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CONTAGION IN THE BANKING ECOSYSTEM: FRACTIONAL ORDER MODELLING

HAMZA AIT TAMERZ[1,*], KARAM ALLALI[2], ADIL MESKAF[1,3]

[1]Experimental Laboratory of Innovation in Technology and Simulations, Faculty of Sciences, Chouaib Doukkali University, El Jadida, 299-24000, Morocco

[2]Laboratory of Mathematics, Computer Science and Applications, Faculty of Sciences and Technologies, Hassan II University of Casablanca, PO Box 146, 20650 Mohammedia, Morocco

[3]Department of SEG, Faculty of Economic and Social Legal Sciences, Chouaib Doukkali University, El Jadida, Morocco

Abstract. This paper examines the dynamics of systemic risk within banking networks through the analysis of equilibrium points and associated stability conditions, using a fractional model to highlight the interactions between distressed and non-distressed banks. Equilibrium points are derived by solving a reduced system of fractional differential equations, accounting for both homogeneous and heterogeneous banking environments. Local and global stability analyses rigorously identify the conditions under which these equilibrium points exhibit stability or instability. Numerical simulations are employed to illustrate the systemic risk dynamics, complementing the theoretical insights. The results contribute to a deeper understanding of systemic financial risk and offer valuable implications for risk management and policy formulation in the banking sector.

Keywords: banking networks; fractional differential equations; risk management; stability.

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[*]Corresponding author

E-mail address: aittamerz.hamza@ucd.ac.ma

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1. INTRODUCTION

The global banking crisis that have been observed during the summer of 2007 brought attention to the vulnerabilities within the financial system [1, 2], particularly highlighting the importance of liquidity risk, which had previously been overshadowed by other types of risks like credit and market risk. It also called into question the existing risk management strategies employed by financial institutions. Interestingly, the international regulatory frameworks, such as Basel Committee on Banking Supervision (BCBS) [3, 5], did not sufficiently address liquidity risk, despite their focus on creating standardized regulations for global banking. Liquidity, defined as the ability to meet obligations as they come due, proved to be a critical yet underappreciated element of financial stability.

This historical context underlines the importance of enhancing models that can effectively capture the spread of financial shocks, such as liquidity crises, through the banking ecosystem, motivating the need for advanced tools like fractional order modelling to better understand and manage systemic risk. Traditional models for analyzing banking contagion often rely on integer-order systems, which may oversimplify the dynamics of contagion propagation. These models typically assume uniform diffusion of shocks across institutions, neglecting the heterogeneous nature of the banking network and the time-varying intensity of contagion events. As a result, there is a need for more sophisticated models that can capture the complexities of contagion with greater accuracy.

This paper aims to address the limitations of traditional contagion models by applying fractional order modelling to the banking ecosystem. Fractional order models offer a more flexible framework for understanding the dynamic behavior of contagion processes, allowing for a more accurate representation of the propagation of financial fluctuations over time and across interconnected institutions. Fractional order modelling has gained significant traction in analyzing complex dynamic systems across diverse fields, including finance and biomedical sciences. In the financial ecosystem, the Susceptible-Infected-Recovered (SIR) model, originally developed for epidemiology, have been adapted to study contagion in banking networks [4]. The SIR model, used to simulate the spread of financial distress among banks, categorizes institutions as susceptible to, infected by, or recovered from financial point of view. In the domain of financial applications, the use of the SIR model to study the spread of liquidity risk contagion has been addressed in [6, 7, 8]. However, traditional SIR models often fail to capture the intricacies of contagion dynamics in a highly interconnected and heterogeneous banking system. This is where fractional order models provide a more robust alternative, offering a refined understanding by accounting for memory effects and long-range dependencies in the spread of financial risks. These advanced models allow for a more accurate representation of how shocks propagate over time, giving deeper insights into systemic risk factors. Similar techniques have been applied in biomathematics, where fractional order models are used to study the progression of diseases such as Hepatitis B and C. For example, fractional models provide enhanced precision in modeling the interactions between the virus and the immune system compared to traditional methods [9, 10, 11]. Likewise, in cancer research, fractional models are used to simulate tumor growth and treatment responses, offering a more nuanced approach to understanding disease progression and therapeutic outcomes [12].

