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GLOBAL STABILITY OF A FRACTIONAL EPIDEMIOLOGICAL SIQR MODEL WITH TWO STRAINS: DELTA AND OMICRON CORONAVIRUS MUTATIONS

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Abstract. In this paper, we suggest a fractional epidemic model *SIQR* (Susceptible-Infectious-Quarantined-Recovered) specifically designed for two strains, Omicron and Delta Mutations. We first formulate the model and show that is well-posed. We then perform global stability to identify the conditions in which the endemic and disease-free equilibria are stable by using suitable Lyapunov functionals and applying LaSalle's invariance principle. Finally we conduct numerical simulations to verify our theoretical conclusions and to illustrate the impact of various parameters on the dynamics of the epidemic.

Keywords: *SIQR* epidemic model; fractional order differential equations; basic reproduction number; global asymptotic stability.

2020 AMS Subject Classification: 92D30, 34A08, 34D23 .

1. INTRODUCTION

The coronavirus strain identified as severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) is the source of COVID-19. 2019 novel coronavirus (2019-nCoV) was the temporary name given to the virus previously, it has also been referred to human coronavirus 2019 (HCoV-19 or hCoV-19). The first determination was in the city of Wuhan, Hubei, China, the World Health Organization (WHO) designated the outbreak a public health emergency of international

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concern from January 30, 2020, to May 5, 2023. It has resulted in more than 6.8 million deaths and over 760 million cases worldwide as of early 2024[1]. The shape of coronaviruses is similar to a crown, with spike proteins sticking out of their surfaces, these spike proteins attach to cell surface receptors to help the virus enter human cells. After getting into the cell, the virus uses its machinery to multiply and create new viral particles that can spread to other cells around it [21]. A wide family of coronaviruses is capable of infecting humans as well as animals.

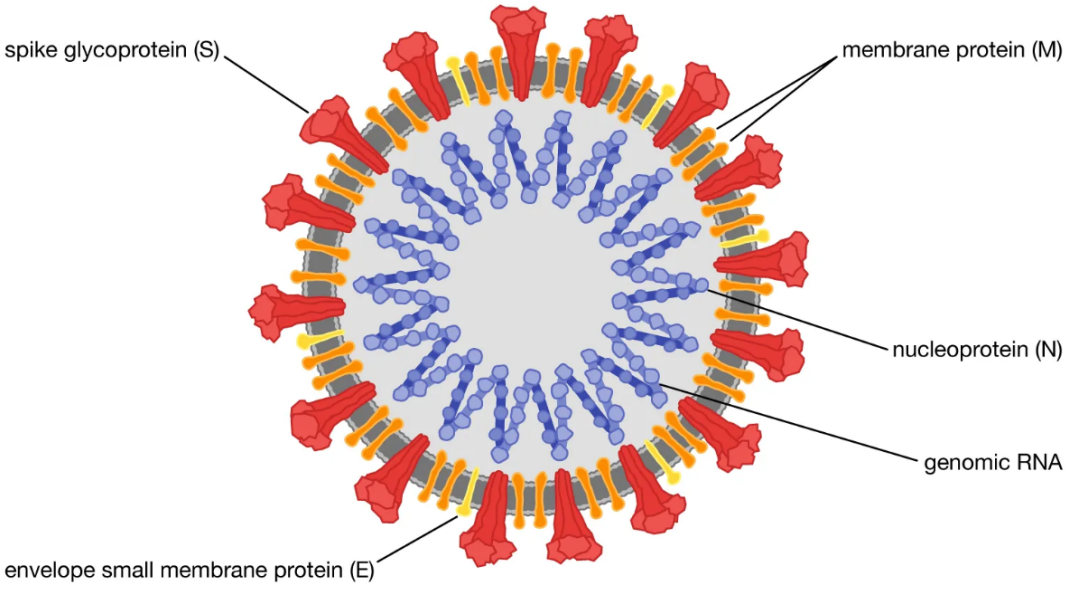

FIGURE 1. Severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2)

Mutations, or small DNA changes can happen when a virus multiplies to create more of itself due to errors during RNA replication, genetic recombination, and selective pressures. These changes frequently have no discernible impact on the virus neither positive nor negative. We may even be able to follow the global spread of viruses thanks to these modifications. For example, the RNA-dependent RNA polymerase enzyme is used in the replication of the SARS-CoV-2 virus, which causes COVID-19. Because this enzyme does not have proofreading abilities, errors occur often, leading to frequent mutations. Genetic recombination can happen when a host cell is simultaneously infected by different strains, like the Alpha and Delta variants of SARS-CoV-2 combine to create a hybrid variant. Furthermore selective pressures, such as immune responses and antiviral treatments, further drive viral evolution. For example the creation of

the Beta variant, which has mutations in the spike protein that help it avoid antibodies. The Omicron variant also shows how selective pressures can occur to a virus with numerous mutations, resulting in higher transmissibility and partial resistance to existing immunity from prior infections or vaccinations [2, 3].

Since they provide a profound knowledge of the transmission and control of infectious diseases, mathematical models are extremely useful in epidemiology. These models provide mathematical models for diverse biological interactions, allowing scientists to determine the development of a disease and evaluate the impact of various strategies for intervention. They offer a systematic approach to comprehending the effects of various factors such as immunity levels, transmission rates, and public health measures [4, 5, 6, 7].

The current study highlights mathematical model incorporating fractional calculus to capture the complex dynamics of Alpha and Omicron mutations of SARS-CoV-2 [8]. By utilizing fractional differential equations, our model effectively describes the unexpected diffusion and memory effects noticed in the transmission of these two significant strains. Furthermore we choose to work with the Caputo fractional derivative, among the several definitions available for fractional derivatives, because it allows for the use of initial conditions in a way close to classical differential equations, the Caputo fractional derivative is especially useful for modeling real-world phenomena, making it more intuitive and applicable in practical scenarios [17, 18, 19, 20]. By using the Caputo fractional derivative, our model benefits from the flexibility and increased accuracy provided by fractional calculus, in order to understanding the spread of COVID-19 variants like Alpha and Omicron.

