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FEAR-DRIVEN DISEASE CONTROL: A TWO-HOST, ONE-PARASITE MODEL WITH SI DYNAMICS AND BEHAVIORAL CHANGE

SHAYMAA H. SALIH[1], ZINA KH. ALABACY[2], SOHAIB KHLIL[3], SUHA N. SHIHAB[1], NADIA M. G. AL-SAIDI[1,*]

[1]Department of Applied Sciences, University of Technology, Baghdad, Iraq

[2]Department of Control and Systems Engineering, University of Technology, Baghdad, Iraq

[3]Technical College-Baghdad, Department of Applied Mechanics, Middle Technical University, Baghdad, Iraq

Abstract: In this paper, we investigate the dynamics of two hosts and one parasite mathematical model with the fear effect and Susceptible-Infected SI disease. The parasite species reproduces by logistic growth law. There is a mutual fear between the first host and the second host. Infecting the second host with SI disease by transmission from the first host by contact or leaving a mark on its surroundings according to the Lotka-Volterra function. The model is studied theoretically, and its validity is studied numerically after obtaining of the local and the global equilibrium points. The parameter's effect on the mathematical model is studied to determine which parameters cause damage and to set appropriate conditions to reduce their effect. This paper explores a novel disease control model where fear of a parasite drives behavioral changes in a two-host, and one-parasite system. The model utilizes the established SI framework to track disease spread alongside fear-induced modifications in host behavior, the disease was combated by controlling unstable balance points and making them stable.

Keywords: hosts-parasite; dynamical systems; SI disease; fear effect; Lotka-Volterra; epidemiological model.

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[*]Corresponding author
E-mail address: nadia.m.ghanim@uotechnology.edu.iq
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1. INTRODUCTION

Mathematical models have been used to give an obvious view of the dynamics to understanding the human population with many diseases that still threaten to be a big cause of death, and due to changed environmental and socio-economic conditions, we have noticed the existence of one of the important diseases that cause human death, the main cause of which is the parasite. It is necessary to study the development and effectiveness of the host-parasite in the biological description.

The aggressive interaction between the parasites and the hosts is very interesting in ecology. This is due to the negative effect of parasites on host health. The study of these interactions is essential in biomedical [1,2].

Many infectious diseases are caused by viruses, bacteria, protozoa, toxins, pasties (worms), etc. The mechanisms of transmission affect the spread of the disease as, direct physical contact, aerosol droplets of an infected individual, passive vectors (water, food, etc.), active vectors rats, etc. [3,4]. Disease was and still is one of people's most important fears, therefore. many researchers have studied it. Some preview investigations in this regard are: Alabacy and Majeed [5] studied the effects of SIS disease that can be cured and reinfected young prey in a predatory prey model with a stage structure including a prey shelter. Tewa, et al. [6] considered the infectious disease SIS that can be cured and reinfected using a predator-prey model. Rahul and Prakash [7] considered a numerical simulation of the curable and immunogenic SIR diseases model for childhood using the fractional Adams-Bashforth method. Buonomo and Giacobbe [8] studied behavioral epidemic models of a curable disease (SIR): the interaction between behavior and hyperexposure.

Hosts-parasites have attracted the attention of many researchers, such as Xu, et.al. [9], who investigated a parasite-host model within an oscillated environment, these interactions have also attracted many other researchers [10-14].

The fear effect is a necessary factor that must be studied ecologically. Many studies investigate the impact of fear on the species. Alabacy and Majeed [15] studied the fear effect, a prey shelter and harvesting effect on a food chain prey-predator model. As well, these effects are studied [16-19]. Rami and Naji [20] investigated a prey-predator model with two hosts, the prey consumed predator by Holling type II functional response.

In addition to their impact on such biological interactions, these systems have long fascinated researchers due to their intricate and often unpredictable behavior. Understanding the dynamics of these systems is crucial for various scientific disciplines, including biology, ecology, engineering, and medicine [21-23].

A mathematical model was created to study the effects of how to reduce parasitic meningitis. It is one of the important and dangerous diseases that caught our attention. This disease represents a great challenge due to its severe complications and it is considered a clear danger to humans due to the high death rates resulting from the infection with are high. Every 1 to 10 infected people is at risk of death. It infects all ages and is spread by parasites, external infection and contact between people, and also with surfaces that have come into contact with the parasites and people.

In this paper, the dynamics of two hosts (for example, rats and humans) and one parasite (for example, rat lungworm) are investigated. The mathematical model of the fear effect and SI disease (for example, Eosinophilic meningitis) is introduced. The parasite species reproduces by logistic growth law. There is a mutual fear between the first host and the second host. Infecting the second host with SI disease by transmission from the first host via contact or by leaving a mark on its surroundings according to the Lotka-Volterra function. The model is studied theoretically, and numerically after the founding of the local and global equilibrium points. The parameter's effects on the mathematical model are also discussed.

2. MATHEMATICAL MODEL

In this section, we proposed an **epidemiological model**, that involves the following: one parasite whose population density at the time T is *P(T)*, interacting with the first host whose population density at time T is $H_1(T)$ and the second host whose population density at the time T is $H_2(T)$. All the assumptions described are presented in Table 1.

Table 1. Parameters description

Parameters	Biological Meaning
g_1, g_3, g_7	The growth rate of the Parasite, the first Host and the second Host respectively.
g_2, g_5	The infection rate from the Parasite to the first Host and from the first Host to the second Host respectively.
g_4, g_8	The internal competition rate between the first Host individuals and the second Host individuals respectively.
g_6	The extermination rate of the first Host
g_9	The death rate of the second Host
K_1, K_2	The fear rate of the first Host from the second Host and the second Host from the first Host respectively.
K	The carrying capacity of the Parasite

Now, the mathematical model is proposed according to the hypotheses given in Table 1 and through the first-order nonlinear differential equations as shown in the system (1), while Figure 1 shows the model graphically.

$$\left.\begin{aligned} \frac{dP}{dT} &= g_1 P\left(1-\frac{P}{K}\right) - g_2 P H_1 = Z_1(P, H_1, H_2) \\ \frac{dH_1}{dT} &= \frac{g_3 H_1}{1+K_1 H_2} - g_4 H_1^2 + g_2 P H_1 - g_5 H_1 H_2 - g_6 H_1 = Z_2(P, H_1, H_2) \\ \frac{dH_2}{dT} &= \frac{g_7 H_2}{1+K_2 H_1} - g_8 H_2^2 + g_5 H_1 H_2 - g_9 H_2 = Z_3(P, H_1, H_2) \end{aligned}\right\} \qquad (1)$$

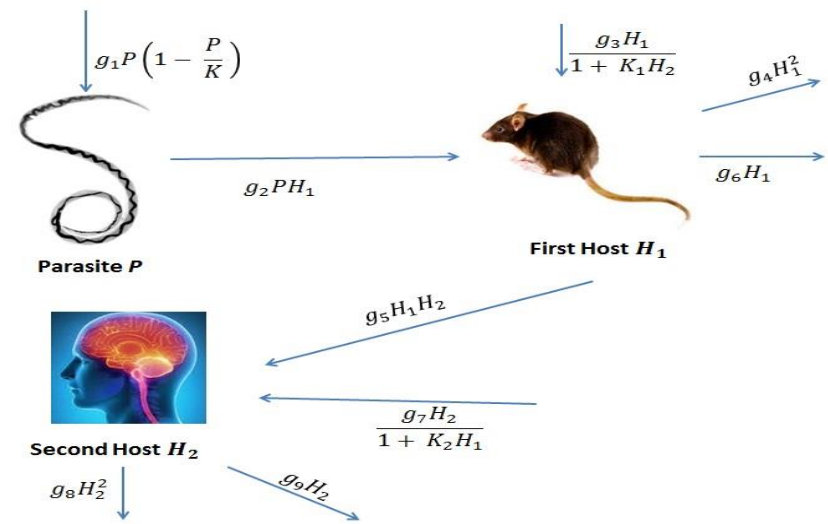


Figure 1: The Block diagram for the model given in system (1)

Parasites may cause a rare type of meningitis called Eosinophilic Meningitis. Brain tapeworm infection or cerebral malaria causes parasitic meningitis. It may quickly turn into a life-threatening disease for the person infected with it.

The main parasite (tapeworm) that causes meningitis usually infects animals (e.g, mice). People usually become infected with this disease by eating foods contaminated with these parasites or by touching surfaces on which the animal has passed. However, parasitic meningitis does not spread between humans. To interpret the parameters of the system (1) in this example as presented in Figure 1, it is as follows:

1- P represents the Tapeworm (parasite): In the phrase $g_1P\left(1-\frac{P}{K}\right)$,where g_1 is the intrinsic growth rate and K is the environment's carrying capacity, refers to the logistic growth of the

Tapeworm population. The Tapeworm interaction g_2PH_1 shows how the Tapeworm parasite on the rat.

2- H_1 represents the rat (first host): The phrase $\frac{g_3H_1}{1+K_1H_2}$, where g_3 is the growth rate and K_1 is the fear rate of the first Host from the second Host. In the phrase $g_4H_1^2$, g_4 refers to the internal competition rate between the first Host individuals. The phrase $g_5H_1H_2$ represent the infection term of the disease of Lotka Volterra type, where g_5 is the infection rate from the first Host to the second Host. The phrase g_6H_1 represent the extermination of the first Host by the second host, and g_6 is the extermination rate of the first Host.

3- H_2 represents the human (second host): In the phrase $\frac{g_7H_2}{1+K_2H_1}$, where g_7 is the growth rate and K_2 is the fear rate of the second Host from the first Host. The phrase $g_8H_2^2$ represents the internal competition between the second host individuals, where g_8 is the internal competition rate between the second Host individuals. The phrase g_9H_2 represent the death term of the second Host by second Host and g_6 is the death rate of the second Host.

In a dynamic epidemiological model, the system and its solutions must be studied in such a way that all the organisms in the system are uniformly bounded. That is, a limited system is a system in which the movement of all organisms in the epidemiological model is limited to a limited area of space. Therefore, the system will be studied in the following theorem.

Theorem: All the solutions of system (1) in R_+^3 are uniformly bounded**.**

Proof: To confirm that the system's (1) solutions are uniformly bounded, we have to suppose a function [2]: $M(T) = P(T) + H_1(T) + H_2(T)$.

Let $(P(T), H_1(T), H_2(T))$ be any solution of system (1) with an initial non-negative condition $(P(0), H_1(0), H_2(0)) \in R_+^3$. Taking the time derivative of $\mathrm{M(T)}$ along the solution of the system (1), we get: $\frac{dM}{dT} \leq N_3 - N_4M$, where $N_3 = \frac{g_1^2K}{2} + \frac{N_1g_3}{g_4} + \frac{N_2g_7}{g_8}$,

$N_4 = \min\{g_1, g_6, g_9\}$,

From the first equation of system (1) and by the comparison theory and the initial point $P(0) = P_0$, we get $\frac{dP}{dT} \leq \frac{g_1K}{4}$. Thus, *Sup*. [24] is $Sup. P(T) = \lim_{T\to\infty} P(T) \leq \frac{g_1K}{4}, \forall T > 0$.

From the second equation of system (1) we have

$\frac{dH_1}{dT} \leq N_1H_1 - g_4H_1^2$, where $N_1 = g_3 + \frac{g_1K}{4}$.

Therefore, using the comparison theory and the initial point $H_1(0) = H_{1,0}$, we get

$H_1(T) \leq \frac{N_1}{g_4 + N_1 c e^{-N_1 T}}$. Thus, $Sup. H_1(T) = \lim_{T\to\infty} H_1(T) \leq \frac{N_1}{g_4}$, $\forall T > 0$.

Now, by using the third equation of system (1), we have

$\frac{dH_2}{dT} \leq g_7 H_2 - g_8 H_2^2 + g_5 H_1 H_2 \leq N_2 H_2 - g_8 H_2^2$.