By introducing fractional order modelling, this research contributes to a deeper understanding of contagion dynamics in the banking ecosystem. The model provides insights into the temporal and spatial diffusion of financial distress, enabling policymakers and regulators to develop more effective tools for monitoring systemic risk and preventing the spread of financial crises.

Recently, [13] studied the contagious banking ecosystem using the following system of differential equations:

$$\begin{cases} \dfrac{dU}{dt} = -\beta UD + \mu E + \theta R, \\ \dfrac{dE}{dt} = \beta UD - (\mu + \gamma)E, \\ \dfrac{dD}{dt} = \gamma E - (\delta_1 + \delta_2) D, \\ \dfrac{dR}{dt} = \delta_1 D - \theta R, \\ \dfrac{dL}{dt} = \delta_2 D. \end{cases} \tag{1}$$

Where $U(t)$ represents the risk-free banks. This category includes undistressed banks that are currently healthy but potentially vulnerable, even though they have not yet experienced distress at time t. $E(t)$ stands for the exposed banks. These are banks that have been interacting with risky banks and have begun to show signs of weakened performance. However, at time t, their expected losses have not become significant. $D(t)$ is the risk-contagious banks. This means that banks in this group are currently distressed due to credit risk at time t and are experiencing potential losses. $R(t)$ represents the recovered Banks. This category consists of banks that have recovered from credit risk and are no longer in distress at time t. Finally, $L(t)$ symbolizes the liquidated banks, which means that banks that have been distressed and subsequently liquidated at time t fall into this category. The parameters of the problem is as follows: β is the rate at which contagion risk spreads due to interactions between undistressed (or risk-free) banks and distressed banks. μ is the rate at which risk-exposed banks transition back to the undistressed state. γ is the rate at which risk-exposed banks move to the distressed class. δ_1 is the rate at which distressed banks recover and transition to the recovered class. δ_2 is the rate at which distressed banks are liquidated. θ is the rate at which recovered banks lose their immunity and revert to the vulnerable class. Figure 1 illustrates the schematic representation of risk contagion within the banking network.

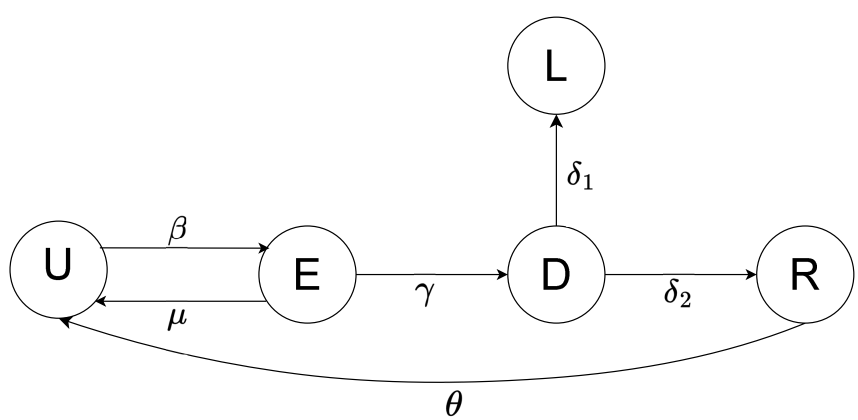

FIGURE 1. Diagram of risk contagion in the banking network.

Most natural phenomena including epidemiological dynamics involve time memory effect and are valuable to demonstrate the facts about nature related processes having non-local dynamics. Models with fractional derivatives handle these issues in better way because non-integral order derivatives contain time-dependent kernels. Many fractional derivatives can be found in the literature but the most common is Caputo fractional derivative. The key advantage of using Caputo fractional derivative is it takes the same form of initial conditions as in the case of classical derivatives, which means it does not require the fractional initial values. Motivated by these useful facts we reformulate the model (1) in fractional form by adopting Caputo fractional time derivative.