The following is the structure of the paper: In section 2, we have described some fundamental properties of the Mittag-Leffler function and the Caputo fractional derivative operator. In section 3, we present the fractional-order $SI_DI_OQ_DQ_OR$ epidemic model . In section 4, to make sure that the model is mathematically and physiologically feasible, we examine the existence, uniqueness, non-negativity, and boundedness of the solutions. In section 5, we determine the basic reproduction number R_0 through the next-generation matrix approach [16], this critical parameter indicates whether the infectious disease will persist or eventually die out. Furthermore, we identify the model's two equilibrium points: the disease-free equilibrium and the endemic

equilibrium. In section 6, we employ LaSalle's invariance principle and Lyapunov functions to conduct a rigorous stability analysis, confirming the global stability of the equilibria. In section 7, we validate the analytical results by performing a range of numerical simulations. These simulations help to confirm the theoretical predictions, offering empirical support for the proposed model and demonstrating its practical relevance and effectiveness in various scenarios.

2. PRELIMINARIES

The Caputo fractional-order derivative is defined in this section, along with certain important lemmas for the next analysis.

The fractional integral of order κ for an integrable function $\Theta(t)$ is defined as

$$\mathbb{I}^{\kappa}\Theta(t) = \frac{1}{\Gamma(\kappa)}\int_0^t (t-r)^{\kappa-1}\Theta(r)\,\mathrm{d}r, \quad t \geq 0.$$

where $\kappa > 0$, $\Gamma(\cdot)$ is the gamma function and $\Gamma(\kappa) = \int_0^{\infty} s^{\kappa-1}e^{-s}\,\mathrm{d}s$.

The fractional derivative in the sens of Caputo of order κ for the function $\Theta(t) \in \mathscr{C}^n([0,\infty),\mathbb{R})$ is provided by

$${}^{c}\mathbb{D}_t^{\kappa}\Theta(t) = \frac{1}{\Gamma(n-\kappa)}\int_0^t \frac{\Theta^{(n)}(r)}{(t-r)^{\kappa-n+1}}\,\mathrm{d}r.$$

where $t \geq 0$, and n is a positive integer such that $n-1 \leq \kappa < n$. Furthermore, when $0 < \kappa < 1$,

$${}^{c}\mathbb{D}_t^{\kappa}\Theta(t) = \frac{1}{\Gamma(1-\kappa)}\int_0^t \frac{\Theta'(r)}{(t-r)^{\kappa}}\,\mathrm{d}r.$$

The function of Mittag-Leffler type with one parameter is defined as follows [9]

$$E_{\kappa}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\kappa+1)}, \qquad \kappa > 0, z \in \mathbb{C}.$$

Now, we declare the existence result for fractional differential equations as follows.

Lemma 1. *([14].) Let $\kappa \in (0,1]$, $\Omega \subset \mathbb{R}^n$ a domain and consider the following fractional order equation*

$$\begin{cases} {}^{c}\mathbb{D}_t^{\kappa}\Theta(t) = \Upsilon(t,\Theta(t)) & t > \tau_0, \\ \Theta(\tau_0) = \Theta_0 \in \Omega. \end{cases} \tag{1}$$

Suppose that Υ satisfies the conditions listed below:

1.: *$\Upsilon(\Theta)$ and $\frac{\partial\Upsilon}{\partial\Theta}$ are continuous for all $\Theta \in \mathbb{R}^n$;*

2.: $\|\Upsilon(\Theta)\| \leq \omega + \lambda\|\Theta\| \quad \forall \Theta \in \mathbb{R}^n$, *with* ω *and* λ *are positive constants.*

Then, system (1) *admits a unique solution on* $[0, +\infty)$.

Lemma 2. *(*[15]*) Let* $\kappa \in (0,1]$ *and consider a continuous function* $\Theta : [\tau_0, \infty) \times \Omega \to \mathbb{R}$ *satisfying the conditions listed below*

$$^{c}\mathbb{D}_{t}^{\kappa}\Theta(t) + \mu\Theta(t) \leq \nu, \quad t > \tau_0,\ \mu, \nu \in \mathbb{R},\ \mu \neq 0.$$

And then there is the inequality.

$$\Theta(t) \leq \left(\Theta(\tau_0) - \frac{\nu}{\mu}\right) E_{\kappa}(-\mu(t-\tau_0)^{\kappa}) + \frac{\nu}{\mu}, \quad \forall t \geq \tau_0.$$

3. Model Formulation

3.1. The classical $SI_DI_OQ_DQ_OR$ model. This model is constructed with six compartments so that, at time $t \geq 0$, the whole population $N(t)$ is split into the following classes: $S(t)$ Individuals who are at risk of contracting the virus. Upon exposure, they can become infected with either the Delta strain $I_D(t)$ or the Omicron strain $I_O(t)$, infected individuals may be quarantined, with Delta-infected individuals moving to the Quarantined Delta $Q_D(t)$ compartment and Omicron-infected individuals moving to the Quarantined Omicron $Q_O(t)$ compartment. In the end, individuals who have recovered from the disease are indicated by the $R(t)$ compartment independent of the strain, and are immunized to infection. The following is an expression for the model :

$$\begin{cases} \frac{dS}{dt} = A_N - \frac{\beta_1 SI_D}{N} - \frac{\beta_2 SI_O}{N} - \mu S(t) \\ \frac{dI_D}{dt} = \frac{\beta_1 SI_D}{N} - (\mu + \gamma_1) I_D(t) \\ \frac{dI_O}{dt} = \frac{\beta_2 SI_O}{N} - (\mu + \gamma_2) I_O(t) \\ \frac{dQ_D}{dt} = \gamma_1 I_D(t) - (\mu + \delta_1) Q_D(t) \\ \frac{dQ_I}{dt} = \gamma_2 I_O(t) - (\mu + \delta_2) Q_O(t) \\ \frac{dR}{dt} = \delta_1 Q_D(t) + \delta_2 Q_O(t) - \mu R(t) \end{cases} \tag{2}$$

The following parameters of system (2) are positive and described as follows

TABLE 1. Biological description of model parameters

Parameters	Description
A_N	Recruitment rate (The rate at which new people join the susceptible population).
μ	The rate at which people die naturally, or as a result of natural causes, within the population.
$\beta_{1,2}$	The transmission rates of the Delta (β_1) and Omicron (β_2) variants, indicating how effectively the infected individuals (I_D) and (I_O) infect susceptible individuals (S).
$\gamma_{1,2}$	The rates at which Delta-infected and Omicron-infected individuals are moving from the (I_D) and (I_O) compartments to the (Q_D) and (Q_O) compartments, respectively.
$\delta_{1,2}$	The recovery rates at which quarantined Delta-infected and Omicron-infected move to the recovered compartment (R).