Here $N_2 = g_7 + \frac{g_1 N_1}{g_4}$.

Therefore, using the comparison theory and the initial point $H_2(0) = H_{2,0}$, we get

$H_2(T) \leq \frac{N_2}{g_8 + N_2 c e^{-N_2 T}}$. Thus, $Sup. H_2(T) = \lim_{T\to\infty} H_2(T) \leq \frac{N_2}{g_8}$, $\forall T > 0$.

Now, for the initial value $M(0) = M_0$, we get:

$M(T) \leq \frac{N_3}{N_4} + c e^{-N_4 T}$. Thus, $\lim_{T\to\infty} M(T) \leq \frac{N_3}{N_4}$, where, $0 \leq M \leq \frac{N_3}{N_4}$, $\forall T > 0$.

Therefore, all solutions are uniformly bounded.

3. EXISTING EQUILIBRIUM POINTS

In this section, all possible equilibrium points of system (1) are found in the following:

- The trivial equilibrium point $E_0(0,0,0)$ is always exists.
- The equilibrium point $E_1(0, \dot{H}_1, 0)$ where $\dot{H}_1 = \frac{g_3 - g_6}{g_4}$ is exists if

  $g_3 > g_6$.
- The equilibrium point $E_2(0,0,\ddot{H}_2)$ where $\ddot{H}_2 = \frac{g_7 - g_9}{g_8}$ is exists if

  $g_7 > g_9$. (2)
- The equilibrium point $E_3(P, H_1, 0)$ is exists if and only if the following two equations have positive solutions:

$$g_1 P \left(1 - \frac{P}{K}\right) - g_2 P H_1 = 0 \quad (3)$$

$$g_3 H_1 - g_4 H_1^2 + g_2 P H_1 - g_6 H_1 = 0 \quad (4)$$

From equation (3) we have,

$$H_1 = \frac{g_1}{g_2}\left(1 - \frac{P}{K}\right) \quad (5)$$

Now, by substituting equation (5) in (4) we obtain:

$$A_1 P^2 + A_2 P + A_3 = 0, \quad (6)$$

where, $A_1 = \frac{g_1(K_2^2 g_2^2 - g_1 g_4)}{g_2^2 K^2}, A_2 = \frac{g_1[2\, g_1 g_4 + g_2 g_6 - g_2(K + g_3)]}{g_2 K}$, *and* $A_3 = \frac{g_1}{g_2^2}[g_2 g_3 - (\ g_2 g_6 + g_1 g_4\ )]$.

The Discarte rule, equation (6) has a unique positive root say $\widehat{\mathrm{P}}$ as long as

$$2g_1g_4 + g_2g_6 < g_2(K + g_3), \tag{7}$$

$$K_2^2 g_2^2 > g_1 g_4, \tag{8}$$

So $\widehat{H}_1 > 0$ if $K > \widehat{P}$, (9)

Hence $\mathrm{E}_3(\widehat{\mathrm{P}}, \widehat{\mathrm{H}}_1, 0)$ is exists under the conditions $(7) - (9)$. Also If we reverse the conditions (7)and(8) with the condition $g_2 g_3 > g_2 g_6 + g_1 g_4$, (10)

So, $E_3(\widehat{P}, \widehat{H}_1, 0)$ exist under conditions (9) and (10).

- The equilibrium point $\mathrm{E}_4(\mathrm{P}, 0, \mathrm{H}_2)$ exists if and only if the following two equations have positive solutions :

$$g_1 P \left(1 - \frac{C}{K}\right) = 0 \tag{11}$$

$$\mathrm{g}_7\mathrm{H}_2 - \mathrm{g}_8\mathrm{H}_2^2 - \mathrm{g}_9\mathrm{H}_2 = 0 \tag{12}$$

From equation (11) we have $\ddot{P} = K > 0$.

From equation (12) we have $\ddot{H}_2 = \frac{1}{g_8}(g_7 - g_9)$.

So $\ddot{\mathrm{H}}_2$ is positive if under condition (2), so $\mathrm{E}_4(\ddot{\mathrm{P}}, 0, \ddot{\mathrm{H}}_2)$ exists.

- The equilibrium point $\mathrm{E}_5(\mathrm{P}, \mathrm{H}_1, \mathrm{H}_2)$ exists if and only if the following three equations have positive solutions:

$$g_1 \left(1 - \frac{P}{K}\right) - g_2 H_1 = 0, \tag{13}$$

$$\frac{g_3}{1 + K_1 H_2} - g_4 H_1 + g_2 P - g_5 H_2 - g_6 = 0, \tag{14}$$

$$\frac{g_7}{1 + K_2 H_1} - g_8 H_2 + g_5 H_1 - g_9 = 0, \tag{15}$$

From equation (13) we get

$$H_1 = \frac{g_1}{g_2}\left(1 - \frac{P}{K}\right)$$

By substituting H_1 in equation (14) we get:

$$P = \frac{K[g_2(g_5 H_2 + g_6) + g_1 g_4 - g_2 g_3]}{K g_2^2 + g_1 g_4}, \tag{16}$$

Now, by substituting H_1 and P in equation (15) we get:

$$B_1 H_2^2 - B_2 H_2 + B_3 = 0,$$

where $B_1 = \frac{K_2 g_5}{K g_2^2 + g_1 g_4}(g_2 g_8 + g_1^2 g_5^2) > 0$,

$$B_2 = g_8 + \frac{K_2(g_1 g_8 + 2g_1^2 g_5^2)}{g_2} + \frac{1}{K_2 g_2^2 + g_1 g_4}[g_8 K_2(g_2[g_6 + g_3] - g_1 g_4) + g_5[g_1(g_5 - g_9 K_2) - 2g_2 g_5 K(g_2(g_2[g_6 + g_3] - g_1 g_4)]]$$

$$B_3 = g_{7-} g_9 + \frac{g_5 K_2 g_1^2}{g_2^2}(g_2(K_2 g_2 - 2[g_6 - g_3] - g_1 g_4) + \frac{1}{K_2 g_2^2 + g_1 g_4}[g_1(g_5 - g_9 K_2)(\, g_2[K_2 g_2 - g_6 + g_3] + g_5 K_2 g_1^2 g_6^2(g_1 g_4 - g_2 g_3)(g_2[2g_6 - g_3] - g_1 g_4)$$

Now $H_{2(1,2)} = \frac{B_2 \pm \sqrt{B_2^2 - 4B_1 B_3}}{2B_1} > 0,$ under the following conditions are hold:

$g_2[g_6 + g_3] > g_1 g_4 > g_2[2g_6 - g_3],$

$\frac{g_5}{g_9} > K_2 > \frac{g_6 - g_3}{g_2},$

$g_1(g_5 - g_9 K_2) > 2g_2 g_5 K(g_2(g_2[g_6 + g_3] - g_1 g_4)],$

$g_1 g_4 > g_2 g_3.$

So we get $\mathrm{H}_1^* = \mathrm{H}_1(\mathrm{H}_2^*) > 0$ and $\mathrm{P}^* = \mathrm{P}(\mathrm{H}_2^*) > 0$, under the following conditions

$K > P^*,$

$g_2(g_5 H_2^* + g_6) + g_1 g_4 > g_2 g_3,$

Therefore, E_5 is exist.

4. LOCAL STABILITY ANALYSIS

In this section, the analysis of the stability of all feasible equilibrium points of system (1) is studied analytically by linearization method [15] as below. Note that, from now onward the characters $\lambda_{iX}, \lambda_{iY}$ and λ_{iZ} represent the eigenvalues of the Jacobian matrix $J_i = J(E_i); i = 0,1,2,3,4,5$ which describes the dynamics in the P, H_1 and H_2 direction respectively. For system (1) it can be written as:

$$J_i = \begin{bmatrix} g_2 H_1 - \frac{g_1 H}{K_1 - 1} & -g_2 P & 0 \\ g_2 H_1 & g_2 P - 2g_4 H_1 - g_5 H_2 - g_6 + \frac{g_3}{H_2 K_1 + 1} & -g_5 H_2 - \frac{g_3 H_1 K_1}{(H_2 K_1 + 1)^2} \\ 0 & g_5 H_2 - \frac{g_7 H_2 K_2}{(H_1 K_2 + 1)^2} & g_5 H_1 - g_9 - 2g_8 H_2 + \frac{g_7}{H_1 K_2 + 1} \end{bmatrix}$$

I. For $\boldsymbol{E_0(0,0,0)}$

$$J_0 = J(E_0) = \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_3 - g_6 & 0 \\ 0 & 0 & g_7 - g_9 \end{bmatrix}.$$

Its corresponding characteristic equation is: $\lambda^3 + (g_6 - g_3 - g_1 - g_7 + g_9)\lambda^2 + [g_1(g_3 - g_6) + (g_7 - g_9)(g_1 + g_3 - g_6)]\lambda - g_1(g_3 - g_6)(g_7 - g_9)$

In which the eigenvalues are: $\lambda_1 = g_1 > 0, \lambda_2 = g_3 - g_6, \lambda_3 = g_7 - g_9$. Hence, E_0 is unstable. Therefore, to make the point stable, control can be made over the conditions of the point to ensure that the environment remains without the presence of any living organism and without the presence of the disease.

***II.* For** $\boldsymbol{E_1(0, \dot{H}_1, 0)}$**, where** $\boldsymbol{\dot{H}_1 = \frac{g_3 - g_6}{g_4}}$**, we have:**

$$J_1 = J(E_1) = \begin{bmatrix} -g_2\dot{H}_1 & 0 & 0 \\ g_2\dot{H}_1 & g_3 - g_6 - 2g_4\dot{H}_1 & -g_5\dot{H}_1 - g_3\dot{H}_1 k_1 \\ 0 & 0 & g_5\dot{H}_1 - g_9 + \dfrac{g_7}{\dot{H}_1 k_2 + 1} \end{bmatrix}.$$

The characteristic equation is given by : $\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3$, where

$\alpha_1 = \left[[g_2 + 2g_4 + g_5]\dot{H}_1 + g_6 + g_9 - g_3 - \frac{g_7}{\dot{H}_1 k_2 + 1}\right]$,

$\alpha_2 = (g_3 - g_6 - [g_2 + 2g_4]\dot{H}_1)(1 + g_5\dot{H}_1 + \frac{g_7}{\dot{H}_1 k_2 + 1} - g_9)$, and

$\alpha_3 = g_2\dot{H}_1(g_3 - g_6 - 2g_4\dot{H}_1)(g_9 - g_5\dot{H}_1 - \frac{g_7}{\dot{H}_1 k_2 + 1})$.

The eigenvalues are: $\lambda_1 = \frac{g_7 + g_5\dot{H}_1(1 + \dot{H}_1 k_2) - g_9(1 + \dot{H}_1 k_2)}{\dot{H}_1 k_2 + 1}$, $\lambda_2 = g_1 - g_2\dot{H}_1$, and $\lambda_3 = g_3 - g_6 - 2g_4\dot{H}_1$. So the equilibrium point is asymptotically stable if the following conditions hold $\frac{g_7 + g_5\dot{H}_1(1 + \dot{H}_1 k_2)}{\dot{H}_1 k_2 + 1} < g_9$, $g_1 < g_2\dot{H}_1$, and $g_3 < g_6 + 2g_4\dot{H}_1$.