In this paper, we continue the investigation in the contagious banking ecosystem by considering the following UEDRL fractional order differential equations:

$$\begin{cases} D^{\alpha}U = -\beta UD + \mu E + \theta R, \\ D^{\alpha}E = \beta UD - (\mu + \gamma)E, \\ D^{\alpha}D = \gamma E - (\delta_1 + \delta_2)D, \\ D^{\alpha}R = \delta_1 D - \theta R, \\ D^{\alpha}L = \delta_2 D. \end{cases} \tag{2}$$

Where α is the fractional order derivative. Figure 1 illustrates the schematic representation of risk contagion within the banking network that we will study in this paper.

The paper is organized as follows. The well-posedness of the model is established in Section 2. The model equilibria is given in Section 3. Section 4 is dedeicated to the stability analysis of the equilibria. Numerical simulations are given in Section 5. The last section concludes the work.

2. Positivity and Boundedness of the Solution

Before study the well-posedness of the model, we assume that:

(1) All banks in the system are assumed to be susceptible to credit risk.
(2) Each bank has an equal probability of being affected by contagious banks upon interaction with a risky bank, leading to potential exposure.
(3) Banks that are exposed to risk may either recover without becoming distressed or progress to a distressed state and move into category $D(t)$.
(4) When a bank is affected by risk, it either recovers through effective banking management or is liquidated. Recovered banks are then categorized as $R(t)$.

(5) Recovered banks can potentially lose their immunity and return to the undistressed state.

Theorem 2.1. *Consider the system of equations given by (2) with initial conditions* $U(0) \geq 0$, $E(0) \geq 0$, $D(0) \geq 0$, $L(0) \geq 0$, *and* $R(0) \geq 0$. *Then, the solutions* $U(t)$, $E(t)$, $D(t)$, $L(t)$, *and* $R(t)$ *remain positive and bounded for all* $t \geq 0$.

Proof. The UEDRL model is employed to represent systemic financial risk within a banking population. It is reasonable to assume that all parameters and variables in the model are non-negative, i.e., $t \geq 0$. We demonstrate that all variables in the model remain nonnegative given nonnegative initial conditions.

From system (2), we have the following:

$$\begin{aligned}
&D^{\alpha}U|_{U=0} = \mu E + \theta R \geq 0, \text{ since } \mu \geq 0, \theta \geq 0, E \geq 0, R \geq 0 \\
&D^{\alpha}E|_{E=0} = \beta UD \geq 0, \text{ since } \beta \geq 0, U \geq 0, D \geq 0 \\
&D^{\alpha}D|_{D=0} = \gamma E \geq 0, \text{ since } \gamma \geq 0, E \geq 0 \\
&D^{\alpha}R|_{R=0} = \delta_1 D \geq 0, \text{ since } \delta_1 \geq 0, D \geq 0 \\
&D^{\alpha}L|_{L=0} = \delta_2 D \geq 0, \text{ since } \delta_2 \geq 0, D \geq 0
\end{aligned} \tag{3}$$

These results confirm that the solution to system (2) remains nonnegative for all $t \geq 0$.

To address boundedness, consider the total number of banks defined as:

$$N(t) = U(t) + E(t) + D(t) + R(t) + L(t). \tag{4}$$

In our model, the total number of banks, is denoted as N.

Taking the derivative of both sides yields:

$$D^{\alpha}N(t) = D^{\alpha}U(t) + D^{\alpha}E(t) + D^{\alpha}D(t) + D^{\alpha}R(t) + D^{\alpha}L(t) \tag{5}$$

From system (2), $D^{\alpha}N(t) = 0$, which implies that $N(t) = N$. Thus, each component of the solution $U(t)$, $E(t)$, $D(t)$, $R(t)$, and $L(t)$ is bounded between zero and the total initial number of banks N.

□

2.1. Feasible Solution. All solutions to the model in system (2) are bounded. The feasible region for the banking population is given by

$$\Omega = \Big\{ (U(t), E(t), D(t), R(t), L(t)) \in \mathbb{R}^5 \mid U(t) + E(t) + D(t) + R(t) + L(t) \leq N \Big\}.$$

The region Ω is positively invariant with respect to the model in system (1). Hence, the model is mathematically well-posed and systemically valid within Ω.