3.2. The fractional $SI_DI_OQ_DQ_OR$ model. Motivated by the classical epidemic system (2), we introduce the following fractional $SI_DI_OQ_DQ_OR$ epidemic model

$$
\begin{cases}
{}^c\mathbb{D}_t^{\alpha} S(t) = A_N^{\alpha} - \frac{\beta_1^{\alpha} S I_D}{N} - \frac{\beta_2^{\alpha} S I_O}{N} - \mu^{\alpha} S(t) \\
{}^c\mathbb{D}_t^{\alpha} I_D(t) = \frac{\beta_1^{\alpha} S I_D}{N} - (\mu^{\alpha} + \gamma_1^{\alpha}) I_D(t) \\
{}^c\mathbb{D}_t^{\alpha} I_O(t) = \frac{\beta_2^{\alpha} S I_O}{N} - (\mu^{\alpha} + \gamma_2^{\alpha}) I_O(t) \\
{}^c\mathbb{D}_t^{\alpha} Q_D(t) = \gamma_1^{\alpha} I_D(t) - (\mu^{\alpha} + \delta_1^{\alpha}) Q_D(t) \\
{}^c\mathbb{D}_t^{\alpha} Q_O(t) = \gamma_2^{\alpha} I_O(t) - (\mu^{\alpha} + \delta_2^{\alpha}) Q_O(t) \\
{}^c\mathbb{D}_t^{\alpha} R(t) = \delta_1^{\alpha} Q_D + \delta_2^{\alpha} Q_O - \mu^{\alpha} R(t)
\end{cases} \tag{3}
$$

The initial conditions for system (3) are as follows:

$$S(0) = S_0 \geq 0 \; ; \; I_D(0) = I_{D_0} \geq 0 \; ; \; I_O(0) = I_{O_0} \geq 0 \; ; \; Q_D(0) = Q_{D_0} \geq 0 \; ; \tag{4}$$

$$Q_O(0) = Q_{O_0} \geq 0 \; ; \; R(0) = R_0 \geq 0. \tag{5}$$

and

$$N(t) = S(t) + I_D(t) + I_O(t) + Q_D(t) + Q_O(t) + R(t).$$

where ${}^c\mathbb{D}_t^{\alpha}$ is the fractional Caputo derivative having order $0 < \alpha \leq 1$ to be able to describe the memory effects in the suggested epidemic model. We assume that the functions S, I_D, I_O, Q_D, Q_O, R and their Caputo fractional derivatives of order $0 < \alpha \leq 1$ are continuous functions.

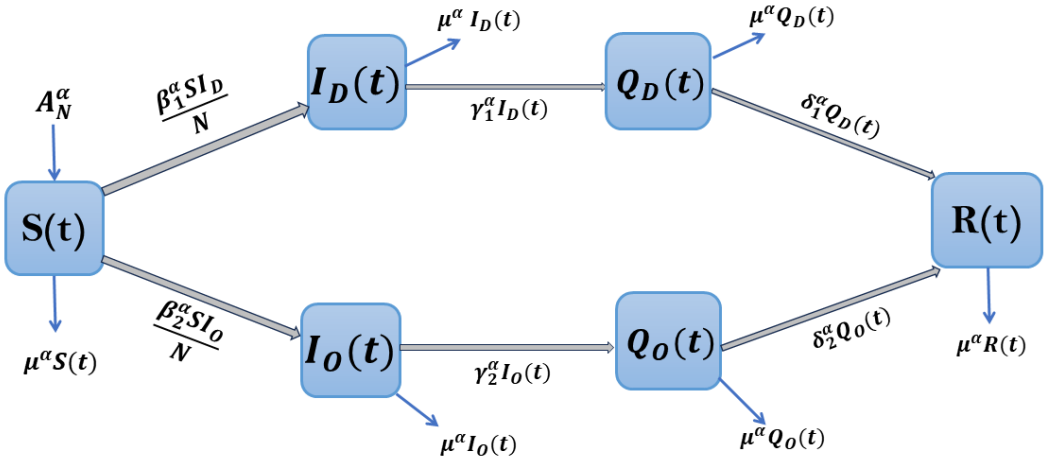

FIGURE 2. Flowchart of $SI_DI_OQ_DQ_OR$ Epidemic model.

4. EXISTENCE, UNIQUENESS, POSITIVITY AND BOUNDARY OF THE SOLUTION

proposition 1. *For any non-negative initial values* $(S_0, I_{D_0}, I_{O_0}, Q_{D_0}, Q_{O_0}, R_0)$*, system (3) has a unique solution in* $\mathbb{R}_+^6$ *for all* $t \geq 0$.

Proof. The following fractional-order system can be examined using the Caputo sense

$$^c\mathbb{D}_t^\alpha w(t) = \Phi(t, w(t)), \text{ for all } t \geq 0, \; w(0) = w_0 = (S_0, I_{D_0}, I_{O_0}, Q_{D_0}, Q_{O_0}, R_0) \in \mathbb{R}_+^6, \tag{6}$$

$$\Phi : \mathbb{R}_+ \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$$

Where

$$(t, w(t)) \longrightarrow \Phi(t, w(t)). \tag{7}$$

and the vector $w(t) = (S(t), I_D(t), I_O(t), Q_D(t), Q_O(t), R(t))$ satisfies the first condition of lemma (1).