***III.* For** $\boldsymbol{E_2(0,0,\ddot{H}_2)}$**, where** $\boldsymbol{\ddot{H}_2 = \frac{g_7 - g_9}{g_8}}$ **we have:**

$$J_2 = J(E_2) = \begin{bmatrix} g_1 & 0 & 0 \\ 0 & \dfrac{g_3}{\ddot{H}_2 K_1 + 1} - g_5\ddot{H}_2 - g_6 & 0 \\ 0 & (g_5 - g_7 K_1)\ddot{H}_2 & g_7 - g_9 - 2g_8\ddot{H}_2 \end{bmatrix},$$

The characteristic equation is given by : $\lambda^3 + \beta_1\lambda^2 + \beta_2\lambda + \beta_3$, where

$\beta_1 = g_7 - g_9 - 2g_8\ddot{H}_2 - g_1 + \frac{\left((1 + \ddot{H}_2 K_1)[g_6 + g_5\ddot{H}_2] - g_3\right)}{\ddot{H}_2 K_1 + 1}$,

$\beta_2 = [g_9 + 2g_8\ddot{H}_2 - g_7 - g_1][\frac{\left((1 + \ddot{H}_2 K_1)[g_6 + g_5\ddot{H}_2] - g_3\right)}{\ddot{H}_2 K_1 + 1} - 1]$, and

$\beta_3 = \frac{g_1(g_9+2g_8\ddot{H}_2-g_7)\left((1+\ddot{H}_2K_1)[g_6+g_5\ddot{H}_2]-g_3\right)}{\ddot{H}_2K_1+1}$.

Therefore, $\lambda_1 = g_1 > 0, \lambda_2 = \frac{(g_3-(1+\ddot{H}_2K_1)[g_6+g_5\ddot{H}_2])}{\ddot{H}_2K_1+1}$, and $\lambda_3 = g_7 - g_9 - 2g_8\ddot{H}_2$.

So, E_2 is unstable. However, to make the point asymptotically stable, we have to control the conditions of the point to ensure that the human remains only in an environment free of parasites that cause the disease and free from the first host that transmits the disease.

***IV.* For $E_3(\hat{P}, \hat{H}_1, 0)$, we have**

$$J_3 = J(E_3) = \begin{bmatrix} g_2\hat{H}_1 - \frac{g_1\hat{P}}{K_1(K_1-1)} & -g_2\hat{P} & 0 \\ g_2\hat{H}_1 & g_3 - g_6 - 2g_4\hat{H}_1 + g_2\hat{P} & -\hat{H}_1(g_3K_1+g_5) \\ 0 & 0 & \frac{g_5\hat{H}_1 - g_9 + g_7}{\hat{H}_1K_2+1} \end{bmatrix},$$

The characteristic equation is given by : $\lambda^3 + \gamma_1\lambda^2 + \gamma_2\lambda + \gamma_3 = 0$, where

$\gamma_1 = \left(g_6 - g_3 - g_1 + g_2[\hat{H}_1 - \hat{P}] + 2g_4\right)\left(\frac{2g_1\hat{P}}{K_1}\right) - \frac{g_5\hat{H}_1(\hat{H}_1K_2+1)+g_7-g_9(\hat{H}_1K_2+1)}{\hat{H}_1k_2+1}$,

$\gamma_2 = \left[g_1\left(g_3 - g_6 - 2g_4\hat{H}_1\right) + g_2\hat{H}_1\left(g_6 - g_3 + 2g_4\hat{H}_1\right)\right]\frac{1}{K_1}\left(g_1\hat{P}\left(2g_6 - 2g_3 + g_2(1-2\hat{P}\right) +\right.$

$\left.4g_4\hat{H}_1)\right) - \frac{1}{K_1(\hat{H}_1K_2+1)}\left(g_7 + g_5\hat{H}_1K_2(1+\hat{H}_1) - g_9\right)\left(K_1\left[g_6 - g_3 - g_1 + 2g_4\hat{H}_1 + g_2(\hat{H}_1 - \hat{P})\right] - 2g_1\hat{P}\right)$,

and $\gamma_3 = \frac{1}{K_1(\hat{H}_1K_2+1)}\left(g_9(\hat{H}_1K_2+1) - g_7 - g_5\hat{H}_1(1+\hat{H}_1K_2)\right)\left(g_1K_1[g_3 - g_6 - 2g_4\hat{H}_1 + g_2\hat{P}] +\right.$

$\left.2g_2\hat{P}[g_6 - g_3 + 2g_4\hat{H}_1 + g_2\hat{P}] + g_2\hat{H}_1K_1[g_6 - g_3 + 2g_4\hat{H}_1]\right)$.

By Routh-Hurwitz principle [25] the roots of the characteristic equation should have negative real parts if and only if $\gamma_1 > 0, \gamma_3 > 0$, and $\Delta = \gamma_4 - \gamma_3 > 0$, where $\gamma_4 = \gamma_1\gamma_2$. Which are satisfied :

$\frac{g_7}{(\hat{H}_1K_2+1)} + g_5\hat{H}_1 < g_9, g_1 + g_3 + g_2\hat{P} < g_6 + (g_2 + 2g_4)\hat{H}_1, c < 2g_2\hat{P}$, and $c_1 > c_2$.

Where: $c = (c_1 - c_2)^{1/2}$

$c_1 = \left[K_1^2\left(g_1[g_1 + 4g_4\hat{H}_1 + 2g_6] + g_2(\hat{P} + \hat{H}_1)(g_2 + 2g_3) + 4g_4\hat{H}_1(g_4\hat{H}_1 + g_6) + g_3^2 +\right.\right.$

$\left.\left.g_6^2\right) + 4\hat{P}g_1\left(\hat{P}[g_1 + g_2K_1] + K_1[g_2\hat{H}_1 + g_3]\right)\right]$, and

$c_2 = \left[2K_1^2\left(g_1[g_2(\hat{P} + \hat{H}_1) + g_3] + g_2\left(\hat{H}_1[\hat{P} + 2g_4(1+\hat{P})] + g_6(\hat{P} + \hat{H}_1) + g_3(2g_4\hat{H}_1 +\right.\right.\right.$

$\left.\left.\left.g_6)\right)\right) + 4g_1k_1\hat{P}\left(g_1 + 2g_4\hat{H}_1 + g_6\right)\right]$.

So, E_3 is asymptotically stable.

***V.* For $E_4(\ddot{P}, 0, \ddot{H}_2)$, we have**

$$J_4 = J(E_4) = \begin{bmatrix} \frac{g_1\ddot{P}}{1-K_1} & -g_2\ddot{P} & 0 \\ 0 & g_2\ddot{P} - g_5\dddot{H}_2 - g_6 + \frac{g_3}{1+K_2\dddot{H}_2} & -g_5\dddot{H}_2 \\ 0 & (g_5 - g_7K_2)\dddot{H}_2 & g_7 - (g_9 + 2g_8\dddot{H}_2) \end{bmatrix},$$

The characteristic equation is: $\lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3 = 0$, where

$\sigma_1 = \left(\frac{g_9 - g_7 + 2g_8\dddot{H}_2}{\dddot{H}_2K_1+1} - \frac{g_1K_1 - 2g_1\ddot{P}}{K_1(\dddot{H}_2K_1+1)}\right)\left(\dddot{H}_2K_1 + 1\right) + \frac{1}{(\dddot{H}_2K_1+1)}(g_6 - g_3 + g_5\dddot{H}_2 - g_2\ddot{P} + g_6\dddot{H}_2K_1 + g_5\dddot{H}_2^2K_1 -$
$g_2\dddot{H}_2K_1\ddot{P})$,

$\sigma_2 = (\frac{(g_9 - g_7 + 2g_8\dddot{H}_2)}{\dddot{H}_2K_1+1} - \frac{g_1K_1 - 2g_1\ddot{P}}{\dddot{H}_2K_1+1} g_6 - g_3 + g_5\dddot{H}_2 - g_2\ddot{P} + g_6\dddot{H}_2K_1 + g_5\dddot{H}_2^2K_1 -$

$g_2\dddot{H}_2K_1\ddot{P})(-\frac{(g_1K_1 - 2g_1\ddot{P})(g_9 - g_7 + 2g_8\dddot{H}_2}{K_1}$, and

$\sigma_3 = -\frac{1}{K_1(\dddot{H}_2K_1+1)}\left(g_1K_1 - 2g_1\ddot{P}\right)\left(g_9 - g_7 + 2g_8\dddot{H}_2\right)(g_6 - g_3 + g_5\dddot{H}_2 - g_2\ddot{P} + g_6\dddot{H}_2K_1 +$

$g_5\dddot{H}_2^2K_1 - g_2\dddot{H}_2K_1\ddot{P})$.

By Routh-Hurwitz principle the roots of the characteristic equation should have negative real parts if and only if $\sigma_1 > 0, \sigma_3 > 0$, and $\Delta = \sigma_4 - \sigma_3 > 0$, where $\sigma_4 = \sigma_1\sigma_2$. where the eigen values are: $\lambda_1 = \frac{g_1}{K_1}(K_1 - 2\ddot{P})$, $\lambda_2 = g_7 - g_9 - 2g_8\overline{H}_2$, and

$\lambda_3 = \frac{g_3 + g_2\ddot{P}(\dddot{H}_2K_1+1) - \dddot{H}_2\,[\,g_5(\dddot{H}_2K_1+1) + g_6K_1]}{(\dddot{H}_2K_1+1)}$.

So, E_4 is asymptotically stable if and only if

$K_1 < 2\ddot{P}$, $g_7 < g_9 + 2\,g_8\dddot{H}_2$, and $g_3 + g_2\ddot{P}\left(\dddot{H}_2K_1 + 1\right) < \overline{H}_2\left[\,g_5\left(\dddot{H}_2K_1 + 1\right) + g_6K_1\right]$.

***VI.* For $E_5(P^*, H_1^*, H_2^*)$, we have**

$J_5 = J(E_5) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$, where

$a_{11} = g_2H_1^* - \frac{g_1P^*}{K_1-1}$, $a_{12} = -g_2P^*$, $a_{21} = g_2H_1^*$, $a_{22} = g_2P^* - 2g_4H_1^* - g_5H_2^* - g_6 + \frac{g_3}{H_2^*K_1+1}$,

$a_{23} = -g_5H_2^* - \frac{g_3H_1^*k_1}{(H_2^*K_1+1)^2}$, $a_{32} = g_5H_2^* - \frac{g_7H_2^*K_2}{(H_1^*K_2+1)^2}$, and $a_{33} = g_5H_1^* - g_9 - 2g_8H_2^* + \frac{g_7}{H_1^*K_2+1}$.

The characteristic equation is: $\lambda^3 + \mathcal{J}_1\lambda^2 + \mathcal{J}_2\lambda + \mathcal{J}_3$, where

$\mathcal{J}_1 = -(a_{11} + a_{22} + a_{33})$, $\mathcal{J}_2 = a_{11}(a_{22} + a_{33}) + a_{33}(a_{22} - a_{12}a_{21}) - a_{23}a_{32}$, and

$\mathcal{J}_3 = a_{12}a_{21}a_{33} - a_{11}(a_{22}a_{33} - a_{23}a_{32})$.

By Routh- Hurwitz the characteristic equation should have negative real parts if and only if: $\mathcal{J}_1 >$

$0,\ \mathcal{J}_3 > 0,$ and $\mathcal{J}_1\mathcal{J}_2 - \mathcal{J}_3 > 0$. Hence, E_5 is asymptotically stable if the following conditions are held:

$2P^* > K_1,\ g_2P^* + \frac{g_3}{H_2^*K_1+1} < 2g_4H_1^* + g_5H_2^* + g_6,\ g_5H_1^* + \frac{g_7}{H_1^*K_2+1} < g_9 + 2g_8H_2^*,$ and

$g_5H_1^*(H_1^*K_2 + 2) > g_7.$

5. The Global Stability Analysis

In this section, we discuss the global stability analysis of the epidemiological model, which is key to understanding the resilience and behavior of epidemiological systems in the long term subject to different environmental conditions. Local and global analysis approaches that complement each other are: global stability provides the most integrated vision and reveals system dynamics over the entire phase space, and local stability substantiates the behavior of the system in the vicinity of equilibrium points. The goal of this section is to find out whether the previously obtained equilibrium states are global.

$$S_i^{\circ}(P,H_1,H_2) = \left(P - P^{\circ} - P^{\circ}\ ln\frac{Z_1}{P^{\circ}}\right) + \left(H_1 - H_1^{\circ} - H_1^{\circ}\ ln\frac{H_1}{H_1^{\circ}}\right) + \left(H_2 - H_2^{\circ} - H_2^{\circ}\ ln\frac{H_2}{H_2^{\circ}}\right). \quad (17)$$

Equation (17) for $i = 1,2,3,4$ [5] is used with the Lyapunov method to study the global stability for all local asymptotically equilibrium points.