3. THE MODEL EQUILIBRIA

3.1. Risk-Free Equilibrium (RFE) Point for Systemic Risk. In order to obtain the equilibrium points of the system, we equate the system of equations to zeros, i.e., $D^{\alpha}U = D^{\alpha}E = D^{\alpha}D = D^{\alpha}R = D^{\alpha}L = 0$. Since the last equation of system (2) is independent of the others, we have the following reduced system:

$$-\beta UD + \mu E + \theta(N - U - E - D) = 0, \tag{6}$$

$$\beta UD - (\mu + \gamma)E = 0, \tag{7}$$

$$\gamma E - (\delta_1 + \delta_2) D = 0, \tag{8}$$

$$\delta_1 D - \theta R = 0. \tag{9}$$

Equilibrium points for risk-free are conditions where there is no systemic risk, that is $E = D = 0$. From equation (6), we have $\theta(N - U) = 0 \Rightarrow U = N$, and then,
the equilibrium point of the risk-free for credit risk is $P_0 = (N, 0, 0, 0)$.

3.2. The Basic Reproduction Number of the UEDR Model for Systemic Risk. First, define the basic reproduction number R_0 as the average number of secondary distressed banks that occur when one distressed bank is interacting with a completely undistressed sample.

lemma 3.1. *The basic reproduction number of system (2) is given by*

$$R_0 = \frac{\gamma\beta U_0}{(\mu + \gamma)(\delta_1 + \delta_2)} \tag{10}$$

where U_0 is the number of undistressed bank at the risk-free equilibrium point.

Proof. The basic reproduction number is determined using the matrix generation method, based on equation (1). Let F and V represent the transitional inflows and the transitional outflows, respectively,

$$F = \begin{pmatrix} 0 & \beta U \\ 0 & 0 \end{pmatrix} \tag{11}$$

$$V = \begin{pmatrix} (\mu+\gamma) & 0 \\ -\gamma & \delta_1+\delta_2 \end{pmatrix}$$

This implies

$$V^{-1} = \frac{1}{(\mu+\gamma)(\delta_1+\delta_2)} \begin{pmatrix} (\delta_1+\delta_2) & 0 \\ \gamma & (\mu+\gamma) \end{pmatrix} \tag{12}$$

Then

$$FV^{-1} = \begin{pmatrix} 0 & \beta U \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{(\mu+\gamma)} & 0 \\ \frac{\gamma}{(\mu+\gamma)(\delta_1+\delta_2)} & \frac{1}{(\delta_1+\delta_2)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\gamma\beta U}{(\mu+\gamma)(\delta_1+\delta_2)} & \frac{\beta U}{\delta_1+\delta_2} \\ 0 & 0 \end{pmatrix} \tag{13}$$

Then by the matrix generation method, we find that

$$R_0 = \frac{\gamma\beta U_0}{(\mu+\gamma)(\delta_1+\delta_2)} \tag{14}$$

□

3.3. **Risk Persistence Equilibrium (RPE) Point.**

In order to indicate the possibility of credit risk spreading, we determine the risk persistence equilibrium point. Since in persistence conditions the risk spreads, the number of banks is $U \neq 0$, $E \neq 0, D \neq 0$, and $R \neq 0$. From equations (6)-(9), we obtain the risk persistence equilibrium point for systemic risk as

$$\begin{aligned} U^* &= \frac{U_0}{R_0} \\ E^* &= \frac{\theta(\delta_1+\delta_2)U_0(R_0-1)}{(\gamma+\theta)(\delta_1+\delta_2)+\gamma\theta} \end{aligned} \tag{15}$$

$$D^* = \frac{\theta\gamma U_0 (R_0 - 1)}{(\gamma+\theta)(\delta_1+\delta_2)+\gamma\theta}$$
$$R^* = \frac{\delta_1\gamma U_0 (R_0 - 1)}{(\gamma+\theta)(\delta_1+\delta_2)+\gamma\theta}$$

4. Stability Analysis of the Model

4.1. Global Stability of the Risk-Free Equilibrium Point.

Theorem 4.1. *The RFE of system (1) is globally asymptotically stable if the basic reproduction number* $R_0 < 1$.

