma)(\mu+\varepsilon)(\Re_0-1), \\ a_4 &= \mu(\mu+\eta)(\mu+\alpha+\gamma)(\mu+\varepsilon)(\Re_0-1). \end{aligned}$$

Using the *Routh-Hurwitz*'s criteria, the eigenvalues $\lambda_i,\ i=1,2,3,4$ of the matrix $\mathscr{J}_{\mathscr{E}_e}$ are negative if $a_1>0, a_1a_2-a_3>0, (a_1a_2-a_3)a_3-a_1^2a_4>0$ and $a_4>0$ and this is satisfied if $\Re_0>1$. This implies $|\arg(\lambda_i)| > \dfrac{\sigma\pi}{2}$ for $i=1,2,3,4$ when $\Re_0>1$. Hence, based on the Theorem 2.2, $\mathscr{E}_e$ is asymptotically stable if $\Re_0>1$ and becomes unstable if $\Re_0<1$.

3.3. Numerical Simulation. In order to show the validity of the results, in the following we present a numerical simulation. For the model (1), let $\mu=0.05, \beta=0.3, \eta=0.4, \alpha=0.003, \gamma=0.2$ and $N=10^6$. The initial conditions are $S(0)=997999, E(0)=1800, I(0)=151$ and $T(0)=50$. Based on these parameter values, we find that the TB-free equilibrium point is $\mathscr{E}_0=(4.3478\times10^5, 0, 0, 5.6522\times10^5)$ and $\Re_0=0.4285$ for $\varepsilon=0,065$. This shows that the TB-free equilibrium point $\mathscr{E}_0$ is asymptotic stable due to $\Re_0<1$. Graphs of the susceptible subpopulation, exposed subpopulation, infected subpopulation, and treated subpopulation for several fractional order are given in Figure 3.