Theorem 5.1. Suppose that $E_1\left(0,\dot{H}_1,0\right)$ of the system (1) is a local asymptotically stable (LAS) in R_+^3. Then E_1 is globally asymptotically stable (GAS) under the conditions (18)-(20):

$$g_2\dot{H}_1 + \frac{g_1P}{K} > g_1, \quad (18)$$

$$H_1 < \dot{H}_1, \quad (19)$$

$$g_5\dot{H}_1 + g_8H_2 + g_9 > g_7. \quad (20)$$

Proof: Let $S_1 = P + \left(H_1 - \dot{H}_1 - \dot{H}_1\ Ln\ \frac{H_1}{\dot{H}_1}\right) + H_2 .$ (21)

Now, equation (21) is driven with respect to time to get

$$\frac{dS_1}{dT} = \frac{dP}{dT} + \frac{(H_1-\dot{H}_1)}{H_1} . \frac{dH_1}{dT} + \frac{dH_2}{dT} .$$

$$\frac{dS_1}{dT} = \frac{g_1\ P}{K}\ (K - P) - g_2P\dot{H}_1 + \frac{g_3}{1+K_1\ H_2}(H_1 - \dot{H}_1) - g_4(H_1 - \dot{H}_1)^2 - g_5\dot{H}_1H_2 +$$

$$\frac{g_7H_2}{1+ K_2H_1} - g_8H_2^2 - g_9H_2 .$$

Therefore,

$\frac{dS_1}{dT} \leq -P\left(g_2 H_1 + \frac{P g_1}{K} - g_1\right) - g_4\left(H_1 - \dot{H}_1\right)^2 + g_3\left(H_1 - \dot{H}_1\right) -$

$H_2\left(g_5 \dot{H}_1 + g_8 H_2 + g_9 - g_7\right) < 0$, if the conditions (18)-(20) , we get $\frac{dS_1}{dT} < 0$. So, E_1 is global.

Theorem 5.2. Suppose that $E_3(\hat{P}, \hat{H}_1, 0)$ of the system (1) is LAS in R_+^3. Then E_3 is GAS under the following conditions:

$$P > \hat{P}, \tag{22}$$

$$H_1 < \hat{H}_1, \tag{23}$$

$$g_8 H_2 + g_9 > g_7 - g_5 \hat{H}_1. \tag{24}$$

Proof: Let $S_2 = \left(P - \hat{P} - P\, ln\, \frac{P}{\hat{P}}\right) + \left(H_1 - \hat{H}_1 - H_1 ln\, \frac{H_1}{\hat{H}_1}\right) + H_2.$ (25)

Now, equation (25) is driven with respect to time to get,

$$\frac{dS_2}{dT} = \frac{(P-\hat{P})}{P}\frac{dP}{dT} + \frac{(H_1-\hat{H}_1)}{H_1}\frac{dH_1}{dT} + \frac{dH_2}{dT} .$$

$$\frac{dS_2}{dT} = -\frac{g_1}{K}\left(P - \hat{P}\right) - g_2\left(P - \hat{P}\right)\left(H_1 - \hat{H}_1\right) + \frac{g_3 K_1 H_2}{1+K_1 H_2}\left(H_1 - \hat{H}_1\right) - g_5 H_2\left(H_1 - \hat{H}_1\right) + g_2\left(P - \hat{P}\right)\left(H_1 - \hat{H}_1\right) - g_4\left(H_1 - \hat{H}_1\right)^2 + \frac{g_7 H_2}{1+K_2 H_2} - g_8 H_2^2 + g_5 H_1 H_2 - g_9 H_2,$$

$$\frac{dS_2}{dT} \leq -\frac{g_1}{K}\left(P - \hat{P}\right)^2 - g_4\left(H_1 - \hat{H}_1\right)^2 + g_3 K_1 H_2\left(H_1 - \hat{H}_1\right) - H_2[\, g_8 H_2 + g_9 - g_7 - g_5 \hat{H}_1]$$

$\frac{dS_2}{dT} < 0$, under the conditions (22)-(24). So, E_3 is GAS.

Theorem 5.2. Suppose that $E_4(\ddot{P}, 0, \ddot{H}_2)$ of the system (1) is LAS in R_+^3. Then E_4 is GAS under the following conditions:

$$P > \ddot{P}, \tag{26}$$

$$H_2 > \ddot{H}_2, \tag{27}$$

$$g_4 H_1 + g_6 + g_5 \ddot{H}_2 > g_2 \ddot{P} - g_3. \tag{28}$$

Proof: Let $S_3 = \left(P - \ddot{P} - \ddot{P}\, ln\, \frac{P}{\ddot{P}}\right) + H_1 + \left(H_2 - \ddot{H}_2 - \ddot{H}_2\, ln\, \frac{H_2}{\ddot{H}_2}\right).$ (29)

Now, equation (29) is driven with respect to time to get,

$$\frac{dS_3}{dT} = \frac{(P - \ddot{P})}{P}\frac{dP}{dT} + \frac{dH_1}{dT} + \frac{(H_2 - \ddot{H}_2)}{H_2}\frac{dH_2}{dT}.$$

$$\frac{dS_3}{dT} = -\frac{g_1}{K}\left(P - \ddot{P}\right)^2 + g_2 \ddot{P} H_1 + \frac{g_3 H_1}{(1+K_1 H_2)} - g_4 H_1^2 - g_6 H_1 - g_8\left(H_2 - \ddot{H}_2\right)^2 - g_5 H_1 \ddot{H}_2 - \frac{(g_7 K_2 H_1)(H_2 - \ddot{H}_2)}{1+K_2 H_1},$$

$\frac{dS_3}{dT} < -\frac{g_1}{K}(P - \ddot{P})^2 - g_8(H_2 - \ddot{H}_2)^2 - H_1[g_4 H_1 + g_6 + g_5 \ddot{H}_2 - g_2 \ddot{P} - g_3] -$

$g_7 K_2 H_1 (H_2 - \ddot{H}_2)$.

$\frac{dS_3}{dT} < 0$, under the conditions (26)-(28). *So,* E_4 is GAS.

Theorem 5.3. Suppose that $E_5(P^*, H_1^*, H_2^*)$ of the system (1) is LAS in R_+^3. Then E_5 is GAS under the following conditions:

$$P > P^*, \quad (30)$$

$$H_1 < H_1^*, \quad (31)$$

$$H_2 > H_2^*, \quad (32)$$

$$g_3 K_1 + g_7 K_2 + K_1 K_2 (g_3 K_2 H_1 H_1^* + g_3(H_1 + H_1^*) + g_7 K_1 K_2 H_2 H_2^* + g_7(H_2 + H_2^*) < 2\sqrt{g_4 g_8}. \quad (33)$$

Proof: Let

$$S_4 = \left(P - P^* - P^* \ln \frac{P}{P^*}\right) + \left(H_1 - H_1^* - H_1 \ln \frac{H_1}{H_1^*}\right) + \left(H_2 - H_2^* - H_2^* \ln \frac{H_2}{H_2^*}\right). (34)$$

Now, equation (34) is driven with respect to time to get,

$\frac{dS_4}{dT} = \frac{(P - P^*)}{P}\frac{dP}{dT} + \frac{(H_1 - H_1^*)}{H_1}\frac{dH_1}{dT} + \frac{(H_2 - H_2^*)}{\overset{*}{H_2}}\frac{dH_2}{dT}$.

$\frac{dS_4}{dT} = -\frac{g_1}{K}(P - P^*)^2 - g_4(H_1 - H_1^*)^2 - g_8(H_2 - H_2^*)^2 - \frac{(H_1 - H_1^*)(H_2 - H_2^*)}{(1+K_1H_2)(1+K_1H_2^*)(1+K_2H_1)(1+K_2H_1^*)}[g_3 K_1 +$

$g_7 K_2 + K_1 K_2 (g_3 K_2 H_1 H_1^* + g_3(H_1 + H_1^*) + g_7 K_1 K_2 H_2 H_2^* + g_7(H_2 + H_2^*)]$,

$\frac{dS_4}{dT} < -\frac{g_1}{K}(P - P^*)^2 - g_4(H_1 - H_1^*)^2 - g_8(H_2 - H_2^*)^2 - (H_1 - H_1^*)(H_2 - H_2^*)[g_3 K_1 +$

$g_7 K_2 + K_1 K_2 (g_3 K_2 H_1 H_1^* + g_3(H_1 + H_1^*) + g_7 K_1 K_2 H_2 H_2^* + g_7(H_2 + H_2^*))]$,

$\frac{dS_4}{dT} < -\frac{g_1}{K}(P - P^*)^2 - (\sqrt{g_4}(H_1 + H_1^*) - \sqrt{g_8}(H_2 - H_2^*))^2$.

$\frac{dS_4}{dT} < 0$, under the conditions (30)-(33). So, E_5 is GAS.

6. CONTROLLING ANALYZING

The stability of this system requires investigating the equilibrium points (where all rates of change become zero) and their local stability properties. This can be done using techniques like Jacobin analysis.

In this specific scenario, we are interested in role stabilization, which refers to the situation where one host species persists at a stable positive population level, while the other host population is driven to extinction. This can occur under certain conditions, depending on the relative values of the parameters like growth and rate of infection rate from the Parasite to the first Host and from the first Host to the second Host.

In some cases, the coexistence of all three species might be desirable, Researchers and experts in controlling insects can create methods to minimize ecological disturbance in the control of two-host, one-parasite systems; such methods were employed in [26-29].

To compare the unstable equilibrium points before and after control, Table 2 was created with Fig.2.

Table 2: Comparison of (Host-Parasitoid) system between instability and stabilization for equilibrium points E_0, E_2

Feature	**Unstable Host-Parasitoid System**	**Stabilized Host-Parasitoid System**
Stability Analysis	1-The set of equilibrium points $E_0(0,0,0), E_2(0,0,\ddot{H}_2)$, where $\ddot{H}_2 = \frac{g_7 - g_9}{g_8}$ 2- the set of eigenvalues $J_{E_0} = \{\lambda_1 = 0.01, \lambda_2 = 0.8900, \lambda_3 = 0.9800\}$ $J_{E_2} = \{\lambda_1 = 0.01, \lambda_2 = 0.8711, \lambda_3 = 0.9600\}$ have unstable equilibrium points	1-The set of equilibrium points $E_0(0,0,0), E_2(0,0,\ddot{H}_2)$, where $\ddot{H}_2 = \frac{g_7 - g_9}{g_8}$ 2- the set of eigenvalues $J_{E_0} = \{\lambda_1 = -0.019, \lambda_2 = -0.009, \lambda_3 = 0.01\}$ $J_{E_2} = \{\lambda_1 = -0.02, \lambda_2 = -0.01, \lambda_3 = 0.01\}$ have stable equilibrium points
Behavior	Small deviations from the equilibrium point lead to larger and diverging population changes (unstable or positive output)	Small deviations from the equilibrium point are dampened and populations return to the equilibrium range (stable or negative output)
Cause	uncontrolled to regulate population (Host-parasitoid) changes around the equilibrium point	Controlled feedback that counteracts deviations (Host-parasitoid) from the equilibrium point.
Consequences	1- Large and unexpected fluctuations in the numbers of hosts and parasites result from increased growth rates of parasites ($g_1 > 0$) representing the worm, which may lead to environmental disasters at the equilibrium point E_0. 2- Large and unexpected fluctuations in the numbers of hosts and parasites result from an increase in the growth rates of the parasites ($g_1 > 0$) and an increase in the growth rates of the first host ($g_3 > 0$), which represents the mouse, that may lead to environmental disasters at the equilibrium point E_2.	1-Relatively small and predictable fluctuations By controlling the growth rate of the parasite g_1 representing the worm at the equilibrium point E_0. 2-Relatively small and predictable fluctuations By controlling the growth rate of the parasite g_1 representing the worm and an increase in the growth rates of the first host g_3, which represents the mouse at the equilibrium point E_2.