Hence

$$\Phi(t, w(t)) = A_0 + \frac{\beta_1 S I_D}{N} A_1 + \frac{\beta_2 S I_O}{N} A_2 + A_3 w(t), \tag{8}$$

$$A_0 = \begin{pmatrix} A_N^\alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$and\, A_3 = \begin{pmatrix} -\mu^\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\mu^\alpha + \gamma_1^\alpha) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\mu^\alpha + \gamma_2^\alpha) & 0 & 0 & 0 \\ 0 & \gamma_1^\alpha & 0 & -(\mu^\alpha + \delta_1^\alpha) & 0 & 0 \\ 0 & 0 & \gamma_2^\alpha & 0 & -(\mu^\alpha + \delta_2^\alpha) & 0 \\ 0 & 0 & 0 & \delta_1^\alpha & \delta_2^\alpha & -\mu^\alpha \end{pmatrix}$$

□

Thus

$$\begin{aligned} {}^c\mathbb{D}_t^\alpha w(t) &= A_0 + \frac{\beta_1 S I_D}{N} A_1 + \frac{\beta_2 S I_O}{N} A_2 + A_3 w(t), \\ \|{}^c\mathbb{D}_t^\alpha w(t)\| &= \left\| A_0 + \frac{\beta_1 S}{N} A_1 w(t) + \frac{\beta_2 S}{N} A_2 w(t) + A_3 w(t) \right\|, \\ &\leq \|A_0\| + \left(\left\| \frac{\beta_1 S}{N} A_1 + \frac{\beta_2 S}{N} A_2 + A_3 \right\| \right) \|w(t)\|, \\ &\leq \|A_0\| + \left(\left\| \frac{\beta_1}{N} \right\| \|S\| \|A_1\| + \left\| \frac{\beta_2}{N} \right\| \|S\| \|A_2\| + \|A_3\| \right) \|w(t)\|, \\ &\leq \lambda_1 + \lambda_2 \|w(t)\|. \end{aligned}$$

Such that $\lambda_1 = \|A_0\|$ and $\lambda_2 = \left\| \frac{\beta_1}{N} \right\| \|S\| \|A_1\| + \left\| \frac{\beta_2}{N} \right\| \|S\| \|A_2\| + \|A_3\|$, which they are bounded and positive. So by Lemma 1 system (3) has a unique solution.

The positivity of the solution is then shown with

$$\begin{aligned} {}^c\mathbb{D}_t^\alpha S(t)|_{S=0} &= A_N^\alpha \geq 0, \\ {}^c\mathbb{D}_t^\alpha I_D(t)|_{I_D=0} &= 0 \geq 0, \\ {}^c\mathbb{D}_t^\alpha I_O(t)|_{I_O=0} &= 0 \geq 0, \end{aligned}$$

$$\begin{aligned} {}^c\mathbb{D}_t^{\alpha} Q_D(t)|_{Q_D=0} &= \gamma_1^{\alpha} I_D(t) \geq 0, \\ {}^c\mathbb{D}_t^{\alpha} Q_O(t)|_{Q_O=0} &= \gamma_2^{\alpha} I_O(t) \geq 0, \\ {}^c\mathbb{D}_t^{\alpha} R(t)|_{R=0} &= \delta_1^{\alpha} Q_D + \delta_2^{\alpha} Q_O \geq 0. \end{aligned}$$

Our objective is to prove that for all $t \geq 0$ the solution belongs to $(\mathbb{R}_0^+)^6$. To produce a contradiction, we suppose that there is a certain moment in time that

$$t^* := inf\{t > 0 \mid (S(t), I_D(t), I_Q(t), Q_D(t), Q_O(t), R(t)) \notin (\mathbb{R}_0^+)^6\}.$$

Thus, $(S(t^*), I_D(t^*), I_Q(t^*), Q_D(t^*), Q_O(t^*), R(t^*)) \in (\mathbb{R}_0^+)^6$ when there is zero in one of the variables. We Suppose that $S(t^*) = 0$, while ${}^c\mathbb{D}_t^{\alpha} S(t^*) = A_N^{\alpha} > 0$ and by the continuity of ${}^c\mathbb{D}_t^{\alpha} S(t^*)$, we determine that for $\varepsilon > 0$, ${}^c\mathbb{D}_t^{\alpha} S([t^*, t^* + \varepsilon[) \subseteq \mathbb{R}_0^+$. Hence from [[11], Theorem 1], $S([t^*, t^* + \varepsilon[) \subseteq \mathbb{R}_0^+$ consequently S is nonnegative.
In a similar manner, we can demonstrate the positivity of the remaining functions I_D, I_O, Q_D, Q_O and R, and obtaining a contradiction. Finally for all $t \geq 0$ the solution belongs to $(\mathbb{R}_0^+)^6$. Which is achieving the desired result.

Finally, we establish the bounds of the solution

proposition 2. *The set*

$$\Theta = \{(S, I_D, I_O, Q_D, Q_O, R) \in \mathbb{R}_+^6 : \ 0 < S + I_D + I_O + Q_D + Q_O + R \leq \frac{A^{\alpha}}{\mu^{\alpha}}\}$$

is a positively invariant and attraction region for system (3).

Proof. We have

$$N(t) = S(t) + I_D(t) + I_O(t) + Q_D(t) + Q_O(t) + R(t).$$

Consequently, adding system(3) equations results in

$$ {}^c\mathbb{D}_t^{\alpha} N(t) = A_N^{\alpha} - \mu^{\alpha} N(t). \tag{9}$$

Applying the Laplace transform to equation (9), we obtain

$$p^{\alpha} \hat{N}(p) - p^{\alpha - 1} N(0) = \frac{A_N^{\alpha}}{p} - \mu^{\alpha} \hat{N}(t).$$

Hence

$$\hat{N}(p) = A_N^\alpha \frac{p^{\alpha-(1+\alpha)}}{p^\alpha + \mu^\alpha} + N(0)\frac{p^{\alpha-1}}{p^\alpha + \mu^\alpha}.$$

In consequence, we have

$$\begin{aligned} N(t) &= A_N^\alpha t^\alpha E_{\alpha,\alpha+1}(-\mu^\alpha t^\alpha) + N(0)E_{\alpha,1}(-\mu^\alpha t^\alpha), \\ &= \frac{A_N^\alpha}{\mu^\alpha} - \frac{A_N^\alpha}{\mu^\alpha}E_{\alpha,1}(-\mu^\alpha t^\alpha) + N(0)E_{\alpha,1}(-\mu^\alpha t^\alpha). \end{aligned}$$

Since $0 \leq E_{\alpha,1}(-\mu^\alpha t^\alpha) \leq 1$ holds and $N(0) \leq \frac{A_N^\alpha}{\mu^\alpha}$, then $N(t) \leq \frac{A_N^\alpha}{\mu^\alpha}$.

Thus, Θ is a positively invariant set, and all initial solutions belong to Θ remain in Θ for all $t > 0$.

