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HUNTING COOPERATION AMONG PREDATORS EFFECTS ON THE DYNAMICS OF FOOD-WEB ECO-EPIDEMIOLOGICAL MODEL WITH ADDITIONAL FOOD TO PREDATORS

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Abstract: In this paper, an eco-epidemiological model with fear and internal competition in the first and second prey populations have been proposed and studied, the susceptible predator is fed on preys using Holling type-II and Lotka-Volterra functional responses in the presence of additional food and hunting cooperation to the susceptible predator. Furthermore, it is assumed that the infected predator is receiving treatment. The existence, uniqueness and boundedness of the solution have been studied. The conditions of local and global stability have been identified. The goal of this study is to determine the fear, additional food, hunting cooperation and treatment effect on the stability of the proposed system, it has been demonstrated that they play a significant role in regulating the system's stability. Finally, our analytical conclusions were verified numerically by Mathematica.

Keywords: eco-epidemiological; fear effect; additional food; hunting cooperation; treatment.

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1. INTRODUCTION

Mathematical modeling is a powerful tool used to understand complex systems and predict their behavior through the use of mathematical frameworks. When applied to ecological and

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biological contexts, such models can illuminate interactions among species, and environmental factors, and many research works have been down in literature [1-4].

In ecological contexts, fear influences predator-prey interactions. For prey species, the perception of risk from predators can alter foraging behavior, habitat use, and social structures. For example, when prey feel threatened, they may change their feeding times or locations to avoid detection, impacting their growth and reproductive success. This behavioral response can also affect predator dynamics, as reduced prey availability can lead to changes in predator hunting strategies and population dynamics, see [5-8] and the reference therein.

Hunting cooperation among predators represents a fascinating aspect of ecological dynamics that significantly influences population structures, species interactions, and ecosystem stability. When predators work together, their collective hunting strategies can enhance their efficiency, leading to profound effects on both prey populations and the broader ecological community. Moreover, the dynamics of hunting cooperation can be affected by environmental factors, such as food availability and habitat structure. When resources are abundant, cooperation may increase, leading to larger predator populations. Conversely, in environments with limited resources, competition may intensify, affecting the stability of predator-prey dynamics, see [9-11] and the reference therein.

When predators are provided with additional food, their foraging behavior may change, potentially leading to increased population densities. With more resources available, predators might exhibit less competitive behavior, allowing them to thrive and even expand their territories. This can lead to cascading effects throughout the food web, as an increase in predator populations can place greater pressure on prey species, potentially leading to declines in their numbers. Understanding the impact of additional food on dynamic systems is essential for effective ecosystem management and conservation strategies. By analyzing these interactions, researchers can better predict how changes in resource availability affect species dynamics, community structure, and overall ecosystem health, see [12-14] and the reference therein.

The ecological interactions such as competition, mutualism, and predation play an important role in population dynamics. However, the size of populations is also impacted by disease infestation. Prey-predator interactions should therefore take this issue seriously. Many field investigations have shown that both predators and prey are infected with disease. Diseases can reduce the ability of infected organisms to survive and reproduce by affecting their internal mechanisms, see [15-17] and the reference therein. Hence, predator-prey relationships when both populations are afflicted should worry us all.

In recent years, infectious disease has emerged as a significant factor, hence modeling transmissible diseases mathematically has become an important tool in analyzing and controlling infectious diseases. Models and simulations can help construct and test hypotheses, evaluate quantitative hypotheses, provide specific answers, establish the impact of parameter changes, and provide parameter estimations. For example, some authors studied diseases that can have infected a human been like COVID-19 and the human immunodeficiency virus, see [18-21] and the reference therein.

In eco-epidemiological models treatment has a significant impact on disease dynamics. Researchers and public health officials can create more effective interventions that take into account medical and environmental factors by comprehending these intricate relationships. In the end, this will enhance public health results and disease control strategies [22, 23].

Recently, Yousef and et al. [24] studied a fractional-order eco-epidemiological model with fear and hunting cooperation but they did not consider the effect of treatment, Ghosh and et al. [25] studied an eco-epidemiological model with the combined effects of fear, hunting cooperation, including treatment for infected prey. Also, they analyzed their model with memory effects but they did not consider the effect of additional food.

The aim of this paper is to study an eco-epidemiological model involving two prey populations and one predator with disease in the presence of fear and internal competition in prey's population, moreover additional food, hunting cooperation and treatment have been proposed in predator population.

2. Mathematical Model

In this work, a mathematical model of first and second preys, susceptible predator and infected predator with hunting cooperation has been proposed for study. whose total population density at time T is denoted by $L_1(T),\ L_2(T),\ L_3(T),\ L_4(T)$ respectively.

$$\left.\begin{aligned}
\frac{dL_1}{dT} &= \frac{a_1 L_1}{1+f_1 L_3} - b_1 L_1^2 - \frac{B_1 L_1 L_3}{C+\alpha\eta A+L_1},\\
\frac{dL_2}{dT} &= \frac{a_2 L_2}{1+f_2 L_3} - b_2 L_2^2 - (\rho + hL_3)L_2 L_3,\\
\frac{dL_3}{dT} &= a_3 L_3\left(1-\frac{L_3+L_4}{K}\right) + \frac{C_1(L_1+\eta A)L_3}{C+\alpha\eta A+L_1} + C_2(\rho+hL_3)L_2L_3 - (d+BL_4)L_3 + \frac{\gamma L_4}{\sigma+L_4},\\
\frac{dL_4}{dT} &= BL_3L_4 - \delta L_4 - \frac{\gamma L_4}{\sigma+L_4}.
\end{aligned}\right\} \qquad (1)$$

Where $L_1(0) \geq 0,\ L_2(0) \geq 0,\ L_3(0) \geq 0$ and $L_4(0) \geq 0$ represent the initial condition (IC). While the biological meaning of the parameters in system (1) described in Table 1.

Table 1: Description of the parameters in system 1

Parameters	**Description**
$a_i, i = 1,2,3$	The intrinsic growth rate of the first prey, second prey and susceptible predator respectively
$b_j, j = 1,2$	Internal competition rate of first prey and second prey respectively
$f_m, m = 1,2$	The fear rate of the first prey and second prey species from susceptible predator
K > 0	The carrying capacity of the susceptible predator population
$B_1 > 0$	The maximum rate of predation
$C \geq 1$	The half saturation value of the predator
α	The ratio of the maximum growth rates of the predator when it consumes the prey and additional food
$\eta > 0$	The ratio of the search rate of the predator for additional food
A > 0	The additional food
ρ	The attack rate of the susceptible predator to the second prey
h	The hunting cooperation in susceptible predator
$C_1 > 0$	The conversion rate by attack the first prey
$0 < C_2 < 1$	The conversion rate by attack the second prey
$d > 0$	The natural death rate of susceptible predator
B	The disease transmission rate
δ	The death rates of infected predator
γ	The maximum medical resource supplied for treatment
σ	Saturation factor that measure the effect of the delay in treatment for the infected individuals

The interaction functions of system (1) are continuous and have continuous partial derivatives on R_+^4 with respect to dependent variables L_1, L_2, L_3 and L_4. Accordingly, they are Lipschitzian functions and hence system (1) has a unique solution for each non-negative initial condition. Additionally, the following theorem demonstrates the system's boundedness.

Theorem 1. All the solutions of system (1) which initiate in R_+^4 are uniformly bounded provided that

$$C \geq 1 \tag{2}$$

Proof. Let $\left(L_1(t),\ L_2(t),\ L_3(t),\ L_4(t)\right)$ be any solution of the system (1) with non-negative initial condition $\left(L_1(0),\ L_2(0),\ L_3(0), L_4(0)\right) \in R_+^4$.

Assume that $M(t) = L_1(t) + L_2(t) + L_3(t) + L_4(t)$ then taken the time derivative of $M(t)$ along the solution of the system (1), we get

$$\frac{dM}{dt} = \frac{a_1 L_1}{1+f_1 L_3} - b_1 L_1^2 - \frac{B_1 L_1 L_3}{C+\alpha\eta A+L_1} + \frac{a_2 L_2}{1+f_2 L_3} - b_2 L_2^2 - (\rho + hL_3)L_2 L_3 + a_3 L_3\left(1 - \frac{L_3+L_4}{K}\right) + \frac{C_1(L_1+\eta A)L_3}{C+\alpha\eta A+L_1} + C_2(\rho + hL_3)L_2L_3 - (d + BL_4)L_3 + \frac{\gamma L_4}{\sigma+L_4} + BL_3L_4 - \delta L_4 - \frac{\gamma L_4}{\sigma+L_4}.$$

Now, by condition (2) with the biological perspective, always $C_1 < B_1$ and $C_2 < 1$ consequently, it is determined that

$$\frac{dM}{dt} \le a_1L_1 - b_1L_1^2 + a_2L_2 - b_2L_2^2 + a_3L_3 + C_1\eta A L_3 - \frac{a_3 L_3^2}{K} - dL_3 - \delta L_4$$

Thus, using comparison theorem, we have

$$\frac{dM}{dt} \le 2a_1L_1\left(1 - \frac{b_1}{2a_1}L_1\right) + 2a_2L_2\left(1 - \frac{b_2}{2a_2}L_2\right) + (a_3 + C_1\eta A)L_3\left(1 - \frac{a_3L_3}{K(a_3+C_1\eta A)}\right) - \mu M,$$

where $\mu = \min\ \{a_1\,, a_2\,, d\,, \delta\}$.

Now since the function $g_1(L_1) = 2a_1L_1\left(1 - \frac{b_1}{2a_1}L_1\right)$ is logistic function with respect to L_1 and hence it is bounded above by the constant $\frac{a_1^2}{b_1}$.

Also, since the function $g_2(L_2) = 2a_2L_2\left(1 - \frac{b_2}{2a_2}L_2\right)$ is logistic function with respect to L_2 and hence it is bounded above by the constant $\frac{a_2^2}{b_2}$.

Finally, since the function $g_3(L_3) = (a_3 + C_1\eta A)L_3\left(1 - \frac{a_3L_3}{K(a_3+C_1\eta A)}\right)$ is logistic function with respect to L_3 and hence it is bounded above by the constant $\frac{\mathrm{K}(a_3+C_1\eta A)^2}{4a_3}$, it is observed that

$$\frac{dM}{dt} + \mu M \le N\ ,\quad where\ N = \left(\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \frac{K(a_3+C_1\eta A)^2}{4a_3}\right).$$

Then by solving the above differential inequality it is obtained that $M(t) \le \frac{N}{\mu} + \left(M_0 - \frac{N}{\mu}\right)\ e^{-\mu t}$.

Then, $\lim\limits_{t\to\infty} M(t) \le \frac{N}{\mu}$.

$So,\ 0 \le M(t) \le \frac{N}{\mu}$. That is the solutions are uniformly bounded.

3. EXISTENCE OF EQUILIBRIUM POINTS (EPs)

In this section, the existence of all possible (EPs) of system (1) has been discussed. It is noticed that, system (1) has fifteen (EPs).

- The trivial (EP) $P_0 = (0,\ 0,\ 0,\ 0)$ is always exists.
- The (EP) $P_1 = \left(\frac{a_1}{b_1},\ 0,\ 0,\ 0\right)$ is always exists.
- The (EP) $P_2 = \left(0,\ \frac{a_2}{b_2},\ 0,\ 0\right)$ is always exists.
- The (EP) $P_3 = (0,\ 0,\ \bar{L}_3,\ 0)$ where $\bar{L}_3 = \frac{K[(a_3-d)(\mathrm{C}+\alpha\eta A)+C_1\eta A]}{a_3(\mathrm{C}+\alpha\eta A)}$ exists if the following condition holds

$$a_3 > d. \tag{3}$$

- The (EP) $P_4 = \left(\frac{a_1}{b_1},\ \frac{a_2}{b_2},\ 0,\ 0\right)$ is always exists.
- The (EP) $P_5 = \left(\bar{\bar{L}}_1,\ 0,\ \bar{\bar{L}}_3,\ 0\right)$ exists if the following equations have a positive solution

$$\frac{a_1}{1+f_1L_3} - b_1L_1 - \frac{B_1L_3}{C+\alpha\eta A+L_1} = 0, \tag{4}$$

$$a_3\left(1-\frac{L_3}{K}\right) + \frac{C_1(L_1+\eta A)}{C+\alpha\eta A+L_1} - d = 0. \tag{5}$$

From Eq.(5), we have

$$L_3 = \frac{K\left[(C_1+a_3-d)L_1+\left((C+\alpha\eta A)(a_3-d)+C_1\eta A\right)\right]}{a_3(C+\alpha\eta A+L_1)}. \tag{6}$$

Now, by substituting Eq.(6) in Eq.(4), we obtain the following equation

$$S_1L_1^4 + S_2L_1^3 + S_3L_1^2 + S_4L_1 + S_5 = 0, \tag{7}$$

where

$$S_1 = -a_3b_1[a_3 + f_1K(C_1 + a_3 - d)] < 0,$$

$$S_2 = a_3\left[a_3\left(a_1 - 3b_1(C+\alpha\eta A)\right) - b_1f_1K\left((C_1 + a_3 - d)[(C+\alpha\eta A)+1] + \left((C+\alpha\eta A)(a_3-d) + C_1\eta A\right)\right)\right],$$

$$S_3 = a_3\left[3a_3(C+\alpha\eta A)[a_1 - b_1(C+\alpha\eta A)] - K[([b_1f_1(C+\alpha\eta A)(a_3-d) + C_1\eta A][(C+\alpha\eta A)+1] + (C_1+a_3-d)[b_1f_1(C+\alpha\eta A)+B_1]) + B_1f_1K(C_1+a_3-d)^2]\right],$$

$$S_4 = a_3\left[a_3(C+\alpha\eta A)^2[3a_1 - b_1(C+\alpha\eta A)] - B_1K\left((C_1+a_3-d)(C+\alpha\eta A) + \left((C+\alpha\eta A)(a_3-d)+C_1\eta A\right)\right)\right] - 2B_1f_1K^2(C_1+a_3-d)[(C+\alpha\eta A)(a_3-d)+C_1\eta A],$$

$$S_5 = a_1{a_3}^2(C+\alpha\eta A)^3 - K\left((C+\alpha\eta A)(a_3-d)+C_1\eta A\right)[a_3(C+\alpha\eta A)[B_1+b_1f_1] + B_1f_1K],$$

which has unique positive solution say $\bar{\bar{L}}_1$ if the next conditions with condition (3) are hold

$$a_1 > 3b_1(C+\alpha\eta A), \tag{8}$$

$$a_3\big(a_1 - 3b_1(C+\alpha\eta A)\big) > b_1 f_1 K\big((C_1 + a_3 - d)[(C+\alpha\eta A)+1] + \big((C+\alpha\eta A)(a_3 - d) + C_1\eta A\big)\big), \quad (9)$$

$$\big[3a_3(C+\alpha\eta A)[a_1 - b_1(C+\alpha\eta A)] > K[([b_1 f_1(C+\alpha\eta A)(a_3-d) + C_1\eta A][(C+\alpha\eta A) + 1] + (C_1 + a_3 - d)[b_1 f_1(C+\alpha\eta A) + B_1]) + B_1 f_1 K(C_1 + a_3 - d)^2]\big], \quad (10)$$

$$a_1 a_3{}^2 (C+\alpha\eta A)^3 > K\big((C+\alpha\eta A)(a_3-d) + C_1\eta A\big)[a_3(C+\alpha\eta A)[B_1 + b_1 f_1] + B_1 f_1 K]. \quad (11)$$

So, the (EP) $P_5 = \big(\bar{\bar{L}}_1,\ 0,\ \bar{\bar{L}}_3\ , 0\big)$, where $\bar{\bar{L}}_3 = L_3\big(\bar{\bar{L}}_1\big)$, exists if in addition to condition (3) conditions $(8-11)$, are hold.

- The (EP) $P_6 = \big(0,\ \hat{L}_2,\ \hat{L}_3,\ 0\big)$ exists if the following equations have a positive solution

$$\frac{a_2}{1+f_2L_3} - b_2L_2 - (\rho + hL_3)L_3 = 0, \quad (12)$$

$$a_3\left(1 - \frac{L_3}{K}\right) + \frac{C_1\eta A}{\mathrm{C}+\alpha\eta A} + C_2(\rho + hL_3)L_2 - d = 0. \quad (13)$$

From Eq.(12), we have

$$L_2 = \frac{a_2 - (1+f_2L_3)(\rho+hL_3)L_3}{b_2(1+f_2L_3)}. \quad (14)$$

Now, by substituting Eq.(14) in Eq.(13), we obtain

$$Q_1L_3^4 + Q_2L_3^3 + Q_3L_3^2 + Q_4L_3 + Q_5 = 0, \quad (15)$$

where

$$Q_1 = -C_2h^2f_2K(\mathrm{C}+\alpha\eta A) < 0,$$
$$Q_2 = -C_2hK(\mathrm{C}+\alpha\eta A)[2\rho f_2 + h] < 0,$$
$$Q_3 = -(\mathrm{C}+\alpha\eta A)[a_3b_2f_2 + C_2\rho K(\rho f_2 + 2h)] < 0,$$
$$Q_4 = (\mathrm{C}+\alpha\eta A)\left[b_2f_2K(a_3-d) + \left(a_2 - \frac{C_2\rho^2K + a_3b_2}{C_2hK}\right)\right],$$
$$Q_5 = K[(\mathrm{C}+\alpha\eta A)[b_2(a_3-d) + C_2a_2\rho] + C_1\eta A b_2].$$

Which has unique positive solution say $\hat{L}_3$ if condition (3) are hold.

