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STABILITY AND BIFURCATION OF PREY-PREDATOR MODEL WITH DISEASE IN PREY INCORPORATING ANTI-PREDATOR AND HUNTING COOPERATION

AMEER M. SAHI, HUDA ABDUL SATAR*

Department of Mathematics, College of Science, University of Baghdad, Baghdad 10071, Iraq

Abstract: This study's main goal is to determine how the prey population's cooperation and anti-predator behavior affect the prey-predator model's dynamic with disease in prey with the fear interaction. For this purpose, a mathematical model with Holling type II functional response was proposed and analyzed. The existence conditions of the equilibrium points with their stability analysis were analyzed to determine the qualitative behavior of the model. Numerical simulations are performed to support our analytical results. It is obtained that fear has a stabilizing role, whereas hunting cooperation has a destabilizing role in the system dynamics. Periodic oscillations were observed in the model through bifurcation.

Keywords: eco-epidemiological; fear; anti-predator; hunting cooperation; bi-stability.

2020 AMS Subject Classification: 92D40, 92D30, 34D20, 37G10.

1. INTRODUCTION

Eco-epidemiology is a prominent branch of biology and applied sciences research. It looks at the dynamics of an ecological system under the influence of epidemiological elements, i.e. the dynamics of a predator-prey system in the presence of an infectious disease. It consists of two primary fields: ecology and epidemiology, which are currently interesting to scientists. When

*Corresponding author
E-mail address: huda.oun@sc.uobaghdad.edu.iq
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disease is introduced into prey or predator populations, a purely ecological system becomes eco-epidemiological. The authors in [1] were the first to use a predator-prey system with disease in the prey population to show how the disease might destabilize the system. The term eco-epidemiology was first coined by [2], which combined the concepts of ecology and epidemiology. After that, an eco-epidemiological system including disease in prey population that applied to the Salton Sea's fish-pelican system was proposed and studied by [3]. The authors in [4] investigated the role of infectious disease in the dynamics of a predator-prey system for different forms of functional responses. Several researchers proposed and studied eco-epidemiological models involving numerous biological factors, see [5-8] when there is a disease in prey, [9-12] regarding the disease in predator, while [13-15] in case of the existence of the disease in both the species. Furthermore, it has been observed that the spread of diseases among a population is the main reason for the species extinction see [16-18] and the references therein.

Ecosystems are also remarkably affected by fear. Numerous researchers have examined the effects of panic on prey species; see [19–21] and the citations therein. In fact, fear can have either immediate or secondary effects; in the former case, the predator kills the prey directly [22]. In the secondary example, on the other hand, the presence of a predator significantly changes the behavior of the prey because they fear being preyed upon [23–25]. In order to understand how prey fear impacts the dynamics of the prey-predator system, a great deal of research has been done [26–28].

Groups of species come together for mutual benefit through cooperation, a basic mechanism that is important to ecology. It has been noted that cooperative hunting, in which predators chase their prey together, is a common practice in nature. A few benefits of cooperative hunting include a shorter chase distance [29], a larger chance of trapping several animals [30], and an enhanced success rate. Furthermore, cooperative conduct makes prey fearful, which facilitates hunting even more. The authors of Alves et al. [31] showed that a stable coexistence equilibrium emerges when hunting cooperation is present in a system. Hunting cooperation in prey-predator interactions has been the subject of numerous studies [32–34]. A mathematical eco-epidemiological model incorporating anti-predator traits and a prey-predator model involving fear induced by hunting cooperation level was recently constructed and studied [35]. Additionally, Alwan and Satar [36] built and analyzed an eco-epidemiological model that included a diseased predator with cooperative hunting and anti-predator properties. The dynamics and ramifications of cooperative hunting in ecological systems are better understood thanks to these investigations.

Harvesting species is an interesting aspect of prey-predator systems from an economical and ecological standpoint. It is very important for resource conservation and societal management. A system can be exploited when there is a lot of prey, predators, or both. Harvesting procedures and a harvesting policy must be put in place for the efficient management of ecological resources to stop such exploitation. To enable systematic resource management, harvesting regulations are essential for regulating the type, time, and intensity of harvesting. Researchers have devoted their time to this field's study during the last few decades. The idea of optimal harvesting in fishing systems was first proposed by Clark [37]. Additional research has been conducted on several predator-prey models that incorporate nonlinear harvesting [38–39]. They looked at numerous kinds of bifurcations in the system and tested the stability of alternative equilibria. These kinds of studies advance our knowledge of the best harvesting practices and how they affect the stability and sustainability of ecological systems.

Anti-predator behavior in prey populations is a distinctive feature of predator-prey interactions and it has been documented by several ecologists over the past decade. This behavior is a type of prey survival strategy that has evolved in prey populations to prevent predation. Many behavioral ecologists have studied different kinds of anti-predator behaviors in various prey species [40-42]. In the presence of a predator, prey species exhibit inducible defense, which is defined as protective behaviors learned from past attacks. Inducible defense triggers chemicals in different body sections of prey to develop new structures or intelligently combat predators.

The actions of every species in a biological system where predators and prey coexist can have an impact on both populations. For instance, prey becomes more difficult to hunt for predators when they are aware of their presence and react fearfully. This fact causes prey to avoid direct predation, which may boost the prey's short-term survival but have the unintended consequence of reducing the prey population over time [43]. In the mathematical modeling approach, many authors examined the roles of hunting cooperation and fear influence in the predator-prey system [33-35] and the references therein.

In this work, we examine an eco-epidemiological system wherein the prey population becomes infected with a disease. In a predator-prey model, this study intends to concurrently examine the effects of hunting cooperation, anti-predator, harvesting, and terror effects. The phenomenon of predator hunting in groups and predator harvesting will be included in our model. We will use numerical simulations to examine the system dynamics, enabling us to evaluate the effects of particular parameters like predator harvesting, prey growth, hunting cooperation, and fear effects. Equilibrium point existence and local stability requirements are derived. For a single set of primary points and a variety of parameter values, the global dynamics are examined both analytically and numerically. There is a brief conclusion at the end of the work.

2. Mathematical Model

In this section, an eco-epidemiological system consisting of prey-predator including anti-predator and hunting cooperation with disease in prey is formulated mathematically for study. It's assumed that there is a harvesting on the prey and predator. So to formulate the dynamics of such a real-life system the following hypotheses are adopted

$$\begin{aligned}
\frac{ds}{dt} &= \frac{rs}{1+\alpha y}\left(1-\frac{s+I}{k}\right) - \delta sI - \frac{(c+my)sy}{1+h(c+my)(s+\beta I)} - q_1 s, \\
\frac{dI}{dt} &= \delta sI - \frac{(c+my)Iy}{1+h(c+my)(s+\beta I)} - d_1 I, \\
\frac{dy}{dt} &= (b_1 s + b_2 I)\frac{(c+my)y}{1+h(c+my)(s+\beta I)} - d_2 y - \mu sy - q_2 y ,
\end{aligned} \tag{1}$$

where $s(0) = s_0 \geq 0, I(0) = I_0 \geq 0$, and $y(0) = y_0 \geq 0$, with $s(t), I(t)$, and $y(t)$ represent the densities at time t for the susceptible prey, Infected prey, and predator, respectively. The biological meaning of all the non-negative parameters in the model (1) in Table (1).

Table 1. Biological explanations of the parameters associated with the model (1).

Parameters	Biological meaning
r	The prey's intrinsic growth rate
k	The carrying capacity of the environment
α	The prey's fear rate
δ	The disease transmission rate
c	the consumption rate by the predator
m	The level of cooperation in predator hunting
β	The preference of infected prey coefficient.
h	The predators handing time of a prey
d_1, d_2	The death rates of the infected prey populations and predator populations.
q_1, q_2	The harvesting constant of susceptible prey and predators respectively.
b_1, b_2	The conversion efficiency from susceptible prey biomass and Infected prey biomass to predator biomass
μ	The rate of anti-predator

Now, rewritten system (1) with (s, I, y), the following system can be obtained

$$\begin{aligned}
\frac{ds}{dt} &= s\left(\frac{r}{1+\alpha y}\left(1-\frac{s+I}{k}\right)-\delta I-\frac{(c+my)y}{1+h(c+my)(s+\beta I)}-q_1\right)=sf_1,\\
\frac{dI}{dt} &= I\left(\delta s-\frac{(c+my)y}{1+h(c+my)(s+\beta I)}-d_1\right)=If_2\ ,\\
\frac{dy}{dt} &= y\left((b_1 s+b_2 I)\frac{(c+my)}{1+h(c+my)(s+\beta I)}-d_2-\mu s-q_2\right)=yf_3\ .
\end{aligned} \tag{2}$$

According to biology, every population in the model ought to have boundaries. A boundedness analysis is important for this. The following theorem on the boundedness of the model (1) solutions was deduced by us. It is simple to verify that the prerequisite for every species in the system (2) to survive is provided by

$$q_1 < r \tag{3}$$

Theorem 1. In system (2) all solutions with ICs are bounded for all time $t > 0$.

Proof. The first equation of the system (2) gives $\frac{ds}{dt}+\frac{r}{K}s^2 \leq (r-q_1)s$ then for $t \to \infty$ we have $s \leq \frac{k(r-q_1)}{r}$. Now let $\mathrm{E}_1 = s + I + y$, then applying some calculations gives that

$\frac{dE_1}{dt} \leq s(r - q_1 + 1) - d_1 I - (d_2 + q_2)y \leq L - \mu E_1.$

Here $\mu = min\{r - q_1 + 1, d_1, d_2 + q_2\}$ and $L = \frac{k(r-q_1)(r-q_1+1)}{r}$.

Then according to the above differential inequality, it's easy to verify that for $t \to \infty$, we have $E_1 \leq \frac{L}{\mu} = \mu_1 as\ t \to \infty$. Thus, every solution of system (2) is bounded in the region

$$\Lambda = \{(s, I, y) \in \mathbb{R}_+^3 : s(t) + I(t) + y(t) \leq \mu_1\}.$$

3. Local Stability Analysis of Equilibrium Points

In this part, the stability analysis of all possible equilibrium points (EPs) is investigated. System (2) has five nonnegative equilibrium point

- The origin equilibrium point (EEP), $\dot{p}_0 = (0,0,0)$ always exists.
- The predator-disease-free point ($PDFEP$), $\hat{p}_1 = (\hat{s}, 0,0)$, where $\hat{s} = \frac{k(r-q_1)}{r}$ which is positive under the condition (1).
- The predator-free point ($PFEP$), $\bar{p}_2 = (\bar{s}, \bar{I}, 0)$ where $\bar{s} = \frac{d_1}{\delta}$, and $\bar{I} = \frac{\delta K(r-q_1)-rd_1}{\delta(r+K\delta)}$. Hence the existence condition can be written as follows

  $rd_1 < \delta K(r - q_1).$
- The disease-free point ($DFEP$) ,$\tilde{p}_3 = (\tilde{s}, 0, \tilde{y})$, where

  $$\tilde{y} = \frac{-ch\mu\tilde{s}^2-(\mu-cb_1+ch(d_2+q_2))\tilde{s}-(d_2+q_2)}{m\tilde{s}(h\mu\tilde{s}-b_1+h(d_2+q_2))} \quad (4)$$