That is what we aimed to demonstrate. □

5. EQUILIBRIUM

5.1. Disease free equilibrium (DFE). For $I_D = I_O = Q_D = Q_O = R = 0$, the system (3) clearly has a unique disease-free equilibrium point, given by

$$P_0 = \left(\frac{A_N^\alpha}{\mu^\alpha}, 0, 0, 0, 0, 0\right). \tag{10}$$

5.2. The basic reproduction number R_0. In a typical compartmental disease transmission model, described by a system of ordinary differential equations, the basic reproduction number R_0 has an important purpose. Specifically, it has been demonstrated that when $R_0 < 1$, the disease-free equilibrium is locally asymptotically stable, indicating the final extinction of the disease , whereas if $R_0 > 1$, then it is unstable. Therefore, R_0 serves as a crucial threshold parameter in determining the model's behavior.

In this section, we determine the basic reproduction number R_0 by employing the next-generation matrix method described in [12].

The basic reproduction rate is the spectral radius of the matrix FV^{-1} . where F and V giving by

$$F = \begin{pmatrix} \frac{\beta_1 A_N^\alpha}{\mu^\alpha N} & 0 \\ 0 & \frac{\beta_2 A_N^\alpha}{\mu^\alpha N} \end{pmatrix}, \; V = \begin{pmatrix} \mu^\alpha + \gamma_1^\alpha & 0 \\ 0 & \mu^\alpha + \gamma_2^\alpha \end{pmatrix},$$

we obtain :

$$R_0 = \rho(FV^{-1}) = \ max\{R_0^1, R_0^2\}. \tag{11}$$

Where

$$R_0^1 = \frac{\beta_1 A_N^\alpha}{\mu^\alpha N(\mu^\alpha + \gamma_1^\alpha)} \ \ and \ \ R_0^2 = \frac{\beta_2 A_N^\alpha}{\mu^\alpha N(\mu^\alpha + \gamma_2^\alpha)}.$$

5.3. Endemic Equilibrium. The endemic equilibrium in an epidemic model represents a stable state where the disease continues to exist within the population at a constant level during an extended period. This is in contrast to the disease-free equilibrium, where no individuals are infected.

theorem 1. *The system (3) has three endemic equilibria P_1^*, P_2^* and P_3^*. Furthermore, we have*

- *The strain 1 endemic equilibrium P_1^* exists when $R_0^1 > 1$.*
- *The strain 2 endemic equilibrium P_2^* exists when $R_0^2 > 1$.*
- *The third endemic equilibrium P_3^* exists when $R_0^1 > 1$ and $R_0^2 > 1$.*

Proof. To identify the endemic equilibrium points, the following equations hold

$$\begin{aligned}
A_N^\alpha - \frac{\beta_1^\alpha S^* I_D^*}{N} - \frac{\beta_2^\alpha S^* I_O^*}{N} - \mu^\alpha S^* &= 0,\\
\frac{\beta_1^\alpha S^* I_D^*}{N} - (\mu^\alpha + \gamma_1^\alpha) I_D^* &= 0,\\
\frac{\beta_2^\alpha S^* I_O^*}{N} - (\mu^\alpha + \gamma_2^\alpha) I_O^* &= 0,\\
\gamma_1^\alpha I_D^* - (\mu^\alpha + \delta_1^\alpha) Q_D^* &= 0,\\
\gamma_2^\alpha I_O^* - (\mu^\alpha + \delta_2^\alpha) Q_O^* &= 0,\\
\delta_1^\alpha Q_D^* + \delta_2^\alpha Q_O^* - \mu^\alpha R^* &= 0.
\end{aligned}$$

Case 1: When $I_D^* \neq 0$ and $I_O^* = 0$, We identify the endemic equilibrium for strain 1, defined as follows:

$$P_1^* = (S_1^*, I_{D_1}^*, 0, Q_{D_1}^*, 0, R_1^*). \tag{12}$$

Where

$$S_1^* = \frac{N(\mu^\alpha + \gamma_1^\alpha)}{\beta_1^\alpha} \; ; \; I_{D_1}^* = \frac{A_N^\alpha}{\mu^\alpha + \gamma_1^\alpha}\left(\frac{R_0^1 - 1}{R_0^1}\right) \; ; \; Q_{D_1}^* = \frac{\gamma_1^\alpha A_N^\alpha}{(\mu^\alpha + \gamma_1^\alpha)(\mu^\alpha + \delta_1^\alpha)}\left(\frac{R_0^1 - 1}{R_0^1}\right) \; ;$$

$$R_1^* = \frac{\delta_1^\alpha \gamma_1^\alpha A_N^\alpha}{\mu^\alpha(\mu^\alpha + \gamma_1^\alpha)(\mu^\alpha + \delta_1^\alpha)}\left(\frac{R_0^1 - 1}{R_0^1}\right). \tag{13}$$

Case 2: When $I_D^* = 0$ and $I_O^* \neq 0$. In a similare way if $R_0^2 > 1$, we find the unique strain 2 endemic equilibrium defined as follows

$$P_2^* = (S_2^*, 0, I_{O_2}^*, 0, Q_{O_2}^*, R_2^*). \tag{14}$$

Where

$$S_2^* = \frac{N(\mu^\alpha + \gamma_2^\alpha)}{\beta_2^\alpha} \; ; \; I_{O_2}^* = \frac{A_N^\alpha}{\mu^\alpha + \gamma_2^\alpha}\left(\frac{R_0^2 - 1}{R_0^2}\right) \; ; \; Q_{O_2}^* = \frac{\gamma_2^\alpha A_N^\alpha}{(\mu^\alpha + \gamma_2^\alpha)(\mu^\alpha + \delta_2^\alpha)}\left(\frac{R_0^2 - 1}{R_0^2}\right) \; ;$$

$$R_2^* = \frac{\delta_2^\alpha \gamma_2^\alpha A_N^\alpha}{\mu^\alpha(\mu^\alpha + \gamma_2^\alpha)(\mu^\alpha + \delta_2^\alpha)}\left(\frac{R_0^2 - 1}{R_0^2}\right). \tag{15}$$