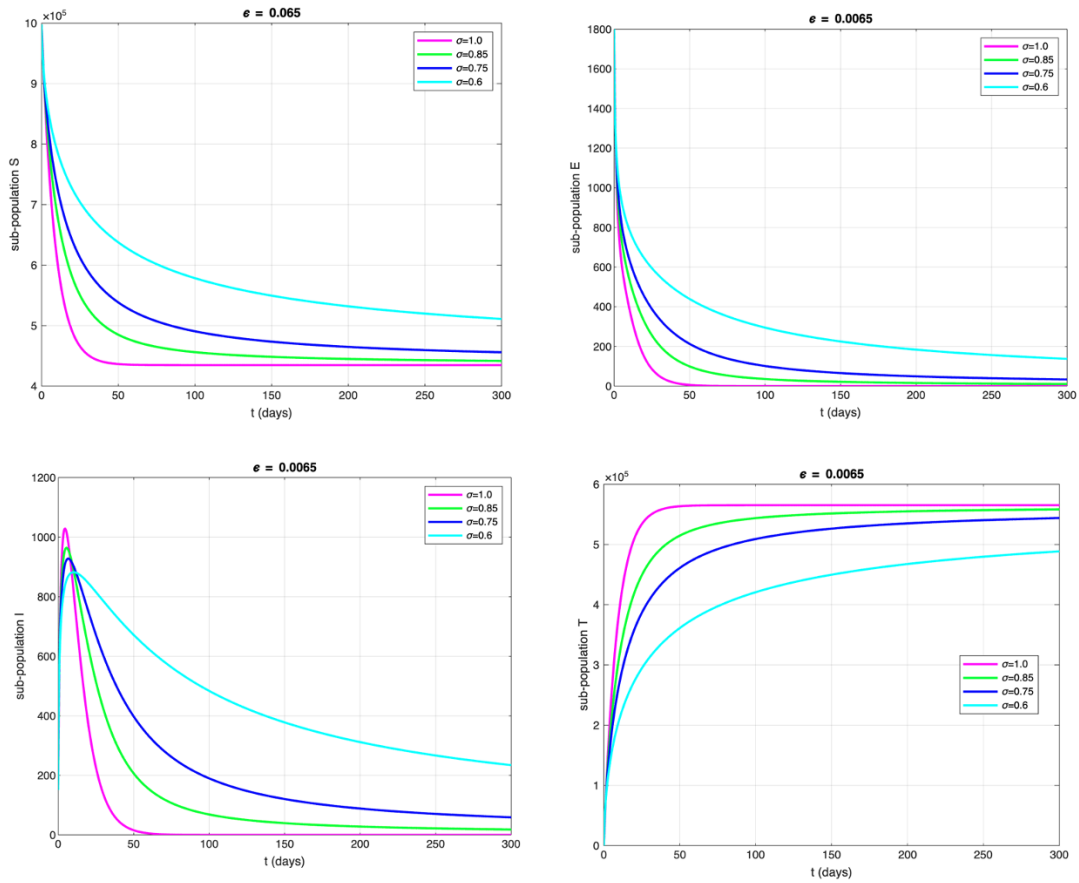

FIGURE 3. The curves of susceptible, exposed, infected and treated for TB-free equilibrium with $\varepsilon = 0.065$

Next, for the above parameters value, if the value of β is replaced with $\beta = 0.75$ we find that

i. $\mathfrak{R}_0 = 2.6350$ and $\mathscr{E}_e = (3.795 \times 10^5, 6.8944 \times 10^4, 1.09 \times 10^5, 4.3601 \times 10^5)$ if $\varepsilon = 0$.

ii. $\mathfrak{R}_0 = 1.598$ and $\mathscr{E}_e = (3.912 \times 10^5, 4.156 \times 10^4, 6.570 \times 10^4, 4.976 \times 10^5)$ if $\varepsilon = 0,03$.

ii. $\mathfrak{R}_0 = 1.0712$ and $\mathscr{E}_e = (4.059 \times 10^5, 7.386 \times 10^3, 1.168 \times 10^4, 5.744 \times 10^5)$ if $\varepsilon = 0,065$.

One can see that the TB-endemic equilibrium $\mathscr{E}_e$ is asymptotic stable due to $\mathfrak{R}_0 > 1$. It can also see that the basic reproduction number and the number of infected individuals decrease as vaccination increases. Graphs of the susceptible subpopulation, exposed subpopulation, infected subpopulation, and treated subpopulation for several fractional order are given in Figure 4, Figure 5, and Figure 6, respectively.