  While $\tilde{s}$ is a positive root for the polynomial

  $$L_0 s^5 + L_1 s^4 + L_2 s^3 + L_3 s^2 + L_4 s + L_5 = 0 \quad (5)$$

  where the coefficients of the equation (5) are given in Appendix A.
- The coexistence point $COEP$, $\check{p}_4 = (\check{s}, \check{I}, \check{y})$, where

  $$\check{I} = \frac{c\check{y}+m\check{y}^2-\delta\check{s}-ch\check{s}^2\delta-hm\check{s}^2\check{y}\delta+d_1+ch\check{s}d_1+hm\check{s}\check{y}d_1}{h(c+m\check{y})\beta(\delta\check{s}-d_1)}, \quad (6)$$

  while $(\check{s}, \check{y})$ represents the positive intersection point of the isoclines

$$\begin{aligned}H_1(s,y) = &-cry - mry^2 - ck\delta y - km\delta y^2 - ck\alpha\delta y^2 - km\alpha\delta y^3 \\ &+(chr\delta + hmr\delta y - chr\beta\delta - hmr\beta\delta y + chK\delta^2 + hkm\delta^2 y \\ &+chk\alpha\delta^2 y + hkm\alpha\delta^2 y^2 - chk\beta\delta^2 - hkm\beta\delta^2 y - chk\alpha\beta\delta^2 y \\ &-hkm\alpha\beta\delta^2 y^2)s^2 - rd_1 - chkr\beta d_1 - hkmr\beta d_1 y - k\delta d_1 \\ &-k\alpha\delta d_1 y - chk\beta d_1^2 - hkm\beta d_1^2 y - chk\alpha\beta d_1^2 y - hkm\alpha\beta d_1^2 y^2 \\ &+chk\beta d_1 q_1 + hkm\beta d_1 q_1 + chky\alpha\beta d_1 q_1 + hkm\alpha\beta d_1 q_1 y^2 + (r\delta \\ &+chkr\beta\delta + hkmrz\beta\delta + k\delta^2 + k\alpha\delta^2 y - chrd_1 - hmrd_1 y + chr\beta d_1 \\ &+hmr\beta d_1 y - chk\delta d_1 - hkm\delta d_1 y - chk\alpha\delta d_1 y - hkm\alpha\delta d_1 y^2 \\ &+2chk\beta\delta d_1 + 2hkm\beta\delta d_1 y + 2chk\alpha\beta\delta d_1 y + 2hkm\alpha\beta\delta d_1 y^2 \\ &-chk\beta\delta q_1 - hkm\beta\delta q_1 y - chk\alpha\beta\delta q_1 y - hkm\alpha\beta\delta q_1 y^2)s = 0,\end{aligned} \tag{7}$$

$$\begin{aligned}H_2(s,y) = &(ch\beta\delta b_1 + hm\beta\delta b_1 y - ch\delta b_2 - hm\delta b_2 y)s^2 + x(-ch\beta\mu y \\ &-hm\beta\mu y^2 - \delta b_2 - ch\beta b_1 d_1 - hm\beta b_1 d_1 y + chb_2 d_1 \\ &+hmb_2 d_1 y) + cb_2 y + mb_2 y^2 + b_2 d_1 \\ &-ch\beta d_2 y - hm\beta d_2 y^2 - ch\beta q_2 y - hm\beta q_2 y^2 = 0\end{aligned} \tag{8}$$

As $y \to 0$, it is obtained that

$$\left.\begin{aligned}H_1(s,0) = &(chr\delta + chk\delta^2)(1-\beta)s^2 + (r\delta + chkr\beta\delta + k\delta^2 \\ &-chrd_1 + chr\beta d_1 - chk\delta d_1 + 2chk\beta\delta d_1 - chk\beta\delta q_1)s \\ &-rd_1 - (r - q_1)chk\beta d_1 - k\delta d_1 - chk\beta d_1^2 = 0 \\ H_2(s,0) = &(ch\beta\delta b_1 - ch\delta b_2)s^2 + (-\delta b_2 - ch\beta b_1 d_1 + chb_2 d_1)s + b_2 d_1 = 0\end{aligned}\right\} \tag{9}$$

Therefore, $H_1(s,0) = 0$ crosses the $s-$axis at the point $s_1 > 0$, however, $H_2(s,0) = 0$ crosses the $s-$axis at the point $s_2 > 0$ under specific conditions. Accordingly, there is a unique positive intersection point of the two isoclines (7) and (8), and then $\check{p}_4$ exists uniquely in the interior of $\mathbb{R}^3_+$ if the next conditions hold.

$$\left.\begin{aligned}ch\beta\delta b_1 &< ch\delta b_2 \\ s_1 &< s_2 \\ \frac{dI}{ds} &= -\frac{\partial H_1/\partial s}{\partial H_1/\partial I} > 0 \\ \frac{dI}{ds} &= -\frac{\partial H_2/\partial s}{\partial H_2/\partial I} < 0\end{aligned}\right\} \tag{10}$$

The local dynamical behaviors are carried out by calculating

$$J\,(s,I,y) = \left[J^*_{ij}\right]_{3\times 3} \tag{11}$$

and then computing the eigenvalues, which specify the stability type of each point, where

$$J^*_{11} = \frac{-rs}{k(1+\alpha y)} + \frac{hsy(c+my)^2}{\left(1+h(c+my)(s+\beta I)\right)^2} + \frac{r}{1+\alpha y}\left(1 - \frac{s+I}{k}\right) - \delta I - \frac{(c+my)y}{1+h(c+my)(s+\beta I)} - q_1,$$

$$J_{12}^{*} = -\frac{rs}{k(1+\alpha y)} - \delta s + \frac{h\beta sy(c+my)^2}{\left(1+h(c+my)(s+\beta I)\right)^2},$$

$$J_{13}^{*} = \frac{-r\alpha s}{(1+\alpha y)^2} - \frac{(c+2my)s}{1+h(c+my)(s+\beta I)} + \frac{hmsy(c+my)(s+\beta I)}{\left(1+h(c+my)(s+\beta I)\right)^2} + \frac{(s+I)s\alpha r}{k(1+\alpha y)^2},$$

$$J_{21}^{*} = \delta I + \frac{hIy(c+my)^2}{\left(1+h(c+my)(s+\beta I)\right)^2},$$

$$J_{22}^{*} = \frac{h\beta Iy(c+my)^2}{\left(1+h(c+my)(s+\beta I)\right)^2} + \delta s - \frac{(c+my)y}{1+h(c+my)(s+\beta I)} - d_1,$$

$$J_{23}^{*} = -\frac{(c+2my)I}{1+h(c+my)(s+\beta I)} + \frac{hmIy(c+my)(s+\beta I)}{\left(1+h(c+my)(s+\beta I)\right)^2},$$

$$J_{31}^{*} = \frac{b_1 y(c+my)}{1+h(c+my)(s+\beta I)} - \frac{hy(c+my)^2(b_1 s+b_2 I)}{\left(1+h(c+my)(s+\beta I)\right)^2} - \mu y,$$

$$J_{32}^{*} = \frac{b_2(c+my)y}{1+h(c+my)(s+\beta I)} - \frac{h\beta y(c+my)^2(b_1 s+b_2 I)}{\left(1+h(c+my)(s+\beta I)\right)^2},$$

$$J_{33}^{*} = \frac{my(b_1 s+b_2 I)}{1+h(c+my)(s+\beta I)} - \frac{hmy(c+my)(s+\beta I)(b_1 s+b_2 I)}{\left(1+h(c+my)(s+\beta I)\right)^2} + (b_1 s + b_2 I)\frac{(c+my)}{1+h(c+my)(s+\beta I)} - d_2 - \mu s - q_2 .$$

The JM at $\dot{p}_0 = (0,0,0)$ is given by

$$J(\dot{p}_0) = \begin{pmatrix} r - q_1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -(d_2 + q_2) \end{pmatrix} \tag{12}$$

So, the eigenvalues of $(\dot{p}_0)$ are $\lambda_{01} = r - q_1, \lambda_{02} = -d_1$ and $\lambda_{03} = -(d_2 + q_2)$, $\dot{p}_0$ is locally asymptotically stable (LAS) if $r < q_1$. Therefore, if the birth rate of prey is less than its death rate, both prey and predator populations will extinct.

The JM examined at $\hat{p}_1 = (\hat{s}, 0,0)$ can be obtained by

$$J(\hat{p}_1) = \begin{pmatrix} \frac{-r\hat{s}}{k} & -\frac{r\hat{s}}{k} - \delta\hat{s} & -r\alpha\hat{s} - \frac{c\hat{s}}{1+hc\hat{s}} + \frac{\hat{s}^2\alpha r}{k} \\ 0 & \delta\hat{s} - d_1 & 0 \\ 0 & 0 & \frac{b_1 c\hat{s}}{1+hc\hat{s}} - d_2 - \mu\hat{s} - q_2 \end{pmatrix} \tag{13a}$$

Thus, $(\hat{p}_1)$ has the following eigenvalues $\lambda_{11} = \frac{-r\hat{s}}{k}$, $\lambda_{12} = \delta\hat{s} - d_1$ and $\lambda_{13} = \frac{b_1 c\hat{s}}{1+hc\hat{s}} - d_2 - \mu s - q_2$ then $\hat{p}_1$ is LAS if

$$\delta\hat{s} < d_1, \tag{13b}$$

$$\frac{b_1 c\hat{s}}{1+hc\hat{s}} < d_2 + \mu\hat{s} + q_2, \tag{13c}$$

The JM examined at $\bar{p}_2 = (\bar{s}, \bar{I}, 0)$ can be obtained by

$$J(\bar{p}_2) = \begin{pmatrix} \frac{-r\bar{s}}{k} & -\frac{r\bar{s}}{k} - \delta\bar{s} & -r\alpha\bar{s} - \frac{c\bar{s}}{1+hc(\bar{s}+\beta\bar{I})} + \frac{(\bar{s}+\bar{I})\bar{s}\alpha r}{k} \\ \delta\bar{I} & 0 & \frac{-c\bar{I}}{1+hc(\bar{s}+\beta\bar{I})} \\ 0 & 0 & \frac{c(b_1\bar{s}+b_2\bar{I})}{1+hc(\bar{s}+\beta\bar{I})} - d_2 - \mu\bar{s} - q_2 \end{pmatrix} \tag{14a}$$

One of the eigenvalues is $\lambda_{23} = \frac{c(b_1\bar{s}+b_2\bar{I})}{1+hc(\bar{s}+\beta\bar{I})} - d_2 - \mu\bar{s} - q_2$ and other two eigenvalues are given by

$$\lambda_{21} = \frac{\bar{T}}{2} + \frac{1}{2}\sqrt{\bar{T}^2 - 4\bar{D}};\ \lambda_{22} = \frac{\bar{T}}{2} - \frac{1}{2}\sqrt{\bar{T}^2 - 4\bar{D}}, \tag{14b}$$

where $\bar{T} = \frac{-r\bar{s}}{k}$, and $\bar{D} = \frac{r\delta\,\bar{s}^2\bar{I}}{k} + \delta^2\bar{s}\bar{I},$ So, the $PFEP$ is a LAS if condition met

$$\frac{c(b_1\bar{s}+b_2\bar{I})}{1+hc(\bar{s}+\beta\bar{I})} < d_2 + \mu\hat{s} + q_2. \tag{14c}$$

The JM examined at $\tilde{p}_3 = (\tilde{s}, 0, \tilde{y})$ can be obtained by

$$J_3(\tilde{p}_3) = (\tilde{a}_{ij}), \tag{15a}$$

where

$$\tilde{a}_{11} = \tilde{s}\left(-\frac{r}{k(1+\alpha\tilde{y})} + \frac{h\tilde{y}(c+m\tilde{y})^2}{(1+h\tilde{s}(c+m\tilde{y}))^2}\right).$$

$$\tilde{a}_{12} = \tilde{s}\left(-\delta - \frac{r}{k(1+\alpha\tilde{y})} + \frac{h\beta\tilde{y}(c+m\tilde{y})^2}{(1+h\tilde{s}(c+m\tilde{y}))^2}\right).$$

$$\tilde{a}_{13} = \tilde{s}\left(-\frac{r\alpha}{(1+\alpha\tilde{y})^2}\left(1 - \frac{\tilde{s}}{K}\right) - \frac{(c+2m\tilde{y})}{1+h(c+m\tilde{y})\tilde{s}} + \frac{hm\tilde{s}\tilde{y}(c+m\tilde{y})}{(1+h(c+m\tilde{y})\tilde{s})^2}\right).$$

$$\tilde{a}_{21} = 0.$$

$$\tilde{a}_{22} = \delta\tilde{y} - \frac{(c+m\tilde{y})\tilde{y}}{1+h\tilde{s}(c+m\tilde{y})} - d_1.$$

$$\tilde{a}_{23} = 0.$$

$$\tilde{a}_{31} = \tilde{y}\left(-\mu + \frac{(c+m\tilde{y})b_1}{(1+h\tilde{s}(c+m\tilde{y}))^2}\right).$$

$$\tilde{a}_{32} = \tilde{y}\left(-\frac{h\beta\tilde{s}(c+m\tilde{y})^2 b_1}{(1+h\tilde{s}(c+m\tilde{y}))^2} + \frac{(c+m\tilde{y})b_2}{1+h\tilde{s}(c+m\tilde{y})}\right).$$

$$\tilde{a}_{33} = \frac{mb_1\tilde{s}\tilde{y}}{(1+h\tilde{s}(c+m\tilde{y}))^2}.$$

Evidently, one of the eigenvalues is $\lambda_{32} = \delta\tilde{y} - \frac{(c+m\tilde{y})\tilde{y}}{1+h\tilde{s}(c+m\tilde{y})} - d_1$, and the other two eigenvalues are given by

$$\lambda_{31} = \frac{\tilde{T}}{2} + \frac{1}{2}\sqrt{\tilde{T}^2 - 4\tilde{D}};\ \lambda_{33} = \frac{\tilde{T}}{2} - \frac{1}{2}\sqrt{\tilde{T}^2 - 4\tilde{D}}, \tag{15b}$$

where $\tilde{T} = \tilde{a}_{11} + \tilde{a}_{33}$, and $\widetilde{D} = \tilde{a}_{11}\tilde{a}_{33} - \tilde{a}_{13}\tilde{a}_{31}$. So, the $DFEP$ is a LAS if the following conditions are met

$$\frac{h\tilde{s}\tilde{y}(c+m\tilde{y})^2}{\left(1+h\tilde{s}(c+m\tilde{y})\right)^2} + \frac{mb_1\tilde{s}\tilde{y}}{\left(1+h\tilde{s}(c+m\tilde{y})\right)^2} < \frac{r\tilde{s}}{k(1+\alpha\tilde{y})}. \tag{15c}$$

$$\mu < \frac{(c+m\tilde{y})b_1}{(1+h\tilde{s}(c+m\tilde{y}))^2}. \tag{15d}$$

$$\tilde{a}_{11}\tilde{a}_{33} - \tilde{a}_{13}\tilde{a}_{31} > 0. \tag{15e}$$

$$\delta\tilde{y} < \frac{(c+m\tilde{y})\tilde{y}}{1+h\tilde{s}(c+m\tilde{y})} + d_1. \tag{15f}$$

Finally, the LAS at $\check{p}_4 = (\check{s}, \check{I}, \check{y})$ can be studied in the following theorem.