So, the (EP) $P_6 = \big(0,\ \hat{L}_2,\ \hat{L}_3, 0\big)$ where $\hat{L}_2 = L_2\big(\hat{L}_3\big)$ exists if in addition to conditions (3) the following condition holds

$$a_2 > \big(1 + f_2\hat{L}_3\big)\big(\rho + h\hat{L}_3\big)\hat{L}_3. \quad (16)$$

- The (EP) $P_7 = \big(0,\ 0,\ \check{L}_3,\ \check{L}_4\big)$ exists if the following equations have a positive solution

$$a_3\left(1 - \frac{L_3+L_4}{K}\right) + \frac{C_1\eta A}{\mathrm{C}+\alpha\eta A} - (d + BL_4) + \frac{\gamma L_4}{(\sigma+L_4)L_3} = 0, \quad (17)$$

$$BL_3 - \delta - \frac{\gamma}{\sigma + L_4} = 0. \tag{18}$$

From Eq.(18), we have

$$L_4 = \frac{\gamma - \sigma(BL_3 - \delta)}{(BL_3 - \delta)}. \tag{19}$$

Now, by substituting Eq.(19) in Eq.(17), we obtain the following equation

$$G_1 L_3^3 + G_2 L_3^2 + G_3 L_3 + G_4 = 0, \tag{20}$$

where

$$G_1 = B(C + \alpha\eta A)[B^2\sigma^2 K + a_3(B\sigma^2 - \gamma)],$$

$$G_2 = \gamma\big((C + \alpha\eta A)[BK(a_3 - d) + a_3(\delta + B\sigma)] + C_1\eta AKB\big),$$

$$G_3 = \Big((C + \alpha\eta A)\big[-\big(K\gamma\delta(a_3 - d) + a_3(\gamma^2 + \gamma\sigma\delta + \sigma^2 B^2)\big) - BK\sigma\delta(\sigma\delta - \gamma)\big] - C_1\eta AK\gamma\delta\Big),$$

$$G_4 = -K\gamma\delta(C + \alpha\eta A)(\gamma + \sigma\delta) < 0.$$

Which has unique positive solution say $\breve{L}_3$ if the next condition with condition (3) are hold

$$\gamma < B\sigma^2. \tag{21}$$

So, the (EP) $P_7 = \big(0,\ 0,\ \breve{L}_3,\ \breve{L}_4\big)$ where $\breve{L}_4 = L_4\big(\breve{L}_3\big)$ exists if in addition to condition (3, 21) the following conditions hold

$$B\breve{L}_3 > \delta, \tag{22}$$

$$\gamma > \sigma\big(B\breve{L}_3 - \delta\big). \tag{23}$$

- The (EP) $P_8 = \big(\tilde{L}_1,\ \tilde{L}_2,\ \tilde{L}_3, 0\big)$ exists if the following equations have a positive solution

$$\frac{a_1}{1 + f_1 L_3} - b_1 L_1 - \frac{B_1 L_3}{C + \alpha\eta A + L_1} = 0, \tag{24}$$

$$\frac{a_2}{1 + f_2 L_3} - b_2 L_2 - (\rho + hL_3)L_3 = 0, \tag{25}$$

$$a_3\left(1 - \frac{L_3}{K}\right) + \frac{C_1(L_1 + \eta A)}{C + \alpha\eta A + L_1} + C_2(\rho + hL_3)L_2 - d = 0. \tag{26}$$

From Eq.(26), we have

$$L_3 = \frac{-K[(C_1 + a_3 - d)L_1 + C_2\rho(C + \alpha\eta A + L_1)L_2 + (a_3 - d)(C + \alpha\eta A) + C_1\eta A]}{(C + \alpha\eta A + L_1)(C_2 hKL_2 - a_3)}. \tag{27}$$

Now, by substituting Eq.(27) in Equations (24 and 25), we obtain the following equations.

$$M_1(L_1, L_2) = (C + \alpha\eta A + L_1)[a_1 - b_1 L_1] + b_1 f_1[(C + \alpha\eta A + L_1)L_1 +$$
$$B_1]\left(\frac{K[(C_1 + a_3 - d)L_1 + C_2\rho(C + \alpha\eta A + L_1)L_2 + (a_3 - d)(C + \alpha\eta A) + C_1\eta A]}{(C + \alpha\eta A + L_1)(C_2 hKL_2 - a_3)}\right) -$$
$$B_1 f_1\left(\frac{-K[(C_1 + a_3 - d)L_1 + C_2\rho(C + \alpha\eta A + L_1)L_2 + (a_3 - d)(C + \alpha\eta A) + C_1\eta A]}{(C + \alpha\eta A + L_1)(C_2 hKL_2 - a_3)}\right)^2 = 0, \tag{28}$$

$$M_2(L_1,L_2) = a_2 - b_2L_2 + [b_2f_2L_2 + \rho]\left(\frac{K[(C_1+a_3-d)L_1+C_2\rho(\mathrm{C}+\alpha\eta A+L_1)L_2+(a_3-d)(\mathrm{C}+\alpha\eta A)+C_1\eta A]}{(\mathrm{C}+\alpha\eta A+L_1)(C_2hKL_2-a_3)}\right) -$$
$$[f_2\rho^2 + h]\left(\frac{-K[(C_1+a_3-d)L_1+C_2\rho(\mathrm{C}+\alpha\eta A+L_1)L_2+(a_3-d)(\mathrm{C}+\alpha\eta A)+C_1\eta A]}{(C+\alpha\eta A+L_1)(C_2hKL_2-a_3)}\right)^2 +$$
$$hf_2\left(\frac{K[(C_1+a_3-d)L_1+C_2\rho(\mathrm{C}+\alpha\eta A+L_1)L_2+(a_3-d)(\mathrm{C}+\alpha\eta A)+C_1\eta A]}{(\mathrm{C}+\alpha\eta A+L_1)(C_2hKL_2-a_3)}\right)^3 = 0. \qquad (29)$$

Now from Eq. (28), we notice that, when $L_1 \to 0$, then $L_2 \to \tilde{L}_2^1$ is a positive root of the equation

$$T_1L_2^2 + T_2L_2 + T_3 = 0, \qquad (30)$$

where

$$T_1 = C_2^2K^2(C+\alpha\eta A)^2[a_1h^2(C+\alpha\eta A) + \rho B_1(h+\rho f_1)] > 0,$$
$$T_2 = C_2K(C+\alpha\eta A)\big[-a_3(C+\alpha\eta A)[2a_1h(C+\alpha\eta A)+\rho B_1] + KB_1(h-2\rho f_1)[(a_3 - d)(C+\alpha\eta A)+C_1\eta A]\big],$$
$$T_3 = \left(a_1a_3(C+\alpha\eta A)^3 + KB_1\left(\frac{Kf_1}{(c+\alpha\eta A)} - a_3\right)[(a_3-d)(C+\alpha\eta A)+C_1\eta A]\right).$$

Moreover, from Eq.(28) we have $\frac{dL_1}{dL_2} = -\left(\left(\frac{\partial M_1}{\partial L_2}\right)/\left(\frac{\partial M_1}{\partial L_1}\right)\right)$. So , $\frac{dL_1}{dL_2} < 0$ if one of the following sets of conditions hold

$$\left(\frac{\partial M_1}{\partial L_2}\right) > 0, \left(\frac{\partial M_1}{\partial L_1}\right) > 0 \quad OR \quad \left(\frac{\partial M_1}{\partial L_2}\right) < 0, \left(\frac{\partial M_1}{\partial L_1}\right) < 0. \qquad (31)$$

As well as, from Eq.(29) we notice that, when $L_1 \to 0$ then $L_2 \to \tilde{L}_2^2$ is a positive root of the equation

$$U_1L_2^4 + U_2L_2^3 + U_3L_2^2 + U_4L_2 + U_5 = 0, \qquad (32)$$

where

$$U_1 = C_2^3K^3h^2b_2(C+\alpha\eta A)[h(C+\alpha\eta A)^2 + \rho f_2] > 0,$$
$$U_2 = C_2^2K^2h\left((C+\alpha\eta A)^3\left[a_2C_2h^2K + a_3b_2\left[\frac{2}{3}\rho f_2 - h\right] + C_2\rho^2\left(h + K\left[\frac{2}{3}\rho f_2 + h\right] + \rho K f_2\right)\right] + hKb_2f_2[(a_3-d)(C+\alpha\eta A)+C_1\eta A]\right),$$
$$U_3 = C_2K\big(a_3(C+\alpha\eta A)^3[h(3a_3b_2 + C_2\rho^2K) - (3a_2C_2Kh^2 + \rho a_3b_2f_2 + C_2K\rho^4f_2)] + (C + \alpha\eta A)[(a_3-d)(C+\alpha\eta A)+C_1\eta A](C_2hK^2\rho(C+\alpha\eta A)[3f_2\rho + h] + 2C_2hK^2\rho a_3[f_2\rho^2 + h])\big),$$
$$U_4 = a_3(\mathrm{C}+\alpha\eta A)^3\Big[3a_2C_2hK - a_3(a_3b_2 + C_2K\rho^2) + [(a_3-d)(\mathrm{C}+\alpha\eta A)+C_1\eta A]\Big((\mathrm{C} + \alpha\eta A)\big[a_3\mathrm{K}b_2f_2(\mathrm{C}+\alpha\eta A) + 2\rho C_2hK^2a_3 + [f_2\rho^2 + h]\big(C_2hK^3 - 2a_2(\mathrm{C}+\alpha\eta A)\big) + 3f_2C_2\rho hK^3\big]\Big)\Big],$$

$U_5 = -a_2 a_3^3(\mathrm{C} + \alpha\eta A) + K[(a_3 - d)(\mathrm{C} + \alpha\eta A) + C_1\eta A]\big(f_2 h + a_3^2\rho(\mathrm{C} + \alpha\eta A)^2 - a_3 K(\rho^2 + h)(\mathrm{C} + \alpha\eta A)\big)$.

Furthermore, from Eq.(29), we have $\frac{dL_1}{dL_2} = -\left(\left(\frac{\partial M_2}{\partial L_2}\right) / \left(\frac{\partial M_2}{\partial L_1}\right)\right)$. So, $\frac{dL_1}{dL_2} > 0$ if one of the following sets of conditions hold

$$\left(\frac{\partial M_2}{\partial L_2}\right) > 0, \left(\frac{\partial M_2}{\partial L_1}\right) < 0 \quad OR \quad \left(\frac{\partial M_2}{\partial L_2}\right) < 0, \left(\frac{\partial M_2}{\partial L_1}\right) > 0. \tag{33}$$

Then the two isoclines (28) and (29) intersect at a unique positive point $(\tilde{L}_1, \tilde{L}_2)$ in the $Int.R_+^2$ of $L_1L_2 -$ plane.

So, the (EP) $P_8 = (\tilde{L}_1, \tilde{L}_2, \tilde{L}_3, 0)$, where $\tilde{L}_3 = L_3(\tilde{L}_1, \tilde{L}_2)$ exists if in addition to condition (3) the following conditions hold

$$h < \mathrm{Min}\left\{2\rho f_1, \quad \frac{2}{3}\rho f_2\right\}, \tag{34}$$

$$a_3 > \mathrm{Max}\left\{\frac{K f_1}{(c+\alpha\eta A)}, \ C_2 h K \tilde{L}_2\right\}, \tag{35}$$

$$h(3a_3b_2 + C_2\rho^2 K) > (3a_2C_2Kh^2 + \rho a_3 b_2 f_2 + C_2 K\rho^4 f_2), \tag{36}$$

$$f_2 h + a_3^2\rho(\mathrm{C} + \alpha\eta A)^2 < a_3 K(\rho^2 + h)(\mathrm{C} + \alpha\eta A), \tag{37}$$

$$\tilde{L}_2^1 > \tilde{L}_2^2. \tag{38}$$

- The (EPs) $P_9 = \left(\tilde{\tilde{L}}_1, 0, \tilde{\tilde{L}}_3, \tilde{\tilde{L}}_4\right)$ *and* $P_{10} = \left(\tilde{\tilde{\tilde{L}}}_1, 0, \tilde{\tilde{\tilde{L}}}_3, \tilde{\tilde{\tilde{L}}}_4\right)$ exists if the next equations have a positive solution

$$\frac{a_1}{1+f_1L_3} - b_1L_1 - \frac{B_1L_3}{\mathrm{C}+\alpha\eta A+L_1} = 0, \tag{39}$$

$$a_3\left(1 - \frac{L_3+L_4}{K}\right) + \frac{C_1(L_1+\eta A)}{\mathrm{C}+\alpha\eta A+L_1} - (d + BL_4) + \frac{\gamma L_4}{(\sigma+L_4)L_3}, \tag{40}$$

$$BL_3 - \delta - \frac{\gamma}{\sigma+L_4} = 0. \tag{41}$$

From Eq. (41), we have

$$L_4 = \frac{\gamma-\sigma(BL_3-\delta)}{(BL_3-\delta)}. \tag{42}$$

Now, by substituting Eq.(42) in Equations (39 and 40), we obtain the following equations.

$$H_1(L_1, L_3) = L_1(a_1 - b_1[(\mathrm{C} + \alpha\eta A + L_1) + f_1(\mathrm{C} + \alpha\eta A)L_3 + f_1L_1L_3]) - B_1(1 + f_1L_3)L_3 + a_1(\mathrm{C} + \alpha\eta A) = 0, \tag{43}$$

$H_2(L_1, L_3) = L_3\big(\gamma\big[-a_3B(\mathrm{C} + \alpha\eta A + L_1)L_3^2 + [(a_3 - d)(\mathrm{C} + \alpha\eta A)KB + (C_1 + a_3 - d)KBL_1 + a_3(\delta + \sigma B)[\mathrm{C} + \alpha\eta A + L_1] + C_1\eta AKB]L_3 - [K\delta(a_3 - d)(\mathrm{C} + \alpha\eta A) + a_3(\mathrm{C} +$

$\alpha\eta A)(\gamma+\sigma\delta)+(a_3(K\delta+\gamma+\sigma\delta)+dK\delta)L_1+C_1K\delta(\eta A+L_1)]\big]+a_3\sigma\delta^2L_1\big)-\gamma K\delta(\gamma+\sigma\delta)[\mathrm{C}+\alpha\eta A+L_1]=0.$ (44)

Now from Eq.(43), we notice that, when $L_1\to 0$, then $L_3\to\tilde{\tilde{L}}_3^1$ is a positive root of the equation

$$f_1B_1L_3^2+B_1L_3-a_1(\mathrm{C}+\alpha\eta A)=0 \tag{45}$$

Moreover, from Eq.(43) we have $\frac{dL_1}{dL_3}=-\left(\left(\frac{\partial H_1}{\partial L_3}\right)/\left(\frac{\partial H_1}{\partial L_1}\right)\right)$. So , $\frac{dL_1}{dL_3}<0$ if one of the following sets of conditions hold

$$\left(\frac{\partial H_1}{\partial L_3}\right)>0,\left(\frac{\partial H_1}{\partial L_1}\right)>0\quad OR\quad\left(\frac{\partial H_1}{\partial L_3}\right)<0,\left(\frac{\partial H_1}{\partial L_1}\right)<0. \tag{46}$$

As well as, from Eq.(44) we notice that, when $L_1\to 0$ then $L_3\to\tilde{\tilde{L}}_3^2$ or $L_3\to\tilde{\tilde{\tilde{L}}}_3^2$ are roots of the equation

$$Y_1L_3^3+Y_2L_3^2+Y_3L_3+Y_4=0, \tag{47}$$

where

$$Y_1=-a_3\gamma B(C+\alpha\eta A)<0,$$
$$Y_2=\gamma\big((C+\alpha\eta A)[(a_3-d)KB+a_3(\delta+\sigma B)]+C_1\eta AKB\big),$$
$$Y_3=-\gamma\big((C+\alpha\eta A)[(a_3-d)K\delta+a_3(\gamma+\sigma\delta)]+C_1\eta AK\delta\big),$$
$$Y_4=-\gamma K\delta(C+\alpha\eta A)(\gamma+\sigma\delta)<0.$$

Clearly, according to the Descartes's rule, either Eq.(47) has no root or there are two roots say $\left(\tilde{\tilde{L}}_3^2,\ \tilde{\tilde{\tilde{L}}}_3^2\right)$, if condition (3) holds.

Furthermore, from Eq.(45), we have $\frac{dL_1}{dL_3}=-\left(\left(\frac{\partial H_2}{\partial L_3}\right)/\left(\frac{\partial H_2}{\partial L_1}\right)\right)$. So, $\frac{dL_1}{dL_3}>0$ if one of the following sets of conditions hold

$$\left(\frac{\partial H_2}{\partial L_3}\right)>0,\left(\frac{\partial H_2}{\partial L_1}\right)<0\quad OR\quad\left(\frac{\partial H_2}{\partial L_3}\right)<0,\left(\frac{\partial H_2}{\partial L_1}\right)>0. \tag{48}$$

Then the two isoclines (43) and (44) intersect at a positive point $\left(\tilde{\tilde{L}}_1,\ \tilde{\tilde{L}}_3\right)$ in the $Int.R_+^2$ of L_1L_3- plane.

Similarly, for $\tilde{\tilde{\tilde{L}}}_3^2$.

So, the (EPs) $P_9=\left(\tilde{\tilde{L}}_1,\ 0,\ \tilde{\tilde{L}}_3,\ \tilde{\tilde{L}}_4\right)$ and $P_{10}=\left(\tilde{\tilde{\tilde{L}}}_1,\ 0,\ \tilde{\tilde{\tilde{L}}}_3,\ \tilde{\tilde{\tilde{L}}}_4\right)$ where $\tilde{\tilde{L}}_4=L_4\big(\tilde{\tilde{L}}_1,\ \tilde{\tilde{L}}_3\big)$ and $\tilde{\tilde{\tilde{L}}}_4=L_4\left(\tilde{\tilde{\tilde{L}}}_1,\ \tilde{\tilde{\tilde{L}}}_3\right)$ exists if in addition to condition (3) the following conditions hold

$$B\tilde{\tilde{L}}_3 > \delta, \tag{49}$$

$$\gamma > \sigma\left(B\tilde{\tilde{L}}_3 - \delta\right), \tag{50}$$

$$\tilde{\tilde{L}}_3^1 > \tilde{\tilde{L}}_3^2. \tag{51}$$

- The (EPs) $P_{11} = \left(0,\ \dot{L}_2,\ \dot{L}_3,\ \dot{L}_4\right)$ and $P_{12} = \left(0,\ \ddot{L}_2,\ \ddot{L}_3,\ \ddot{L}_4\right)$ exists "if and only if" the next equations have a positive solution

$$\frac{a_2}{1+f_2L_3} - b_2L_2 - (\rho + hL_3)L_3 = 0, \tag{52}$$

$$a_3\left(1 - \frac{L_3+L_4}{K}\right) + \frac{C_1\eta A}{C+\alpha\eta A} + C_2(\rho + hL_3)L_2 - (d + BL_4) + \frac{\gamma L_4}{(\sigma+L_4)L_3} = 0, \tag{53}$$

$$BL_3 - \delta - \frac{\gamma}{\sigma+L_4} = 0. \tag{54}$$

From Eq. (52), we have

$$L_2 = \frac{a_2-(1+f_2L_3)(\rho+hL_3)L_3}{b_2(1+f_2L_3)}. \tag{55}$$

And from Eq. (54), we have

$$L_4 = \frac{\gamma-\sigma(BL_3-\delta)}{(BL_3-\delta)}. \tag{56}$$

Now, by substituting Equations (55 and 56) in Eq.(53), we obtain the following equation

$$Z_1L_3^7 + Z_2L_3^6 + Z_3L_3^5 + Z_4L_3^4 + Z_5L_3^3 + Z_6L_3^2 + Z_7L_3 + Z_8 = 0, \tag{57}$$

where

$$Z_1 = -C_2f_2K\sigma h^2B^2(C + \alpha\eta A)[1 + \sigma] < 0,$$

$$Z_2 = C_2hKB(C + \alpha\eta A)[B\sigma(f_2\rho + h)[\sigma - 1] - hf_2],$$

$$Z_3 = C_2hK(C + \alpha\eta A)\left[\rho\sigma B^2(\sigma - 1) + \gamma f_2\left(\frac{\delta}{B} - 2\rho\right) - \gamma B\right],$$

$$Z_4 = (C + \alpha\eta A)[-B(a_3b_2f_2\gamma + C_2K\sigma[2h\gamma + \rho f_2\gamma + 4h\sigma\delta]) + C_2hK\delta(h\gamma + f_2\rho[2\gamma + \sigma\delta])],$$

$$Z_5 = \gamma(C + \alpha\eta A)\left[(a_3 - d)KBb_2f_2 + \left(\frac{\delta}{B} - \frac{1}{f_2}\right)[a_3f_2 + C_2\rho^2K] + a_3b_2B\delta f_2 + C_2hK(2\rho\delta + a_2B)\right],$$

$$Z_6 = -(C + \alpha\eta A)\left[b_2\gamma\left(a_3\left[K\left(\frac{\delta}{B} - \frac{1}{f_2}\right) + \sigma\left(\frac{\delta}{B} - \frac{1}{2f_2}\right) + \left(f_2 - \frac{\delta}{\gamma}\right)\right] + BK\sigma(B - f_2) - dK\left(\frac{\delta}{B} - \frac{1}{f_2}\right)\right) + C_2K(\delta(ha_2\gamma + 2\rho^2\sigma + \rho^2\gamma) - a_2\rho\gamma B)\right] - C_1\eta AKb_2\gamma\left(\frac{\delta}{B} - \frac{1}{f_2}\right),$$

$$Z_7 = -\gamma\left((C+\alpha\eta A)\left[a_2 b_2(\gamma+\sigma\delta) + K\delta\left[b_2(a_3-d) + C_2\rho a_2 + b_2\sigma\left(\frac{\delta}{B}-\frac{1}{f_2}\right) + \gamma b_2 f_2\right]\right] + C_1\eta A K b_2 \delta\right),$$

$$Z_8 = -\gamma K b_2 \delta(C+\alpha\eta A)(\gamma+\sigma\delta) < 0.$$

Clearly, by Descartes's rule, Eq. (57) has no root or there are two roots say $(\dot{L}_3,\ \ddot{L}_3)$, if in addition to condition (3) the following conditions hold

$$\sigma < 1, \tag{58}$$

$$\frac{1}{f_2} < \frac{\delta}{B} < 2\rho, \tag{59}$$

$$C_2 \mathrm{hK}\delta(h\gamma + f_2\rho[2\gamma+\sigma\delta]) < B(a_3 b_2 f_2 \gamma + C_2 \mathrm{K}\sigma[2h\gamma + \rho f_2\gamma + 4h\sigma\delta]), \tag{60}$$

So, the (EP) $P_{11} = (0,\ \dot{L}_2,\ \dot{L}_3,\ \dot{L}_4)$ where $\dot{L}_2 = L_2(\dot{L}_3)$ and $\dot{L}_4 = L_4(\dot{L}_3)$ exists if in addition to conditions $(3, 58-60)$ the following conditions hold

$$a_2 > (1+f_2\dot{L}_3)(\rho + h\dot{L}_3)\dot{L}_3, \tag{61}$$

$$B\dot{L}_3 > \delta, \tag{62}$$

$$\gamma > \sigma(B\dot{L}_3 - \delta). \tag{63}$$

Similarly for $P_{12} = (0,\ \ddot{L}_2,\ \ddot{L}_3,\ \ddot{L}_4)$.