Proof. Let the Lyapunov function $L_f : \Omega \longrightarrow \mathbb{R}$ be defined as follows:

$$L_f(t) = aE + bD.$$

Then,

$$\begin{aligned} D^\alpha L_f(t) &\leq aD^\alpha E + bD^\alpha D \\ &\leq a(\beta UD - (\mu+\gamma)E) + b(\gamma E - (\delta_1+\delta_2)D) \\ &\leq a\beta U_0 D - a(\mu+\gamma)E + b\gamma E - b(\delta_1+\delta_2)D \end{aligned} \tag{16}$$

Let the coefficient of E correspond to zero, and the values of a and b are given by

$$b = \frac{a(\mu+\gamma)}{\gamma}, \quad \forall a \in \mathbb{R}_+ \tag{17}$$

Then, combining equations (16) and (17), $D^\alpha L_f(t)$ can be written as

$$\begin{aligned} D^\alpha L_f(t) &\leq \beta U_0 D - \frac{(\mu+\gamma)(\delta_1+\delta_2)D}{\gamma} \\ &\leq \frac{\gamma\beta U_0 D - (\mu+\gamma)(\delta_1+\delta_2)D}{\gamma} \\ &\leq \frac{(\mu+\gamma)(\delta_1+\delta_2)(R_0-1)D}{\gamma} \end{aligned} \tag{18}$$

Then, this implies that $D^\alpha L_f(t) \leq 0$ if $R_0 \leq 1$. Hence, it follows from Lasalle's invariance principle that the system is globally asymptotically stable at P_0.

□

4.2. Global Stability of the Risk Persistence Equilibrium Point.

Theorem 4.2. *The RPE of system (2) is globally asymptotically stable.*

Proof. Let a Lyapunov function be defined as

$$L_1(t) = \left(U - U^* - U^* \ln \frac{U}{U*}\right) + \left(E - E^* - E^* \ln \frac{E}{E^*}\right) + \left(D - D^* - D^* \ln \frac{D}{D^*}\right) \tag{19}$$

Then,

$$D^\alpha L_1(t) \leq \left(1 - \frac{U^*}{U}\right) D^\alpha U + \left(1 - \frac{E^*}{E}\right) D^\alpha E + \left(1 - \frac{D^*}{D}\right) D^\alpha D$$

$$\begin{aligned} D^\alpha L_1(t) \leq& \left(1 - \frac{U^*}{U}\right)(-\beta UD + \mu E + \theta R) + \left(1 - \frac{E^*}{E}\right)(\beta UD - (\mu+\gamma)E) \\ &+ \left(1 - \frac{D^*}{D}\right)(\gamma E - (\delta_1 + \delta_2) D) \\ &\leq (-\beta UD + \mu E + \theta R) - \frac{U^*}{U}(-\beta UD + \mu E + \theta R) + \beta UD - (\mu+\gamma)E \\ &- \frac{E^*}{E}(\beta UD - (\mu+\gamma)E) + (\gamma E - (\delta_1+\delta_2) D) - \frac{D^*}{D}(\gamma E - (\delta_1+\delta_2) D). \end{aligned} \tag{20}$$

Summing the term with D but without D^* or E^*, we have

$$\beta U^* D - (\delta_1 + \delta_2) D = 0 \Rightarrow \beta U^* = (\delta_1 + \delta_2) \tag{21}$$

Inserting (20) in (21) yields

(22)

$$\begin{aligned} D^\alpha L_1(t) &\leq \theta R - \frac{U^*}{U}(\mu E + \theta R) - \frac{E^*}{E}(\beta UD - (\mu+\gamma)E) - \frac{D^*}{D}(\gamma E - (\delta_1+\delta_2) D) \\ &\leq \theta R - \frac{U^*}{U}\left(\left(\frac{\beta U^* D^* - \theta R^*}{E^*}\right) E + \theta R\right) - \beta UD\frac{E^*}{E} + (\mu+\gamma)E^* - \frac{\gamma D^* E}{D} \\ &+ (\delta_1+\delta_2) D^* \\ &\leq \theta R - \frac{\beta U^{*2} D^* E}{UE^*} + \theta \frac{U^* R^* E}{UE^*} - \theta \frac{U^* R}{U} - \frac{\beta UDE^*}{E} + \beta U^* D^* - \frac{\gamma D^* E}{D} + \beta U^* D^* \\ &\leq \beta U^* D^* \left(2 - \frac{U^* E}{UE^*} - \frac{UDE^*}{UD^* E}\right) + \theta R + \theta \frac{U^* R^* E}{UE^*} - \frac{\theta U^* R}{U} - \frac{\gamma D^* E}{D} \\ &\leq \gamma E^* \beta U^* D^* \left(2 - \frac{U^* E}{UE^*} - \frac{UDE^*}{U^* D^* E}\right) + \theta R\left(\frac{U - U^*}{U}\right) + \gamma E\left(\frac{U^* D - UD^*}{UD}\right) \end{aligned}$$