Theorem 2. The $COEP$ of the system (2) is LAS if

$$\breve{a}_{11} + \breve{a}_{22} < 0. \tag{16a}$$

$$\breve{a}_{11} + \breve{a}_{33} < 0. \tag{16b}$$

$$\breve{a}_{31} > 0. \tag{16c}$$

$$\breve{a}_{32} > 0. \tag{16d}$$

$$\breve{a}_{21}(\breve{a}_{12}\breve{a}_{33} - \breve{a}_{13}\breve{a}_{32}) + \breve{a}_{31}(\breve{a}_{22}\breve{a}_{13} - \breve{a}_{12}\breve{a}_{23}) > 0. \tag{10e}$$

$$\breve{a}_{11}\breve{a}_{22} - \breve{a}_{12}\breve{a}_{21} > 0. \tag{16f}$$

$$\breve{a}_{11}\breve{a}_{33} - \breve{a}_{13}\breve{a}_{31} > 0. \tag{16g}$$

$$\breve{a}_{12}\breve{a}_{23}\breve{a}_{31} + \breve{a}_{13}\breve{a}_{21}\breve{a}_{32} > 0. \tag{16h}$$

$where$ $\breve{a}_{ij}$ are the JM coefficients that are given in the proof.

Proof. The JM examined at $\check{p}_4$ can be written as

$$\breve{a}_{11} = \check{s}\left(-\frac{r}{k(1+\alpha\check{y})} + \frac{h\check{y}(c+m\check{y})^2}{(1+h(c+m\check{y})(\check{s}+\beta\check{I}))^2}\right).$$

$$\breve{a}_{12} = \check{s}\left(-\frac{r}{k(1+\alpha\check{y})} + \frac{h\beta\check{y}(c+m\check{y})^2}{(1+h(c+m\check{y})(\check{s}+\beta\check{I}))^2} - \delta\right).$$

$$\breve{a}_{13} = \check{s}\left(-\frac{r\alpha}{(1+\alpha\check{y})^2}\left(1 - \frac{(\check{s}+\check{I})}{k}\right) + \frac{hm\check{y}(c+m\check{y})(\check{s}+\beta\check{I})}{(1+h(c+m\check{y})(\check{s}+\beta\check{I}))^2} - \frac{(c+2m\check{y})}{1+h(c+m\check{y})(\check{s}+\beta\check{I})}\right).$$

$\breve{a}_{21} = \breve{I}\left(\frac{h\breve{y}(c+m\breve{y})^2}{(1+h(c+m\breve{y})(\breve{s}+\beta\breve{I}))^2} + \delta\right)$.

$\breve{a}_{22} = \frac{h\beta\breve{I}\breve{y}(c+m\breve{y})^2}{(1+h(c+m\breve{y})(\breve{s}+\beta\breve{I}))^2}$.

$\breve{a}_{23} = \breve{I}\left(\frac{hm\breve{y}(c+m\breve{y})(\breve{s}+\beta\breve{I})}{(1+h(c+m\breve{y})(\breve{s}+\beta\breve{I}))^2} - \frac{mz}{1+h(c+m\breve{z})(\breve{x}+\breve{y}\beta)} - \frac{(c+2m\breve{y})}{1+h(c+m\breve{y})(\breve{s}+\beta\breve{I})}\right)$.

$\breve{a}_{31} = \breve{y}\left(-\mu + \frac{(c+m\breve{y})b_1}{1+h(c+m\breve{y})(\hat{s}+\beta\hat{I})} - \frac{h(c+m\breve{y})^2(b_1\breve{s}+b_2\breve{I})}{(1+h(c+m\breve{y})(\hat{s}+\beta\hat{I}))^2}\right)$.

$\breve{a}_{32} = \breve{y}\left(\frac{b_2(c+m\breve{y})}{1+h(c+m\breve{y})(\breve{s}+\beta\breve{I})} - \frac{h\beta(c+m\breve{y})^2(b_1\breve{s}+b_2\breve{I})}{(1+h(c+m\breve{y})(\breve{s}+\beta\breve{I}))^2}\right)$.

$\breve{a}_{33} = \frac{m(b_1\breve{s}+b_2\breve{I})\breve{y}}{\left(1+h(c+m\breve{y})(\breve{s}+\beta\breve{I})\right)^2}$.

As a result, the characteristic equation of $J(\breve{p}_4)$

$$\lambda_3^3 + \theta_1\lambda_3^2 + \theta_2\lambda_3 + \theta_3 = 0 \tag{16i}$$

where,

$\theta_1 = -(\breve{a}_{11} + \breve{a}_{22} + \breve{a}_{33})$.

$\theta_2 = (\breve{a}_{11}\breve{a}_{22} - \breve{a}_{12}\breve{a}_{21}) + (\breve{a}_{11}\breve{a}_{33} - \breve{a}_{13}\breve{a}_{31}) + (\breve{a}_{22}\breve{a}_{33} - \breve{a}_{23}\breve{a}_{32})$.

$\theta_3 = -[\breve{a}_{11}(\breve{a}_{22}\breve{a}_{33} - \breve{a}_{23}\breve{a}_{32}) - \breve{a}_{21}(\breve{a}_{12}\breve{a}_{33} - \breve{a}_{13}\breve{a}_{32}) - \breve{a}_{31}(\breve{a}_{22}\breve{a}_{13} - \breve{a}_{12}\breve{a}_{23})]$

with

$$\Delta = \theta_1\theta_2 - \theta_3 = -(\breve{a}_{11} + \breve{a}_{22})(\breve{a}_{11}\breve{a}_{22} - \breve{a}_{12}\breve{a}_{21}) - (\breve{a}_{11} + \breve{a}_{33})(\breve{a}_{11}\breve{a}_{33} - \breve{a}_{13}\breve{a}_{31}) - (\breve{a}_{22} + \breve{a}_{33})(\breve{a}_{22}\breve{a}_{33} - \breve{a}_{23}\breve{a}_{32}) + \breve{a}_{12}\breve{a}_{23}\breve{a}_{31} + \breve{a}_{13}\breve{a}_{21}\breve{a}_{32}$$

According to the Routh–Hurwitz criterion, the sign of the real part of the roots of equations (16i) will be negative when θ_1, θ_3, and $\Delta = \theta_1\theta_2 - \theta_3$ have positive signs. Moreover, direct computation shows that due to the given conditions Routh–Hurwitz conditions are satisfied and hence the proof is done.

4. PERSISTENCE

The system's (2) persistence is examined in this part. Therefore, if every species is shown for every positive moment, system (2) survives. We shall outline certain conditions for the system's consistent persistence in the sections that follow (2).

The subsystems in the positive quadrant of the sI-plane and the sy-plane of the system (2) can be written respectively as follows

$$\begin{aligned} \frac{ds}{dt} &= s\left[r - \frac{r(s+I)}{k} - \delta I - q_1\right] = \Upsilon_1(s,I), \\ \frac{dI}{dt} &= I[\delta s - d_1] = \Upsilon_2(s,I), \end{aligned} \tag{17}$$

and

$$\begin{aligned} \frac{ds}{dt} &= s\left[\frac{r}{1+\alpha y}\left(1 - \frac{s}{k}\right) - \frac{(c+my)y}{1+h(c+my)s} - q_1\right] = \Upsilon_3(s,y), \\ \frac{dy}{dt} &= y\left[\frac{b_1 s(c+my)}{1+h(c+my)s} - d_2 - \mu s - q_2\right] = \Upsilon_4(s,y). \end{aligned} \tag{18}$$

The subsystems (17) and (18) have a positive equilibrium point coinciding with $\bar{p}_2 = (\bar{s}, \bar{I}, 0)$ and $\tilde{p}_3 = (\tilde{s}, 0, \tilde{y})$ of the system (2) in the interior of the positive quadrant of the sI-plane and the sy -plane. Dulac's function $\Gamma_1(s,I) = \frac{1}{sI}$, and $\Gamma_2(s,y) = \frac{1}{sy}$, which are continuously differentiable functions in the $\mathbb{R}_+^2$. Moreover, it is obtained

$$\Delta_1(s,I) = \frac{\partial}{\partial s}(\Gamma_1.\Upsilon_1) + \frac{\partial}{\partial I}(\Gamma_1.\Upsilon_2) = -\frac{r}{kI}.$$

$$\Delta_2(s,y) = \frac{\partial}{\partial s}(\Gamma_2.\Upsilon_3) + \frac{\partial}{\partial y}(\Gamma_2.\Upsilon_4) = -\frac{r}{ky(1+\alpha y)} + \frac{h(c+my)^2 + b_1 m}{[1+h(c+my)s]^2}.$$

Clearly, $\Delta_1(s,I) < 0$ for any point and $\Delta_2(s,I) < 0$ under the condition $\frac{h(c+my)^2+1}{[1+h(c+my)s]^2} < \frac{r}{ky}$. Consequently, there are no periodic dynamics in the interior of the positive quadrants of the sI −plane and sy −plane.

Theorem3. Assume that there are no periodic dynamics in the boundary planes, then the system (2) is uniformly persistent provided that

$$r > q_1. \tag{19a}$$

$$\left.\begin{array}{c} \delta\hat{s} > d_1 \\ or \\ \frac{b_1 c\hat{s}}{1+hc\hat{s}} > d_2 + \mu\hat{s} + q_2 \end{array}\right\}. \tag{19b}$$

$$\frac{(b_1\bar{s} + b_2\bar{I})c}{1+hc(\bar{s}+\beta\bar{I})} > d_2 + \mu\bar{s} + q_2, \tag{19c}$$

$$\delta\tilde{s} > \frac{(c+m\tilde{y})\tilde{y}}{1+h\tilde{s}(c+m\tilde{y})} + d_1. \tag{19d}$$

Proof. Define the function $\sigma(s,I,y) = s^{\rho_1} I^{\rho_2} y^{\rho_3}$, where ρ_1, ρ_2, ρ_3 are positive constants, and $\sigma(s,I,y) > 0$ for all $(s,I,y) \in \mathbb{R}_+^3$ with $\sigma(s,I,y) \to 0$ if either s, I or y goes to zero. Now, let

$$\ell(s,I,y)=\frac{\sigma'(s,I,y)}{\sigma(s,I,y)}=\rho_1 f_1+\rho_2 f_2+\rho_3 f_3$$

where the functions f_1, f_2 and f_3 are given in the system (2).

Now, according to the average Lyapunov method, we must show that the function $\ell(s,I,y)>0$ for all boundary equilibrium points. Then,

$$\begin{aligned}\ell(s,I,y)=\rho_1&\left[\frac{r}{1+\alpha y}\left(1-\frac{s+I}{k}\right)-\delta I-\frac{(c+my)y}{1+h(c+my)(s+\beta I)}-q_1\right]\\ &+\rho_2\left[\delta s-\frac{(c+my)y}{1+h(c+my)(s+\beta I)}-d_1\right]\\ &+\rho_3\left[(b_1 s+b_2 I)\frac{(c+my)}{1+h(c+my)(s+\beta I)}-d_2-\mu s-q_2\right]\end{aligned} \tag{20}$$

Therefore,

$$\ell(\dot{p}_0)=\rho_1[r-q_1]+\rho_2[-d_1]+\rho_3[-d_2-q_2],$$

then we will obtain that $\ell(p_0)>0$ for suitable choice of ρ_1,ρ_2, and ρ_3 with condition (19a).

$$\ell(\hat{p}_1)=\rho_2[\delta\hat{s}-d_1]+\rho_3\left[\frac{b_1 c\hat{s}}{1+hc\hat{s}}-d_2-\mu\hat{s}-q_2\right],$$

which satisfies $\ell(\hat{p}_1)>0$ for suitable choice of ρ_2 and ρ_3 with condition (19b).

$$\ell(\bar{p}_2)=\rho_3\left[\frac{(b_1\bar{s}+b_2\bar{I})c}{1+hc(\bar{s}+\beta\bar{I})}-d_2-\mu\bar{s}-q_2\right].$$

$$\ell(\tilde{p}_3)=\rho_2\left[\delta\tilde{s}-\frac{(c+m\tilde{y})\tilde{y}}{1+h\tilde{s}(c+m\tilde{y})}-d_1\right].$$

Thus, the system (2) is uniformly persistent due to given conditions.