- The (EPs) $P_{13} = (L_1^*, L_2^*, L_3^*, L_4^*)$ and $P_{14} = (L_1^{**}, L_2^{**}, L_3^{**}, L_4^{**})$ exists "if and only if" the next equations have a positive solution

$$\frac{a_1}{1+f_1L_3} - b_1L_1 - \frac{B_1L_3}{C+\alpha\eta A + L_1} = 0, \tag{64}$$

$$\frac{a_2}{1+f_2L_3} - b_2L_2 - (\rho + hL_3)L_3 = 0, \tag{65}$$

$$a_3\left(1-\frac{L_3+L_4}{K}\right) + \frac{C_1(L_1+\eta A)}{C+\alpha\eta A+L_1} + C_2(\rho+hL_3)L_2 - (d+BL_4) + \frac{\gamma L_4}{(\sigma+L_4)L_3} = 0, \tag{66}$$

$$BL_3 - \delta - \frac{\gamma}{\sigma+L_4} = 0. \tag{67}$$

from Eq.(65), we have

$$L_2 = \frac{a_2 - (1+f_2L_3)(\rho+hL_3)L_3}{b_2(1+f_2L_3)}. \tag{68}$$

And from Eq.(67), we have

$$L_4 = \frac{\gamma - \sigma(BL_3-\delta)}{(BL_3-\delta)}. \tag{69}$$

Now, by substituting Equations $(68,\ 69)$ in Eq.(66) we obtain the following equations

$$A_1(L_1,L_3) = L_1(a_1 - b_1[(C+\alpha\eta A + L_1) + f_1(C+\alpha\eta A)L_3 + f_1L_1L_3]) - B_1(1+f_1L_3)L_3 + a_1(C+\alpha\eta A) = 0, \tag{70}$$

$$A_2(L_1,L_3) = \gamma L_3\Bigg(-C_2f_2KBh^2[c+\alpha\eta A+L_1]L_3^5 + C_2hK\left(2\rho f_2B + \left(\frac{\delta}{B}-\frac{1}{f_2}\right)\right)[c+\alpha\eta A + L_1]L_3^4 - \left(a_2b_2f_2B + C_2K(\rho^2f_2B - h^2\delta) - 2C_2K\rho h\left(\frac{\delta}{B}-\frac{1}{f_2}\right)\right)[c+\alpha\eta A+L_1]L_3^3 + \left(\left[Kb_2f_2B(a_3-d) + \left(\frac{\delta}{B}-\frac{1}{f_2}\right)[a_3b_2 + C_2K\rho^2] + C_2Kh[2\rho\delta + a_2B]\right][c+\alpha\eta A+L_1] + C_1b_2f_2BK[L_1+\eta A]\right)L_3^2 - \Bigg((\mathrm{C}+\alpha\eta A)\left[\left(\frac{\delta}{B}-\frac{1}{f_2}\right)\left(Kb_2(a_3-d)(1-\sigma) + C_2K\big(\rho^2\delta + a_2(\rho B - h\delta)\big) - a_3b_2\big(\delta + f_2(\gamma+\sigma\delta)\big)\right)\right] + \left(-\left(\frac{\delta}{B}-\frac{1}{f_2}\right)[(a_3-d)Kb_2 + 2C_1Kb_2] + a_3b_2\big[\delta + f_2\big(\sigma B - (\gamma+\sigma\delta)\big)\big] + C_2K\rho(Ba_2+\rho\delta)\right)L_1 - C_1\eta Ab_2K\left(\frac{\delta}{B}-\frac{1}{f_2}\right)\Bigg)L_3 + \Bigg((\mathrm{C}+\alpha\eta A)\left[-Kb_2\left(\frac{\delta}{B}-\frac{1}{f_2}\right)(\gamma+\sigma\delta) - a_3b_2(\gamma+\sigma\delta) - K(C_2a_2\rho\delta + Bb_2\gamma)\right] - \left[Kb_2\delta\sigma f_2\left(\frac{\delta}{B}-\frac{1}{f_2}\right) + Kb_2\delta(a_3-d) + a_2b_2(\gamma+\sigma\delta)\right]L_1 - C_1\eta Ab_2K\delta\Bigg)\Bigg) - Kb_2\delta(\gamma+\sigma\delta)[c+\alpha\eta A+L_1], \tag{71}$$

Now from Eq. (70), we notice that, when $L_1 \to 0$, then $L_3 \to L_3^{*1}$ is a positive root of the equation that follows

$$f_1B_1L_3^2 + B_1L_3 - a_1(C+\alpha\eta A) = 0 \tag{72}$$

Moreover, from Eq.(70) we have $\frac{dL_1}{dL_3} = -\left(\left(\frac{\partial A_1}{\partial L_3}\right)/\left(\frac{\partial A_1}{\partial L_1}\right)\right)$. So , $\frac{dL_1}{dL_3} < 0$ if one of the following sets of conditions hold

$$\left(\frac{\partial A_1}{\partial L_3}\right) > 0, \left(\frac{\partial A_1}{\partial L_1}\right) > 0 \quad OR \quad \left(\frac{\partial A_1}{\partial L_3}\right) < 0, \left(\frac{\partial A_1}{\partial L_1}\right) < 0. \tag{73}$$

As well as, from Eq.(71) we notice that, when $L_1 \to 0$, then $L_3 \to L_3^{*2}$ or $L_3 \to L_3^{**2}$, are roots of the equation

$$F_1L_3^6 + F_2L_3^5 + F_3L_3^4 + F_4L_3^3 + F_5L_3^2 + F_6L_3 + F_7 = 0, \tag{74}$$

where

$$F_1 = -C_2f_2KB\gamma h^2(C+\alpha\eta A) < 0,$$

$$F_2 = -C_2hK(C+\alpha\eta A)\left(2\rho f_2B - h\left(\frac{\delta}{B}-\frac{1}{f_2}\right)\right),$$

$F_3 = -(C+\alpha\eta A)\left(a_2b_2f_2B + C_2K(\rho^2 f_2B - h^2\delta) + 2C_2K\rho h\left(\frac{1}{f_2}-\frac{\delta}{B}\right)\right),$

$F_4 = \gamma\left[(C+\alpha\eta A)\left(Kb_2f_2B(a_3-d) - \left(\frac{1}{f_2}-\frac{\delta}{B}\right)[a_3b_2 + C_2K\rho^2] + C_2Kh[2\rho\delta + a_2B]\right) + C_1\eta Ab_2f_2BK\right],$

$F_5 = \gamma\left[(C+\alpha\eta A)\left[\left(\frac{1}{f_2}-\frac{\delta}{B}\right)\left((a_3-d)(1-\sigma)Kb_2 + C_2K\big(\rho^2\delta + a_2(\rho B - h\delta)\big) - a_3b_2\big(\delta + f_2(\gamma+\sigma\delta)\big)\right)\right] + C_1\eta Ab_2K(B-\delta f_2)\right]$

$F_6 = \gamma\left[(C+\alpha\eta A)\left[-Kb_2\left(\frac{1}{f_2}-\frac{\delta}{B}\right)(\gamma+\sigma\delta) + a_3b_2(\gamma+\sigma\delta) + K(C_2a_2\rho\delta + Bb_2\gamma)\right] - C_1\eta Ab_2K\delta\right]$

$F_7 = -Kb_2\delta(C+\alpha\eta A)(\gamma+\sigma\delta) < 0$

Clearly, according to the Descartes's rule, either Eq.(74) has no root or there are two roots say $(L_3^{*2},\ L_3^{**2})$, if in addition to reversing condition (59), conditions (3, 58) and the following conditions hold

$$B > Max\left\{\frac{h^2\delta}{\rho^2 f_2},\ \frac{h\delta}{\rho}\right\}, \tag{75}$$

$$Kb_2f_2B(a_3-d) + C_2Kh(2\rho\delta + a_2B) > \left(\frac{1}{f_2}-\frac{\delta}{B}\right)[a_3b_2 + C_2K\rho^2], \tag{76}$$

$$(a_3-d)(1-\sigma)\left(\frac{1}{f_2}-\frac{\delta}{B}\right)Kb_2 + C_2K\big(\rho^2\delta + a_2(\rho B - h\delta)\big) > a_3b_2\big(\delta + f_2(\gamma+\sigma\delta)\big), \tag{77}$$

Furthermore, from Eq. (71), we have $\frac{dL_1}{dL_3} = -\left(\left(\frac{\partial H_2}{\partial L_3}\right)/\left(\frac{\partial H_2}{\partial L_1}\right)\right)$. So, $\frac{dL_1}{dL_3} > 0$ if one of the following sets of conditions hold

$$\left(\frac{\partial A_2}{\partial L_3}\right) > 0, \left(\frac{\partial A_2}{\partial L_1}\right) < 0 \quad OR \quad \left(\frac{\partial A_2}{\partial L_3}\right) < 0, \left(\frac{\partial A_2}{\partial L_1}\right) > 0. \tag{78}$$

Then from the two isoclines (70) and (71) intersect at a unique positive point (L_1^*, L_3^*) in the $Int.R_+^2$ of L_1L_3-plane.

Similarly, for (L_1^{**}, L_3^{**})

So, the (EPs) $P_{13} = (L_1^*, L_2^*, L_3^*, L_4^*)$ and $P_{14} = (L_1^{**}, L_2^{**}, L_3^{**}, L_4^{**})$ where $L_2^* = L_2(L_3^*)$, $L_4^* = L_4(L_3^*)$ and $L_2^{**} = L_2(L_3^{**})$, $L_4^{**} = L_4(L_3^{**})$ exists if in addition to reversing condition (59), condition $(3, 58, 75-77)$ the following conditions hold

$$a_2 > (1 + f_2L_3^*)(\rho + hL_3^*)L_3^*. \tag{79}$$

$$BL_3^* > \delta, \tag{80}$$

$$\gamma > \sigma(BL_3^* - \delta), \tag{81}$$

$$L_3^{*1} > L_3^{*2}. \tag{82}$$

4. LOCAL STABILITY ANALYSIS (LSA)

In this section, the (LSA) around each of the (EPs) is discussed by calculating the Jacobian matrix (JM), $J(L_1, L_2, L_3, L_4)$ and found the eigenvalues of system (1) at each of them.

Note that, we used the symbols λ_{iL_1}, λ_{iL_2}, λ_{iL_3} and λ_{iL_4} to symbolize the eigenvalues of (JM) J_i; $i = 0, 1, \ldots$ 14 that represent the dynamics in L_1-direction , L_2-direction, L_3-direction and L_4-direction respectively, where the (JM), $J(L_1, L_2, L_3, L_4)$ of the system (1) at each of them can be written as $J = [b_{ij}]_{4\times4}$,

$$b_{11} = \frac{a_1}{1+f_1L_3} - 2b_1L_1 - \frac{(C+\alpha\eta A)B_1L_3}{(C+\alpha\eta A+L_1)^2}, \quad b_{12} = 0, \quad b_{13} = \frac{-a_1f_1L_1}{(1+f_1L_3)^2} - \frac{B_1L_1}{(C+\alpha\eta A+L_1)} < 0, \quad b_{14} = 0,$$

$$b_{21} = 0, \quad b_{22} = \frac{a_2}{1+f_2L_3} - 2b_2L_2 - (\rho + hL_3)L_3, \quad b_{23} = \frac{-a_2f_2L_2}{(1+f_2L_3)^2} - (\rho + 2hL_3)L_2, \quad b_{24} = 0,$$

$$b_{31} = \frac{C_1(C+\eta A(\alpha-1))L_3}{(C+\alpha\eta A+L_1)^2}, \quad b_{32} = C_2(\rho + hL_3)L_3,$$

$$b_{33} = a_3 - \frac{a_3}{K}(2L_3 + L_4) + \frac{C_1(L_1+\eta A)}{(C+\alpha\eta A+L_1)} + C_2(\rho + 2hL_3)L_2 - (d + BL_4),$$

$$b_{34} = -\left(\frac{a_3}{K} + B\right)L_3 + \frac{\sigma\gamma}{(\sigma+L_4)^2}, \quad b_{41} = 0, \quad b_{42} = 0, \quad b_{43} = BL_4, \quad b_{44} = BL_3 - \delta - \frac{\sigma\gamma}{(\sigma+L_4)^2}.$$

- It is easy to verify that, the (JM) of system (1) at $P_0 = (0,0,0,0)$ can be written as:

$$J_0 = J(P_0) = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} - d & \frac{\gamma}{\sigma} \\ 0 & 0 & 0 & -\left(\delta + \frac{\gamma}{\sigma}\right) \end{bmatrix}. \tag{83}$$

Then the characteristic equation of J_0 is given by

$$(a_1 - \lambda)(a_2 - \lambda)\left(a_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} - d - \lambda\right)\left(-\left(\delta + \frac{\gamma}{\sigma}\right) - \lambda\right) = 0$$

then the eigenvalues of J_0 are

$$\lambda_{0L_1} = a_1 > 0, \quad \lambda_{0L_2} = a_2 > 0, \ \lambda_{0L_3} = a_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} - d \quad \text{and} \ \lambda_{0L_4} = -\left(\delta + \frac{\gamma}{\sigma}\right) < 0,$$

so, the equilibrium point P_0 is unstable.

- The (JM) of system (1) at $P_1 = \left(\frac{a_1}{b_1}, 0, 0, 0\right)$ can be written as

$$J_1 = J(P_1) = \begin{bmatrix} -a_1 & 0 & \frac{-a_1{}^2}{b_1} - \frac{a_1 B_1}{a_1+(C+\alpha\eta A)b_1} & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 + \frac{C_1(a_1+\eta A b_1)}{a_1+(C+\alpha\eta A)b_1} - d & \frac{\gamma}{\sigma} \\ 0 & 0 & 0 & -\left(\delta + \frac{\gamma}{\sigma}\right) \end{bmatrix}. \quad (84)$$

Then the characteristic equation of J_1 is given by

$$(-a_1 - \lambda)(a_2 - \lambda)\left(a_3 + \frac{C_1(a_1+\eta A b_1)}{a_1+(\mathrm{C}+\alpha\eta A)b_1} - d - \lambda\right)\left(-\left(\delta + \frac{\gamma}{\sigma}\right) - \lambda\right) = 0$$

then the eigenvalues of J_1 are

$$\lambda_{1L_1} = -a_1 < 0\,,\ \lambda_{1L_2} = a_2 > 0,\ \lambda_{1L_3} = a_3 + \frac{C_1(a_1+\eta A b_1)}{a_1+(C+\alpha\eta A)b_1} - d \text{ and } \lambda_{1L_4} = -\left(\delta + \frac{\gamma}{\sigma}\right) < 0,$$

so, the equilibrium point P_1 is unstable.

- The (JM) of system (1) at $P_2 = \left(0,\ \frac{a_2}{b_2},\ 0,\ 0\right)$ can be written as

$$J_2 = J(P_2) = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & -a_2 & \frac{-a_2(a_2 f_2+\rho)}{b_2} & 0 \\ 0 & 0 & \left(a_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} + \frac{a_2 C_2 \rho}{b_2} - d\right) & \frac{\gamma}{\sigma} \\ 0 & 0 & 0 & -\left(\delta + \frac{\gamma}{\sigma}\right) \end{bmatrix}. \quad (85)$$

Then the characteristic equation of J_3 is given by

$$(a_1 - \lambda)(-a_2 - \lambda)\left(a_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} + \frac{a_2 C_2 \rho}{b_2} - d - \lambda\right)\left(-\left(\delta + \frac{\gamma}{\sigma}\right) - \lambda\right) = 0$$

then the eigenvalues of J_3 are

$$\lambda_{2L_1} = a_1 > 0,\ \lambda_{2L_2} = -a_2 < 0,\ \lambda_{2L_3} = \left(a_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} + \frac{a_2 C_2 \rho}{b_2} - d\right) \text{ and } \lambda_{2L_4} = -\left(\delta + \frac{\gamma}{\sigma}\right) < 0,$$

so, the equilibrium point P_2 is unstable.