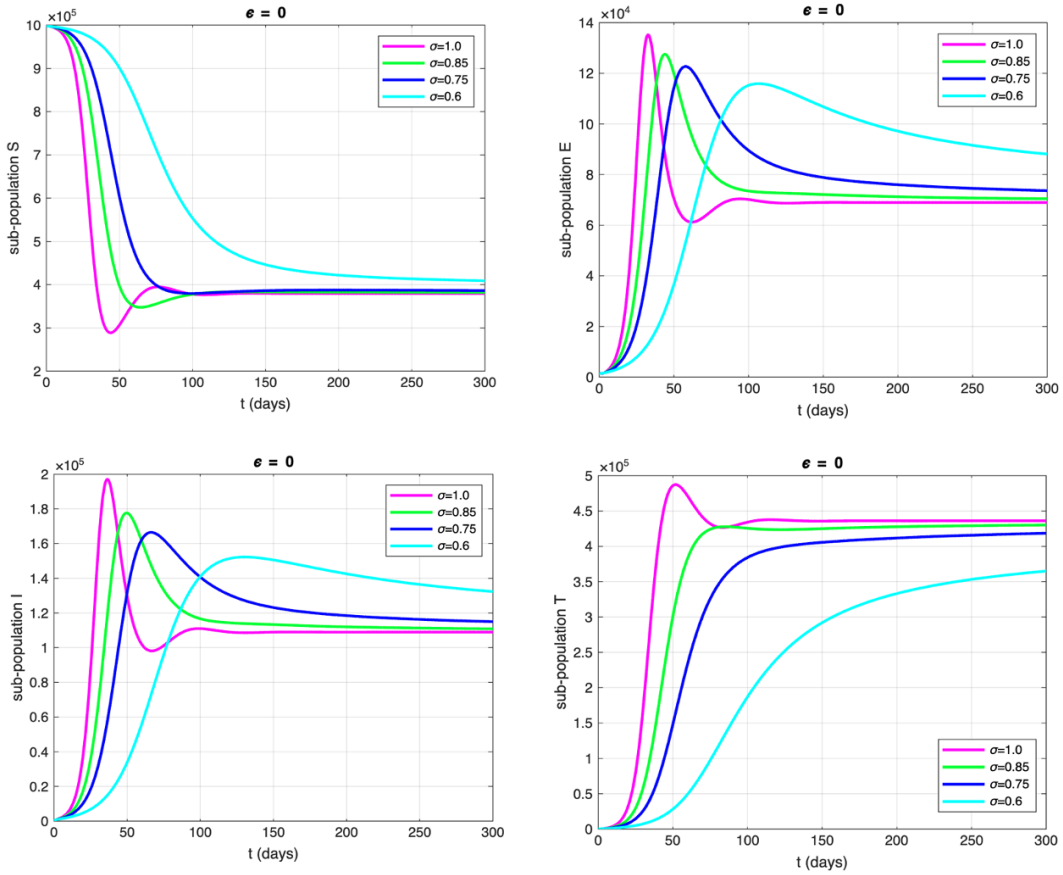

FIGURE 4. The curves of susceptible, exposed, infected and treated for $\varepsilon = 0$

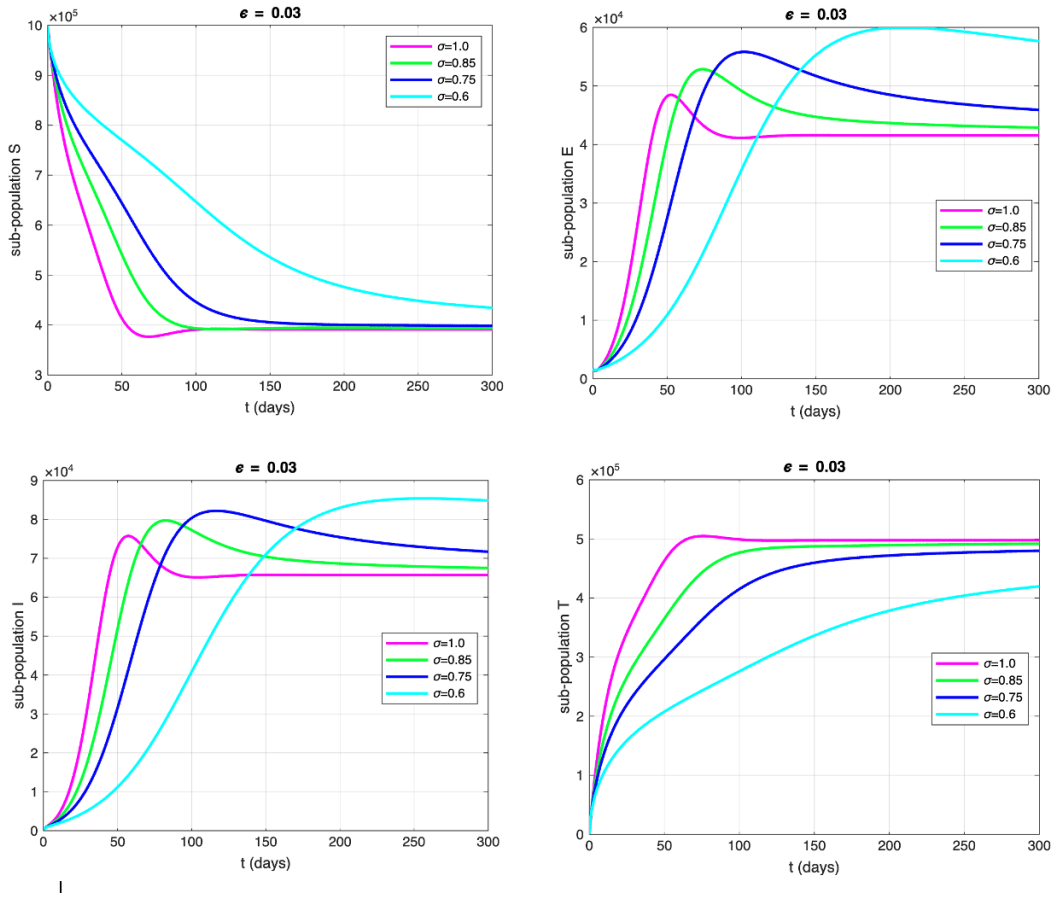

FIGURE 5. The curves of susceptible, exposed, infected and treated for $\varepsilon = 0.03$