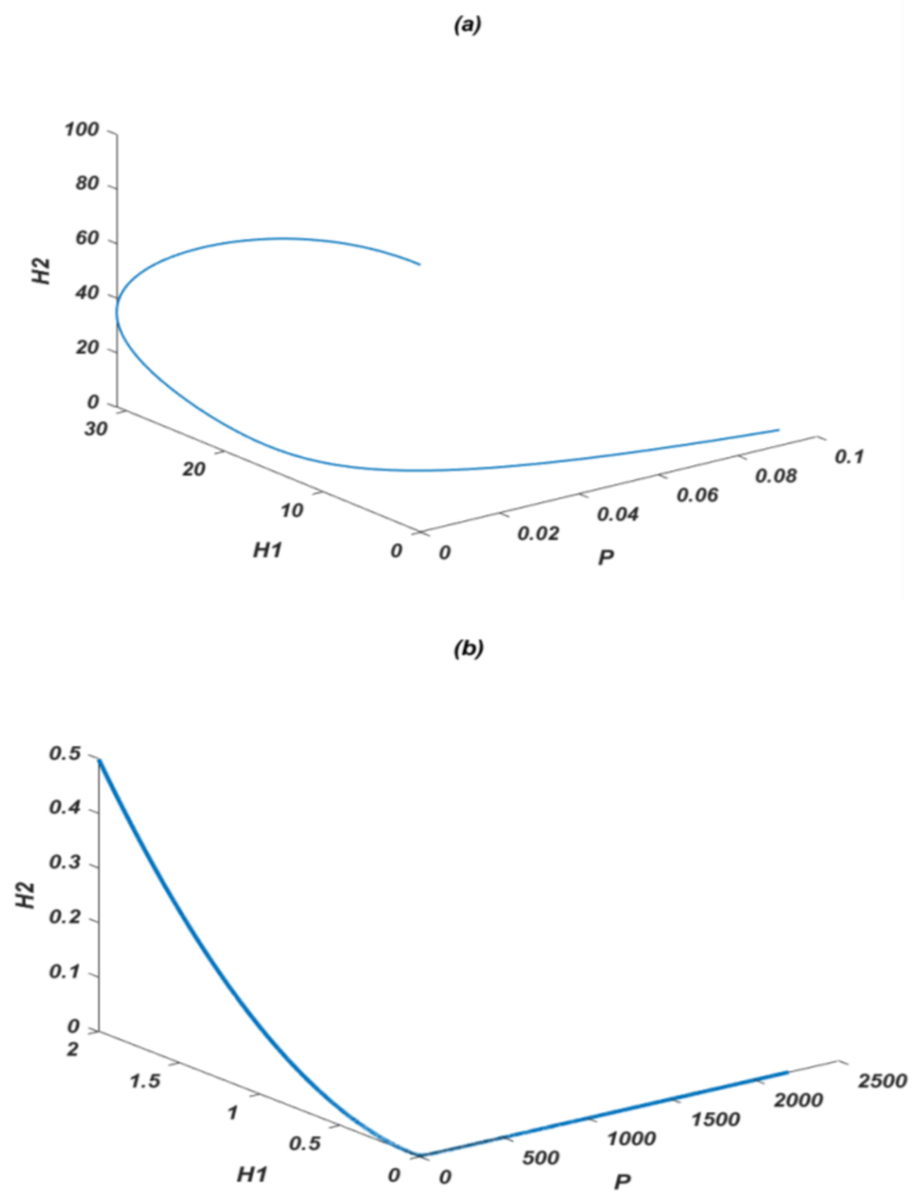


Figure 2: (a) The for equilibrium point E_2 before control (b) The for equilibrium point E_2 after control.

7. NUMERICAL SIMULATION

In previous sections, the system (1) has been studied theoretically. Now, to prove the validity of (1), MATLAB code [25] has been used to consider the system numerically. The effectiveness of the parameters has been shown in the dynamics of the model by observing the parameters set given in (35) which achieves the positive equilibrium point stability conditions, as seen in Fig.3(a-e). The solution converges asymptotically to $E_5 = (2.79,2.21,98.05)$ starting from three initial points $(0.1,2,0.5)$, $(0.3,4,0.7)$ and $(0.1,0.4,1)$, which proves that the system is valid. Where three randomly initial points are selected and from all of them, the solution converges to one positive equilibrium point E_5.

$$\left.\begin{array}{c} g_1 = g_7 = 1, K = 5, g_2 = 0.2, g_3 = 0.9, \\ g_4 = g_5 = K_1 = g_8 = g_6 = K_2 = 0.01, g_9 = 0.02 \end{array}\right\}. \tag{35}$$

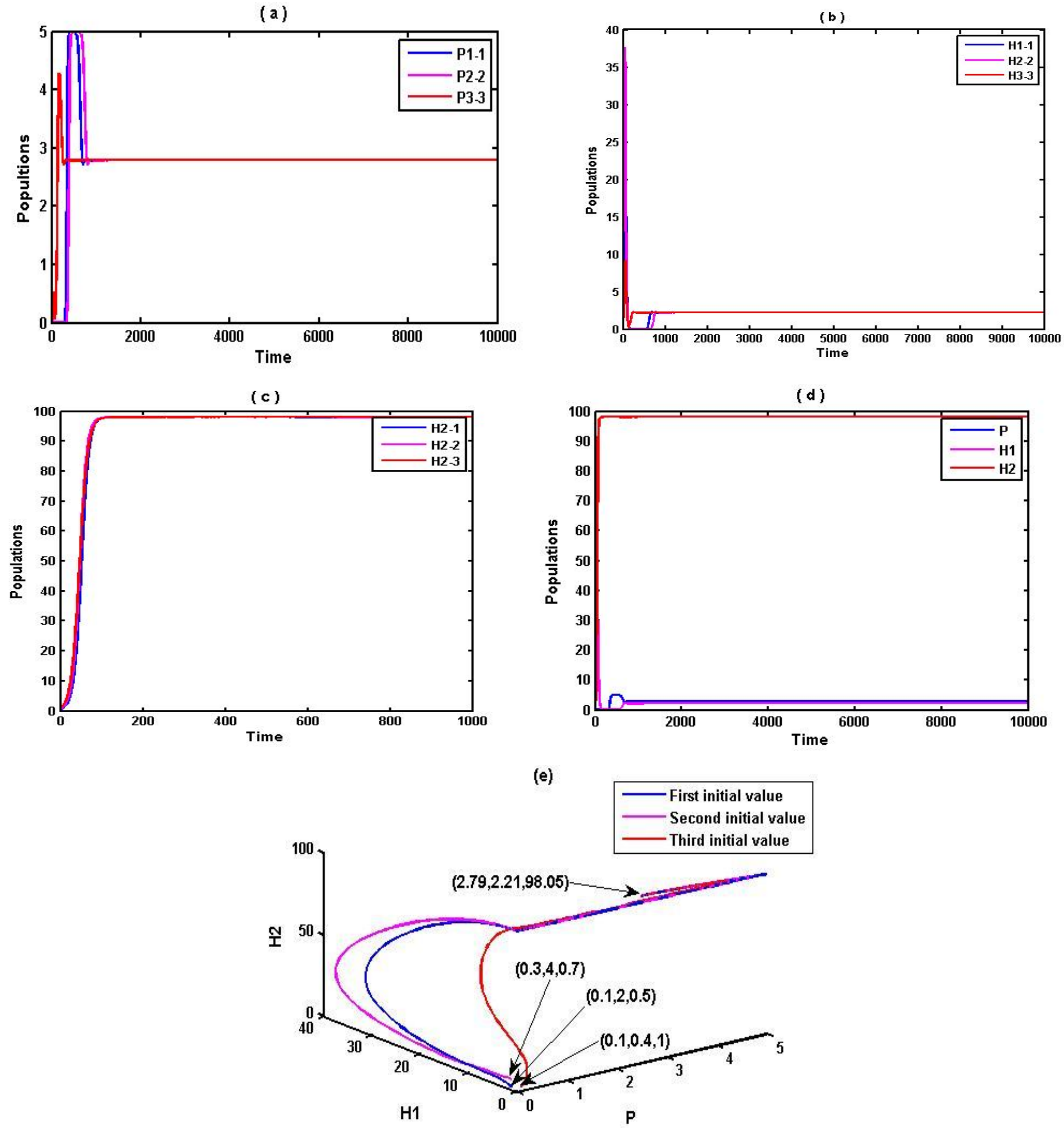


Figure 3: Time series (TS) of system's (1) (a) Trajectories of P, (b) Trajectories of H_1, (c) Trajectories of H_2, (d) TS of the system's (1) solution converges to $E_5 = (2.79,2.21,98.05)$, (e) the Phase portraits of the model (PPM).

To argue the impacts of the system's (1) parameters on the dynamic system behavior, one parameter is changed each time for data given in (35).

Changing the parameter g_1(the growth rate of the Parasite), it is seen that in the range of $0.1 \leq g_1 \leq 1$, system's (1) path converges to E_5 and this means that changing this parameter did not cause the extinction of this food chain, see Fig.4 (a,b), for the perfect value $g_1 = 0.5$.

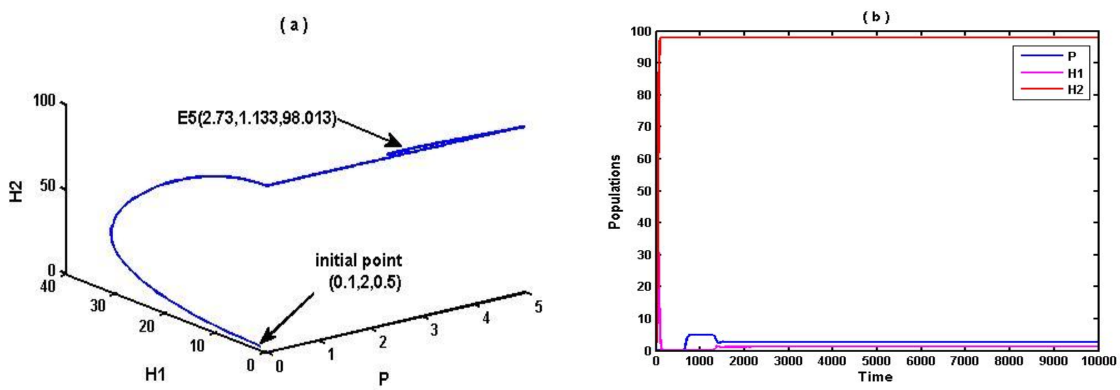


Figure 4: (a) PPM , (b) TS of the system's (1) solution converges to $E_5 = (2.73,1.133,98.013)$ for perfect value $g_1 = 0.5$.

Changing the parameter K (the carrying capacity of the Parasite), it is seen that in the range of $2.7 \leq K \leq 6$, the system's (1) path converges to E_5, so this parameter was unaffected and did not cause the extinction of this food chain, see Fig. 5(a,b), for the perfect value $K = 3$.