- The (JM) of system (1) at $P_3 = (0\,,0\,,\bar{L}_3, 0\,)$ can be written as

$$J_3 = J(P_3) = [\Omega_{ij}]_{4\times 4} \quad (86)$$

$$\Omega_{11} = \frac{a_1}{(1+f_1\bar{L}_3)} - \frac{B_1\bar{L}_3}{(\mathrm{C}+\alpha\eta A)},\quad \Omega_{12} = 0,\quad \Omega_{13} = 0,\ \ \Omega_{14} = 0,\ \ \Omega_{21} = 0,$$

$$\Omega_{22} = \frac{a_2}{(1+f_2\bar{L}_3)} - (\rho + h\bar{L}_3)\bar{L}_3,\ \ \Omega_{23} = 0,\ \ \Omega_{24} = 0,\ \ \Omega_{31} = \frac{C_1\big(\mathrm{C}+\eta A(\alpha-1)\big)\bar{L}_3}{(\mathrm{C}+\alpha\eta A)^2},$$

$$\Omega_{32} = C_2(\rho + h\bar{L}_3)\bar{L}_3,\ \ \Omega_{33} = a_3 - 2\frac{a_3}{K}\bar{L}_3 + \frac{C_1\eta A}{(\mathrm{C}+\alpha\eta A)} - d\,,\quad \Omega_{34} = -\left(\frac{a_3}{K} + B\right)\bar{L}_3 + \frac{\gamma}{\sigma},$$

$$\Omega_{41} = 0,\quad \Omega_{42} = 0,\quad \Omega_{43} = 0,\quad \Omega_{44} = B\bar{L}_3 - \left(\delta + \frac{\gamma}{\sigma}\right).$$

Then the characteristic equation of J_3 is given by:

$$\left(\frac{a_1}{(1+f_1\bar{L}_3)}-\frac{B_1\bar{L}_3}{(\mathrm{C}+\alpha\eta A)}-\lambda\right)\left(\frac{a_2}{(1+f_2\bar{L}_3)}-(\rho+h\bar{L}_3)\bar{L}_3-\lambda\right)$$
$$\left(a_3-2\frac{a_3}{K}\bar{L}_3+\frac{C_1\eta A}{(\mathrm{C}+\alpha\eta A)}-d-\lambda\right)\left(B\bar{L}_3-\left(\delta+\frac{\gamma}{\sigma}\right)-\lambda\right)=0,$$

then the eigenvalues of J_3 are

$$\lambda_{3L_1}=\frac{a_1}{(1+f_1\bar{L}_3)}-\frac{B_1\bar{L}_3}{(\mathrm{C}+\alpha\eta A)},\qquad \lambda_{3L_2}=\frac{a_2}{(1+f_2\bar{L}_3)}-(\rho+h\bar{L}_3)\bar{L}_3,\ \ \lambda_{3L_3}=a_3-2\frac{a_3}{K}\bar{L}_3+\frac{C_1\eta A}{(\mathrm{C}+\alpha\eta A)}-d$$

and $\lambda_{3L_4}=B\bar{L}_3-\left(\delta+\frac{\gamma}{\sigma}\right)$ then P_3 is (LAS) if in addition to condition (3) and the following conditions hold

$$\frac{a_1}{(1+f_1\bar{L}_3)}<\frac{B_1\bar{L}_3}{(\mathrm{C}+\alpha\eta A)}, \tag{87}$$

$$\frac{a_2}{(1+f_2\bar{L}_3)}<(\rho+h\bar{L}_3)\bar{L}_3, \tag{88}$$

$$a_3-d+\frac{C_1\eta A}{(\mathrm{C}+\alpha\eta A)}<\frac{2a_3}{K}\bar{L}_3, \tag{89}$$

$$B\bar{L}_3<\left(\delta+\frac{\gamma}{\sigma}\right). \tag{90}$$

otherwise P_3 is unstable.

- The (JM) of system (1) at $P_4=\left(\frac{a_1}{b_1},\ \frac{a_2}{b_2},\ 0,\ 0\right)$ can be written as

$$J_4=J(P_4)=\begin{bmatrix}-a_1 & 0 & \frac{-a_1{}^2f_1}{b_1}-\frac{a_1B_1}{a_1+(\mathrm{C}+\alpha\eta A)b_1} & 0\\ 0 & -a_2 & \frac{-a_2(a_2f_2+\rho)}{b_2} & 0\\ 0 & 0 & \left(a_3+\frac{C_1(a_1+\eta Ab_1)}{a_1+(C+\alpha\eta A)b_1}+\frac{a_2C_2\rho}{b_2}-d\right) & \frac{\gamma}{\sigma}\\ 0 & 0 & 0 & -\left(\delta+\frac{\gamma}{\sigma}\right)\end{bmatrix}. \tag{91}$$

Then the characteristic equation of J_4 is given by

$$(-a_1-\lambda)(-a_2-\lambda)\left(a_3+\frac{C_1(a_1+\eta Ab_1)}{a_1+(C+\alpha\eta A)b_1}+\frac{a_2C_2\rho}{b_2}-d-\lambda\right)\left(-\left(\delta+\frac{\gamma}{\sigma}\right)-\lambda\right)=0$$

then the eigenvalues of J_4 are

$\lambda_{4L_1}=-a_1<0,\quad \lambda_{4L_2}=-a_2<0,\quad \lambda_{4L_3}=\left(a_3+\frac{C_1(a_1+\eta Ab_1)}{a_1+(C+\alpha\eta A)b_1}+\frac{a_2C_2\rho}{b_2}-d\right)$ and

$\lambda_{4L_4}=-\left(\delta+\frac{\gamma}{\sigma}\right)<0,$ then P_4 is (LAS) if the following condition holds

$$a_3+\frac{C_1(a_1+\eta Ab_1)}{a_1+(\mathrm{C}+\alpha\eta A)b_1}+\frac{a_2C_2\rho}{b_2}<d, \tag{92}$$

otherwise P_4 is unstable.

- The (JM) of system (1) at $P_5 = \left(\bar{\bar{L}}_1,\ 0,\ \bar{\bar{L}}_3,\ 0\right)$ can be written as

$$J_5 = J(P_5) = \left[c_{ij}\right]_{4\times4}. \tag{93}$$

$c_{11} = \frac{a_1}{1+f_1\bar{\bar{L}}_3} - 2b_1\bar{\bar{L}}_1 - \frac{(\mathrm{C}+\alpha\eta A)B_1\bar{\bar{L}}_3}{\left(\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1\right)^2}$, $c_{12} = 0$, $c_{13} = \frac{-a_1 f_1\bar{\bar{L}}_1}{\left(1+f_1\bar{\bar{L}}_3\right)^2} - \frac{B_1\bar{\bar{L}}_1}{\left(\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1\right)} < 0$,

$c_{14} = 0$, $c_{21} = 0$, $c_{22} = \frac{a_2}{1+f_2\bar{\bar{L}}_3} - \left(\rho + h\bar{\bar{L}}_3\right)\bar{\bar{L}}_3$, $c_{23} = 0$, $c_{24} = 0$,

$c_{31} = \frac{C_1\left(\mathrm{C}+\eta A(\alpha-1)\right)\bar{\bar{L}}_3}{\left(\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1\right)^2}$, $c_{32} = C_2\left(\rho + h\bar{\bar{L}}_3\right)\bar{\bar{L}}_3$, $c_{33} = a_3 - 2\frac{a_3}{K}\bar{\bar{L}}_3 + \frac{C_1\left(\bar{\bar{L}}_1+\eta A\right)}{\left(\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1\right)} - d$,

$c_{34} = -\left(\frac{a_3}{K} + B\right)\bar{\bar{L}}_3 + \frac{\gamma}{\sigma}$, $c_{41} = 0$, $c_{42} = 0$, $c_{43} = 0$, $c_{44} = B\bar{\bar{L}}_3 - \delta - \frac{\gamma}{\sigma}$.

Then the characteristic equation of J_5 is given by

$$(c_{22}-\lambda)(c_{44}-\lambda)[\lambda^2 - (c_{11}+c_{33})\lambda + c_{11}c_{33} - c_{13}c_{31}] = 0, \tag{94}$$

Then, either $(c_{22}-\lambda) = 0$ or $(c_{44}-\lambda) = 0$, which gives $\lambda_{5L_2} = c_{22}$, $\lambda_{5L_4} = c_{44}$.

Or $[\lambda^2 - (c_{11}+c_{33})\lambda + c_{11}c_{33} - c_{13}c_{31}] = 0$, which gives

$$\lambda_{5L_1} + \lambda_{5L_3} = \left(\frac{a_1}{1+f_1\bar{\bar{L}}_3} - 2b_1\bar{\bar{L}}_1 - \frac{(C+\alpha\eta A)B_1\bar{\bar{L}}_3}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^2}\right) + \left(a_3 - 2\frac{a_3}{K}\bar{\bar{L}}_3 + \frac{C_1\left(\bar{\bar{L}}_1+\eta A\right)}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)} - d\right)$$

$$\lambda_{5L_1}.\lambda_{5L_3} = \left(\frac{a_1}{1+f_1\bar{\bar{L}}_3} - 2b_1\bar{\bar{L}}_1 - \frac{(C+\alpha\eta A)B_1\bar{\bar{L}}_3}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^2}\right).\left(a_3 - 2\frac{a_3}{K}\bar{\bar{L}}_3 + \frac{C_1\left(\bar{\bar{L}}_1+\eta A\right)}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)} - d\right) +$$

$$\left(\frac{a_1 f_1\bar{\bar{L}}_1}{\left(1+f_1\bar{\bar{L}}_3\right)^2} + \frac{B_1\bar{\bar{L}}_1}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)}\right).\left(\frac{C_1\left(C+\eta A(\alpha-1)\right)\bar{\bar{L}}_3}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^2}\right)$$

then P_5 is (LAS) if the next conditions hold

$$\frac{a_2}{1+f_2\bar{\bar{L}}_3} < \left(\rho + h\bar{\bar{L}}_3\right)\bar{\bar{L}}_3, \tag{95}$$

$$B\bar{\bar{L}}_3 < \delta + \frac{\gamma}{\sigma}, \tag{96}$$

$$\frac{a_1}{1+f_1\bar{\bar{L}}_3} < 2b_1\bar{\bar{L}}_1 + \frac{(C+\alpha\eta A)B_1\bar{\bar{L}}_3}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^2}, \tag{97}$$

$$a_3 - d + \frac{C_1\left(\bar{\bar{L}}_1+\eta A\right)}{\left(\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1\right)} < 2\frac{a_3}{K}\bar{\bar{L}}_3, \tag{98}$$

$$\alpha > 1. \tag{99}$$

Therefore, under the above conditions, all of J_5's eigenvalues have negative real parts , and as a result P_5 is (LAS), otherwise P_5 is unstable.

- The (JM) of system (1) at $P_6 = \left(0,\ \hat{L}_2,\ \hat{L}_3,\ 0\right)$ can be written as

$$J_6 = J(P_6) = \left[d_{ij}\right]_{4\times4},\ \text{where} \tag{100}$$

$d_{11} = \frac{a_1}{1+f_1\hat{L}_3} - \frac{B_1\hat{L}_3}{(\mathrm{C}+\alpha\eta A)}$, $d_{12} = 0$, $d_{13} = 0$, $d_{14} = 0$, $d_{21} = 0$,

$d_{22} = \frac{a_2}{1+f_2\hat{L}_3} - 2b_2\hat{L}_2 - (\rho + h\hat{L}_3)\hat{L}_3$, $d_{23} = -\left(\frac{a_2 f_2\hat{L}_2}{(1+f_2\hat{L}_3)^2} + (\rho + 2h\hat{L}_3)\hat{L}_2\right) < 0$,

$d_{24} = 0$, $d_{31} = \frac{C_1(\mathrm{C}+\eta A(\alpha-1))\hat{L}_3}{(\mathrm{C}+\alpha\eta A)^2}$, $d_{32} = C_2(\rho + h\hat{L}_3)\hat{L}_3$,

$d_{33} = a_3 - 2\frac{a_3}{K}\hat{L}_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} + C_2(\rho + 2h\hat{L}_3)\hat{L}_2 - d$, $d_{34} = -\left(\frac{a_3}{K} + B\right)\hat{L}_3 + \frac{\gamma}{\sigma}$,

$d_{41} = 0$, $d_{42} = 0$, $d_{43} = 0$, $d_{44} = B\hat{L}_3 - \delta - \frac{\gamma}{\sigma}$.

Then the characteristic equation of J_6 is given by

$$(d_{11} - \lambda)(d_{44} - \lambda)[\lambda^2 - (d_{22} + d_{33})\lambda + d_{22}d_{33} - d_{23}d_{32}] = 0. \quad (101)$$

Then, either $(d_{11} - \lambda) = 0$ or $(d_{44} - \lambda) = 0$, which give $\lambda_{6L_1} = d_{11}$, $\lambda_{6L_4} = d_{44}$.

Or $[\lambda^2 - (d_{22} + d_{33})\lambda + d_{22}d_{33} - d_{23}d_{32}] = 0$, which give

$\lambda_{6L_2} + \lambda_{6L_3} = \left(\frac{a_2}{1+f_2\hat{L}_3} - 2b_2\hat{L}_2 - (\rho + h\hat{L}_3)\hat{L}_3\right) + \Big(a_3 - 2\frac{a_3}{K}\hat{L}_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} + C_2(\rho +$

$2h\hat{L}_3)\hat{L}_2 - d\Big)$,

$$\lambda_{6L_2}.\lambda_{6L_3} = \left(\frac{a_2}{1+f_2\hat{L}_3} - 2b_2\hat{L}_2 - (\rho + h\hat{L}_3)\hat{L}_3\right)\left(a_3 - 2\frac{a_3}{K}\hat{L}_3 + \frac{C_1\eta A}{(C+\alpha\eta A)} + C_2(\rho + 2h\hat{L}_3)\hat{L}_2 - d\right)$$
$$+\left(\frac{a_2 f_2\hat{L}_2}{(1+f_2\hat{L}_3)^2} + (\rho + 2h\hat{L}_3)\hat{L}_2\right)\left(C_2(\rho + h\hat{L}_3)\hat{L}_3\right),$$

then P_6 is (LAS) if in addition to condition $(3, 16)$ the next conditions hold

$$\frac{a_1}{1+f_1\hat{L}_3} < \frac{B_1\hat{L}_3}{(\mathrm{C}+\alpha\eta A)}, \quad (102)$$

$$B\hat{L}_3 < \delta + \frac{\gamma}{\sigma}, \quad (103)$$

$$\frac{a_2}{1+f_2\hat{L}_3} - (\rho + h\hat{L}_3)\hat{L}_3 < 2b_2\hat{L}_2, \quad (104)$$

$$a_3 - d + \frac{C_1\eta A}{(\mathrm{C}+\alpha\eta A)} + C_2(\rho + 2h\hat{L}_3)\hat{L}_2 < 2\frac{a_3}{K}\hat{L}_3. \quad (105)$$

Otherwise P_6 is unstable.

- The (JM) of system (1) at $P_7 = (0,\ 0,\ \check{L}_3,\ \check{L}_4)$ can be written as

$J_7 = J(P_7) = [e_{ij}]_{4\times4}$, where (106)

$e_{11} = \frac{a_1}{1+f_1\check{L}_3} - \frac{B_1\check{L}_3}{(C+\alpha\eta A)}$, $e_{12} = e_{13} = e_{14} = e_{21} = 0$, $e_{22} = \frac{a_2}{1+f_2\check{L}_3} - (\rho + h\check{L}_3)\check{L}_3$,

$e_{23} = 0$, $e_{24} = 0$, $e_{31} = \frac{C_1(C+\eta A(\alpha-1))\check{L}_3}{(C+\alpha\eta A)^2}$, $e_{32} = C_2(\rho + h\check{L}_3)\check{L}_3$,

$e_{33} = a_3 - \frac{a_3}{K}\left(2\check{L}_3 + \check{L}_4\right) + \frac{C_1\eta A}{(C+\alpha\eta A)} - \left(d + B\check{L}_4\right),\ \ e_{34} = -\left(\frac{a_3}{K} + B\right)\check{L}_3 + \frac{\gamma\sigma}{(\sigma+\check{L}_4)^2},$

$e_{41} = 0, \qquad e_{42} = 0, \qquad e_{43} = B\check{L}_4 > 0, \qquad e_{44} = B\check{L}_3 - \delta - \frac{\gamma\sigma}{(\sigma+\check{L}_4)^2}.$

Then the characteristic equation of J_7 is given by

$$(e_{11} - \lambda)(e_{22} - \lambda)[\lambda^2 - (e_{33} + e_{44})\lambda + e_{33}e_{44} - e_{34}e_{43}] = 0 \tag{107}$$

Then, either $(e_{11} - \lambda) = 0$ or $(e_{22} - \lambda) = 0$, which give $\lambda_{7L_1} = e_{11},\ \lambda_{7L_2} = e_{22}$.

Or $[\lambda^2 - (e_{33} + e_{44})\lambda + e_{33}e_{44} - e_{34}e_{43}] = 0$,

which gives

$$\lambda_{7L_3} + \lambda_{7L_4} = \left(a_3 - \frac{a_3}{K}\left(2\check{L}_3 + \check{L}_4\right) + \frac{C_1\eta A}{(\mathrm{C}+\alpha\eta A)} - \left(d + B\check{L}_4\right)\right) + \left(B\check{L}_3 - \delta - \frac{\gamma\sigma}{(\sigma+\check{L}_4)^2}\right),$$

$$\lambda_{7L_3}.\lambda_{7L_4} = \left(a_3 - \frac{a_3}{K}\left(2\check{L}_3 + \check{L}_4\right) + \frac{C_1\eta A}{(\mathrm{C}+\alpha\eta A)} - \left(d + B\check{L}_4\right)\right)\left(B\check{L}_3 - \delta - \frac{\gamma\sigma}{(\sigma+\check{L}_4)^2}\right)$$
$$- \left(-\left(\frac{a_3}{K} + B\right)\check{L}_3 + \frac{\gamma\sigma}{(\sigma+\check{L}_4)^2}\right)\left(B\check{L}_4\right),$$

then P_7 is (LAS) if in addition to condition $(3, 22)$ the next conditions hold

$$\frac{a_1}{1+f_1\check{L}_3} < \frac{B_1\check{L}_3}{(\mathrm{C}+\alpha\eta A)}, \tag{108}$$

$$\frac{a_2}{1+f_2\check{L}_3} < \left(\rho + h\check{L}_3\right)\check{L}_3, \tag{109}$$

$$a_3 - d + \frac{C_1\eta A}{(\mathrm{C}+\alpha\eta A)} < 2\frac{a_3}{K}\check{L}_3 + \frac{a_3}{K}\check{L}_4 + B\check{L}_4, \tag{110}$$

$$B\check{L}_3 - \delta < \frac{\gamma\sigma}{(\sigma+\check{L}_4)^2} < \left(\frac{a_3}{K} + B\right)\check{L}_3. \tag{111}$$

otherwise P_7 is unstable.