Case 3: When $I_D^* \neq 0$ and $I_O^* \neq 0$. The third endemic equilibrium can be identified and described as follows:

$$P_3^* = (S_3^*, I_{D_3}^*, I_{O_3}^*, Q_{D_3}^*, Q_{O_3}^*, R_3^*). \tag{16}$$

Where

$$S_3^* = \frac{N(\mu^\alpha + \gamma_1^\alpha)}{\beta_1^\alpha} = \frac{N(\mu^\alpha + \gamma_2^\alpha)}{\beta_2^\alpha} \; ; \; I_{O_3}^* = \frac{A_N^\alpha}{(\mu^\alpha + \gamma_2^\alpha)} - \frac{\mu^\alpha N}{\beta_2^\alpha} - \frac{(\mu^\alpha + \gamma_1^\alpha)}{(\mu^\alpha + \gamma_2^\alpha)} I_{D_3}^* \; ;$$

$$Q_{D_3}^* = \frac{\gamma_1^\alpha}{(\mu^\alpha + \delta_1^\alpha)} I_{D_3}^* \; ; Q_{O_3}^* = \frac{\gamma_2^\alpha}{(\mu^\alpha + \delta_2^\alpha)} I_{O_3}^* \; ; \; R_3^* = \frac{1}{\mu^\alpha}(\delta_1^\alpha Q_{D_3}^* + \delta_2^\alpha Q_{O_3}^*). \tag{17}$$

□

6. Global Stability

6.1. Global Stability of Disease-Free Equilibrium P_0.

In this section, we examine the global stability of the disease-free equilibrium P_0 and the endemic equilibrium P^* for system (3), by constructing appropriate Lyapunov functions.

We propose a function $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varpi(\xi(t)) = \xi(t) - \xi^* - \xi^* ln\frac{\xi(t)}{\xi^*}, \; for \; all \; t \geq 0.$$

It is important to note that $\varpi(\xi)$ is a non-negative function for any $\xi > 0$, reaching its global minimum at $\xi = 1$.

theorem 2. *The disease-free equilibrium P_0 of system (3) is globally asymptotically stable on Ω if $R_0 \leq 1$. Conversely, P_0 becomes unstable when $R_0 > 1$.*

Proof. The expression of a Lyapunov function is as follows:

$$V_1(t) = \varpi(S(t)) + I_D(t) + I_O(t).$$

The function V_1 attains a global minimum since it is non-negative with respect to the disease-free steady state P_0. Applying the Caputo fractional derivative, we obtain

$$\begin{aligned}
{}^c\mathbb{D}_t^\alpha V_1(t) &= {}^c\mathbb{D}_t^\alpha \varpi(S(t)) + {}^c\mathbb{D}_t^\alpha I_D(t) + {}^c\mathbb{D}_t^\alpha I_O(t), \\
&\leq \left(1 - \frac{S_0}{S}\right) {}^c\mathbb{D}_t^\alpha S(t) + {}^c\mathbb{D}_t^\alpha I_D(t) + {}^c\mathbb{D}_t^\alpha I_O(t), \\
&\leq \left(1 - \frac{S_0}{S}\right) \left(A_N^\alpha - \frac{\beta_1^\alpha S I_D}{N} - \frac{\beta_2^\alpha S I_O}{N} - \mu^\alpha S(t)\right) + \left(\frac{\beta_1^\alpha S I_D}{N} - (\mu^\alpha + \gamma_1^\alpha) I_D(t)\right) \\
&+ \left(\frac{\beta_2^\alpha S I_O}{N} - (\mu^\alpha + \gamma_2^\alpha) I_O(t)\right),
\end{aligned}$$

Considering that $S_0 = \frac{A_N^\alpha}{\mu^\alpha}$, we get

$$\begin{aligned}
{}^c\mathbb{D}_t^\alpha V_1(t) &\leq -\frac{\mu^\alpha (S - S_0)^2}{S} - \left(1 - \frac{S_0}{S}\right) \frac{\beta_1^\alpha S I_D}{N} + \frac{\beta_1^\alpha S I_D}{N} - \left(1 - \frac{S_0}{S}\right) \frac{\beta_2^\alpha S I_O}{N} + \frac{\beta_2^\alpha S I_O}{N} \\
&- (\mu^\alpha + \gamma_1^\alpha) I_D(t) - (\mu^\alpha + \gamma_2^\alpha) I_0, \\
&\leq -\frac{\mu^\alpha (S - S_0)^2}{S} - I_D(\mu^\alpha + \gamma_1^\alpha) \left(1 - \frac{\beta_1^\alpha A_N^\alpha}{\mu^\alpha N(\mu^\alpha + \gamma_1^\alpha)}\right) - I_O(\mu^\alpha + \gamma_2^\alpha) \left(1 - \frac{\beta_2^\alpha A_N^\alpha}{\mu^\alpha N(\mu^\alpha + \gamma_2^\alpha)}\right), \\
&\leq -\frac{\mu^\alpha (S - S_0)^2}{S} - I_D(\mu^\alpha + \gamma_1^\alpha) \left(1 - R_0^1\right) - I_O(\mu^\alpha + \gamma_2^\alpha) \left(1 - R_0^2\right).
\end{aligned}$$

As a result, $R_0 \leq 1$ confirms that ${}^c\mathbb{D}_t^\alpha V_1(t) \leq 0$.

Additionally, it can be easily verified that ${}^c\mathbb{D}_t^\alpha V_1(t) = 0$ if and only if $S(t) = S_0$, $I_D = 0$ and $I_0 = 0$. Hence we deduce that $\{P_0\}$ is the largest invariant set where ${}^c\mathbb{D}_t^\alpha V_1(t) = 0$, and according to LaSalle's invariance principle[13], it follows that $\{P_0\}$ is globally asymptotically stable.