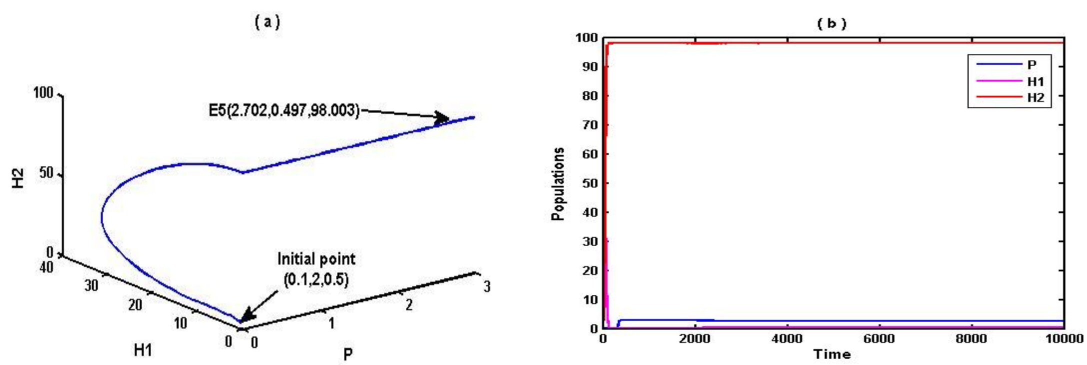


Figure 5: (a) PPM, (b) TS of the system's (1) solution converges to $E_5 = (2.702,0.497,98.003)$ perfect value $K = 3$.

The effect of varying the parameter g_2, (the infection rate from the Parasite to the first Host) while keeping the other parameters as given in (35) has been studied. It is observed that the system's (1) solution converges to E_4 for $0.01 \leq g_2, \leq 0.1$, it means that changing this parameter causes an extinction for the rat as seen in Fig.6(a_1-b_1) for perfect value $g_2 = 0.05$, whereas, for $0.11 \leq g_2 \leq 0.34$, the solution converges to E_5, so the parameter was ineffective as seen in Fig.6(a_2-b_2) for the perfect value $g_2 = 0.3$.

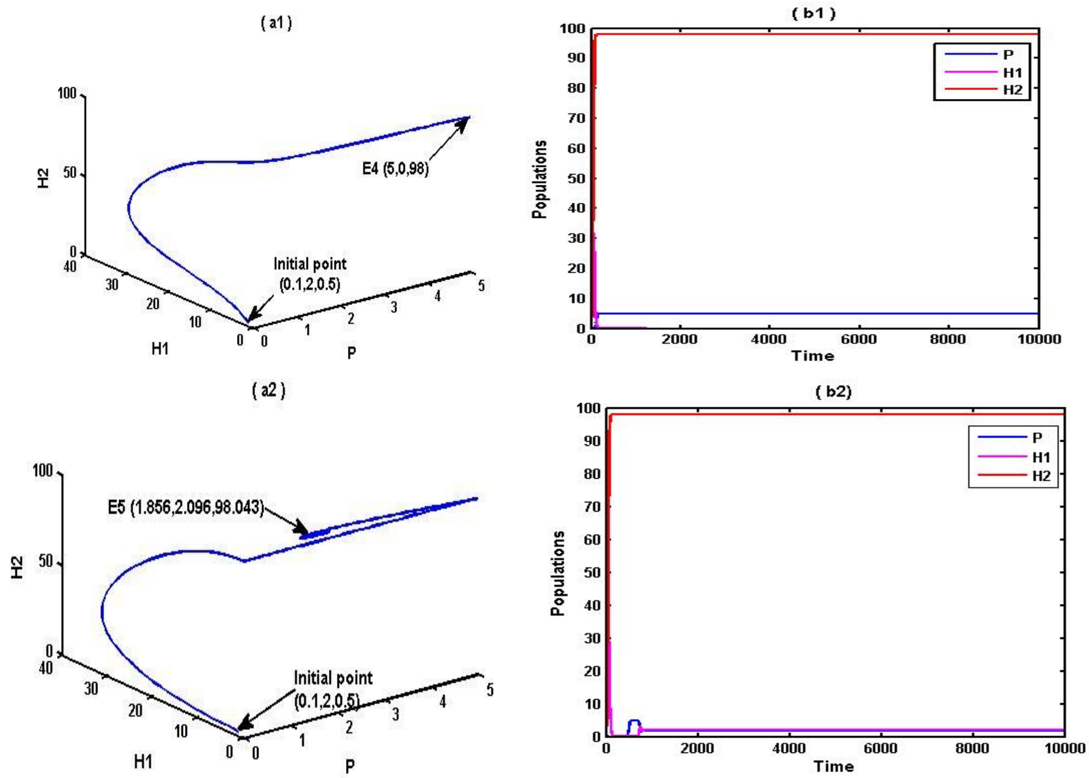


Figure 6: (a_1) PPM, (b_1) TS of the system's (1) solution converges to $E_4 = (5,0,98)$ for perfect value $g_2 = 0.05$, (a_2) PPM, (b_2) TS of the system's (1) solution converges to $E_5 = (4.87,0.206,98)$ for perfect value $g_2 = 0.3$.

Changing the parameter g_3(the growth rate of the first Host), it is seen that in the range of $0.01 \leq g_3 \leq 0.99$, system's (1) path converges to E_5 and this means that changing this parameter did not cause the extinction of this food chain, see Fig.7(a,b), for the perfect value $g_3 = 0.5$.

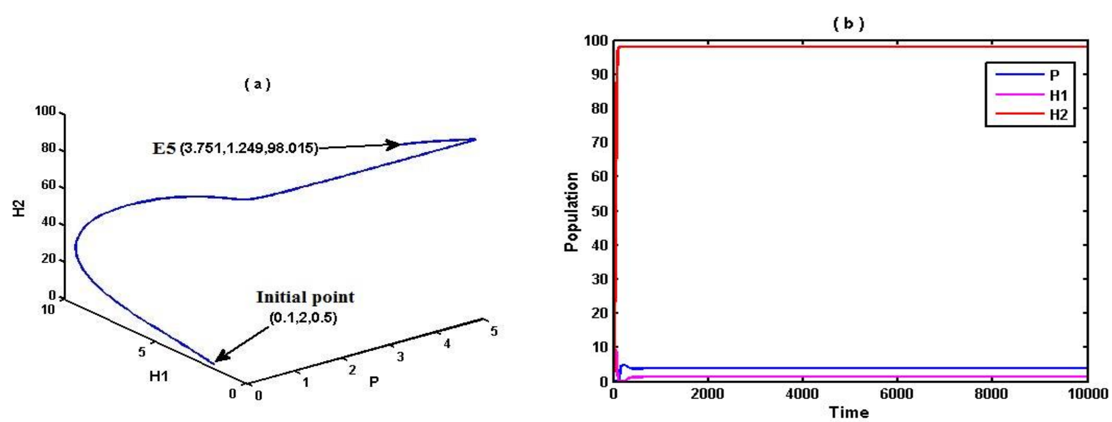


Figure 7: (a) PPM, (b)TS of the system's (1) solution converges to $E_5 = (3.751,1.249,98.015)$ for perfect value $g_3 = 0.5$.

Changing the parameter K_1(the fear rate of the first Host from the second Host), it is seen that in the range of $0.01 \leq K_1 \leq 0.99$, the system's (1) path converges to E_5 which means changing this parameter did not cause the extinction of this food chain, see Fig.8(a,b), for the perfect value $K_1 = 0.5$.

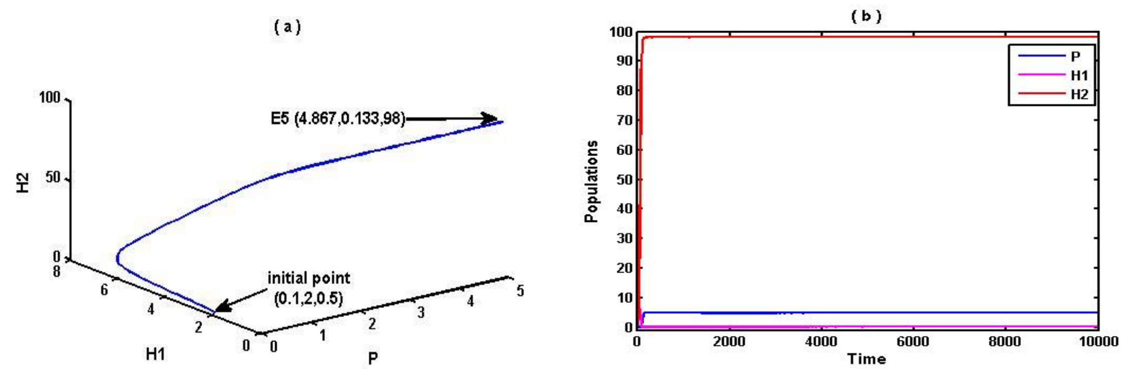


Figure 8: (a) PPM, (b)TS of the system's (1) solution converges to $E_5 = (4.887,0.133,98)$ for perfect value $K_1 = 0.5$.

Changing the parameter g_4(the internal competition rate between the first Host individuals), it is seen that in the range of $0.01 \leq g_4 \leq 0.9$, system's (1) path converges to E_5 and this means that changing this parameter did not cause the extinction of this food chain, see Fig.9(a,b), for the perfect value $g_4 = 0.3$.

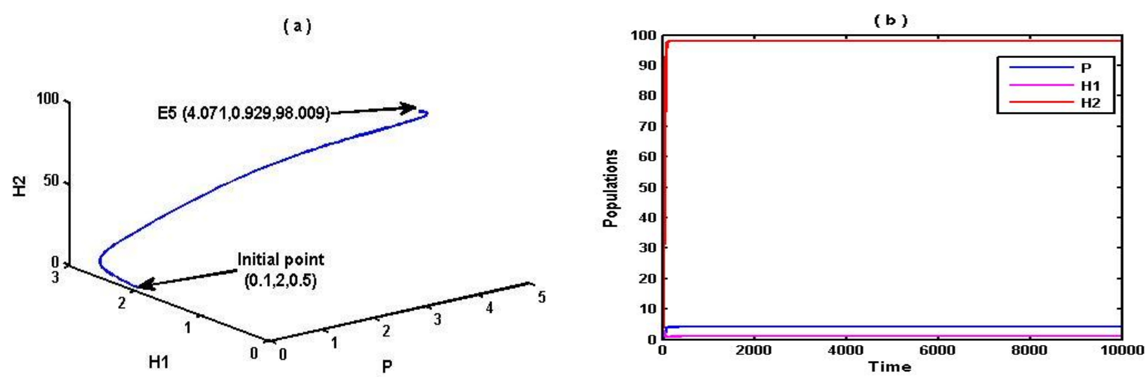


Figure 9: (a) PPM, (b) TS of the system's (1) solution converges to $E_5 = (4.887,0.133,98)$ for perfect value $g_4 = 0.3$.

The effect of varying the parameter g_5, (the infection rate from the first Host to the second Host) has been studied. It is observed that the system's (1) solution converges to E_4 for $0.005 \leq g_5 \leq 0.014$, so the parameter was effective as only the rats disappeared as seen in Fig.10(a_1-b_1) for perfect value $g_5 = 0.011$, whereas for $0.015 \leq g_5 \leq 0.9$, the solution converges to E_5, it means that changing this parameter keeps this food chain free from extinction as seen in Fig.10(a_2-b_2) for perfect value $g_5 = 0.5$.