Let us denote $x_1 = \dfrac{U^* E}{UE^*}$ and $x_2 = \dfrac{UDE^*}{U^* D^* E}$. If $D = D^* \Rightarrow x_1 x_2 = 1$, using the relation

(23) $$\frac{x_1+x_2}{2} \geq \sqrt{x_1 x_2}, \qquad x_1 x_2 \geq 0$$

where this implies that $x_1 + x_2 \geq 2$ with the equality attained

if $x_1 = x_2 = 1$. Hence, we obtain $D^{\alpha} L_1(t) \leq 0$ for $U = U^*$, with $D^{\alpha} L_1(t) = 0$ on the set $\{(U,E,D); U = U^*, D = D^*, E = E^*\}$. Therefore, it follows from Lasalle's invariance principle that the system is globally asymptotically stable at the risk persistence equilibrium point. □

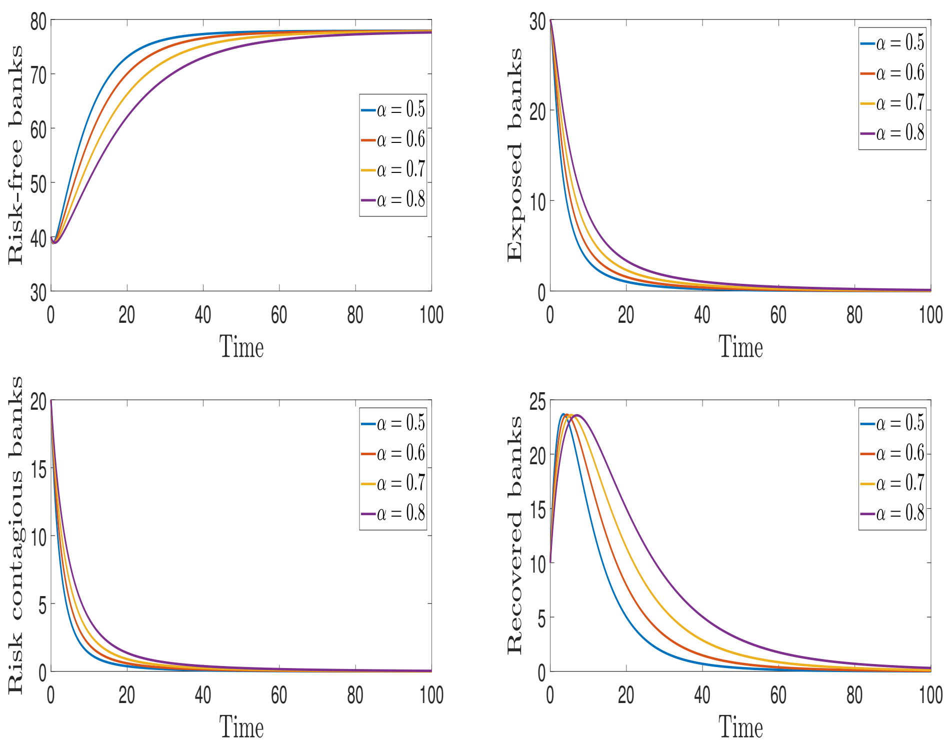

FIGURE 2. Banking risk-free dynamics. Risk-free banks (top left), exposed banks (top right), risk contagious banks (bottom left) and recovered banks (bottom right) for different values of the fractional derivative order.