□

6.2. Global Stability of Endemic Equilibrium P^*.

theorem 3. *The endemic equilibrium point P^* of system (3) is globally asymptotically stable on Ω when $R_0 > 1$. Conversely, it becomes unstable.*

Proof. We define a non-negative and continuous Lyapunov function V_2 as follows:

$$V_2(t) = \varpi(S(t)) + \varpi(I_D(t)) + \varpi(I_O(t)) \text{ for all } t \geq 0.$$

By applying the Caputo fractional derivative, we obtain

$$\begin{aligned}
{}^c\mathbb{D}_t^{\alpha} V_2(t) &= {}^c\mathbb{D}_t^{\alpha}\varpi(S(t)) + {}^c\mathbb{D}_t^{\alpha}\varpi(I_D(t)) + {}^c\mathbb{D}_t^{\alpha}\varpi(I_O(t)),\\
&\leq \left(1-\frac{S^*}{S}\right)\left(A_N^{\alpha} - \frac{\beta_1^{\alpha} S I_D}{N} - \frac{\beta_2^{\alpha} S I_O}{N} - \mu^{\alpha} S(t)\right) + \left(1-\frac{I_D^*}{I_D}\right)\left(\frac{\beta_1^{\alpha} S I_D}{N} - (\mu^{\alpha}+\gamma_1^{\alpha}) I_D(t)\right)\\
&+ \left(1-\frac{I_O^*}{I_O}\right)\left(\frac{\beta_2^{\alpha} S I_O}{N} - (\mu^{\alpha}+\gamma_2^{\alpha}) I_O(t)\right),
\end{aligned}$$

We use the endemic condition : $A_N^{\alpha} = \frac{\beta_1^{\alpha} S^* I_D^*}{N} + \frac{\beta_2^{\alpha} S^* I_O^*}{N} + \mu^{\alpha} S^*$, we get

$$\begin{aligned}
{}^c\mathbb{D}_t^{\alpha} V_2(t) \leq & -\frac{\mu^{\alpha}(S-S^*)^2}{S} + \frac{\beta_1^{\alpha} S^* I_D^*}{N} + \frac{\beta_2^{\alpha} S^* I_O^*}{N} - \frac{\beta_1^{\alpha} S I_D}{N} - \frac{\beta_2^{\alpha} S I_O}{N} - \frac{\beta_1^{\alpha} I_D^* (S^*)^2}{NS} - \frac{\beta_2^{\alpha} I_O^* (S^*)^2}{NS}\\
&+ \frac{\beta_1^{\alpha} S^* I_D}{N} + \frac{\beta_2^{\alpha} S^* I_O}{N} + \frac{\beta_1^{\alpha} S I_D}{N} - (\mu^{\alpha}+\gamma_1^{\alpha}) I_D - \frac{\beta_1^{\alpha} S I_D^*}{N} + (\mu^{\alpha}+\gamma_1^{\alpha}) I_D^* + \frac{\beta_2^{\alpha} S I_O}{N}\\
&- (\mu^{\alpha}+\gamma_2^{\alpha}) I_O - \frac{\beta_2^{\alpha} S I_O^*}{N} + (\mu^{\alpha}+\gamma_2^{\alpha}) I_O^*,
\end{aligned}$$

Taking into account that

$$\begin{cases} (\mu^{\alpha}+\gamma_1^{\alpha}) I_D^* = \frac{\beta_1^{\alpha} S^* I_D^*}{N} \\ (\mu^{\alpha}+\gamma_2^{\alpha}) I_O^* = \frac{\beta_2^{\alpha} S^* I_O^*}{N} \end{cases} ; \begin{cases} (\mu^{\alpha}+\gamma_1^{\alpha}) I_D = \frac{\beta_1^{\alpha} S^* I_D}{N} \\ (\mu^{\alpha}+\gamma_2^{\alpha}) I_O = \frac{\beta_2^{\alpha} S^* I_O}{N} \end{cases}$$

and after performing some adjustments, we arrive at

$${}^c\mathbb{D}_t^{\alpha} V_2(t) \leq -\frac{\mu^{\alpha}(S-S^*)^2}{S} + \frac{\beta_1^{\alpha} S^* I_D^*}{N}\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + \frac{\beta_2^{\alpha} S^* I_O^*}{N}\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right).$$

Given that the arithmetic mean of non-negative real numbers is greater than the geometric mean

$$t_1 + t_2 + t_3 + \ldots + t_n \geq n\sqrt[n]{t_1 t_2 \ldots t_n}, \text{ For } t_1, t_2, t_3, \ldots, t_n \geq 0. \tag{18}$$

We get

$$\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) \leq 0.$$

As a result, ${}^{c}\mathbb{D}_t^{\alpha}V_2(t) \leq 0$. Additionally, ${}^{c}D_t^{\alpha}V_2(t) = 0$ if and only if $S = S^*$, $I_D = I_D^*$, and $I_O = I_O^*$. Therefore, the largest compact invariant set contained in $\{(S, I_D, I_O, Q_D, Q_O, R) | {}^{c}\mathbb{D}_t^{\alpha}V_2(t) = 0\}$ is the singleton set $\{P^*\}$. Hence according to the Lasalle invariance principle , P^* is globally asymptotically stable for $R_0 > 1$. □

7. Numerical Simulations

To validate the theoretical results discussed in the earlier sections, we introduce a numerical simulation of our model incorporating fractional derivatives of different orders to analyze how varying memory effects influence transmission dynamics. Additionally, we explore the impact of the basic reproduction number on the epidemic's spread and the behavior of the endemic equilibrium. By adjusting the reproduction number, we evaluate how the system transitions between disease-free and endemic equilibrium.

First, we illustrate the the Disease-Free Equilibrium (DFE) along with the following parameter values: $A_N^{\alpha} = 10$, $\beta_1^{\alpha} = 0.0005$, $\beta_2^{\alpha} = 0.005$, $\delta_1^{\alpha} = 0.05$, $\delta_2^{\alpha} = 0.005$, $\gamma_1^{\alpha} = 0.12$, $\gamma_2^{\alpha} = 0.06$, $\mu^{\alpha} = 0.03$.