5. Global Stability (GS)

In this section, the Lyapunov method is used to investigate the GS or specify the basin of attraction of each EP for all LSEs as shown in the following theorems

Theorem 4. The EEP, of the system (2) is GAS whenever it is LAS.

Proof. We select an appropriate positive definite function about $\dot{p}_0$ as

$$L_0=\varepsilon_1 s+\varepsilon_2 I+\varepsilon_3 y$$

Then, the derivative $\frac{dL_0}{dt}$ can be determined as

$$\frac{dL_0}{dt}=\left[\frac{rs}{1+\alpha y}\left(1-\frac{s+I}{k}\right)-\delta sI-\frac{(c+my)sy}{1+h(c+my)(s+\beta I)}-q_1 s\right]$$
$$+\left[\delta sI-\frac{(c+my)Iy}{1+h(c+my)(s+\beta I)}-d_1 I\right] .$$
$$+\left[(b_1 s+b_2 I)\frac{(c+my)y}{1+h(c+my)(s+\beta I)}-d_2 y-\mu sy-q_2 y\right]$$

Hence

$$\frac{dL_0}{dt}\leq (r-q_1)s-d_1 I-(d_2+q_2)y.$$

So, The function $\frac{dL_0}{dt}$ is negative definite whenever $\dot{p}_0$ is LAS. Hence, $\dot{p}_0$ is GAS.

Theorem 5. The $PDFEP$ of the system (2) is GAS if the forthcoming inequalities hold.

$$q_2+d_2>\frac{\alpha r}{(1+\alpha y)}\left(1-\frac{\hat{s}}{k}\right)+(c+my)\hat{s} \tag{21a}$$

$$d_1>\delta\hat{s}+\frac{r\hat{s}}{k} \tag{21b}$$

$$k>\hat{s} \tag{21c}$$

Proof. By selecting an appropriate positive definite function about $\hat{p}_1$ as

$L_1=\left[s-\hat{s}-\hat{s}ln\left(\frac{s}{\hat{s}}\right)\right]+I+y$ which is a real-valued function.

Then, the derivative $\frac{dL_1}{dt}$ can be determined as

$$\frac{dL_1}{dt}=(s-\hat{s})\left[\frac{r}{A}\left(1-\frac{s+I}{k}\right)-\delta I-\frac{(c+my)y}{B}-q_1\right]$$
$$+\left[\delta sI-\frac{(c+my)Iy}{B}-d_1 I\right]$$
$$+\left[(b_1 s+b_2 I)\frac{(c+my)y}{B}-d_2 y-\mu sy-q_2 y\right].$$

Hence

$$\frac{dL_1}{dt}\leq -\frac{r}{kA}(s-\hat{s})^2-\frac{\alpha r}{A}\left[1-\frac{\hat{s}}{k}\right]sy-\left[q_2+d_2-(c+my)\hat{s}-\frac{\alpha r}{A}(1-\frac{\hat{s}}{k})\right]y$$
$$-\left[d_1-\delta\hat{s}-\frac{r\hat{s}}{k}\right]I,$$

where $A=1+\alpha y$ and $B=1+h(c+my)(s+\beta I)$. Therefore, if the inequalities in (21a)-(21c) hold, $\frac{dL_1}{dt}$ is negative definite. Hence, $\hat{p}_1$ is GAS.

Theorem 6. The $PFEP$ of the system (2) is GAS if the forthcoming inequalities hold.

$$\bar{s} + \bar{I} < k. \tag{22a}$$

$$\frac{1}{2}\left(\frac{r}{k} + \delta - \delta\bar{I}\right)^2 < \frac{r}{(1+\alpha y_{max})k}. \tag{22b}$$

$$\delta s_{max} + \frac{1}{2} < d_1. \tag{22c}$$

$$r\alpha\bar{s}\left(1 - \frac{\bar{s}+\bar{I}}{k}\right) + (\bar{s} + (\bar{I} + b_2)I_{max} + b_1 s_{max})(c + m y_{max}) < d_2 + q_2. \tag{22d}$$

Proof. By selecting an appropriate positive definite function about $\bar{p}_2$ as

$$L_2 = \left[s - \bar{s} - \bar{s}\ln\left(\frac{s}{\bar{s}}\right)\right] + \frac{(I-\bar{I})^2}{2} + y.$$

Then, the derivative $\frac{dL_2}{dt}$ can be determined as

$$\begin{aligned}\frac{dL_2}{dt} = (s-\bar{s})&\left[-\frac{r\alpha y}{A}\left(1 - \frac{\bar{s}+\bar{I}}{k}\right) - \frac{r}{Ak}(s-\bar{s}) - \left(\frac{r}{Ak} + \delta\right)(I-\bar{I}) - \frac{(c+my)y}{B}\right] \\ &+(I-\bar{I})\left[\delta(s(I-\bar{I}) + (s-\bar{s})\bar{I}) - \frac{(c+my)Iy}{B} - d_1(I-\bar{I})\right] \\ &+\left[(b_1 s + b_2 I)\frac{(c+my)y}{B} - d_2 y - \mu s y - q_2 y\right]\end{aligned}$$

Hence

$$\begin{aligned}\frac{dL_2}{dt} < &-\left[\frac{r}{Ak} - \frac{1}{2}\left(\frac{r}{Ak} + \delta - \delta\bar{I}\right)^2\right](s-\bar{s})^2 - \left[d_1 - \delta s - \frac{1}{2}\right](I-\bar{I})^2 \\ &-\left[d_2 + q_2 - r\alpha\bar{s}\left(1 - \frac{\bar{s}+\bar{I}}{k}\right) - (\bar{s} + (\bar{I}+b_2)I_{max} + b_1 s_{max})(c + m y_{max})\right]y\end{aligned}.$$

Using the inequalities in (22a)-(22d), make $\frac{dL_2}{dt}$ negative definite. Hence, $\bar{p}_2$ is GAS.

Theorem 7. The $DFEP$ of the system (2) is GAS if the forthcoming inequalities hold

$$b_1\tilde{y} < 1. \tag{23a}$$

$$b_2 y_{max} < 1 + \frac{h b_1 \beta (c + m\tilde{y})\tilde{s}\tilde{y}}{\tilde{B}} + b_2\tilde{y}. \tag{23b}$$

$$\frac{r\tilde{s}}{k} + \delta\tilde{s} < d_1. \tag{23c}$$

$$\frac{h\tilde{y}(c+m\tilde{y})(c+m y_{max})}{\tilde{B}} + \frac{(Q_{max})^2}{2} < \frac{r}{k(1+\alpha y_{max})}. \tag{23d}$$

$$\frac{\tilde{s}}{k\tilde{A}} < 1. \tag{23e}$$

$\frac{b_1 s_{max}}{\tilde{B}}[hc\tilde{s}y_{max}(m+c)+m\tilde{y}+(c+my_{max})(1+mh\tilde{s}\tilde{y})]+\frac{1}{2}<d_2+q_2+\mu\tilde{s},$ (23f)

where $Q_{max}=\mu y_{max}+r\alpha\left(1-\frac{\tilde{s}}{k\tilde{A}}\right)+\frac{(c+m\tilde{y})(1+hc\tilde{s}-b_1\tilde{y})+m(1+hc\tilde{s}+hm\tilde{y}\tilde{s})y_{max}}{\tilde{B}}$ while s_{max} and y_{max} are the upper bound for the s and y respectively.

Proof. By selecting an appropriate positive definite function about $\tilde{p}_3$ as

$$L_3=\left[s-\tilde{s}-\tilde{s}ln\left(\frac{s}{\tilde{s}}\right)\right]+I+\frac{(y-\tilde{y})^2}{2}.$$

Then, the derivative $\frac{dL_2}{dt}$ can be determined as

$$\begin{aligned}\frac{dL_3}{dt}=&-\frac{r\alpha(s-\tilde{s})(y-\tilde{y})}{A}-\frac{r\tilde{A}(s-\tilde{s})^2}{kA\tilde{A}}+\frac{r\alpha\tilde{s}(s-\tilde{s})(y-\tilde{y})}{kA\tilde{A}}\\&-\frac{r(s-\tilde{s})I}{kA}+\delta\tilde{s}I-\frac{c(1+hc\tilde{s})(s-\tilde{s})(y-\tilde{y})}{B\tilde{B}}+\frac{ch\tilde{y}(c+my)(s-\tilde{s})^2}{B\tilde{B}}\\&+\frac{c\tilde{y}h(c+my)(s-\tilde{s})\beta I}{B\tilde{B}}-\frac{[m(1+hc\tilde{s})(y+\tilde{y})+hm^2y\tilde{y}\tilde{s}](s-\tilde{s})(y-\tilde{y})}{B\tilde{B}}\\&+\frac{mh(c+my)\tilde{y}^2(s-\tilde{s})^2}{B\tilde{B}}+\frac{mh\tilde{y}^2(c+my)\beta(s-\tilde{s})I}{B\tilde{B}}-\frac{(c+my)Iy}{B}-d_1I\\&+\frac{b_1s(c+c^2h\tilde{s}y+m^2h\tilde{s}\tilde{y}y)(y-\tilde{y})^2}{B\tilde{B}}+\frac{b_1\tilde{y}(c+m\tilde{y})(s-\tilde{s})(y-\tilde{y})}{B\tilde{B}}\\&+\frac{mb_1s(1+hc\tilde{s})(y+\tilde{y})(y-\tilde{y})^2}{B\tilde{B}}-\frac{h(c+my)(cb_1\tilde{s}\tilde{y}+mb_1\tilde{s}\tilde{y}^2)\beta(y-\tilde{y})I}{B\tilde{B}}\\&-(d_2+q_2+\mu\tilde{s})(y-\tilde{y})^2-\mu(s-\tilde{s})(y-\tilde{y})y+\frac{(c+my)yb_2(y-\tilde{y})I}{B}\end{aligned},$$

where $\tilde{A}=1+\alpha\tilde{y}$, and $\tilde{B}=1+h(c+m\tilde{y})\tilde{s}$.

Hence

$$\begin{aligned}\frac{dL_3}{dt}<&-\left[\frac{r}{kA}-\frac{h\tilde{y}(c+m\tilde{y})(c+my)}{B\tilde{B}}-\frac{Q^2}{2}\right](s-\tilde{s})^2\\&-\left[d_2+q_2+\mu\tilde{s}-\frac{b_1s}{B\tilde{B}}\left(hc\tilde{s}y(m+c)+m\tilde{y}+(c+my)(1+mh\tilde{s}\tilde{y})\right).\right.\\&\left.-\frac{1}{2}\right](y-\tilde{y})^2-\left[d_1-\frac{r\tilde{s}}{kA}-\delta\tilde{s}\right]I\end{aligned}$$

Therefore, $\frac{dL_3}{dt}$ is a negative definite function, Thus, $\tilde{p}_3$ is GAS under the given conditions.

Theorem 8. The *COEP* of the system (2) is GAS if the forthcoming inequalities hold.

$$\frac{(N_{12})^2}{2}+\frac{(N_{13})^2}{2}<N_{11}. \tag{24a}$$

$$\frac{(N_{23})^2}{2}+\frac{1}{2}<N_{22}. \tag{24b}$$

$$1<N_{33}. \tag{24c}$$

Where N_{ij} are given in the proof.