- The (JM) of system (1) at $P_8 = \left(\tilde{L}_1,\ \tilde{L}_2,\ \tilde{L}_3,\ 0\right)$ can be written as

$$J_8 = J(P_8) = \left[f_{ij}\right]_{4\times 4}, \text{ where} \tag{112}$$

$f_{11} = \frac{a_1}{1+f_1\tilde{L}_3} - 2b_1\tilde{L}_1 - \frac{(\mathrm{C}+\alpha\eta A)B_1\tilde{L}_3}{(\mathrm{C}+\alpha\eta A+\tilde{L}_1)^2},\ \ f_{12} = 0,\ \ f_{13} = \frac{-a_1 f_1\tilde{L}_1}{(1+f_1\tilde{L}_3)^2} - \frac{B_1\tilde{L}_1}{(\mathrm{C}+\alpha\eta A+\tilde{L}_1)} < 0, \quad f_{14} = 0,$

$f_{21} = 0,\ \ f_{22} = \frac{a_2}{1+f_2\tilde{L}_3} - 2b_2\tilde{L}_2 - \left(\rho + h\tilde{L}_3\right)\tilde{L}_3,\ \ f_{23} = -\left(\frac{a_2 f_2\tilde{L}_2}{(1+f_2\tilde{L}_3)^2} + \left(\rho + 2h\tilde{L}_3\right)\tilde{L}_2\right) < 0,$

$f_{24} = 0, \qquad f_{31} = \frac{C_1\left(\mathrm{C}+\eta A(\alpha-1)\right)\tilde{L}_3}{(\mathrm{C}+\alpha\eta A+\tilde{L}_1)^2},\ \ f_{32} = C_2\left(\rho + h\tilde{L}_3\right)\tilde{L}_3 > 0,$

$f_{33} = a_3 - 2\frac{a_3}{K}\tilde{L}_3 + \frac{C_1(\tilde{L}_1+\eta A)}{(\mathrm{C}+\alpha\eta A+\tilde{L}_1)} + C_2\left(\rho + 2h\tilde{L}_3\right)\tilde{L}_2 - d\,, \quad f_{34} = -\left(\frac{a_3}{K} + B\right)\tilde{L}_3 + \frac{\gamma}{\sigma},$

$f_{41} = f_{42} = f_{43} = 0, \quad f_{44} = B\tilde{L}_3 - \delta - \frac{\gamma}{\sigma}.$

Then the characteristic equation of J_8 is given by

$$(f_{44}-\lambda)[\lambda^3+\theta_1\lambda^2+\theta_2\lambda+\theta_3]=0, \tag{113}$$

then either $(f_{44}-\lambda)=0$ which gives $\lambda_{8L_4}=f_{44}$, or $[\lambda^3+\theta_1\lambda^2+\theta_2\lambda+\theta_3]=0$

where

$$\theta_1=-(f_{11}+f_{22}+f_{33}),$$
$$\theta_2=f_{11}(f_{22}+f_{33})+f_{22}f_{33}-f_{13}f_{31}-f_{23}f_{32},$$
$$\theta_3=-(f_{11}f_{22}f_{33}-f_{11}f_{13}f_{32}-f_{22}f_{13}f_{31}).$$

Hence, according to the Routh-Hurwitz criterion, all the eigenvalues of Eq.(113), have negative real parts if and only if $\theta_i>0$, $i=1,3$ and $\tilde{\Delta}=(\theta_1\theta_2-\theta_3)\theta_3=\tilde{\Delta}_1\theta_3$,

where $\tilde{\Delta}_1=\theta_1\theta_2-\theta_3>0$.

Clearly, we have $\theta_1>0$, $\theta_3>0$, and

$$\tilde{\Delta}_1=-f_{11}^2(f_{22}+f_{33})-f_{22}^2(f_{11}+f_{33})-f_{33}^2(f_{11}+f_{22})-2f_{11}f_{22}f_{33}+f_{13}f_{31}(f_{11}+f_{33})+f_{23}f_{32}(f_{22}+f_{33})>0$$

If in addition to conditions $(3,99)$ and the following conditions hold

$$B\tilde{L}_3<\delta+\frac{\gamma}{\sigma}, \tag{114}$$

$$\frac{a_1}{1+f_1\tilde{L}_3}<2b_1\tilde{L}_1+\frac{(\mathrm{C}+\alpha\eta A)B_1\tilde{L}_3}{(\mathrm{C}+\alpha\eta A+\tilde{L}_1)^2}, \tag{115}$$

$$\frac{a_2}{1+f_2\tilde{L}_3}<2b_2\tilde{L}_2+(\rho+h\tilde{L}_3)\tilde{L}_3, \tag{116}$$

$$a_3-d+\frac{C_1(\tilde{L}_1+\eta A)}{(\mathrm{C}+\alpha\eta A+\tilde{L}_1)}+C_2(\rho+2h\tilde{L}_3)\tilde{L}_2<2\frac{a_3}{K}\tilde{L}_3. \tag{117}$$

Therefore, under the above conditions, all of J_8's eigenvalues have negative real parts , and as a result P_8 is (LAS), otherwise P_8 is unstable.

- The (JM) of system (1) at $P_9=\left(\tilde{\tilde{L}}_1,\ 0,\ \tilde{\tilde{L}}_3,\ \tilde{\tilde{L}}_4\right)$ and can be written as

$$J_9=J(P_9)=[\,g_{ij}\,]_{4\times4}, \text{ where} \tag{118}$$

$$g_{11}=\frac{a_1}{1+f_1\tilde{\tilde{L}}_3}-2b_1\tilde{\tilde{L}}_1-\frac{(\mathrm{C}+\alpha\eta A)B_1\tilde{\tilde{L}}_3}{\left(\mathrm{C}+\alpha\eta A+\tilde{\tilde{L}}_1\right)^2},\ g_{12}=0,\ g_{13}=\frac{-a_1f_1\tilde{\tilde{L}}_1}{\left(1+f_1\tilde{\tilde{L}}_3\right)^2}-\frac{B_1\tilde{\tilde{L}}_1}{\left(\mathrm{C}+\alpha\eta A+\tilde{\tilde{L}}_1\right)}<0,\quad g_{14}=0,$$

$$g_{21}=0,\ g_{22}=\frac{a_2}{1+f_2\tilde{\tilde{L}}_3}-\left(\rho+h\tilde{\tilde{L}}_3\right)\tilde{\tilde{L}}_3,\ g_{23}=g_{24}=0,\qquad g_{31}=\frac{C_1\left(\mathrm{C}+\eta A(\alpha-1)\right)\tilde{\tilde{L}}_3}{\left(\mathrm{C}+\alpha\eta A+\tilde{\tilde{L}}_1\right)^2},$$

$$g_{32}=C_2\left(\rho+h\tilde{\tilde{L}}_3\right)\tilde{\tilde{L}}_3>0,\quad g_{33}=a_3-2\frac{a_3}{K}\tilde{\tilde{L}}_3-\frac{a_3}{K}\tilde{\tilde{L}}_4+\frac{C_1\left(\tilde{\tilde{L}}_1+\eta A\right)}{\left(\mathrm{C}+\alpha\eta A+\tilde{\tilde{L}}_1\right)}-\left(d+B\tilde{\tilde{L}}_4\right),$$

$$g_{34}=-\left(\frac{a_3}{K}+B\right)\tilde{\tilde{L}}_3+\frac{\gamma\sigma}{\left(\sigma+\tilde{\tilde{L}}_4\right)^2},\quad g_{41}=g_{42}=0,\quad g_{43}=B\tilde{\tilde{L}}_4,\quad g_{44}=B\tilde{\tilde{L}}_3-\delta-\frac{\gamma\sigma}{\left(\sigma+\tilde{\tilde{L}}_4\right)^2}.$$

Then the characteristic equation of J_9 is given by

$$(g_{22}-\lambda)[\lambda^3+K_1\lambda^2+K_2\lambda+K_3]=0 \tag{119}$$

Then, either $(g_{22}-\lambda)=0$, which gives $\lambda_{9L_2}=g_{22}$.

Or $\left[\lambda^3+K_1\lambda^2+K_2\lambda+K_3\right]=0$,

where

$$K_1=-(g_{11}+g_{33}+g_{44})$$
$$K_2=g_{11}(g_{33}+g_{44})+g_{33}g_{44}-g_{13}g_{31}-g_{34}g_{43}$$
$$K_3=-(g_{11}[g_{44}-g_{34}g_{43}]-g_{13}g_{31}g_{44})$$

Hence, according to the Routh-Hurwitz criterion, all the eigenvalues of Eq.(119), have negative real parts if and only if $K_i>0$, $i=1{,}3$ and $\tilde{\tilde{\Delta}}=(K_1K_2-K_3)K_3=\tilde{\tilde{\Delta}}_1K_3$,

where $\tilde{\tilde{\Delta}}_1=K_1K_2-K_3>0$.

Clearly, we have $K_1>0$, $K_3>0$ and

$$\tilde{\tilde{\Delta}}_1=-g_{11}^2(g_{33}+g_{44})-g_{33}^2(g_{11}+g_{44})-g_{44}^2(g_{11}+g_{33})-2g_{11}g_{44}g_{33}+g_{13}g_{31}(g_{11}+g_{33}+2g_{44})+g_{34}g_{43}(g_{33}+g_{44})>0,$$

If in addition to conditions $(3,49,99)$ the following conditions hold

$$\frac{a_2}{1+f_2\tilde{\tilde{L}}_3}<\left(\rho+h\tilde{\tilde{L}}_3\right)\tilde{\tilde{L}}_3, \tag{120}$$

$$\frac{a_1}{1+f_1\tilde{\tilde{L}}_3}<2b_1\tilde{\tilde{L}}_1+\frac{(\mathrm{C}+\alpha\eta A)B_1\tilde{\tilde{L}}_3}{\left(\mathrm{C}+\alpha\eta A+\tilde{\tilde{L}}_1\right)^2}, \tag{121}$$

$$a_3-d+\frac{C_1\left(\tilde{\tilde{L}}_1+\eta A\right)}{\left(\mathrm{C}+\alpha\eta A+\tilde{\tilde{L}}_1\right)}<2\frac{a_3}{K}\tilde{\tilde{L}}_3+\frac{a_3}{K}\tilde{\tilde{L}}_4+B\tilde{\tilde{L}}_4, \tag{122}$$

$$B\tilde{\tilde{L}}_3-\delta<\frac{\gamma\sigma}{\left(\sigma+\tilde{\tilde{L}}_4\right)^2}<\left(\frac{a_3}{K}+B\right)\tilde{\tilde{L}}_3. \tag{123}$$

Therefore, under the above conditions, all of J_9’s eigenvalues have negative real parts , and as a result P_9 is (LAS). Similarly for (EP) $P_{10}=\left(\tilde{\tilde{\tilde{L}}}_1,0,\ \tilde{\tilde{\tilde{L}}}_3,\ \tilde{\tilde{\tilde{L}}}_4\right)$.

Otherwise, P_9 and P_{10} are unstable.

- The (JM) of system (1) at $P_{11}=\left(0,\ \dot{L}_2,\ \dot{L}_3,\ \dot{L}_4\right)$ can be written as

$$J_{11}=J(P_{11})=\left[n_{ij}\right]_{4\times 4}, \text{where} \tag{124}$$

$$n_{11}=\frac{a_1}{1+f_1\dot{L}_3}-\frac{B_1\dot{L}_3}{(C+\alpha\eta A)},\quad n_{12}=n_{13}=n_{14}=0,\qquad n_{21}=0,$$

$n_{22} = \frac{a_2}{1+f_2\dot{L_3}} - 2b_2\dot{L}_2 - (\rho + h\dot{L_3})\dot{L_3}$, $n_{23} = \frac{-a_2 f_2 \dot{L}_2}{(1+f_2\dot{L_3})^2} - (\rho + 2h\dot{L_3})\dot{L_2}$, $n_{24} = 0$,

$n_{31} = \frac{C_1(C+\eta A(\alpha-1))\dot{L_3}}{(C+\alpha\eta A)^2}$, $n_{32} = C_2(\rho + h\dot{L_3})\dot{L_3} > 0$,

$n_{33} = a_3 - 2\frac{a_3}{K}\dot{L_3} - \frac{a_3}{K}\dot{L_4} + \frac{C_1\eta A}{(C+\alpha\eta A)} + C_2(\rho + 2h\dot{L_3})\dot{L_2} - (d + B\dot{L_4})$,

$n_{34} = -\left(\frac{a_3}{K} + B\right)\dot{L_3} + \frac{\gamma\sigma}{(\sigma+\dot{L_4})^2}$, $n_{41} = n_{42} = 0$, $n_{43} = B\dot{L_4}$, $n_{44} = B\dot{L_3} - \delta - \frac{\gamma\sigma}{(\sigma+\dot{L_4})^2}$.

Then the characteristic equation of J_{11} is given by

$$(n_{11} - \lambda)[\lambda^3 + N_1\lambda^2 + N_2\lambda + N_3] = 0. \quad (125)$$

Then, either $(n_{11} - \lambda) = 0$, which gives $\lambda_{11L_1} = n_{11}$.

Or $\left[\lambda^3 + N_1\lambda^2 + N_2\lambda + N_3\right] = 0$,

where

$N_1 = -(n_{22} + n_{33} + n_{44})$,

$N_2 = n_{22}(n_{33} + n_{44}) + n_{33}n_{44} - n_{23}n_{32} - n_{34}n_{43}$,

$N_3 = -(n_{22}(n_{33}n_{44} - n_{34}n_{43}) - n_{23}n_{32}n_{44})$.

Hence, according to the Routh-Hurwitz criterion, all the eigenvalues of the second part of Eq.(125), have negative real parts if and only if $N_i > 0$, $i = 1,3$ and

$\dot{\Delta} = (N_1N_2 - N_3)N_3 = \dot{\Delta}_1 N_3 > 0$ where $\dot{\Delta}_1 = N_1N_2 - N_3$

Clearly, we have $N_1 > 0$, $N_3 > 0$ and

$\dot{\Delta}_1 = -n_{22}^2(n_{33} + n_{44}) - n_{33}^2(n_{22} + n_{44}) - n_{44}^2(n_{22} + n_{33}) - 2n_{22}n_{33}n_{44} + n_{23}n_{32}(n_{22} + n_{33}) + n_{34}n_{43}(n_{33} + n_{44}) > 0$, provided that condition $(3,61,62)$ and the following conditions hold

$$\frac{a_1}{1+f_1\dot{L_3}} < \frac{B_1\dot{L_3}}{(C+\alpha\eta A)}, \quad (126)$$

$$\frac{a_2}{1+f_2\dot{L_3}} - (\rho + h\dot{L_3})\dot{L_3} < 2b_2\dot{L}_2, \quad (127)$$

$$a_3 - d + \frac{C_1\eta A}{(C+\alpha\eta A)} + C_2(\rho + 2h\dot{L_3})\dot{L_2} < 2\frac{a_3}{K}\dot{L_3} + \frac{a_3}{K}\dot{L_4} + B\dot{L_4}, \quad (128)$$

$$B\dot{L_3} - \delta < \frac{\gamma\sigma}{(\sigma+\dot{L_4})^2} < \left(\frac{a_3}{K} + B\right)\dot{L_3}. \quad (129)$$

Therefore, under the above conditions, all of J_{11}'s eigenvalues have negative real parts , and as a result P_{11} is (LAS). Similarly for (EP) $P_{12} = (0,\ \ddot{L}_2,\ \ddot{L}_3,\ \ddot{L}_4)$.

Otherwise, P_{11} and P_{12} are unstable.

- Finally, the (JM) of system (1) at $P_{13} = (L_1^*, L_2^*, L_3^*, L_4^*)$ can be written as

$J_{13} = J(P_{13}) = [\chi_{ij}]_{4\times 4}$, where (130)

$\chi_{11} = \frac{a_1}{1+f_1 L_3^*} - 2b_1 L_1^* - \frac{(\mathrm{C}+\alpha\eta A)B_1 L_3^*}{(\mathrm{C}+\alpha\eta A+L_1^*)^2}$, $\chi_{12} = 0$, $\chi_{13} = \frac{-a_1 f_1 L_1^*}{(1+f_1 L_3^*)^2} - \frac{B_1 L_1^*}{(\mathrm{C}+\alpha\eta A+L_1^*)} < 0$, $\chi_{14} = 0$,

$\chi_{21} = 0$, $\chi_{22} = \frac{a_2}{1+f_2 L_3^*} - 2b_2 L_2^* - (\rho + hL_3^*)L_3^*$, $\chi_{23} = \frac{-a_2 f_2 L_2^*}{(1+f_2 L_3^*)^2} - (\rho + 2hL_3^*)L_2^*$, $\chi_{24} = 0$,

$\chi_{31} = \frac{C_1(\mathrm{C}+\eta A(\alpha-1))L_3^*}{(\mathrm{C}+\alpha\eta A+L_1^*)^2}$, $\chi_{32} = C_2(\rho + hL_3^*)L_3^* > 0$,

$\chi_{33} = a_3 - 2\frac{a_3}{K}L_3^* - \frac{a_3}{K}L_4^* + \frac{C_1(L_1^*+\eta A)}{(\mathrm{C}+\alpha\eta A+L_1^*)} + C_2(\rho + 2hL_3^*)L_2^* - (d + BL_4^*)$,

$\chi_{34} = -\left(\frac{a_3}{K} + B\right)L_3^* + \frac{\gamma\sigma}{(\sigma+L_4^*)^2}$, $\chi_{41} = \chi_{42} = 0$, $\chi_{43} = BL_4^*$, $\chi_{44} = BL_3^* - \delta - \frac{\gamma\sigma}{(\sigma+L_4^*)^2}$.

Then the characteristic equation of J_{13} is given by

$\lambda^4 + E_1\lambda^3 + E_2\lambda^2 + E_3\lambda + E_4 = 0$, (131)

where

$E_1 = -(\chi_{11} + \chi_{22} + \chi_{33} + \chi_{44})$

$E_2 = \chi_{11}(\chi_{22} + \chi_{33} + \chi_{44}) + \chi_{22}(\chi_{33} + \chi_{44}) + \chi_{33}\chi_{44} - \chi_{13}\chi_{31} - \chi_{23}\chi_{32} - \chi_{34}\chi_{43}$

$E_3 = -\big(\chi_{11}\chi_{22}(\chi_{33} + \chi_{44}) + \chi_{33}\chi_{44}(\chi_{11} + \chi_{22}) - \chi_{13}\chi_{31}(\chi_{22} + \chi_{44}) - \chi_{23}\chi_{32}(\chi_{11} + \chi_{44}) - \chi_{34}\chi_{43}(\chi_{11} + \chi_{22})\big)$

$E_4 = \chi_{11}\chi_{22}(\chi_{33}\chi_{44} - \chi_{34}\chi_{43}) - \chi_{44}(\chi_{11}\chi_{23}\chi_{32} + \chi_{22}\chi_{13}\chi_{31})$

Hence, according to the Routh-Hurwitz criterion, all the eigenvalues of Eq.(131), have negative real parts if and only if $E_1 > 0$, $E_3 > 0$, $E_4 > 0$ and $\Delta_2^* = (E_1E_2 - E_3)E_3 - E_1^2E_2 > 0$.