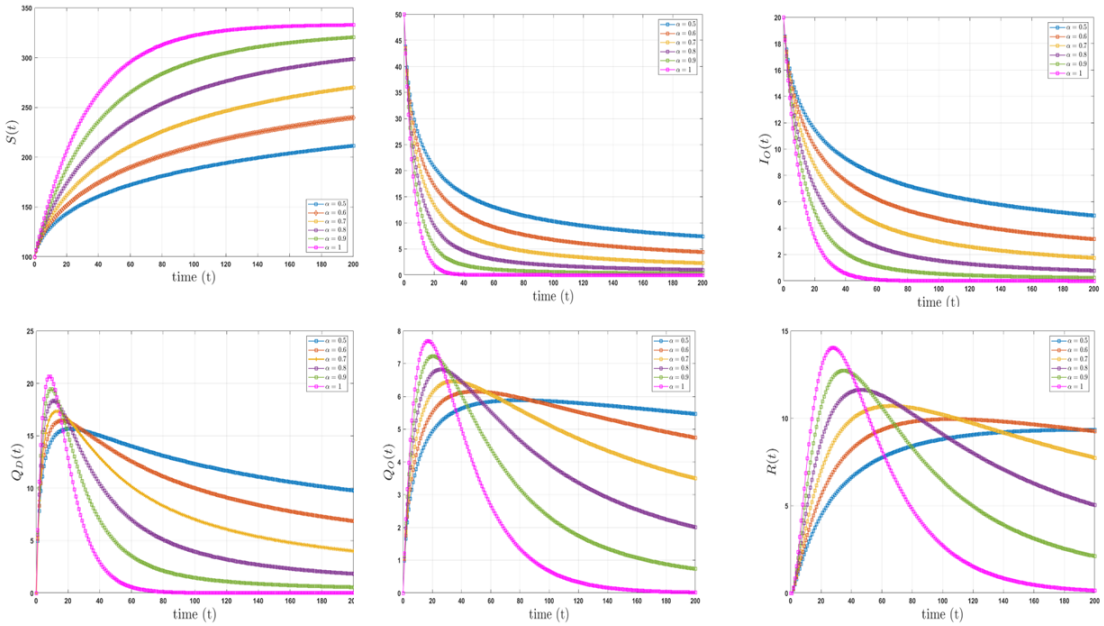

FIGURE 3. The dynamic behavior of compartments S, I_D, I_O, Q_D, Q_O and R presenting the stability of the system 3 at the disease-free equilibrium for $\alpha = 0.5$, $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$ and $\alpha = 1$.

The used parameters in figure 3 show that the basic reproduction number for both strains is less than one ($max\{R_0^1, R_0^2\} = 0.1008 < 1$), where both strains ($I_D$ and I_O) die out. Which, concerning the stability of the disease-free equilibrium, is in good agreement with the theoretical results. We have observed that while increasing the value of parameter α (e.g., 1) we get more rapid convergence to the DFE. In contrast, lower α value (e.g., 0.5) has enhanced memory effects, resulting in a more gradual response to changes and potentially slower stabilization toward the Disease-Free Equilibrium (DFE).

Further, we use the following parameter values to illustrate the Endemic Equilibrium (EE) $A_N^\alpha = 10$, $\beta_1^\alpha = 0.06$, $\beta_2^\alpha = 0.18$, $\delta_1^\alpha = 0.05$, $\delta_2^\alpha = 0.005$, $\gamma_1^\alpha = 0.02$, $\gamma_2^\alpha = 0.06$, $\mu^\alpha = 0.001$.

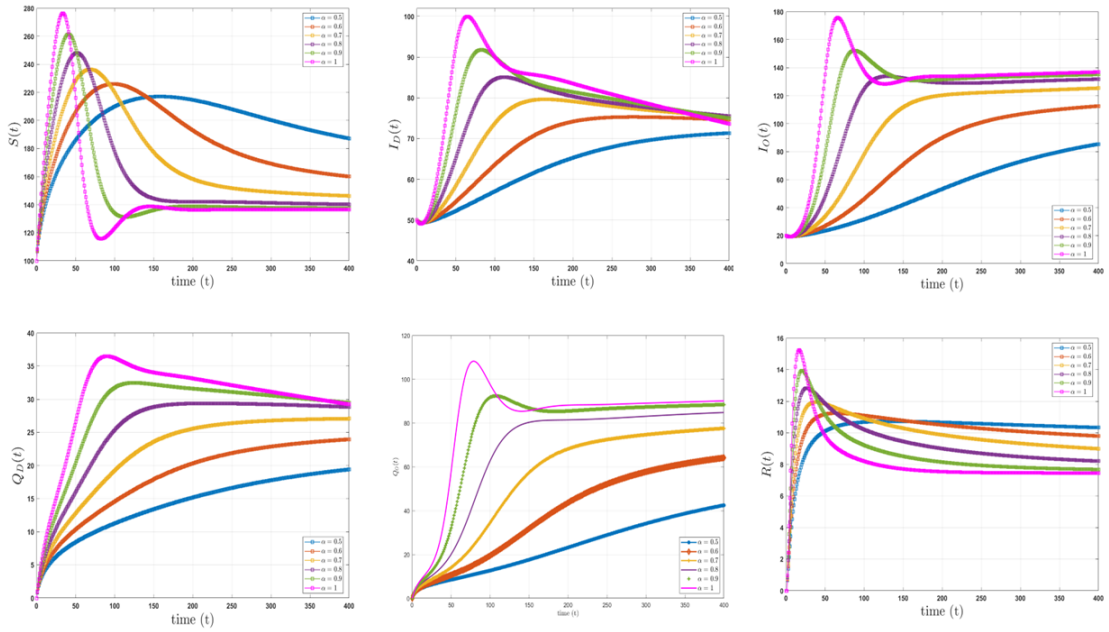

FIGURE 4. The dynamic behavior of compartments S, I_D, I_O, Q_D, Q_O and R presenting the stability of the system 3 at the endemic equilibrium for $\alpha = 0.5$, $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$ and $\alpha = 1$.

Fig.(4) shows a situation in which both strains persist, with the basic reproduction numbers for each strain being greater than one ($max\{R_0^1, R_0^2\} = 7.340 > 1$). The system approaches a stable endemic state which confirms our theoretical results on the stability of endemic equilibrium.

However, we notice again a similar effect of the fractional order α on the rate of convergence to the equilibrium state.

8. Conclusion

The rapid spread of COVID-19 and its mutations, particularly the Delta and Omicron variants, illustrates the urgent need for an immediate epidemiological analysis to enhance awareness and lead effective intervention strategies. In this context, fractional-order derivatives offer an essential role in modeling the disease dynamics. This study has explored the impact of fractional calculus on the behavior of the two viral strains, showing that changes in the fractional order α affects both the rate of convergence and the stability of the system. Higher α values encourage faster stabilization, while lower values slow the adjustment process due to stronger memory effects. These insights highlight the importance of fractional calculus in comprehending and controlling the propagation of complex viral mutations.

Conflict of Interests

The authors declare that there is no conflict of interests.

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