Proof. Consider the appropriate positive definite function about $\check{p}_4$ that is given by

$$L_4 = \left[s - \check{s} - \check{s}ln\left(\frac{s}{\check{s}}\right)\right] + \frac{(I-\check{I})^2}{2} + \frac{(y-\check{y})^2}{2}.$$

Then the derivative of L_4 can be determined by

$$\begin{aligned}\frac{dL_4}{dt} &= (s-\check{s})\left[\frac{r}{A}\left(1-\frac{s+I}{k}\right) - \delta I - \frac{(c+my)y}{B} - q_1\right] \\ &+\left(I-\check{I}\right)\left[\delta sI - \frac{(c+my)Iy}{B} - d_1 I\right] \\ &+(y-\check{y})\left[\frac{(b_1 s + b_2 I)(c+my)y}{B} - (d_2+q_2)y - \mu sy\right].\end{aligned}$$

Hence

$$\begin{aligned}\frac{dL_4}{dt} &= -N_{11}(s-\check{s})^2 - N_{22}\left(I-\check{I}\right)^2 - N_{33}(y-\check{y})^2 - N_{12}(s-\check{s})\left(I-\check{I}\right) \\ &-N_{13}(s-\check{s})(y-\check{y}) - N_{23}\left(I-\check{I}\right)(y-\check{y}),\end{aligned}$$

where

$$N_{11} = \frac{r\check{A}}{kA\check{A}} + h\check{y}(c+my)(c+m\check{y}),$$

$$N_{22} = d_1 + \frac{1}{B\check{B}}\left(\check{y}(c+m\check{y})\left(1+h\check{s}(c+my)\right)\right) - \delta s,$$

$$N_{33} = d_2 + q_2 + \mu s - \frac{(b_1 s + b_2 I)[c + h(c+m\check{y})(c+my)(\check{s}+\beta\check{I}) + m(y+\check{y})]}{B\check{B}},$$

$$N_{12} = \frac{r\check{A}}{kA\check{A}} + \delta + \beta h\check{y}(c+my)(c+m\check{y}) - \delta\check{I} + \frac{1}{B\check{B}}h\check{I}\check{y}(c+m\check{y})(c+my),$$

$$\begin{aligned}N_{13} &= \frac{\alpha r}{A\check{A}}\left(1-\frac{(\check{s}+\check{I})}{k}\right) + \frac{\left(c\left(1+hc(\check{s}+\beta\check{I})\right)+hm^2\check{y}y(\check{s}+\beta\check{I})\right)}{B\check{B}} \\ &-m\left(1+hc\left(\check{s}+\beta\check{I}\right)\right)(y+\check{y}) + \mu\check{y} \\ &-\frac{(c+m\check{y})\check{y}[b_1-(c+my)h\check{I}(b_2-b_1\beta)]}{B\check{B}},\end{aligned}$$

$$\begin{aligned}N_{23} &= \frac{\left(cI + hI(\check{s}+\beta\check{I})(c^2+m^2\check{y}y) + mI\left(1+hc(\check{s}+\beta\check{I})\right)(y+\check{y})\right)}{B\check{B}} \\ &-\frac{(c+m\check{y})\check{y}[b_2+(b_2-b_1\beta)h\check{s}(c+my)]}{B\check{B}},\end{aligned}$$

with $\check{A} = 1 + \alpha\check{y}$; and $\check{B} = 1 + h(c+m\check{y})(\check{s}+\beta\check{I})$. Now further computation gives that

$$\frac{dL_4}{dt} < -\left[N_{11} - \frac{(N_{12})^2}{2} - \frac{(N_{13})^2}{2}\right](s-\check{s})^2 - \left[N_{22} - \frac{(N_{23})^2}{2} - \frac{1}{2}\right]\left(I-\check{I}\right)^2 - [N_{33}-1](y-\check{y})^2.$$

Accordingly, if the inequalities in (24a)-(24c) hold, $\frac{dL_4}{dt}$ is negative definite. Hence, $\check{p}_4$ is GAS.

6. LOCAL BIFURCATION

The occurrence of local bifurcation is investigated in this section using the Sotomayor theorem Recall that a non-hyperbolic equilibrium point represents a necessary but not sufficient condition for a local bifurcation to occur. Now, the next theorems examine whether or not LB might occur near a non-hyperbolic EP.

Theorem 9. System (2) at the EEP undergoes a transcritical bifurcation (TB) whenever the parameter q_1 passes the value $q_1{}^*$.

Proof. The JM at $(\dot{p}_0, q_1{}^*)$ can be represented by

$$J_0 = J_{(\dot{p}_0, q_1{}^*)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -(d_2 + q_2) \end{pmatrix}$$

Therefore, the eigenvalues of J_0 are given by $\lambda_{01}^* = 0, \lambda_{02}^* = -d_2,\ \lambda_{03}^* = -(d_2 + q_2)$. So, the EEP is a non-hyperbolic point.

Suppose, $\boldsymbol{\varpi_0} = (\varpi_{01}, \varpi_{02}, \varpi_{03})^T$ be the eigenvectors corresponding to $\lambda_{01}^* = 0$. Then it is obtained from $J_0 \boldsymbol{\varpi_0} = 0$ that $\boldsymbol{\varpi_0} = (\varpi_{01}, 0, 0)^T$, with ($\varpi_{01} \neq 0$).

Now, let $\boldsymbol{U_0} = (u_{01}, u_{02}, u_{03})^T$ refers to the eigenvector corresponding $\lambda_{01}^* = 0$ for $J_0{}^T$. Then $J_0{}^T \boldsymbol{U_0} = 0$ gives $\boldsymbol{U_0} = (u_{01}, 0, 0)^{\mathrm{T}}$, with ($u_{01} \neq 0$). Since $F_{q_1} = (-s, 0, 0)^T$. Hence $F_{q_1}(\dot{p}_0, q_1{}^*) = (0, 0, 0)^T$. Then $\boldsymbol{U_0}^T F_{q_1}(\dot{p}_0, q_1{}^*) = 0$.

Following the "Sotomayor theorem" at the EEP, the first requirement for TB is satisfied.

Now, we have $\boldsymbol{U_0}^T \left[DF_{q_1}(\dot{p}_0, q_1{}^*) \boldsymbol{\varpi_0} \right] = -\varpi_{01} u_{01} \neq 0$.

Also, $\boldsymbol{U_0}^T \left[D^2 F(\dot{p}_0, q_1{}^*)(\boldsymbol{\varpi_0}, \boldsymbol{\varpi_0}) \right] = \frac{-2r}{k} \varpi_{01}{}^2 u_{01}{}^2 \neq 0$, where $D^2 F$ is given in Appendix B.

Hence, TB takes place near $\dot{p}_0$.

Theorem 10. The system (2) undergoes a TB near $PDFEP$ when the parameter d_1 crosses the value $d_1^* = \delta \hat{s}$ if the following condition holds.

$$\frac{2}{(1 + hc\hat{s})^3} \neq (1 + hc\hat{s})^3 \delta \pi_1 \tag{25}$$

Proof. The JM at $(\hat{p}_1, d_1^*)$ can be represented by

$$J_1 = J(\hat{p}_1, d_1^*)\begin{pmatrix} \frac{-r\hat{s}}{k} & -\frac{r\hat{s}}{k} - \delta\hat{s} & -r\alpha\hat{s} - \frac{c\hat{s}}{1+hc\hat{s}} + \frac{\hat{s}^2\alpha r}{k} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{b_1 c\hat{s}}{1+hc\hat{s}} - d_2 - \mu\hat{s} - q_2 \end{pmatrix}$$

Thus, the eigenvalues of J_1 are given by $\lambda_{11}^* = \frac{-r\hat{s}}{k}$, $\lambda_{12}^* = 0$, and $\lambda_{13}^* = \frac{b_1 c\hat{s}}{1+hc\hat{s}} - d_2 - \mu\hat{s} - q_{\mathbf{2}}$. So, the $PDFEP$ is a non-hyperbolic point.

Let $\boldsymbol{\varpi_1} = (\varpi_{11}, \varpi_{12}, \varpi_{13})^T$ be the eigenvectors corresponding to $\lambda_{12}^* = 0$. Thus $J_1\boldsymbol{\varpi_1} = 0$ gives $\boldsymbol{\varpi_1} = (\pi_1\varpi_{12}, \varpi_{12}, 0)^T$, where $(\varpi_{12} \neq 0)$ with $\pi_1 = -(1 + \frac{\delta k}{r}) < 0$.

Now, let $\boldsymbol{U_1} = (u_{11}, u_{12}, u_{13})^T$ refers to the eigenvector corresponding to $\lambda_{12}^* = 0$ of ${J_1}^T$ then ${J_1}^T\ \boldsymbol{U_1} = 0$ gives $\boldsymbol{U_1} = (0, u_{12}, 0)^T$, where $(u_{12} \neq 0)$.

Since $F_{d_1} = (0, -I, 0)^T$, as result we find that $F_{d_1}(\hat{p}_1,\ d_1^*) = (0,0,0)^T$.

So, ${\boldsymbol{U_1}}^T F_{d_1}(\hat{p}_1,\ d_1^*) = 0$. Hence, system (2) has no SNB due to the Sotomayor theorem.

Now, we have ${\boldsymbol{U_1}}^T\left[DF_{d_1}(\hat{p}_1,\ d_1^*)\boldsymbol{\varpi_1}\right] = -\varpi_{12}\, u_{12} \neq\ 0$.

Additionally, ${\boldsymbol{U_1}}^T\ [D^2F(\hat{p}_1, d_1^*)(\boldsymbol{\varpi_1}, \boldsymbol{\varpi_1})] = \left[\frac{2}{(1+hc\hat{s})^3} + (1+hc\hat{s})^3\delta\pi_1\right]{\varpi_{12}}^2 u_{12} \neq\ 0$ under the condition (25). Hence, TB takes place near $\hat{p}_1$.

Theorem 11. The system (2) undergoes under the following a TB near $PFEP$ when the parameter q_2 crosses the value ${q_2}^* = \frac{c(b_1\bar{s}+b_2\bar{I})}{1+hc(\bar{s}+\beta\bar{I})} - d_2 - \mu\bar{s}$, condition

$$\mathrm{M}_1 \neq 0, \tag{26}$$

where M_1 is defined in the proof.

Proof. The JM at $(\bar{p}_2, {q_2}^*)$ can be represented by

$$J_2 = J_{(\bar{p}_2, {q_2}^*)} = \begin{pmatrix} \frac{-r\bar{s}}{k} & -\frac{r\bar{s}}{k} - \delta\bar{s} & -r\alpha\bar{s} - \frac{c\bar{s}}{1+hc(\bar{s}+\beta\bar{I})} + \frac{(\bar{s}+\bar{I})\bar{s}\alpha r}{k} \\ \delta\bar{I} & 0 & \frac{-c\bar{I}}{1+hc(\bar{s}+\beta\bar{I})} \\ 0 & 0 & 0 \end{pmatrix} = (\bar{a}_{ij}).$$

Then the matrix has two eigenvalues having negative real parts, while the third is zero.

Hence it is a non-hyperbolic point.

Let $\boldsymbol{\varpi_2} = (\varpi_{21}, \varpi_{22}, \varpi_{23})^T$ be the eigenvectors corresponding to $\lambda_{23}^* = 0$. Thus $J_2\boldsymbol{\varpi_2} = 0$

gives that $\boldsymbol{\varpi_2} = (\pi_2\varpi_{23}, \pi_3\varpi_{23}, \varpi_{23})^T$, where $\pi_2 = \frac{-\bar{a}_{23}}{\bar{a}_{21}} > 0$ and $\pi_3 = \frac{\bar{a}_{23}\bar{a}_{11}-\bar{a}_{21}\bar{a}_{13}}{\bar{a}_{12}\bar{a}_{21}}$ and, $(\varpi_{23} \neq 0)$.

Now, let $\boldsymbol{U_2} = (u_{21}, u_{22}, u_{23})^T$ represents the eigenvector corresponding to $\lambda_{23}^* = 0$ of ${J_2}^T$ then ${J_2}^T\ \boldsymbol{U_2} = 0$ yields $\boldsymbol{U_2} = (0, 0, u_{23})^T$, where $(u_{23} \neq 0)$.

Since $F_{q_2} = (0, 0, -y)^T$. Hence $F_{q_2}(\bar{p}_2, {q_2}^*) = (0, 0, 0)^T$

Therefore, ${\boldsymbol{U_2}}^T F_{q_2}(\bar{p}_2, {q_2}^*) = 0$. Hence system (2) has no SNB with the Sotomayor theorem.Now,

${\boldsymbol{U_2}}^T\left[DF_{q_2}(\bar{p}_2, {q_2}^*)\boldsymbol{\varpi_2}\right] = -\varpi_{23}u_{23} \neq 0.$

Also,

$$\begin{aligned}{\boldsymbol{U_2}}^T[D^2F(\bar{p}_2, {q_2}^*)(\boldsymbol{\varpi_2}, \boldsymbol{\varpi_2})] = &-\frac{b_1hc^2}{\left(1 + hc(\bar{s} + \beta\bar{I})\right)^2}{\pi_2}^2u_{23}{\varpi_{23}}^2 \\ &+\left[\frac{b_1c}{1 + hc(\bar{s} + \beta\bar{I})} - 2\mu + \frac{b_1hc^2\bar{s} - b_2hIc^2}{\left(1 + hc(\bar{s} + \beta\bar{I})\right)^2}\right]\pi_2u_{23}{\varpi_{23}}^2 \\ &-\left[\frac{b_1h\beta c^2\bar{s} - b_2h\beta c^2\bar{I}}{\left(1 + hc(s + \beta I)\right)^2} - \frac{2cb_2}{1 + hc(\bar{s} + \beta\bar{I})}\right]\pi_3u_{23}{\varpi_{23}}^2 \\ &+\left[\frac{2b_1m\bar{s} + b_2m\bar{I}}{\left(1 + hc(\bar{s} + \beta\bar{I})\right)} - \frac{2(hmc\bar{s} + hmc\beta\bar{I})(b_1\bar{s} + b_2\bar{I})}{\left(1 + hc(\bar{s} + \beta\bar{I})\right)^2}\right]u_{23}{\varpi_{23}}^2 = \mathrm{M}_1u_{23}{\varpi_{23}}^2\end{aligned}$$

Consequently, ${\boldsymbol{U_2}}^T[D^2F(\bar{p}_2, {q_2}^*)(\boldsymbol{\varpi_2}, \boldsymbol{\varpi_2})] \neq 0$. Hence, TB occurs near $\bar{p}_2$ under the condition (26).