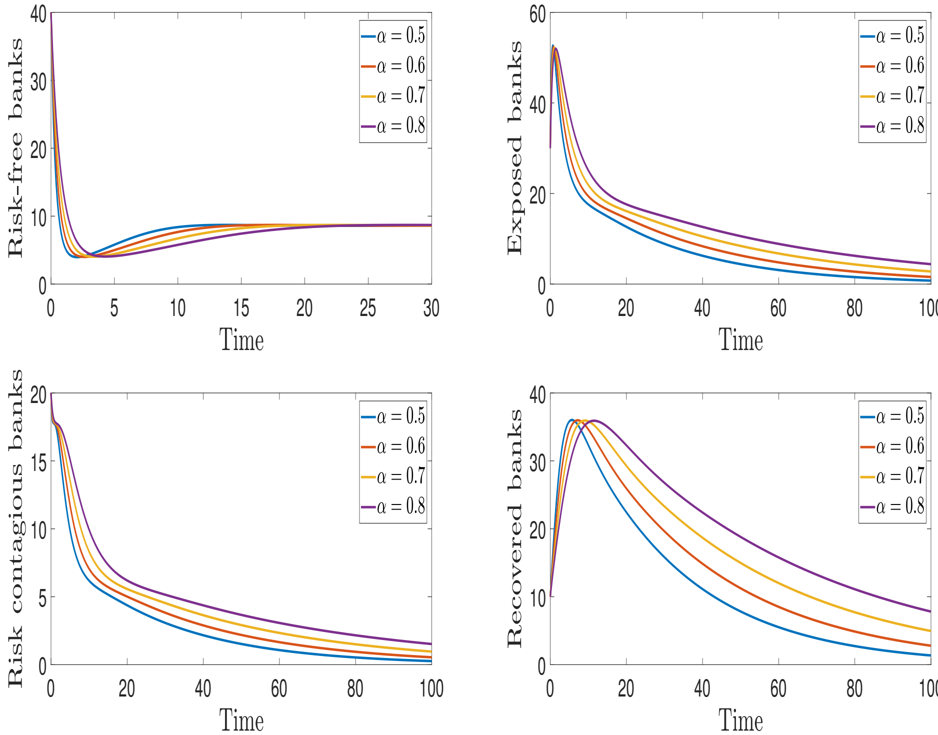

FIGURE 3. Banking risk persistence dynamics. Risk-free banks (top left), exposed banks (top right), risk contagious banks (bottom left) and recovered banks (bottom right) for different values of the fractional derivative order.

5. NUMERICAL SIMULATIONS

This section is devoted to numerical simulation of our suggested mathematical model. To this end we will choose the following initial conditions:

$$U(1) = 40, E(1) = 30, D(1) = 20 \text{ and } R(1) = 10.$$

Figure 2 shows the temporal evolution of the number of the risk-free banks, the exposed banks, the risk contagious banks and the recovered banks for the following parameters: $\beta = 0.01$, $\mu = 0.2$, $\theta = 0.05$, $\gamma = 0.1$, $\delta_1 = 0.2$ and $\delta_2 = 0.1$. This figure illustrate the banking risk-free dynamics. Indeed, the number of risk-free banks reaches their maximal level. However, the other banks compartments vanish. In addition, by decreasing the fractional order derivative the convergence towards the equilibrium is more quick.

Figure 3 shows the evolution dynamics of the number of the risk-free banks, the exposed banks, the risk contagious banks and the recovered banks for the following parameters: $\beta =$

0.04, $\mu = 0.03$, $\theta = 0.05$, $\gamma = 0.1$, $\delta_1 = 0.2$ and $\delta_2 = 0.1$. This figure illustrate the banking risk persistence dynamics. Indeed, the number of all acting compartments remains at constant level. Moreover, we conclude that by decreasing the fractional order derivative the convergence towards the equilibrium is more quick.

6. CONCLUSION

In this paper, we have used a fractional model to represent mainly the interactions between distressed and non-distressed banks. We have analyzed the equilibrium points and related stability criteria to investigate the dynamics of systemic risk within banking networks. In order to account for both homogeneous and heterogeneous banking environments, equilibrium points are obtained through the solution of a simplified system of fractional differential equations. These equilibrium points' stable and unstable conditions are thoroughly determined by local and global stability results. The systemic risk dynamics are illustrated by numerical simulations, which serve to support the theoretical results. The findings provide important new insights into systemic financial risk and have important ramifications for risk management and the development of banking sector regulation.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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