Clearly, we have $E_1 > 0$, $E_3 > 0$, $E_4 > 0$ and

$\Delta_2^* = (\omega_1 + \omega_2)(\omega_3 + \omega_4) > 0$,

where

$\omega_1 = -(\chi_{11} + \chi_{22} + \chi_{33} + \chi_{44})^2\big(-\chi_{13}\chi_{31}\chi_{22}\chi_{44} - \chi_{11}(\chi_{22}\chi_{34}\chi_{43} + \chi_{23}\chi_{32}\chi_{44} - \chi_{22}\chi_{33}\chi_{44})\big)$,

$\omega_2 = \big((\chi_{11} + \chi_{22})[\chi_{34}\chi_{43} - \chi_{33}\chi_{44}] + \chi_{23}\chi_{32}(\chi_{11} + \chi_{44}) + \chi_{13}\chi_{31}(\chi_{22} + \chi_{44}) - \chi_{11}\chi_{22}(\chi_{33} + \chi_{44})\big)$,

$\omega_3 = -(\chi_{11} + \chi_{22})[\chi_{34}\chi_{43} - \chi_{33}\chi_{44}] - \chi_{23}\chi_{32}(\chi_{11} + \chi_{44}) - \chi_{13}\chi_{31}(\chi_{22} + \chi_{44}) + \chi_{11}\chi_{22}(\chi_{33} + \chi_{44})$,

$\omega_4 = -(\chi_{11} + \chi_{22} + \chi_{33} + \chi_{44})\big(-\chi_{13}\chi_{31} - \chi_{23}\chi_{32} + \chi_{22}\chi_{33} - \chi_{34}\chi_{43} + \chi_{22}\chi_{44} + \chi_{33}\chi_{44} + \chi_{11}(\chi_{22} + \chi_{33} + \chi_{44})\big)$,

if in addition to conditions $(3, 79, 80, 99)$ the following conditions hold

$$\frac{a_1}{1+f_1L_3^*} < 2b_1L_1^* + \frac{(C+\alpha\eta A)B_1L_3^*}{(C+\alpha\eta A+L_1^*)^2}, \tag{132}$$

$$\frac{a_2}{1+f_2L_3^*} - (\rho + hL_3^*)L_3^* < 2b_2L_2^*, \tag{133}$$

$$a_3 - d + \frac{C_1(L_1^* + \eta A)}{(C + \alpha\eta A + L_1^*)} + C_2(\rho + 2hL_3^*)L_2^* < 2\frac{a_3}{K}L_3^* + \frac{a_3}{K}L_4^* + BL_4^*, \tag{134}$$

$$BL_3^* - \delta < \frac{\gamma\sigma}{(\sigma+L_4^*)^2} < \left(\frac{a_3}{K} + B\right)L_3^*, \tag{135}$$

$$\omega_4 > \omega_3. \tag{136}$$

Therefore, under the above conditions, all of J_{13}'s eigenvalues have negative real parts , and as a result P_{13} is (LAS). Similarly for (EP) $P_{14} = (L_1^{**}, L_2^{**}, L_3^{**}, L_4^{**})$.

Otherwise, P_{13} and P_{14} are unstable.

5. Global Stability Analysis (GSA)

In this section the (GSA) for the (EPs), which are (LAS) of system (1) have been determined analytically through the application of Lyapunov method demonstrated by the following theorems.

Theorem 2. The (EP) $P_3 = (0,\ 0,\ \bar{L}_3,\ 0)$ of system (1) is (GAS) in the sub region $\psi_1 \subset R_+^4$ if the following condition holds

$$\bar{\varphi}_1 < \bar{\varphi}_2, \tag{137}$$

where

$$\bar{\varphi}_1 = \frac{a_1L_1}{1+f_1L_3} + \frac{a_2L_2}{1+f_2L_3} + \frac{C_1\eta AL_3}{C+\alpha\eta A+L_1} + \left(\frac{a_3}{K} + B\right)\bar{L}_3L_4 + \frac{C_1\eta A\bar{L}_3}{C+\alpha\eta A},$$

$$\bar{\varphi}_2 = \frac{a_3L_3L_4}{K} + \frac{C_1(L_1+\eta A)\bar{L}_3}{C+\alpha\eta A+L_1} + C_2(\rho + hL_3)L_2\bar{L}_3 + \frac{C_1\eta AL_3}{C+\alpha\eta A} + \left(\frac{\gamma\bar{L}_3}{(\sigma+L_4)L_3} + \delta\right)L_4.$$

Proof. Consider the following function $V_1(L_1, L_2, L_3, L_4) = L_1 + L_2 + \left(L_3 - \bar{L}_3 - \bar{L}_3\ln\frac{L_3}{\bar{L}_3}\right) + L_4$

Clearly, $V_1: R_+^4 \to R$ is a C^1 positive definite function.

Now, using some algebraic manipulation and differentiating V_1 with respect to time t, provides that

$$\frac{dV_1}{dt} = \frac{a_1L_1}{1+f_1L_3} - b_1L_1^2 - \frac{(B_1-C_1)L_1L_3}{C+\alpha\eta A+L_1} + \frac{a_2L_2}{1+f_2L_3} - b_2L_2^2 - (1 - C_2)(\rho + hL_3)L_2L_3 +$$
$$(L_3 - \bar{L}_3)\left[\left(\frac{-a_3L_3 - a_3L_4}{K}\right) - BL_4 + \frac{a_3\bar{L}_3}{K} - \frac{C_1\eta A}{C+\alpha\eta A}\right] + \frac{C_1L_1\bar{L}_3}{C+\alpha\eta A+L_1} + C_2(\rho + hL_3)L_2\bar{L}_3 +$$
$$\frac{\gamma L_4}{(\sigma+L_4)L_3} + BL_3L_4 - \delta L_4 - \frac{\gamma L_4}{\sigma+L_4}$$

Now, due to the biological facts $C_1 < B_1$, $C_2 < 1$ so

$$\frac{dV_1}{dt} < \left(\frac{a_1L_1}{1+f_1L_3} + \frac{a_2L_2}{1+f_2L_3} + \frac{C_1\eta AL_3}{C+\alpha\eta A+L_1} + \left(\frac{a_3}{K} + B\right)\bar{L}_3L_4 + \frac{C_1\eta A\bar{L}_3}{C+\alpha\eta A}\right) - \left(\frac{a_3L_3L_4}{K} + \frac{C_1(L_1+\eta A)\bar{L}_3}{C+\alpha\eta A+L_1} + C_2(\rho + hL_3)L_2\bar{L}_3 + \frac{C_1\eta AL_3}{C+\alpha\eta A} + \left(\frac{\gamma\bar{L}_3}{(\sigma+L_4)L_3} + \delta\right)L_4\right),$$

$$= \bar{\varphi}_1 - \bar{\varphi}_2,$$

Hence by condition (137), $\frac{dV_1}{dt} < 0$ in the region ψ_1, then $V_1$is strictly Lyapunov function (LF).

Consequently, P_3 is a (GAS) in the region $\psi_1 \subset R^4_+$.

Theorem 3. The EP $P_4 = \left(\frac{a_1}{b_1},\ \frac{a_2}{b_2},\ 0,\ 0\right)$ of system (1) is (GAS) in the sub region $\psi_2 \subset R^4_+$ if the following condition holds

$$\breve{\varphi}_1 < \breve{\varphi}_2, \tag{138}$$

where

$$\breve{\varphi}_1 = \frac{a_1L_1}{1+f_1L_3} + \frac{a_1^2}{b_1} + \frac{a_2L_2}{1+f_2L_3} + \frac{a_2^2}{b_2} + \frac{a_1B_1L_3}{b_1(C+\alpha\eta A+L_1)} + \frac{a_2}{b_2}(\rho + hL_3)L_3 + \left(a_3 + \frac{C_1\eta A}{C+\alpha\eta A+L_1}\right)L_3 + \left(\frac{\gamma}{\sigma+L_4} + B\right)L_3L_4,$$

$$\breve{\varphi}_2 = \left(\frac{a_1^2}{b_1(1+f_1L_3)} + a_1L_1 + \frac{a_2^2}{b_2(1+f_1L_3)} + a_2L_2 + dL_3 + \left(\frac{a_3}{K} + B\right)L_3L_4 + \left(\delta + \frac{\gamma}{\sigma+L_4}\right)L_4\right).$$

Proof. Consider the following function

$$V_2(L_1, L_2, L_3, L_4) = \left(L_1 - \breve{L}_1 - \breve{L}_1 \ln\frac{L_1}{\breve{L}_1}\right) + \left(L_2 - \breve{L}_2 - \breve{L}_2 \ln\frac{L_2}{\breve{L}_2}\right) + L_3 + L_4$$

Where $\breve{L}_1 = \frac{a_1}{b_1}$, $\breve{L}_2 = \frac{a_2}{b_2}$.

Clearly, $V_2: R^4_+ \to R$ is a C^1 positive definite function.

Now, using some algebraic manipulation and differentiating V_2 with respect to time t, provides that

$$\frac{dV_2}{dt} = \frac{a_1(L_1-\breve{L}_1)}{1+f_1L_3} - b_1(L_1 - \breve{L}_1)^2 - \frac{(B_1-C_1)L_1L_3}{C+\alpha\eta A+L_1} + \frac{B_1\breve{L}_1L_3}{C+\alpha\eta A+L_1} - a_1(L_1 - \breve{L}_1) + \frac{a_2(L_2-\breve{L}_2)}{1+f_2L_3} - b_2(L_2 - \breve{L}_2)^2 - (1 - C_2)(\rho + hL_3)L_2L_3 + (\rho + hL_3)\breve{L}_2L_3 - a_2(L_2 - \breve{L}_2) + a_3L_3 - \frac{a_3}{K}L_3^2 - \frac{a_3}{K}L_3L_4 + \frac{C_1\eta AL_3}{C+\alpha\eta A+L_1} - dL_3 - BL_3L_4 + \frac{\gamma L_3L_4}{\sigma+L_4} + BL_3L_4 - \delta L_4 - \frac{\gamma L_4}{\sigma+L_4}$$

Now, due to the biological facts $C_1 < B_1$, $C_2 < 1$ so

$$\frac{dV_2}{dt} < \left(\frac{a_1L_1}{1+f_1L_3} + \frac{a_1^2}{b_1} + \frac{a_2L_2}{1+f_2L_3} + \frac{a_2^2}{b_2} + \frac{a_1B_1L_3}{b_1(C+\alpha\eta A+L_1)} + \frac{a_2}{b_2}(\rho + hL_3)L_3 + \left(a_3 + \frac{C_1\eta A}{C+\alpha\eta A+L_1}\right)L_3 + \left(\frac{\gamma}{\sigma+L_4} + B\right)L_3L_4\right) - \left(\frac{a_1^2}{b_1(1+f_1L_3)} + a_1L_1 + \frac{a_2^2}{b_2(1+f_1L_3)} + a_2L_2 + dL_3 + \left(\frac{a_3}{K} + B\right)L_3L_4 + \left(\delta + \frac{\gamma}{\sigma+L_4}\right)L_4\right),$$

$$= \breve{\varphi}_1 - \breve{\varphi}_2,$$

Hence by condition (138), $\frac{dV_2}{dt} < 0$ in the region ψ_2, then V_2 is strictly (LF). Consequently, P_4 is a (GAS) in the region $\psi_2 \subset R^4_+$.

Theorem 4. The EP $P_5 = \left(\bar{\bar{L}}_1,\ 0,\ \bar{\bar{L}}_3\ ,0\right)$ of system (1) is (GAS) in the sub region $\psi_3 \subset R^4_+$. If the following condition holds

$$\bar{\bar{\varphi}}_1 < \bar{\bar{\varphi}}_2, \tag{139}$$

where

$$\bar{\bar{\varphi}}_1 = \frac{a_1 L_1}{1+f_1 L_3} + \frac{a_2 L_2}{1+f_2 L_3} + \frac{\left(B_1\bar{\bar{L}}_1 + C_1\eta A\right)L_3}{(\mathrm{C}+\alpha\eta A+L_1)} + \frac{a_1\bar{\bar{L}}_1}{1+f_1\bar{\bar{L}}_3} + \frac{a_3}{K}L_4\bar{\bar{L}}_3 + B\bar{\bar{L}}_3 L_4 + \frac{(C_1\eta A+B_1 L_1)\bar{\bar{L}}_3}{\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1},$$

$$\bar{\bar{\varphi}}_2 = \frac{a_1\bar{\bar{L}}_1}{1+f_1 L_3} + \frac{a_1 L_1}{1+f_1\bar{\bar{L}}_3} + \frac{a_3}{K}L_3 L_4 + \frac{C_1(L_1+\eta A)\bar{\bar{L}}_3}{\mathrm{C}+\alpha\eta A+L_1} + \frac{C_1\left(\bar{\bar{L}}_1+\eta A\right)L_3}{\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1} + C_2(\rho+hL_3)L_2\bar{\bar{L}}_3 + \frac{\gamma}{(\sigma+L_4)L_3}\bar{\bar{L}}_3 L_4.$$

Proof. Consider the following function

$$V_3(L_1,L_2,L_3,L_4) = \left(L_1 - \bar{\bar{L}}_1 - \bar{\bar{L}}_1\ ln\frac{L_1}{\bar{\bar{L}}_1}\right) + L_2 + \left(L_3 - \bar{\bar{L}}_3 - \bar{\bar{L}}_3\ ln\frac{L_3}{\bar{\bar{L}}_3}\right) + L_4$$

Clearly, $V_3\colon R^4_+ \to R$ is a C^1 positive definite function.

Now, using some algebraic manipulation and differentiating V_3 with respect to time t, provides that

$$\frac{dV_3}{dt} = \frac{a_1\left(L_1-\bar{\bar{L}}_1\right)}{1+f_1 L_3} - b_1\left(L_1-\bar{\bar{L}}_1\right)^2 - \frac{(B_1-C_1)L_1L_3}{\mathrm{C}+\alpha\eta A+L_1} + \frac{B_1\bar{\bar{L}}_1 L_3}{\mathrm{C}+\alpha\eta A+L_1} - \frac{a_1\left(L_1-\bar{\bar{L}}_1\right)}{1+f_1\bar{\bar{L}}_3} - \frac{(B_1-C_1)\bar{\bar{L}}_1\bar{\bar{L}}_3}{\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1} + \frac{a_2 L_2}{1+f_2 L_3} - b_2 {L_2}^2 -$$

$$(1-C_2)(\rho+hL_3)L_2L_3 - \frac{a_3}{K}\left(L_3-\bar{\bar{L}}_3\right)^2 - \frac{a_3}{K}\left(L_3-\bar{\bar{L}}_3\right)L_4 + \frac{C_1\eta A\left(L_3-\bar{\bar{L}}_3\right)}{\mathrm{C}+\alpha\eta A+L_1} - \frac{C_1 L_1\bar{\bar{L}}_3}{\mathrm{C}+\alpha\eta A+L_1} -$$

$$C_2(\rho+hL_3)L_2\bar{\bar{L}}_3 + B\bar{\bar{L}}_3 L_4 - \frac{\gamma\bar{\bar{L}}_3 L_4}{(\sigma+L_4)L_3} - \frac{C_1\left(\bar{\bar{L}}_1+\eta A\right)L_3}{\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1} + \frac{(C_1\eta A+B_1L_1)\bar{\bar{L}}_3}{\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1} - \delta L_4$$

Now, due to the biological facts $C_1 < B_1,\ C_2 < 1$ so

$$\frac{dV_3}{dt} < \left(\frac{a_1 L_1}{1+f_1 L_3} + \frac{a_2 L_2}{1+f_2 L_3} + \frac{\left(B_1\bar{\bar{L}}_1 + C_1\eta A\right)L_3}{(C+\alpha\eta A+L_1)} + \frac{a_1\bar{\bar{L}}_1}{1+f_1\bar{\bar{L}}_3} + \frac{a_3}{K}L_4\bar{\bar{L}}_3 + B\bar{\bar{L}}_3 L_4 + \frac{(C_1\eta A+B_1L_1)\bar{\bar{L}}_3}{C+\alpha\eta A+\bar{\bar{L}}_1}\right) - \left(\frac{a_1\bar{\bar{L}}_1}{1+f_1 L_3} + \right.$$

$$\left.\frac{a_1 L_1}{1+f_1\bar{\bar{L}}_3} + \frac{a_3}{K}L_3L_4 + \frac{C_1(L_1+\eta A)\bar{\bar{L}}_3}{\mathrm{C}+\alpha\eta A+L_1} + \frac{C_1\left(\bar{\bar{L}}_1+\eta A\right)L_3}{\mathrm{C}+\alpha\eta A+\bar{\bar{L}}_1} + C_2(\rho+hL_3)L_2\bar{\bar{L}}_3 + \frac{\gamma}{(\sigma+L_4)L_3}\bar{\bar{L}}_3 L_4\right),$$

$$= \bar{\bar{\varphi}}_1 - \bar{\bar{\varphi}}_2,$$

hence by condition (139), $\frac{dV_3}{dt} < 0$ in the region ψ_3, then V_3 is strictly (LF). Consequently, P_5 is a (GAS) in the region $\psi_3 \subset R^4_+$.

Theorem 5. The EP $P_6 = \left(0\ ,\hat{L}_2\ ,\hat{L}_3, 0\ \right)$ of system (1) is (GAS) in the sub region $\psi_4 \subset R^4_+$ if the following condition holds

$$\hat{\varphi}_1 < \hat{\varphi}_2, \tag{140}$$

where

$$\hat{\varphi}_1=\frac{a_1L_1}{1+f_1L_3}+\frac{a_2L_2}{1+f_2L_3}+\left(\rho+h\hat{L}_3\right)L_2\hat{L}_3+\left(\rho+hL_3\right)\hat{L}_2L_3+\left(\frac{a_2}{1+f_2\hat{L}_3}-\left(\rho+h\hat{L}_3\right)\hat{L}_3\right)\hat{L}_2+\frac{C_1\eta AL_3}{\mathrm{C}+\alpha\eta A+L_1}+$$

$$\frac{a_3}{K}\hat{L}_3L_4+\mathrm{B}\hat{L}_3L_4+\frac{C_1\eta A\hat{L}_3}{\mathrm{C}+\alpha\eta A}+C_2\left(\rho+h\hat{L}_3\right)\hat{L}_2,$$

$$\hat{\varphi}_2=\frac{a_2L_2}{1+f_2\hat{L}_3}+\frac{a_2\hat{L}_2}{1+f_2L_3}+\frac{a_3}{K}L_3L_4+\frac{C_1\eta AL_3}{\mathrm{C}+\alpha\eta A}+C_2\left(\rho+h\hat{L}_3\right)L_3\hat{L}_3+\frac{C_1(L_1+\eta A)\hat{L}_3}{\mathrm{C}+\alpha\eta A+L_1}+C_2\left(\rho+hL_3\right)L_2\hat{L}_3+$$

$$\frac{\gamma}{(\sigma+L_4)L_3}\hat{L}_3L_4+\delta L_4.$$

Proof. Consider the following function

$$V_4(L_1,L_2,L_3,L_4)=L_1+\left(L_2-\hat{L}_2-\hat{L}_2\ln\frac{L_2}{\hat{L}_2}\right)+\left(L_3-\hat{L}_3-\hat{L}_3\ln\frac{L_3}{\hat{L}_3}\right)+L_4$$

Clearly, $\mathrm{V}_4:\mathrm{R}^4_+\to\mathrm{R}$ is a C^1 positive definite function.