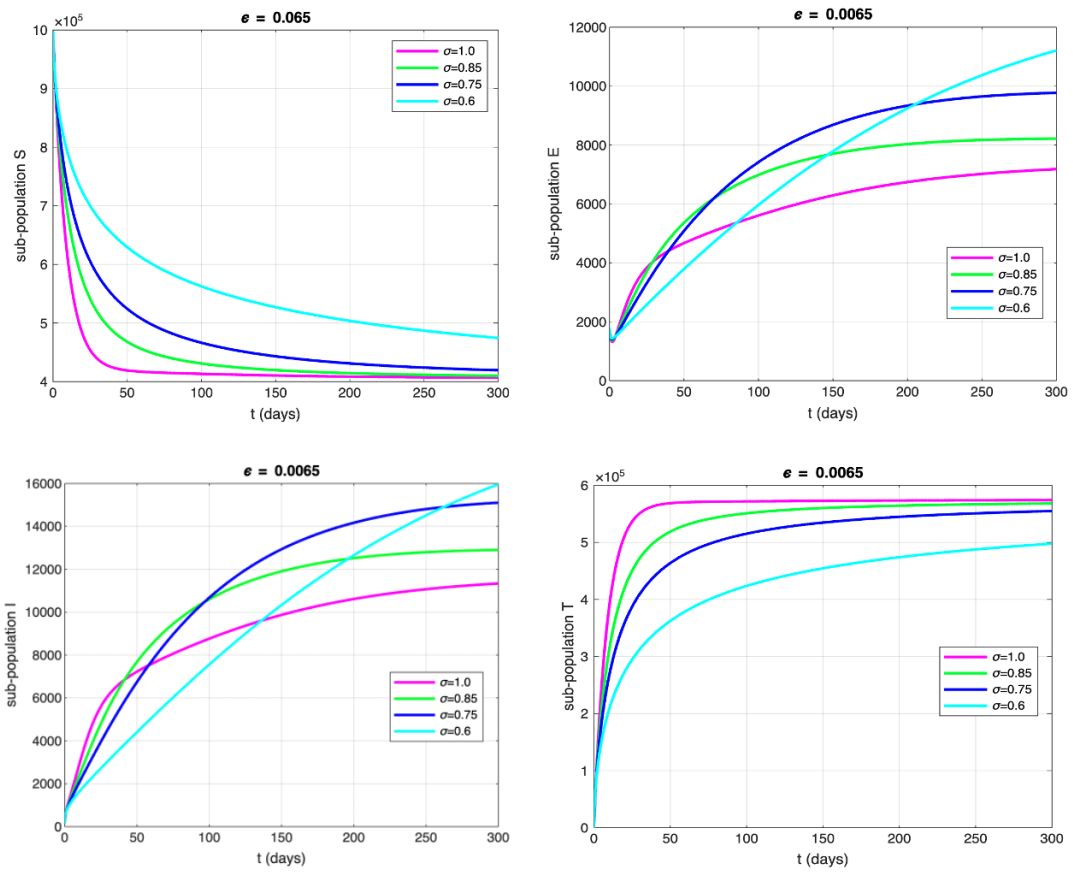

FIGURE 6. The curves of susceptible, exposed, infected and treated for $\varepsilon = 0.065$

4. CONCLUSION

We have examine the stability of the endemic and disease-free equilibriums of the new *SEIT* model (1). In addition, we also studied the effect of vaccination on the number of infected individuals. The analysis shows that the basic reproduction number and the number of infected individuals decrease as vaccination increases, so that the *SEIT* model give the adequate information about spread of tuberculosis.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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