Theorem 12. The system (2) undergoes a TB near $DFEP$ when the parameter $\delta = \delta^* = \frac{(c+m\tilde{y})}{\left(1+h\tilde{s}(c+m\tilde{y})\right)} + \frac{d_1}{\tilde{y}}$, if the following condition holds.

$$\mathrm{M}_2 \neq 0, \tag{27}$$

where M_2 is defined in the proof.

Proof. The JM at $(\tilde{p}_3, \delta^*)$ can be represented by

$$J_3 = J_{(\tilde{p}_3, \delta^*)} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ 0 & 0 & 0 \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix}$$

Then the matrix has two eigenvalues having negative real parts, while the third is zero.

Hence it is a non-hyperbolic point.

Let $\boldsymbol{\varpi_3} = (\varpi_{31}, \varpi_{32}, \varpi_{33})^T$ be the eigenvectors corresponding to $\lambda_{32}^* = 0$. Thus $J_3\boldsymbol{\varpi_3} = 0$ gives that $\boldsymbol{\varpi_3} = (\pi_4\varpi_{32}, \varpi_{32}, \pi_5\varpi_{32})^T$, where $\pi_4 = \frac{\tilde{a}_{13}\tilde{a}_{32}-\tilde{a}_{12}\tilde{a}_{13}}{\tilde{a}_{11}\tilde{a}_{33}-\tilde{a}_{13}\tilde{a}_{31}}$ and $\pi_5 = \frac{\tilde{a}_{31}\tilde{a}_{12}-\tilde{a}_{11}\tilde{a}_{32}}{\tilde{a}_{11}\tilde{a}_{33}-\tilde{a}_{13}\tilde{a}_{31}}$ and, $(\varpi_{32} \neq 0)$.

Now, let $\boldsymbol{U_3} = (u_{31}, u_{32}, u_{33})^T$ represents the eigenvectors corresponding to $\lambda_{32}^* = 0$ for ${J_3}^T$ then ${J_3}^T\ \boldsymbol{U_3} = 0$ leads to $\boldsymbol{U_3} = (0, u_{32}, 0)^T$, where $(u_{32} \neq 0)$.

Since $F_\delta = (-sI, sI, 0)^T$. Hence, we obtain that $F_\delta(\tilde{p}_3, \delta^*) = (0,0,0)^T$

Therefore, ${\boldsymbol{U_3}}^T F_\delta(\tilde{p}_3, \delta^*) = 0$, which shows that system (2) has no SNB.

Now, we have ${\boldsymbol{U_3}}^T[DF_\delta(\tilde{p}_3, \delta^*)\boldsymbol{\varpi_3}] = \tilde{s}\varpi_{32} \neq 0$.

Also,

$$\begin{gathered}{\boldsymbol{U_3}}^T[D^2F(\tilde{p}_3, \delta^*)(\boldsymbol{\varpi_3}, \boldsymbol{\varpi_3})] = [h\tilde{y}(c+m\tilde{y})^2(1+h\tilde{s}(c+m\tilde{y}))\beta \\ -((c+c^3h^2\tilde{s}^2) + m\tilde{y}(2+hm\tilde{s}\tilde{y}(3+hm\tilde{s}\tilde{y})) \\ +c^2h(\tilde{s}(2+3hm\tilde{s}\tilde{y})) + chm\tilde{s}\tilde{y}(5+3hm\tilde{s}\tilde{y}))\pi_5 \\ +(h\tilde{y}(c+m\tilde{y})^2(1+h\tilde{s}(c+m\tilde{y})) + (1+h(c+m\tilde{y})\tilde{s})^3\delta)\pi_4 \\ +m\tilde{y}(3+hm\tilde{s}\tilde{y}))\pi_4\pi_5]{\varpi_{32}}^2 = \mathrm{M}_2{\varpi_{32}}^2\end{gathered}$$

Then ${\boldsymbol{U_3}}^T[D^2F(\tilde{p}_3, \delta^*)(\boldsymbol{\varpi_3}, \boldsymbol{\varpi_3})] \neq 0$ under the condition (27). Hence, TB takes place near $\tilde{p}_3$.

Theorem 13. The system (2) undergoes an SNB near *COEP* when the parameter $\mu = \mu^* = \left[\frac{-\breve{a}_{11}(\breve{a}_{22}\breve{a}_{33}-\breve{a}_{23}\breve{a}_{32})+\breve{a}_{21}(\breve{a}_{12}\breve{a}_{33}-\breve{a}_{13}\breve{a}_{32})}{(\breve{a}_{22}\breve{a}_{13}-\breve{a}_{12}\breve{a}_{23})} + (\frac{(c+m\breve{y})b_1}{(1+h(c+m\breve{y})(\hat{s}+\beta\hat{I})} - \frac{h(c+m\breve{y})^2(b_1\breve{s}+b_2\breve{I})}{(1+h(c+m\breve{y})(\hat{s}+\beta\hat{I}))^2\breve{y}})\right]$, if the following condition holds.

$$m_{11}\tau_1u_{43} + m_{21}\tau_2u_{43} + m_{31}u_{43} \neq 0, \tag{28}$$

where m_{11}, m_{21}, and m_{31} are given in the Appendix B.

Proof. The JM of the system (2) at $(\breve{p}_4, \mu^*)$ can be represented by

$$J_4 = J_{(\breve{p}_4,\mu^*)} = (\breve{a}_{ij})_{(\breve{p}_4,\mu^*)}$$

Therefore, it is straightforward to check that the coefficient $\theta_3 = 0$ at $\mu = \mu^*$ in equation (16g). Hence the characteristic equation has a zero root

Let $\boldsymbol{\varpi_4} = (\varpi_{41}, \varpi_{42}, \varpi_{43})^T$ be the eigenvectors corresponding to $\lambda_{41}^* = 0$. Thus $J_4\boldsymbol{\varpi_4} = 0$ gives that $\boldsymbol{\varpi_4} = (\pi_6\varpi_{43}, \varpi_{43}, \pi_7\varpi_{43})^T$, where $\pi_6 = \frac{\breve{a}_{12}\breve{a}_{23}-\breve{a}_{22}\breve{a}_{13}}{\breve{a}_{11}\breve{a}_{22}-\breve{a}_{12}\breve{a}_{21}}$ and $\pi_7 = \frac{\breve{a}_{21}\breve{a}_{13}-\breve{a}_{11}\breve{a}_{23}}{\breve{a}_{11}\breve{a}_{22}-\breve{a}_{12}\breve{a}_{21}}$ and, $(\varpi_{43} \neq 0)$.

Now, let $\boldsymbol{\varpi_4} = (u_{41}, u_{42}, u_{43})^T$ represents the eigenvectors corresponding to $\lambda_{41}^* = 0$ of ${J_4}^T$ then ${J_4}^T \boldsymbol{U_4} = 0$ yields $\boldsymbol{U_4} = (\tau_1 u_{43}, \tau_2 u_{43}, u_{43})^T$, where $, (u_{43} \neq 0)$ with $\tau_1 = \frac{\breve{a}_{21}\breve{a}_{32} - \breve{a}_{22}\breve{a}_{31}}{\breve{a}_{11}\breve{a}_{22} - \breve{a}_{12}\breve{a}_{21}}$, $\tau_2 = \frac{\breve{a}_{12}\breve{a}_{31} - \breve{a}_{11}\breve{a}_{32}}{\breve{a}_{11}\breve{a}_{22} - \breve{a}_{12}\breve{a}_{21}}$.

Since $F_\mu = (0, 0, -\hat{s}\hat{y})^T$. Hence we obtain that $F_\mu(\breve{p}_4, \mu^*) = (0,0, -\hat{s}\hat{y})^T$

Therefore, ${\boldsymbol{U_4}}^T F_\mu(\breve{p}_4, \mu^*) = -\hat{s}\hat{y}u_{43} \neq 0$.

Also, by using the condition (28) it is obtained that

$$ {\boldsymbol{U_4}}^T [D^2 F(\breve{p}_4, \mu^*)(\boldsymbol{\varpi_4}, \boldsymbol{\varpi_4})] = m_{11}\tau_1 u_{43} + m_{21}\tau_2 u_{43} + m_{31} u_{43} \neq 0. $$

Hence, SNB takes place near $\breve{p}_4$.

7. NUMERICAL SIMULATIONS

This section explores the aspects of system (2) dynamics. The primary objective is understanding how the system behaves as its parameters are altered.

The obtained theoretical results are verified utilizing numerical simulation. System (2) is solved numerically utilizing Matlab version R2021a.

$$ r = 2.8;\ \alpha = 0.3;\ K = 10\,;\ \delta = 0.3;\ c = 0.6\,;\ m = 0.02;\ b_1 = 0.5;\ b_2 = 0.25; $$
$$ h = 0.12;\ d_1 = 0.1;\ d_2 = 0.2;\ \beta = 0.1;\ \mu = 0.2;\ q_1 = 0.1;\ q_2 = 0.05; \qquad (29) $$

Now, utilizing the above data starting from different initial points, system (2) approaches asymptotically to *COEP*, $\breve{p}_4 = (3.21, 0.60, 1.70)$, as illustrated in Figure (1).

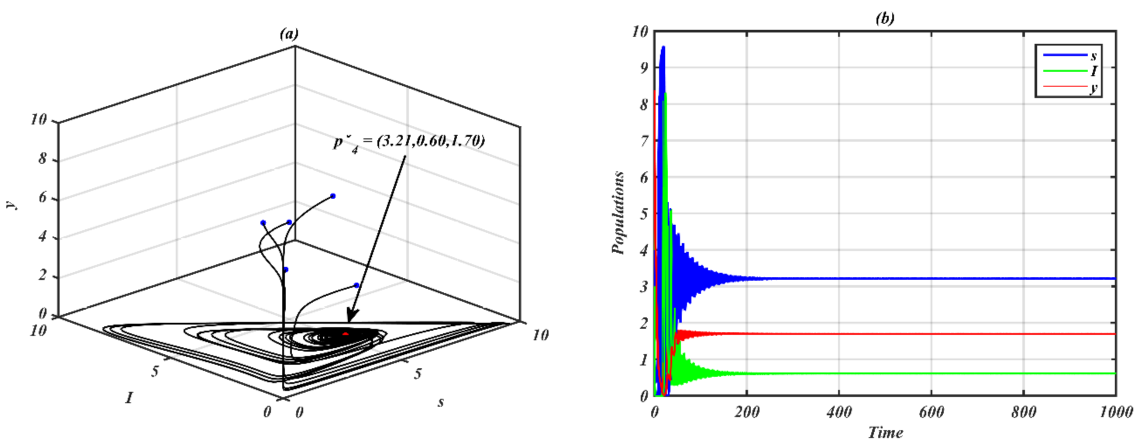

Figure 1. System's trajectory utilizes the set of data (29) and starting from various initial points approach asymptotically to $\breve{p}_4 = (3.20, 0.60, 1.70)$.(a) Phase portrait of system (2). (b) Evolution of time series.

As seen in Fig. 1, it is found that the system's (2) trajectories approach the COEP asymptotically given the following set of hypothetical parameter values, stsrting from various initial conditions. According to Fig. 1, the obtained theoretical finding of theorem 8 is confirmed.

Now, the influence of r is studied in Fig. 2, it is observed that for $r \in [0.1,0.5]$ the system approaches the $PFEP$, while for $r \leq 0.09$ the system approach to EEP, as shown in Fig. 2 however, for $r \geq 0.6$ it approaches the $COEP$, as illustrated in Fig. 1.