Now, using some algebraic manipulation and differentiating V_4 with respect to time t, provides that

$$\frac{dV_4}{dt}=\frac{a_1L_1}{1+f_1L_3}-b_1L_1^2-\frac{(B_1-C_1)L_1L_3}{C+\alpha\eta A+L_1}+\frac{a_2(L_2-\hat{L}_2)}{1+f_2L_3}-b_2\left(L_2-\hat{L}_2\right)^2-(1-C_2)(\rho+hL_3)L_2L_3-\frac{a_2(L_2-\hat{L}_2)}{1+f_2\hat{L}_3}+$$

$$\left(\rho+h\hat{L}_3\right)\left(L_2-\hat{L}_2\right)\hat{L}_3+(\rho+hL_3)\hat{L}_2L_3-\frac{a_3}{K}\left(L_3-\hat{L}_3\right)^2-\frac{a_3}{K}\left(L_3-\hat{L}_3\right)L_4+\frac{C_1\eta A(L_3-\hat{L}_3)}{C+\alpha\eta A+L_1}-$$

$$\frac{C_1\eta A(L_3-\hat{L}_3)}{C+\alpha\eta A}-\frac{C_1L_1\hat{L}_3}{C+\alpha\eta A+L_1}-C_2\left(\rho+h\hat{L}_3\right)\left(L_3-\hat{L}_3\right)\hat{L}_3-C_2(\rho+hL_3)L_2\hat{L}_3-\frac{\gamma}{\sigma+L_4}\hat{L}_3L_4+\left(B\hat{L}_3-\delta\right)L_4$$

Now, due to the biological facts $C_1<B_1$, $C_2<1$ so

$$\frac{d\mathrm{V}_4}{dt}<\left(\frac{a_1L_1}{1+f_1L_3}+\frac{a_2L_2}{1+f_2L_3}+\left(\rho+h\hat{L}_3\right)L_2\hat{L}_3+(\rho+hL_3)\hat{L}_2L_3+\left(\frac{a_2}{1+f_2\hat{L}_3}-\left(\rho+h\hat{L}_3\right)\hat{L}_3\right)\hat{L}_2+\frac{C_1\eta AL_3}{\mathrm{C}+\alpha\eta A+L_1}+\right.$$

$$\left.\frac{a_3}{K}\hat{L}_3L_4+\mathrm{B}\hat{L}_3L_4+\frac{C_1\eta A\hat{L}_3}{\mathrm{C}+\alpha\eta A}+C_2\left(\rho+h\hat{L}_3\right)\hat{L}_2\right)-\left(\frac{a_2L_2}{1+f_2\hat{L}_3}+\frac{a_2\hat{L}_2}{1+f_2L_3}+\frac{a_3}{K}L_3L_4+\frac{C_1\eta AL_3}{\mathrm{C}+\alpha\eta A}+\quad C_2\left(\rho+\right.\right.$$

$$\left.\left.h\hat{L}_3\right)L_3\hat{L}_3+\frac{C_1(L_1+\eta A)\hat{L}_3}{\mathrm{C}+\alpha\eta A+L_1}+C_2(\rho+hL_3)L_2\hat{L}_3+\frac{\gamma}{(\sigma+L_4)L_3}\hat{L}_3L_4+\delta L_4\right),$$

$$=\hat{\varphi}_1-\hat{\varphi}_2,$$

hence in addition to condition (16) condition (140) holds, $\frac{d\mathrm{V}_4}{dt}<0$ in the region ψ_4, then V_4 is strictly (LF). Consequently, P_6 is a (GAS) in the region $\psi_4\subset\mathrm{R}^4_+$.

Theorem 6. The EP $P_7=\left(0,\ 0,\ \check{L}_3,\ \check{L}_4\right)$ of system (1) is (GAS) in the sub region $\psi_5\subset R^4_+$ if the following condition holds

$$\check{\varphi}_1<\check{\varphi}_2, \tag{141}$$

where

$$\check{\varphi}_1=\frac{a_1L_1}{1+f_1L_3}+\frac{a_2L_2}{1+f_2L_3}+\frac{a_3(\check{L}_3L_4+L_3\check{L}_4)}{K}+\frac{C_1\eta AL_3}{C+\alpha\eta A+L_1}+\frac{C_1\eta A\check{L}_3}{C+\alpha\eta A}+\frac{\gamma\check{L}_4}{\sigma+L_4}+\frac{\gamma L_4}{\sigma+\check{L}_4},$$

$$\breve{\varphi}_2=\frac{a_3(L_3L_4+\breve{L}_3\breve{L}_4)}{K}+\frac{C_1\eta AL_3}{\mathrm{C}+\alpha\eta A}+\frac{\gamma L_3\breve{L}_4}{(\sigma+\breve{L}_4)\breve{L}_3}+\frac{C_1(L_1+\eta A)\breve{L}_3}{\mathrm{C}+\alpha\eta A+L_1}+C_2(\rho+hL_3)L_2\breve{L}_3+\frac{\gamma\breve{L}_3L_4}{(\sigma+L_4)L_3}.$$

Proof. Consider the following function

$$V_5(L_1,L_2,L_3,L_4)=L_1+L_2+\left(L_3-\breve{L}_3-\breve{L}_3\ln\frac{L_3}{\breve{L}_3}\right)+\left(L_4-\breve{L}_4-\breve{L}_4\ln\frac{L_4}{\breve{L}_4}\right)$$

Clearly, $\mathrm{V}_5:\mathrm{R}_+^4\to\mathrm{R}$ is a C^1 positive definite function.

Now, using some algebraic manipulation and differentiating V_5 with respect to time t, provides that

$$\frac{d\mathrm{V}_5}{dt}=\frac{a_1L_1}{1+f_1L_3}-b_1L_1^2-\frac{(B_1-C_1)L_1L_3}{\mathrm{C}+\alpha\eta A+L_1}+\frac{a_2L_2}{1+f_2L_3}-b_2L_2^2-(1-C_2)(\rho+hL_3)L_2L_3-\frac{a_3}{K}\left(L_3-\breve{L}_3\right)^2+$$
$$\left(L_3-\breve{L}_3\right)\left[\frac{-a_3L_4}{K}+\frac{C_1\eta A}{\mathrm{C}+\alpha\eta A+L_1}+\frac{a_3\breve{L}_4}{K}-\frac{C_1\eta A}{\mathrm{C}+\alpha\eta A}\right]-\frac{C_1L_1\breve{L}_3}{\mathrm{C}+\alpha\eta A+L_1}-C_2(\rho+hL_3)L_2\breve{L}_3-\frac{\gamma\breve{L}_3L_4}{(\sigma+L_4)L_3}-\frac{\gamma L_3\breve{L}_4}{(\sigma+\breve{L}_4)\breve{L}_3}+$$
$$\frac{\gamma\breve{L}_4}{\sigma+L_4}+\frac{\gamma L_4}{\sigma+\breve{L}_4}$$

Now, due to the biological facts $C_1<B_1$, $C_2<1$ so

$$\frac{d\mathrm{V}_5}{dt}<\left(\frac{a_1L_1}{1+f_1L_3}+\frac{a_2L_2}{1+f_2L_3}+\frac{a_3(\breve{L}_3L_4+L_3\breve{L}_4)}{K}+\frac{C_1\eta AL_3}{\mathrm{C}+\alpha\eta A+L_1}+\frac{C_1\eta A\breve{L}_3}{\mathrm{C}+\alpha\eta A}+\frac{\gamma\breve{L}_4}{\sigma+L_4}+\frac{\gamma L_4}{\sigma+\breve{L}_4}\right)-\left(\frac{a_3(L_3L_4+\breve{L}_3\breve{L}_4)}{K}+\frac{C_1\eta AL_3}{\mathrm{C}+\alpha\eta A}+\right.$$
$$\left.\frac{\gamma L_3\breve{L}_4}{(\sigma+\breve{L}_4)\breve{L}_3}+\frac{C_1(L_1+\eta A)\breve{L}_3}{\mathrm{C}+\alpha\eta A+L_1}+C_2(\rho+hL_3)L_2\breve{L}_3+\frac{\gamma\breve{L}_3L_4}{(\sigma+L_4)L_3}\right),$$
$$=\breve{\varphi}_1-\breve{\varphi}_2,$$

hence by condition (141), $\frac{d\mathrm{V}_5}{dt}<0$ in the region ψ_5, then V_5 is strictly (LF). Therefore, P_7 is a (GAS) in the region $\psi_5\subset\mathrm{R}_+^4$.

Theorem 7. The EP $P_8=\left(\tilde{L}_1,\ \tilde{L}_2,\ \tilde{L}_3,\ 0\right)$ of system (1) is (GAS) in the sub region $\psi_6\subset R_+^4$ if the following condition holds

$$\tilde{\varphi}_1<\tilde{\varphi}_2, \tag{142}$$

where

$$\tilde{\varphi}_1=\frac{a_1L_1}{1+f_1L_3}+\frac{a_1\tilde{L}_1}{1+f_1\tilde{L}_3}+\frac{B_1L_1\tilde{L}_3}{\mathrm{C}+\alpha\eta A+\tilde{L}_1}+\frac{a_2L_2}{1+f_2L_3}+(\rho+h\tilde{L}_3)L_2\tilde{L}_3+(\rho+hL_3)\tilde{L}_2L_3+\frac{a_2\tilde{L}_2}{1+f_2\tilde{L}_3}+\frac{(C_1\eta A+B_1\tilde{L}_1)L_3}{\mathrm{C}+\alpha\eta A+L_1}+$$
$$\frac{C_1\eta A\tilde{L}_3}{\mathrm{C}+\alpha\eta A+\tilde{L}_1}+\frac{a_3\tilde{L}_3L_4}{K}+B\tilde{L}_3L_4,$$

$$\tilde{\varphi}_2=\frac{a_1L_1}{1+f_1\tilde{L}_3}+\frac{a_1\tilde{L}_1}{1+f_1L_3}+\frac{a_2L_2}{1+f_2\tilde{L}_3}+\frac{a_2\tilde{L}_2}{1+f_2L_3}+\frac{a_3L_3L_4}{K}+\frac{C_1(\tilde{L}_1+\eta A)L_3}{\mathrm{C}+\alpha\eta A+\tilde{L}_1}+\frac{C_1(a_1\tilde{L}_1+\eta A)\tilde{L}_3}{\mathrm{C}+\alpha\eta A+L_1}+C_2(\rho+h\tilde{L}_3)\tilde{L}_2L_3+$$
$$C_2(\rho+hL_3)L_2\tilde{L}_3+\frac{\gamma\tilde{L}_3L_4}{(\sigma+L_4)L_3}+\delta L_4.$$

Proof. Consider the following function

$$V_6(L_1,L_2,L_3,L_4)=\left(L_1-\tilde{L}_1-\tilde{L}_1\ln\frac{L_1}{\tilde{L}_1}\right)+\left(L_2-\tilde{L}_2-\tilde{L}_2\ln\frac{L_2}{\tilde{L}_2}\right)+\left(L_3-\tilde{L}_3-\tilde{L}_3\ln\frac{L_3}{\tilde{L}_3}\right)+L_4$$

Clearly, $V_6: R_+^4 \to R$ is a C^1 positive definite function.

Now, using some algebraic manipulation and differentiating V_6 with respect to time t, provides that

$$\frac{dV_6}{dt} = (L_1 - \tilde{L}_1)\left[\frac{a_1}{1+f_1L_3} - \frac{a_1}{1+f_1\tilde{L}_3}\right] - b_1(L_1-\tilde{L}_1)^2 - \frac{(B_1-C_1)L_1L_3}{C+\alpha\eta A+L_1} + \frac{B_1L_1\tilde{L}_3}{C+\alpha\eta A+\tilde{L}_1} + \frac{B_1\tilde{L}_1L_3}{C+\alpha\eta A+L_1} - \frac{(B_1-C_1)\tilde{L}_1\tilde{L}_3}{C+\alpha\eta A+\tilde{L}_1} +$$
$$(L_2-\tilde{L}_2)\left[\frac{a_2}{1+f_2L_3} - \frac{a_2}{1+f_2\tilde{L}_3}\right] - b_2(L_2-\tilde{L}_2)^2 - (1-C_2)(\rho+hL_3)L_2L_3 + (\rho+h\tilde{L}_3)L_2\tilde{L}_3 +$$
$$(\rho+hL_3)\tilde{L}_2L_3 - (1-C_2)(\rho+h\tilde{L}_3)\tilde{L}_2\tilde{L}_3 - \frac{a_3}{K}(L_3-\tilde{L}_3)^2 - (L_3-\tilde{L}_3)\left[\frac{a_3L_4}{K} - \frac{C_1\eta A}{C+\alpha\eta A+L_1}\right] -$$
$$\frac{C_1(\tilde{L}_1+\eta A)L_3}{C+\alpha\eta A+\tilde{L}_1} - C_2(\rho+h\tilde{L}_3)\tilde{L}_2L_3 - C_2(\rho+hL_3)L_2\tilde{L}_3 + \left(B - \frac{\gamma}{(\sigma+L_4)L_3}\right)\tilde{L}_3L_4 + \frac{C_1\eta A\tilde{L}_3}{C+\alpha\eta A+\tilde{L}_1} + \frac{C_1\tilde{L}_1\tilde{L}_3}{C+\alpha\eta A+L_1} - \delta L_4$$

Now, due to the biological facts $C_1 < B_1$, $C_2 < 1$ so

$$\frac{dV_6}{dt} < \left(\frac{a_1L_1}{1+f_1L_3} + \frac{a_1\tilde{L}_1}{1+f_1\tilde{L}_3} + \frac{B_1L_1\tilde{L}_3}{C+\alpha\eta A+\tilde{L}_1} + \frac{a_2L_2}{1+f_2L_3} + (\rho+h\tilde{L}_3)L_2\tilde{L}_3 + (\rho+hL_3)\tilde{L}_2L_3 + \frac{a_2\tilde{L}_2}{1+f_2\tilde{L}_3} + \frac{(C_1\eta A+B_1\tilde{L}_1)L_3}{C+\alpha\eta A+L_1} +\right.$$
$$\left.\frac{C_1\eta A\tilde{L}_3}{C+\alpha\eta A+\tilde{L}_1} + \frac{a_3\tilde{L}_3L_4}{K} + B\tilde{L}_3L_4\right) - \left(\frac{a_1L_1}{1+f_1\tilde{L}_3} + \frac{a_1\tilde{L}_1}{1+f_1L_3} + \frac{a_2L_2}{1+f_2\tilde{L}_3} + \frac{a_2\tilde{L}_2}{1+f_2L_3} + \frac{a_3L_3L_4}{K} + \frac{C_1(\tilde{L}_1+\eta A)L_3}{C+\alpha\eta A+\tilde{L}_1} +\right.$$
$$\left.\frac{C_1(\tilde{L}_1+\eta A)\tilde{L}_3}{C+\alpha\eta A+L_1} + C_2(\rho+h\tilde{L}_3)\tilde{L}_2L_3 + C_2(\rho+hL_3)L_2\tilde{L}_3 + \frac{\gamma\tilde{L}_3L_4}{(\sigma+L_4)L_3} + \delta L_4\right),$$
$$= \tilde{\varphi}_1 - \tilde{\varphi}_2,$$

hence by condition (142), $\frac{dV_6}{dt} < 0$ in the region ψ_6, then V_6 is strictly (LF). Consequently, P_8 is a (GAS) in the region $\psi_6 \subset R_+^4$.

Moreover, two equilibrium points (P_9 and P_{10}) that are situated inside R_+^4 and have different beginning point neighborhoods but the same local stability criteria cannot be examined for global stability using the Lyapunov function. As a result, we will examine it numerically rather than analytically, as shown in the earlier theorems.

Similarly for (P_{11} and P_{12}), and (P_{13} and P_{14}).

6. NUMERICAL ANALYSES

In order to confirm our obtained analytical results and examine the impact of changing the values of each parameter on the dynamical behavior of the system, the dynamical behavior of system (1) is numerically analyzed using Mathematica.

Figure (1) illustrates that system (1) has a (GAS) positive equilibrium point for the subsequent set of default parameters that satisfy the positive (EP) stability requirements.

$$\left.\begin{array}{l} a_1 = 0.7,\ a_2 = 0.7,\ a_3 = 0.8,\ f_1 = 0.3,\ f_2 = 0.5,\ b_1 = 0.2,\ b_2 = 0.2, \\ B_1 = 0.7,\ \mathrm{c} = 1,\ \alpha = 2,\ \eta = 0.5,\ A = 0.03,\ \rho = 0.4,\ h = 0.1,\ K = 1.7, \\ C_1 = 0.6,\ C_2 = 0.2,\ d = 0.5,\ B = 0.4,\ \gamma = 0.01,\ \sigma = 0.3,\ \delta = 0.15 \end{array}\right\} \quad (143)$$

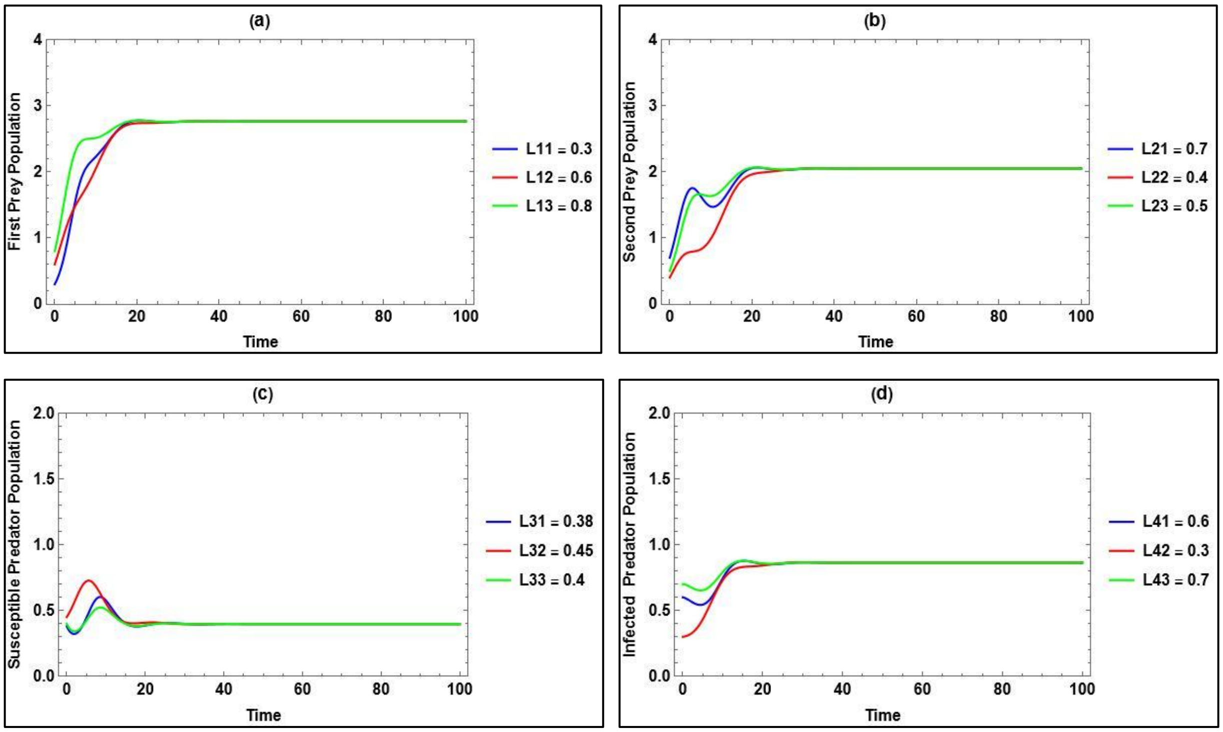

Figure -1 The trajectories of system (1) that started from three different initial points $(0.3,\ 0.7,\ 0.38,\ 0.6)$, $(0.6,\ 0.4,\ 0.45,\ 0.3)$ and $(0.8,\ 0.5,\ 0.4,\ 0.7)$ for the data given in (143). (a) the trajectory of L_1 as a function of time, (b) trajectory of L_2 as a function of time, (c) trajectory of L_3 as a function of time, (d) the trajectory of L_4 as a function of time, approaches to $P_{13} = (\,2.761,\ 2.05,\ 0.396,\ 0.863)$.