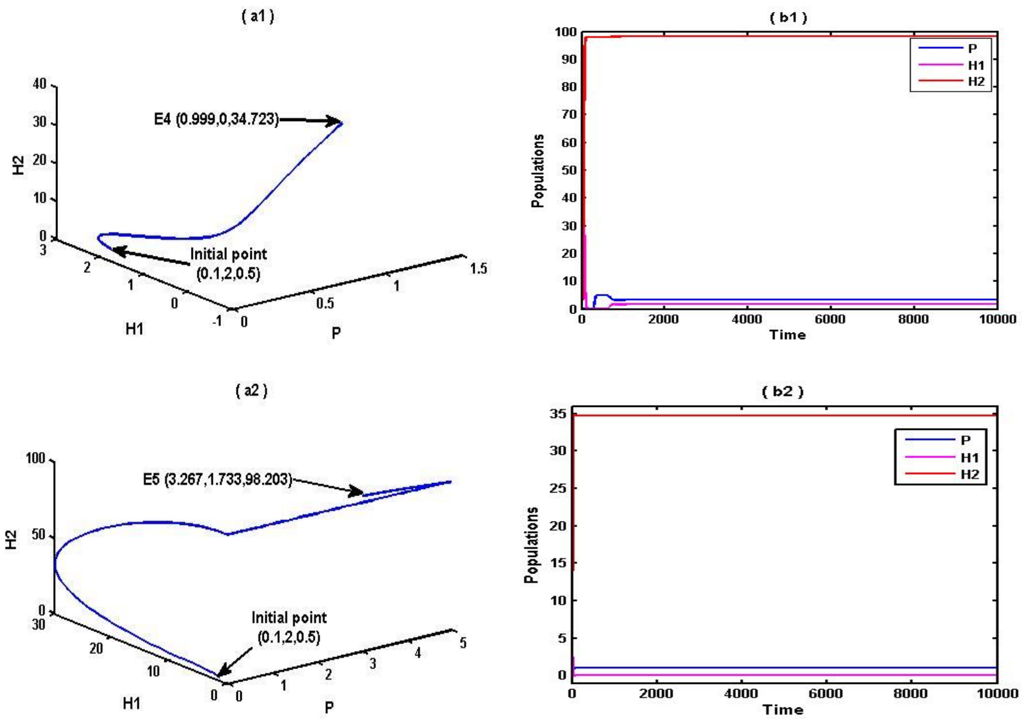
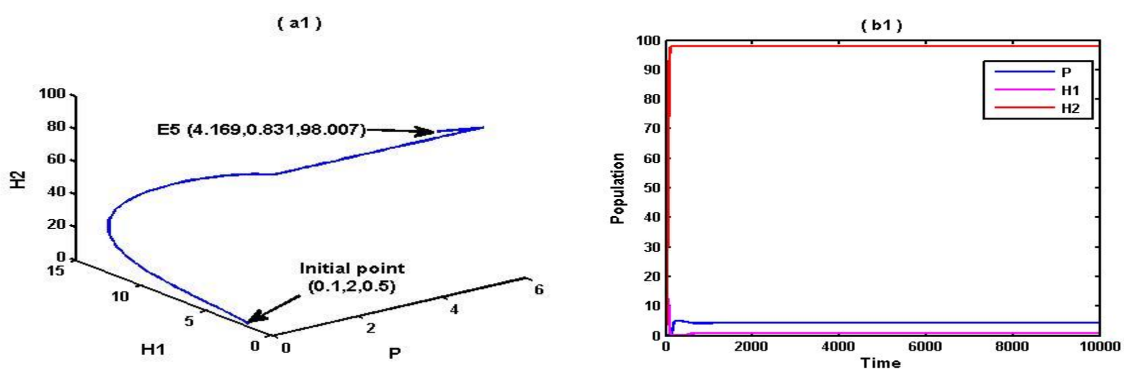


Figure 10: (a_1) PPM, (b_1) TS of the system's (1) solution converges to $E_4 = (0.999,0,34,723)$ for perfect value $g_5 = 0.011$, (a_2) PPM, (b_2) TS of the system's (1) solution converges to $E_5 = (3.267,1.733,98.203)$ for perfect value $g_5 = 0.5$.

The effect of varying only the parameter g_6, (the extermination rate of the first Host) has been studied. It is observed that the system's (1) solution converges to E_5 for $0.001 \leq g_6 \leq 0.47$, thus means that changing this parameter keeps this food chain free from extinction as seen in Fig.11(a_1-b_1) for perfect value $g_6 = 0.3$, whereas for $0.48 \leq g_6 \leq 0.99$, the solution converges to E_4, so the parameter was effective as the warms and the humans remained as seen in Fig.11(a_2-b_2) for perfect value $g_6 = 0.6$.



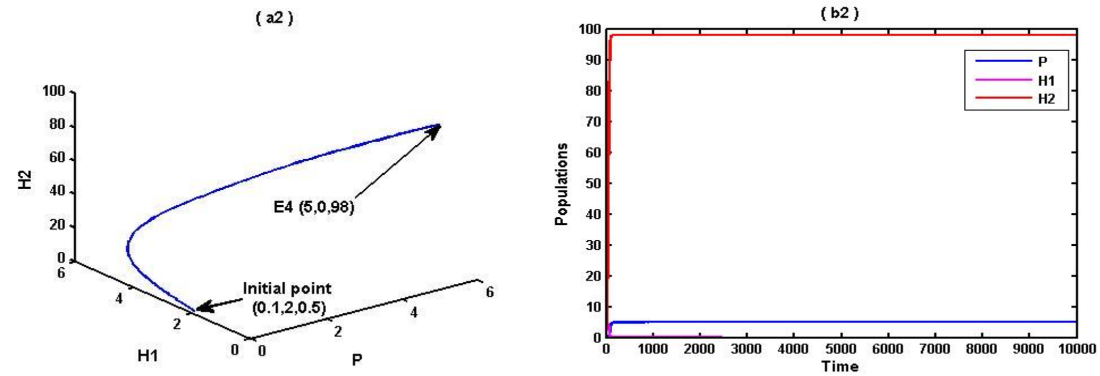


Figure 11: (a_1) PPM, (b_1) TS of the system's (1) solution converges to $E_5 = (4.169,0.831,98,007)$ for perfect value $g_6 = 0.3$, (a2) PPM, (b2) TS of the system's (1) solution converges to $E_4 = (5,0,98)$ for perfect value $g_6 = 0.6$.

Changing the parameter g_7(the growth rate of the second Host), it is seen that in the range of $0.7 \le g_7 \le 1,$ system's (1) path converges to E_5 which means changing this parameter did not cause the extinction of this food chain, see Fig.12(a,b), for the perfect value $g_7 = 0.8$.

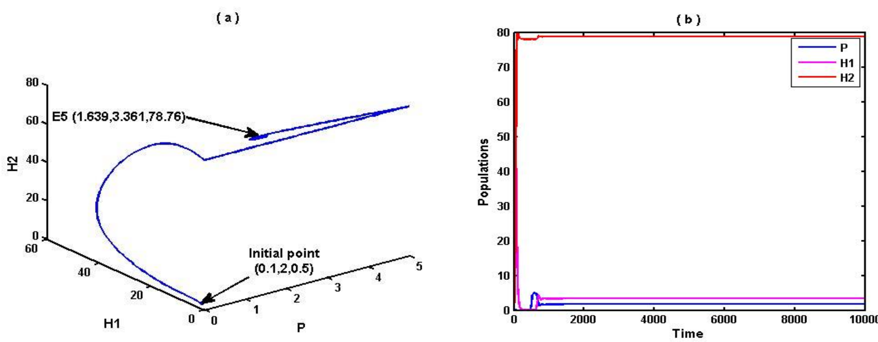


Figure 12: (a) PPM, (b) TS of the system's (1) solution converges to $E_5 = (1.639,3.361,78.76)$ for perfect value $g_7 = 0.8$.

Changing the parameter g_8(the internal competition rate between the second Host individuals), it is seen that in the range of $0.001 \le g_8 \le 0.007,$ system's (1) path converges to E_4 and this means that changing this parameter causes the extinction of the first host, see Fig.13(a_1-b_1), for the perfect value $g_8 = 0.005$, then in the range $0.008 \le g_8 \le 0.018,$ system's (1) path converges to E_5 and this means that changing this parameter did not cause the extinction of the food chain, see Fig.13(a_2-b_2), for the perfect value $g_8 = 0.017$.

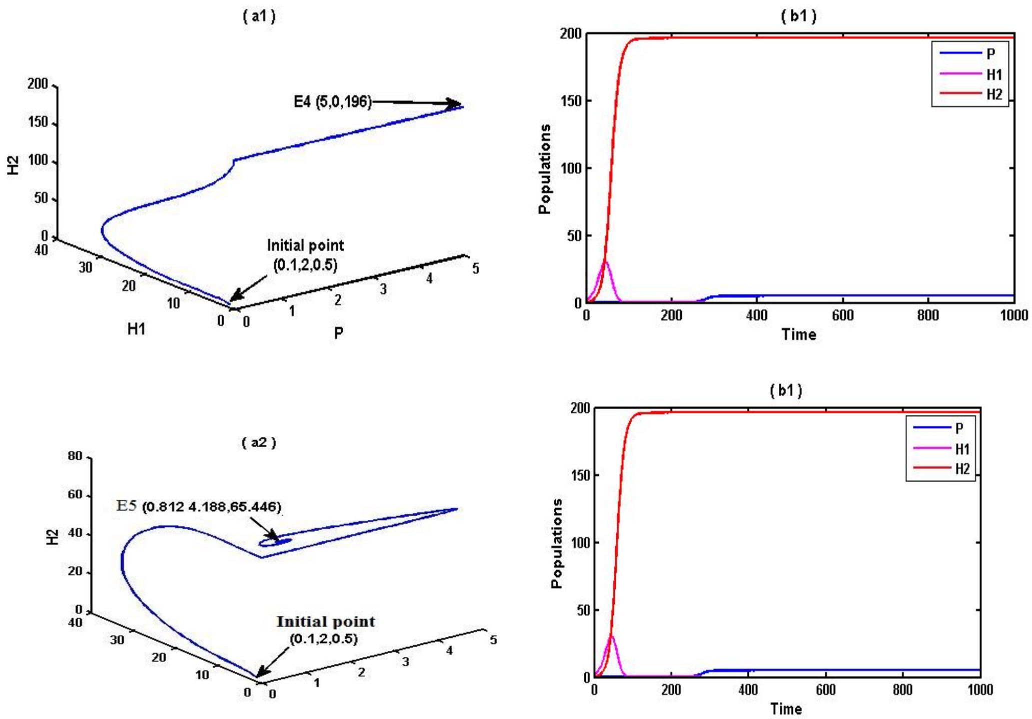


Figure 13: (a_1) PPM, (b_1) TS of the system's (1) solution converges to $E_4 = (5,0,196)$ for perfect value $g_8 = 0.005$,(a_2) PPM, (b_1) TS of the system's (1) solution converges to $E_5 = (0.812,4.188,65.446)$ for perfect value $g_8 = 0.015$.

Changing the parameter K_2(the fear rate of the second Host from the first Host), it is seen that in the range of $0.001 \leq K_2 \leq 0.07$, system's (1) path converges to E_5 and this means that changing this parameter did not cause the extinction of this food chain, see Fig.14, for the perfect value $K_2 = 0.03$.

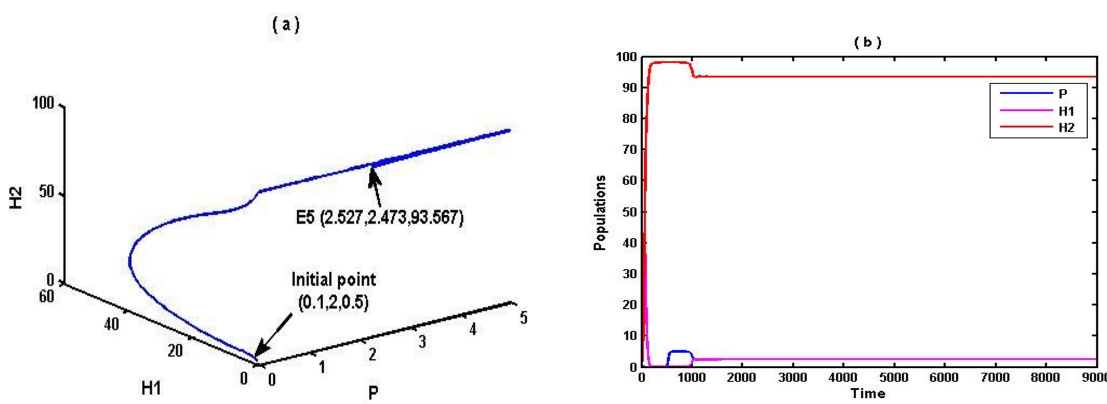


Figure 14: (a) PPM, (b) TS of the system's (1) solution converges to $E_5 = (1.639,3.361,78.76)$ for perfect value $K_2 = 0.03$.