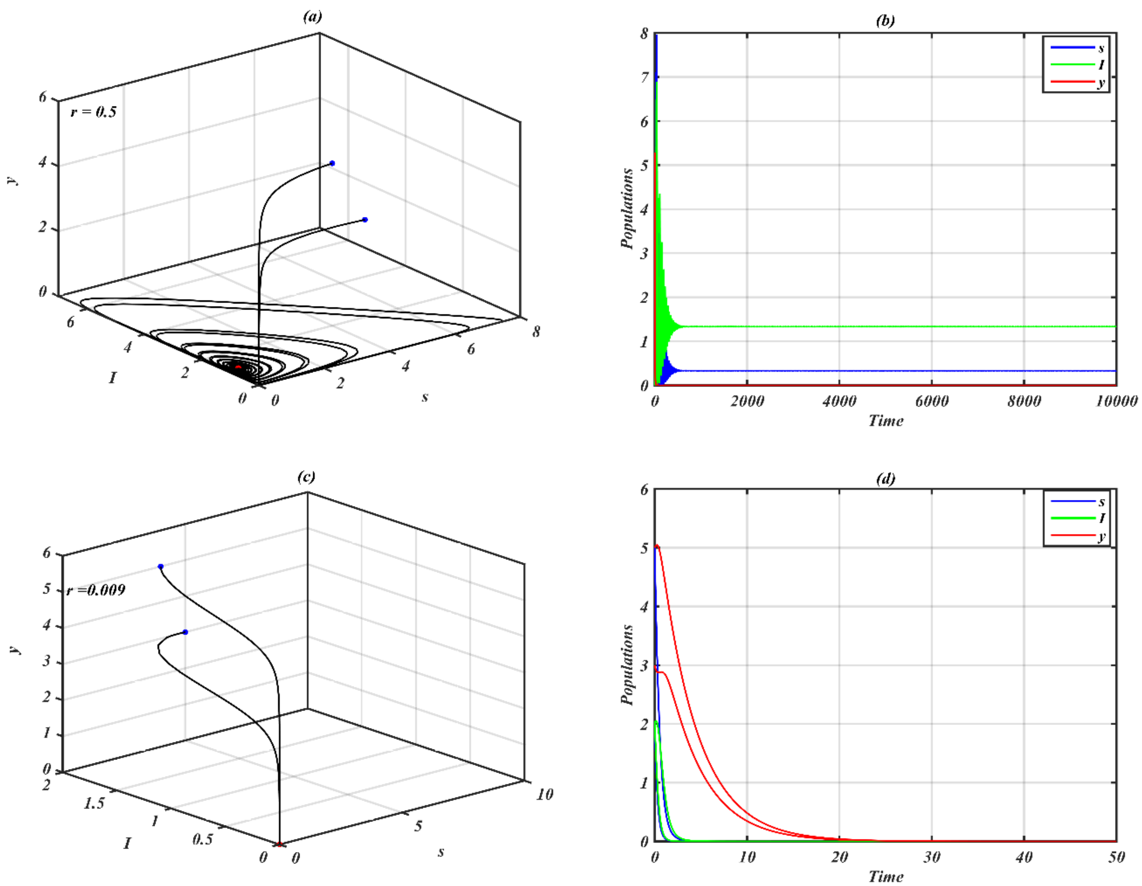

Figure 2. The system's trajectories (2) with data set (29) and different values of r. (a) The system approaches asymptotically to $\bar{p}_2 = (0.33,1.09,0)$ when $r = 0.5$. (b) Time series for $r = 0.5$. (c)The system approaches asymptotically to $\dot{p}_0$ for $r = 0.009$. (d) Time series for $r = 0.009$.

The effect of varying the value of k of system (2) is investigated. Obviously, for $k \leq 0.35$ then the system's trajectory (2) approaches to $PDFEP$, Also for $k \in [0.36,2.3]$ then the system's trajectory (2) approaches to $PFEP$, and for $k \geq 11$ then the system's trajectory converges to $3D$ period attractor as illustrated in Fig. 3. While for $k \in [2.4,10]$ the system's trajectory approaches $COEP$, as illustrated in Fig. 1.

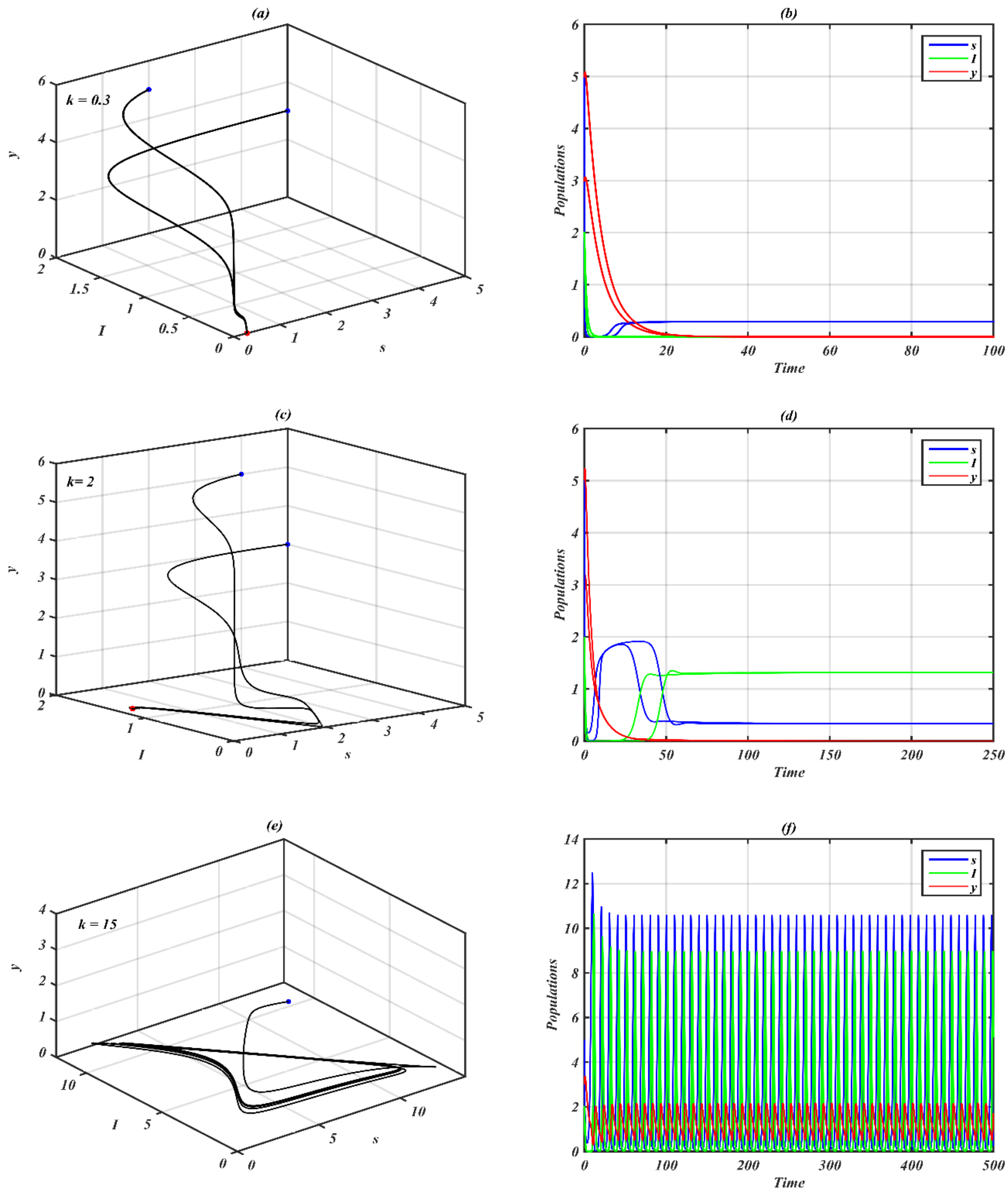

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Figure 3. The system's trajectories (2) with Eq. (29) with different values of k. (a) Approaches asymptotically to $\hat{p}_1 = (0.28,0,0)$ for $k = 0.3$. (b) Time series for $k = 0.3$. (c) The system approaches asymptotically to $\bar{p}_2 = (0.33,1.31,0)$ for $k = 2$. (d) Time series for $k = 2$. (e) Approaches to periodic attractor in $\mathcal{R}_+^3$ for $k = 15$. (d) Time series for $k = 15$.

Moreover, for the parameter α, it observed that for $\alpha \leq 0.01$ the system's trajectory(2) approaches to $PDFEP$, for $\alpha \in [0.02,0.2]$ the system's trajectory converges to $3D$ period attractor as illustrated in Fig 4, while $\alpha \geq 0.3$ the system approaches the $COEP$, as illustrated in Fig. 1.

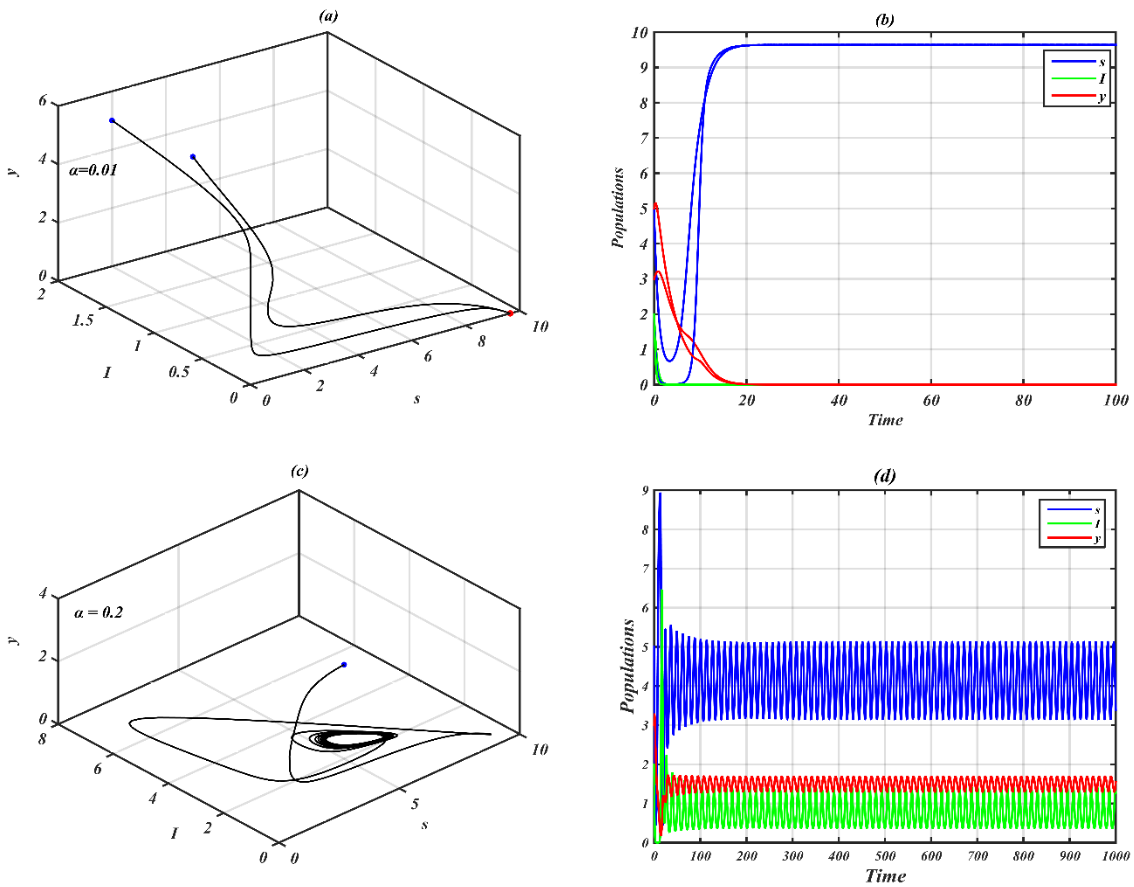

Figure 4. The system's trajectories(2) with Eq. (29) with different values of α. (a) Approaches asymptotically to $\hat{p}_1 = (9.64,0,0)$ for $\alpha = 0.01$. (b) Time series for $\alpha = 0.01$. (c) Approaches to periodic attractor in $\mathcal{R}_+^3$ for $\alpha = 0.2$. (d) Time series for $\alpha = 0.2$.

The influence of varying the value c of the system (2) is numerically investigated with data (29), clearly, for $c \leq 0.2$ the system's trajectory(2) approaches to $PFEP$, Morevere, for $c \in [0.7,0.9]$ then the system's trajectory approaches to $DFEP$, while for $c \geq 0.95$ the system's trajectory approaches periodic attractor in 2D as illustrated in Fig. 5, and where $c \in [0.3,0.6]$ the system approaches the $COEP$, as illustrated in Fig. 1.