As the solution of system (1) approaches asymptotically to the positive equilibrium point $P_{13} = (\,2.761,\ 2.049,\ 0.396,\ 0.863)$, Figure (2) clearly demonstrates that system (1) possesses a (GAS).

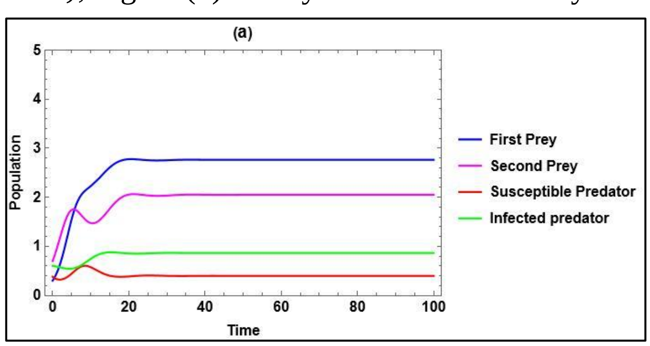

Figure -2 The trajectories of system (1) for the data given in (143) has a (GAS) positive equilibrium point $P_{13} = (\,2.761,\ 2.05,\ 0.396,\ 0.863)$.

Now, to discuss how the parameters, affect the system's dynamical behavior, we changed one parameter at a time while maintaining the other parameters as data in (143). The results are displayed in Table 2.

Table 2. The dynamic behavior of system (1) at every parameter

Range of Parameter	**Stable Point**
$0.01 \leq a_1 < 0.306$ $0.306 \leq a_1 \leq 2$	P_{11} P_{13}
$0.01 \leq a_2 < 0.23$ $0.23 \leq a_2 \leq 2$	P_9 P_{13}
$0.01 \leq a_3 \leq 5$	P_{13}
$0.001 \leq f_1 < 3.869$ $3.869 \leq f_1 \leq 5$	P_{13} P_{11}
$0.01 \leq f_2 < 7.3$ $7.3 \leq f_2 < 10$	P_{13} P_9
$0.01 \leq b_1 \leq 1$	P_{13}
$0.01 \leq b_2 \leq 1$	P_{13}
$0.61 \leq B_1 \leq 1$	P_{13}
$0.1 \leq \alpha \leq 2$	P_{13}
$0.001 \leq \eta \leq 1$	P_{13}
$0.001 \leq A \leq 1$	P_{13}
$0.001 \leq K < 0.02$ $0.02 \leq K < 0.42$ $0.42 \leq K < 3$	P_4 P_8 P_{13}
$0.01 \leq h < 2.62$ $2.62 \leq h < 3$	P_{13} P_9
$1 \leq \mathrm{c} \leq 2$	P_{13}
$0.001 \leq C_1 \leq 0.69$	P_{13}
$0.0001 \leq C_2 < 1$	P_{13}
$0.01 \leq \rho \leq 1.41$ $1.41 \leq \rho \leq 2$	P_{13} P_9
$0.1 \leq d < 1$	P_{13}
$0.001 \leq B < 0.1716$ $0.1716 \leq B < 0.175$ $0.175 \leq B \leq 1$	P_8 P_9 P_{13}
$0.001 \leq \delta < 0.3623$ $0.3623 \leq \delta < 0.174$ $0.174 \leq \delta < 1$	P_{13} P_9 P_8
$0.001 \leq \gamma < 0.143$ $0.143 \leq \gamma < 1$	P_{13} P_8
$0.01 \leq \sigma \leq 3$	P_{13}

The effect of varying the growth rate of the first prey population in the range $0.01 \leq a_1 < 0.306$ is examined, it is found that system (1) approach to P_{11}, however increasing this parameter further $0.306 \leq a_1 \leq 2$ it is observed that the system still approach asymptotically to the positive equilibrium point P_{13} as shown in Figure (3).

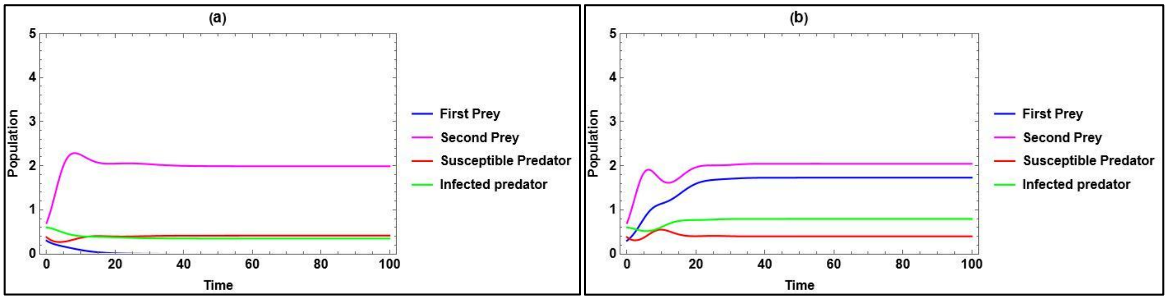

Figure -3. (a) Time series of the solution of system (1) which approaches to $P_{11} = (0,\ 1.987, 0.413,\ 0.347)$ when $a_1 = 0.1$, (b) Time series of the solution of system (1) which approaches $P_{13} = (1.728, 2.044, 0.397, 0.792)$ when a_1=0.5.

The effect of varying the conversion rate of food to the predator and top predator respectively in the range $0.001 \leq B < 0.1716$ is examined, it is found that system (1) approach to P_8, however increasing this parameter further $0.1716 \leq B < 0.175$ system (1) approach to P_9, moreover increasing this parameter in the range $0.175 \leq B \leq 1$ system (1) approach to P_{13}, as shown in Figure (4).

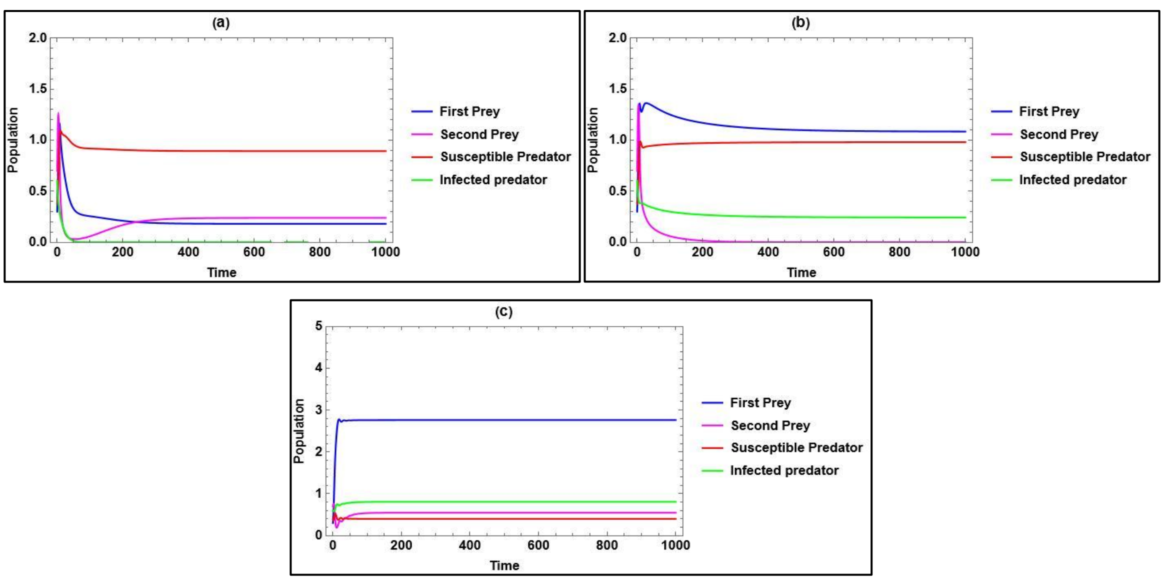

Figure -4. (a) Time series of the solution of system (1) which still approaches to $P_8 = (0.179,$ $0.238, 0.892, 0)$ when $B = 0.1$, (b) Time series of solution of system (1) which approaches $P_9 = (1.083,\ 0, 0.979, 0.242)$ when $B = 0.172$, (c) Time series of the solution of system (1) which approaches to $P_{13} = (3.2067,\ 2.90627, 0.160455, 0.656438)$ when $B = 1$.

The effect of varying the death rates of infected predator in the range $0.001 \leq \delta < 0.3623$ is examined, it is found that system (1) still approach to P_{13}, however increasing this parameter further $0.3623 \leq \delta < 0.374$ system (1) approach to P_9, moreover increasing this parameter in the range $0.374 \leq \delta < 1$ system (1) approach to P_{13}, as shown in Figure (5).

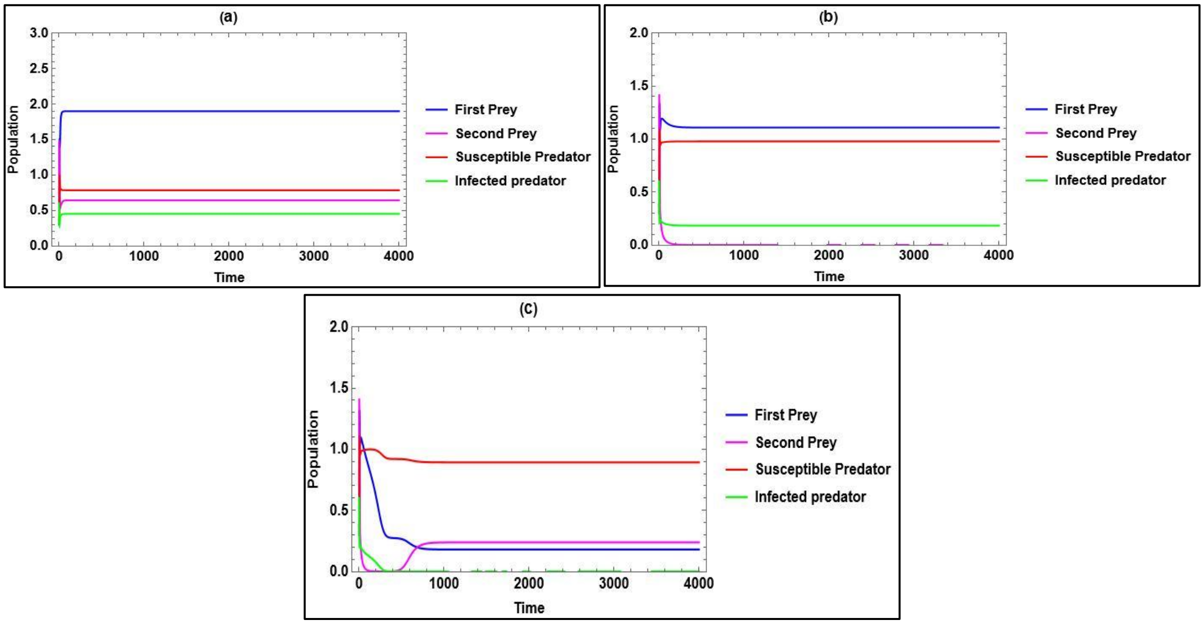

Figure -5. (a) Time series of the solution of system (1) which still approaches to $P_{13} = (1.897, 0.641, 0.783, 0.451)$ when $\delta = 0.3$, (b) Time series of the solution of system (1) which approaches to $P_9 = (1.107, 0, 0.976, 0.182)$ when $\delta = 0.37$, (c) Time series of solution of system (1) which approaches to $P_8 = (0.179, 0.238, 0.892, 0)$ when $\delta = 0.38$.

The effect of varying the growth rate of the first and second prey populations and the death rate of the infected predator is examined, and it is found that system (1) will approach to the (EP) P_3 as it shows in Figure (6).

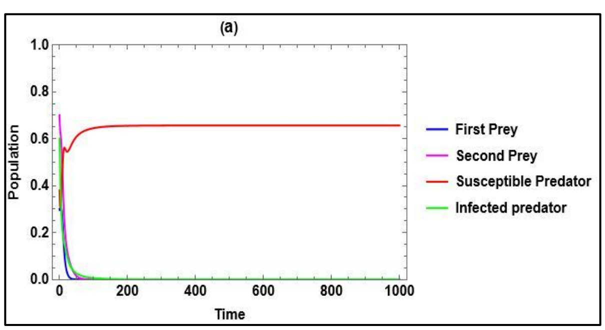

Figure -6 (a)-Time series of solution of system (1) for the data given in (143) approaches to $P_3 = (0, 0, 0.656, 0)$ when $a_1 = 0.3$, $a_2 = 0.2$, and $\delta = 0.374$.

The effect of varying the growth rate of the susceptible predator population, the death rate of the susceptible predator, and the internal competition rate of first prey is examined, and it is found that system (1) will approach to the (EP) P_4 as it shows in Figure (7).

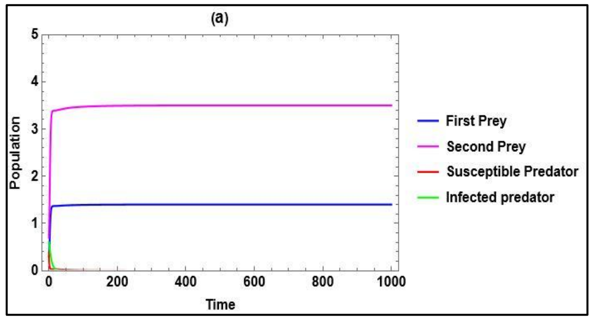

Figure -7 (a)-Time series of solution of system (1) for the data given in (143) approaches to $P_4 = (1.4,\ 3.5,\ 0,\ 0)$ when $a_3 = 0.3,\ d = 0.95,$ and $b_1 = 0.5$.

The effect of varying the growth rate of the susceptible second prey population, the internal competition rate of the first prey, the attack rate of the susceptible predator to the second prey, and the death rate of the infected predator is examined, and it is found that system (1) will approach to the (EP) P_5 as it shows in Figure (8).

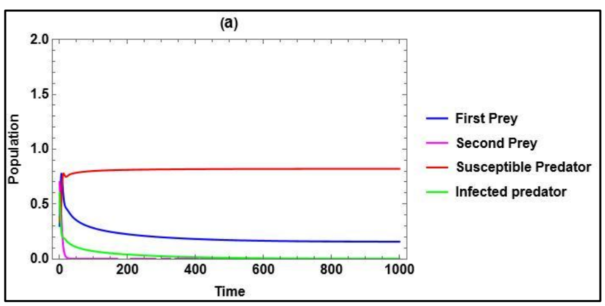

Figure -8 (a)-Time series of solution of system (1) for the data given in (143) approaches to $P_5 = (0.155,\ 0,\ 0.819,\ 0)$ when $a_2 = 0.2,\ b_1 = 0.5,\ \rho = 0.5$ and $\delta = 0.3$.

The effect of varying the growth rate of the first prey and second prey population, the attack rate of the susceptible predator to the second prey, the maximum growth rates of the predator when it consumes the prey and additional food, and the death rate of the infected predator is examined, and it is found that system (1) will approach to the (EP) P_6 as it shows in Figure (9).

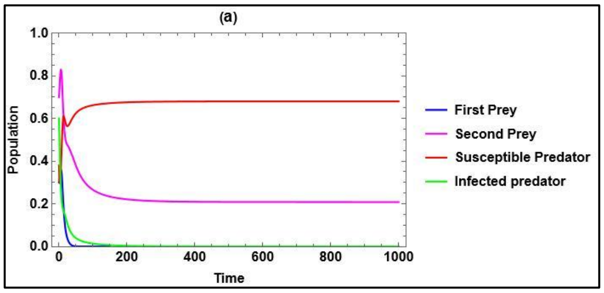

Figure -9 (a)-Time series of solution of system (1) for the data given in (143) approaches to $P_6 = (0,\ 0.208,\ 0.680,\ 0)$ when $a_1 = 2.9,\ a_2 = 0.3,\ \rho = 0.2,\ \alpha = 1.1$ and $\delta = 0.25$.

The effect of varying the growth rate of the first prey and second prey populations, and the attack rate of the susceptible predator to the second prey is examined, and it is found that system (1) will approach to the (EP) P_7 as it shows in Figure (10).

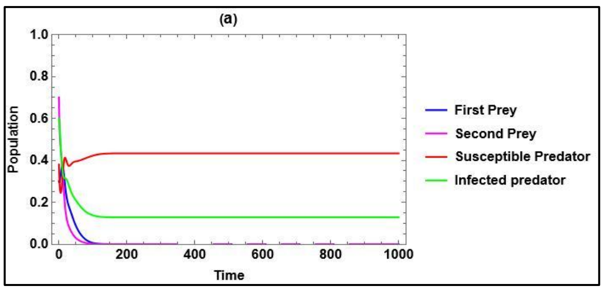

Figure -10 (a)-Time series of solution of system (1) for the data given in (143) approaches to $P_7 = (0, 0,\ 0.433,\ 0)$ when $a_1 = 2.25,\ a_2 = 0.2,$ and $\rho = 0.5$.

7. The Conclusions and Discussions

In our present work, we proposed and analyzed an eco-epidemiological model consisting of four species two prey's, one predator with disease, the first and second prey are growth logistic with fear of the susceptible predator, while the susceptible predator's population growth logistic with additional food and hunting cooperation and the infected predator involving treatment. The objectives of our work are to study:

- The impact of fear in the first and second prey populations on the survival of prey's and predator.
- The contribution of additional food and hunting cooperation to the survival of susceptible predator.
- The benefit of treatment for the infected predator and its impact on the rate of recovery from the disease.

It is observed that system (1) has fifteen equilibria, three are unstable (P_0, P_1, P_2), while the other are locally and globally asymptotically stable under suitable conditions. Finally, we verified our analytic result numerically for the data given in (143) and which are summarized.

1. The parameters a_1, a_2, f_1, f_2, K, ρ , B, h, γ, and δ have an important effect in controlling the stability of system (1).
2. The parameters a_3, C, b_1, b_2, B_1, C_1, C_2, α, η, A, σ, and d the solutions still approach to the positive equilibrium point.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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