Changing the parameter g_9(the death rate of the second Host), it is seen that in the range of $0.001 \leq g_9 \leq 0.4$, system's (1) path converges to E_5 and this means that changing this parameter did not cause the extinction of this food chain, see Fig.15(a,b), for the perfect value $g_9 = 0.3$.

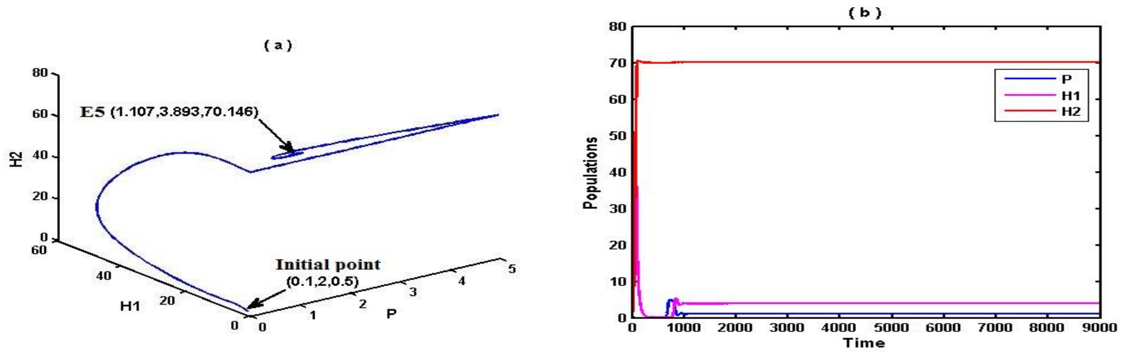


Figure 15: (a) PPM, (b) TS of the system's (1) solution converges to $E_5 = (1.107,3.893,70.146)$ for perfect value $g_9 = 0.3$.

By changing the parameters g_5, g_7, g_8, g_9, K_2 (the infection rate from the first Host to the second Host, the growth rate of the second Host, the internal competition rate between the second Host individuals, the death rate of the second Host, the fear rate of the second Host from the first Host respectively) in the range $0.0001 < g_5$, $g_7 \leq 0.002$, $3.5 < g_8 \leq 10, 0.955 < g_9 \leq 0.9999$, $0.5 < K_2 \leq 6$, it is seen that the system's (1) path converges to E_1 and this means that changing these parameters keeps only the first Host (rats) alive, see Fig.16(a,b), for the perfect values $g_5 = g_7 = 0.001$, $g_8 = 5, g_9 = 0.99, K_2 = 3$.

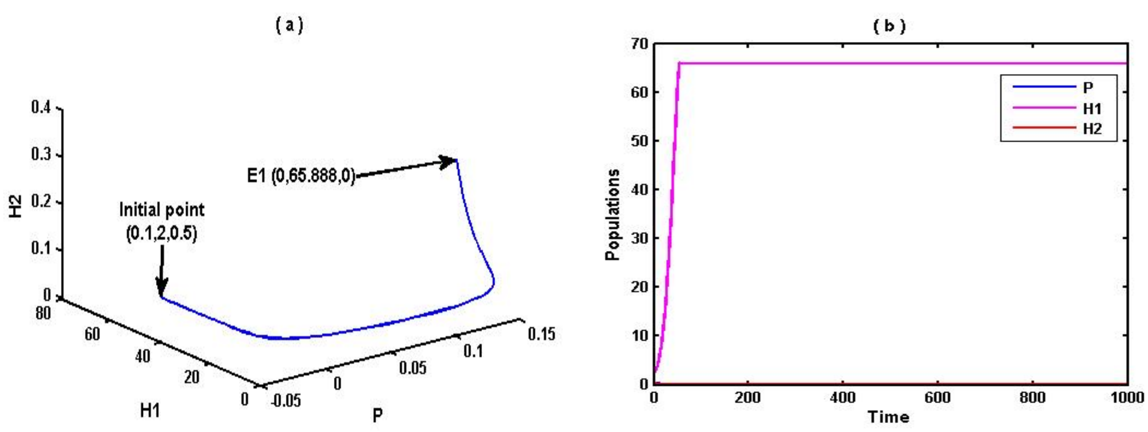


Figure 16: (a) PPM, (b) TS of the system's (1) solution converges to $E_1 = (0,65.888,0)$ for perfect values $g_5 = g_7 = 0.001$, $g_8 = 5, g_9 = 0.99, K_2 = 3$.

By changing the parameters $g_1, g_2, g_5, g_7, g_8, g_9, K$ (the growth rate of the Parasite, infection rate from the Parasite to the first Host, the infection rate from the first Host to the second Host, growth rate the second Host, the internal competition rate between the second Host individuals, the death rate of the second Host, the Carrying capacity of the Parasite respectively) in the range $2 < g_1 \leq 10, 0.01 < g_2 \leq 0.4, 0.0001 < g_5 \leq 0.003, 0.001 < g_7 \leq 0.03, 3.5 < g_8 \leq 11, 0.955 < g_9 \leq 0.9999, 0.1 < \mathrm{K} \leq 8$, it is seen that system's (1) path converges to E_3 and this means that changing these parameters keep only the Parasite and first Host (rats) alive, see Fig.17(a,b), for the perfect values $g_1 = 4, g_2 = 0.02, g_5 = 0.002, g_7 = 0.001, g_8 = K = 6, g_9 = 0.99$.

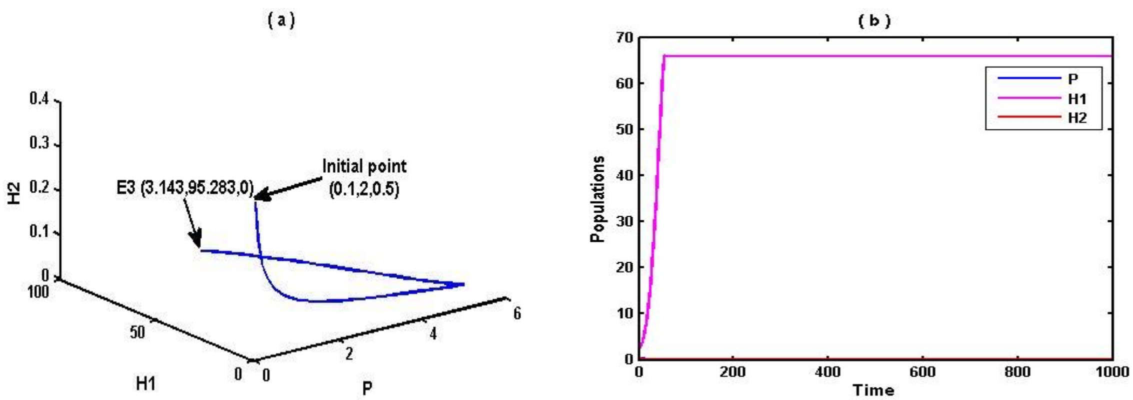


Figure 17: (a) PPM, (b) TS of the system's (1) solution converges to $E_3 = (3.142,95.283,0)$ for perfect values $g_1 = 4, g_2 = 0.02, g_5 = 0.002, g_7 = 0.001, g_8 = K = 6, g_9 = 0.99$.

So, the most effective parameters are shown in Table 3. Whereas, Table 4 shows the ineffective parameters that only converge to E_5. Whereas Table 5 shows the parameters in which the bifurcation appeared. Table 6 shows the effective parameters that only converge to E_1. Table 7 shows the effective parameters that only converge to E_3.

Table 3. The most effective parameters

Parameter	Converge	Parameter	Converge
$0.01 \leq g_2 \leq 0.1$	E_4	$0.001 \leq g_6 \leq 0.47$	E_5
$0.11 \leq g_2 \leq 0.34$	E_5	$0.48 \leq g_6 \leq 0.99$	E_4
$0.005 \leq g_5 \leq 0.014$	E_4	$0.001 \leq g_8 \leq 0.007$	E_4
$0.015 \leq g_5 \leq 0.9$	E_5	$0.008 \leq g_8 \leq 0.018$	E_5

Table 4. The ineffective parameters converge to E_5

Parameter	Parameter
$0.1 \leq g_1 \leq 1$	$0.7 \leq g_7 \leq 1$
$0.01 \leq g_4 \leq 0.9$	$0.001 \leq K_2 \leq 0.07$
$0.01 \leq g_3 \leq 0.99$	$0.001 \leq g_9 \leq 0.4$
$0.01 \leq K_1 \leq 0.99$	$2.7 \leq K \leq 6$

Table 5. The bifurcation parameters

Parameter	Converge	Bifurcation
$0.1 \leq g_1 \leq 1$	E_5	
$0.01 \leq g_2 \leq 0.1$ $0.1 < g_2 \leq 0.34$	E_4 E_5	$g_2 = 0.1$
$0.01 \leq g_3 \leq 0.99$	E_5	
$0.01 \leq g_4 \leq 0.9$	E_5	
$0.005 \leq g_5 \leq 0.009$ $0.009 < g_5 \leq 0.9$	E_4 E_5	$g_5 = 0.009$
$0.001 \leq g_6 \leq 0.47$ $0.47 < g_6 \leq 0.99$	E_5 E_4	$g_6 = 0.47$
$0.7 \leq g_7 \leq 1$	E_5	
$0.001 \leq g_8 \leq 0.007$ $0.007 < g_8 \leq 0.018$	E_4 E_5	$g_8 = 0.007$
$0.001 \leq g_9 \leq 0.4$	E_5	
$2.7 \leq K \leq 6$	E_5	
$0.01 \leq K_1 \leq 0.99$	E_5	
$0.001 \leq K_2 \leq 0.07$	E_5	

Table.6. The most effective parameters that together converge to E_1

Parameter	Parameter
$0.0001 < g_5 \leq 0.002$	$0.955 < g_9 \leq 0.9999$
$3.5 < g_8 \leq 10$	$0.5 < K_2 \leq 6$

Table 7. The most effective parameters that together converge to E_3

Parameter	Parameter
$0.01 < g_2 \leq 0.4$	$0.001 < g_7 \leq 0.03$
$3.5 < g_8 \leq 10$	$3.5 < g_8 \leq 11$
$0.0001 < g_5 \leq 0.003$	$0.955 < g_9 \leq 0.9999$
$0.1 < K \leq 8$	

8. Conclusions and Discussion

In this work, we investigate the dynamics of two hosts and one parasite mathematical model with the fear effect and SI disease. The parasite species reproduces by logistic growth law. There is a mutual fear between the first host and the second host. Infecting the second host with SI disease through transmission from the first host through contact or by leaving a mark on its surroundings according to the Lotka-Volterra function. The model has been studied theoretically and its validity has been studied numerically after the founding of the local and global equilibrium points. The effects of parameters on the mathematical model are studied. This research also explores a new model of disease control, where disease is combated by controlling unstable equilibrium points and making them stable.

Therefore, the model is solved numerically for the given set of parameters in (35) with three initial points. The following observations were obtained:

1- The model has four global equilibrium points.

2- The model has one kind of attraction in Int. R_+^3 for the data given in (35).

3- The solution of the model converges asymptotically to $E_5 = (2.79, 2.21, 98.05)$ for the data given in (35).

4- The most effective parameters g_2, g_5, g_6, g_8.

5- The ineffective parameters $g_1, g_3, g_7, g_4, g_9, K, K_1, K_2$.

6- We conclude that changing only the parameters g_5, g_7, g_8, g_9, K_2, it is seen that the system's (1) path converges to E_1. This means that changing these parameters keeps only the first Host (rats) alive.

7- Changing only the parameters $g_1, g_2, g_5, g_7, g_8, g_9, K$, it is seen that the system's (1) path converges to E_3 and this means that changing these parameters keeps only the Parasite and first Host (rats) alive.

8-Around the equilibrium point E_0, E_2, as indicated by Table 2, stabilization work.

Conflict of Interests

The author(s) declare that there is no conflict of interest.

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