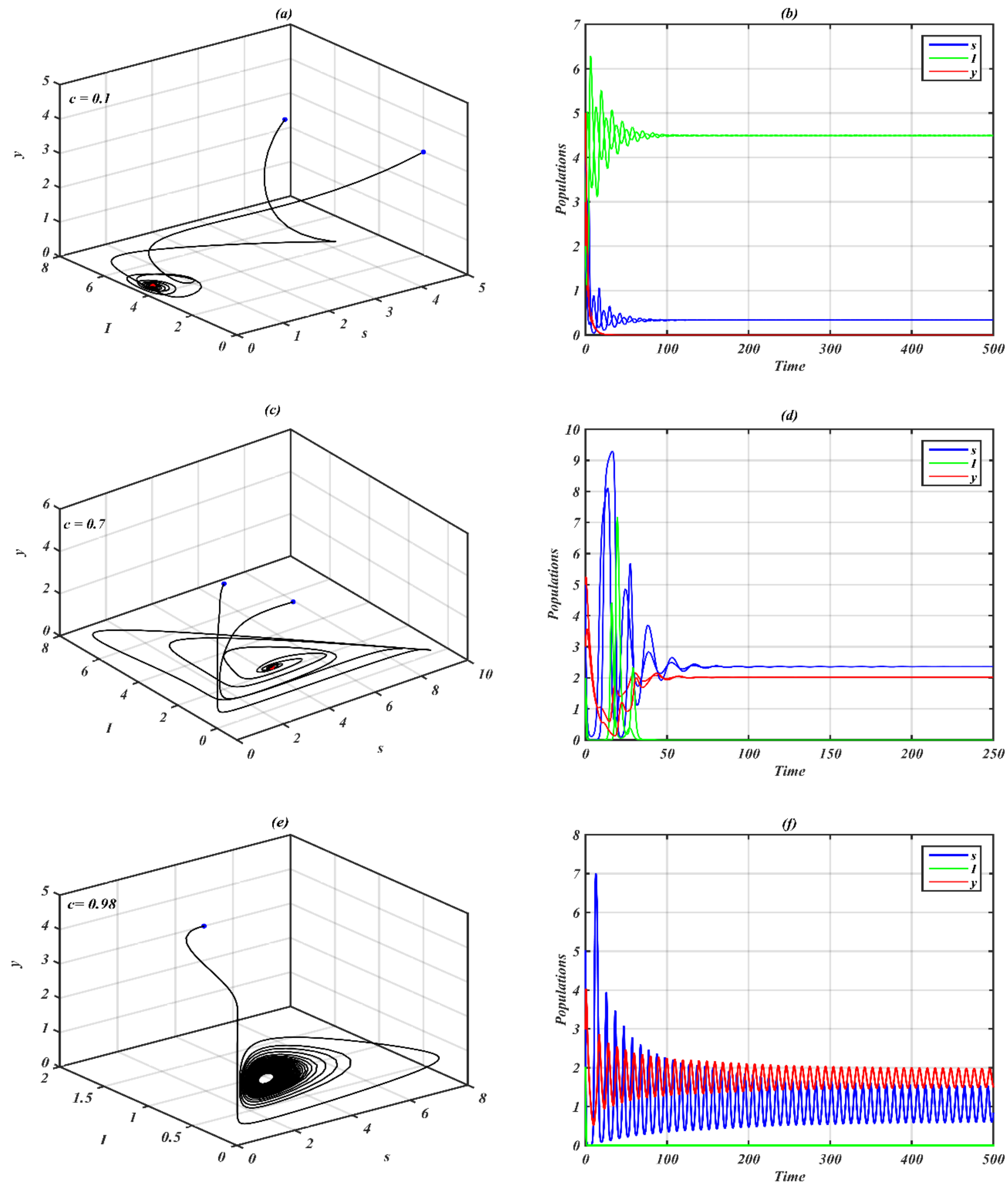

Figure 5. The system's trajectories (2) with Eq. (29) and different values of c. (a) Approaches asymptotically to $\bar{p}_2 = (0.33,4.4,0)$ for $c = 0.1$. (b) Time series for $c = 0.1$. (c) The system approaches asymptotically to $\tilde{p}_3 = (2.35,0,2.01)$ for $c = 0.7$. (d) Time series for $c = 0.7$. (e) Approaches to periodic attractor in $\mathcal{R}_+^3$ for $c = 0.98$. (f) Time series for $\alpha = 0.98$.

Moreover, for the parameter in the range $m \in [0.04,0.053]$ and $m \geq 0.2$ then the system's trajectory converges to $3D$ period attractor, Also, for $m \in [0.06,0.08]$ the system approaches $DFEP$. While for $m \in [0.09,0.1]$ the system's trajectory converges to $2D$ period attractor as illustrated in Fig. 6. When $m \leq 0.03$ the system's approaches to the $COEP$, as illustrated in Fig. 1.

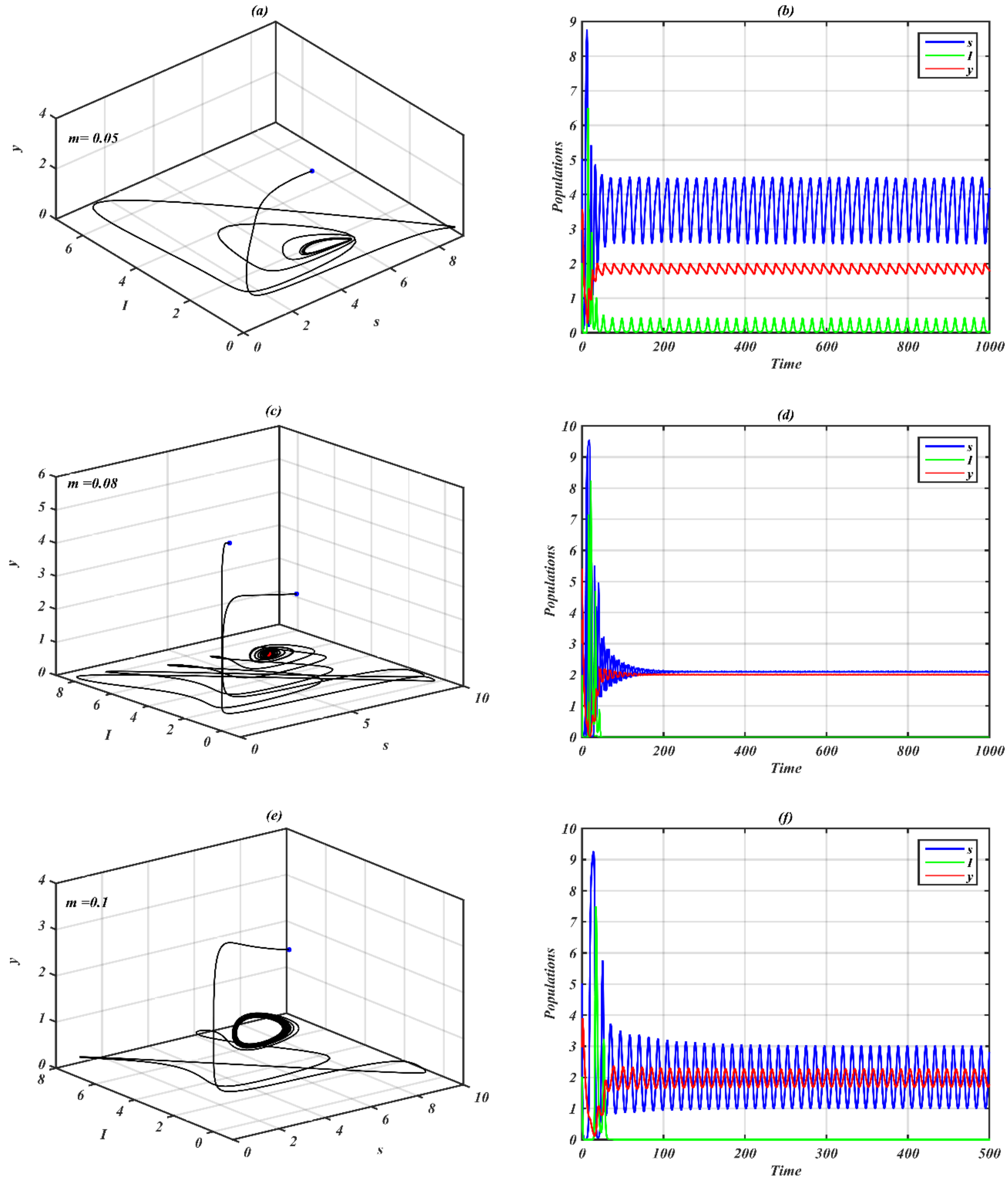

Figure 6. The system's trajectories (2) with Eq. (29) and different values of m. (a) Approaches periodic attractor in $\mathcal{R}_+^3$ for $m = 0.05$. (b) Time series for $m = 0.05$. (c) The system approaches asymptotically to $\tilde{p}_3 = (2.10,0,2.01)$ for $m = 0.08$. (d) Time series for $m = 0.08$. (e) Appeoches periodic attractor in $SY-$plane for $m = 0.1$. (f) Time series for $m = 0.1$.

Now, the effect of varying the value of b_1 is investigated. Clearly, for $b_1 \leq 0.4$ the system's trajectory converges to $3D$ period attractor. Moreover, for $b_1 \geq 0.54$ the system's approaches to $DFEP$. as illustration in Fig. 7. Otherwise, the $COEP$ of the system(2) is a GAS as illustrated in Fig. 1.

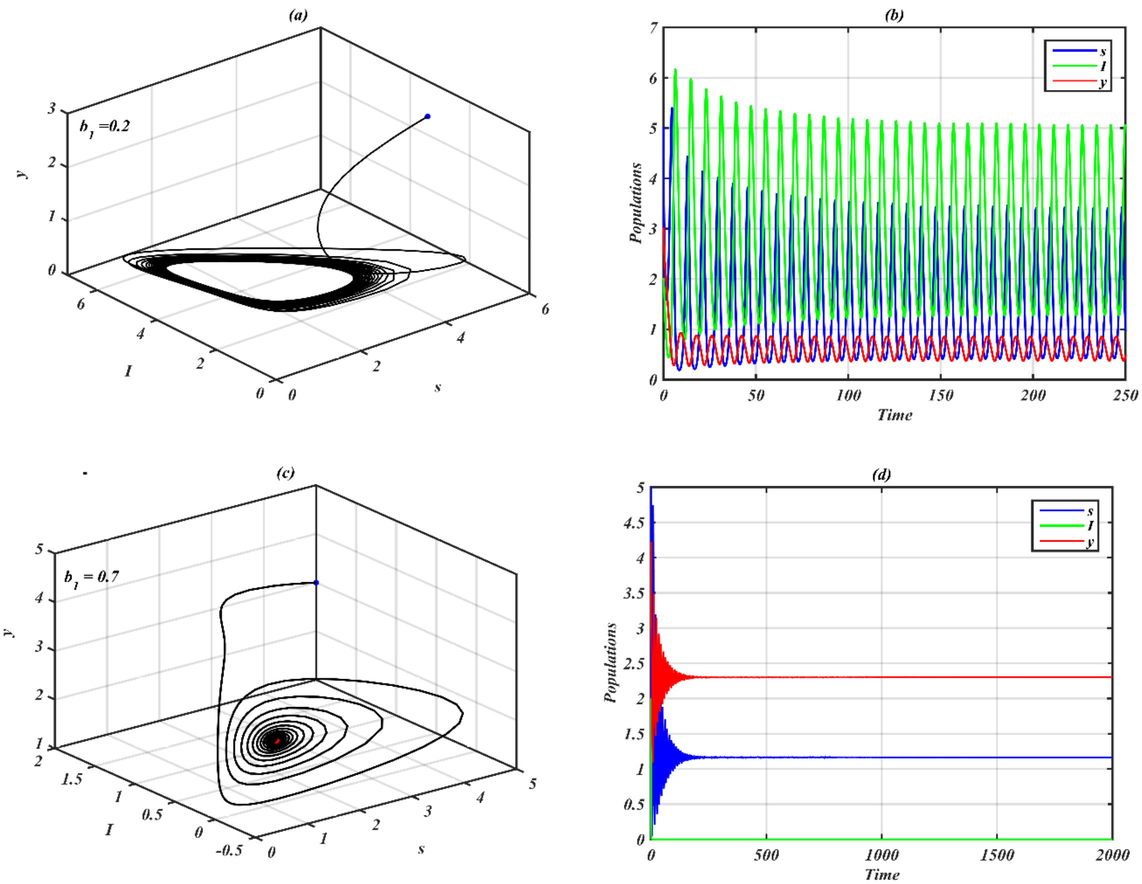

Figure 7. The system's trajectories (2) with Eq. (29) and different values of b_1. (a) Approaches Periodic attractor in $\mathcal{R}_+^3$ for $b_1 = 0.2$. (b) Time series for $b_1 = 0.2$. (c) The system approaches asymptotically to $\tilde{p}_3 = (1.16,0,2.30)$ for $b_1 = 0.7$. (d) Time series for $b_1 = 0.7$.

The impact of value b_2 of the system is studied. For $b_2 \leq 0.08$ the system's approaches to $DFEP$. Also, for $b_2 \geq 0.27$. The system approaches a stable limit cycle as illustration in Fig. 8. Otherwise, the$COEP$ of the system (2) is a GAS, as illustrated in Fig. 1.

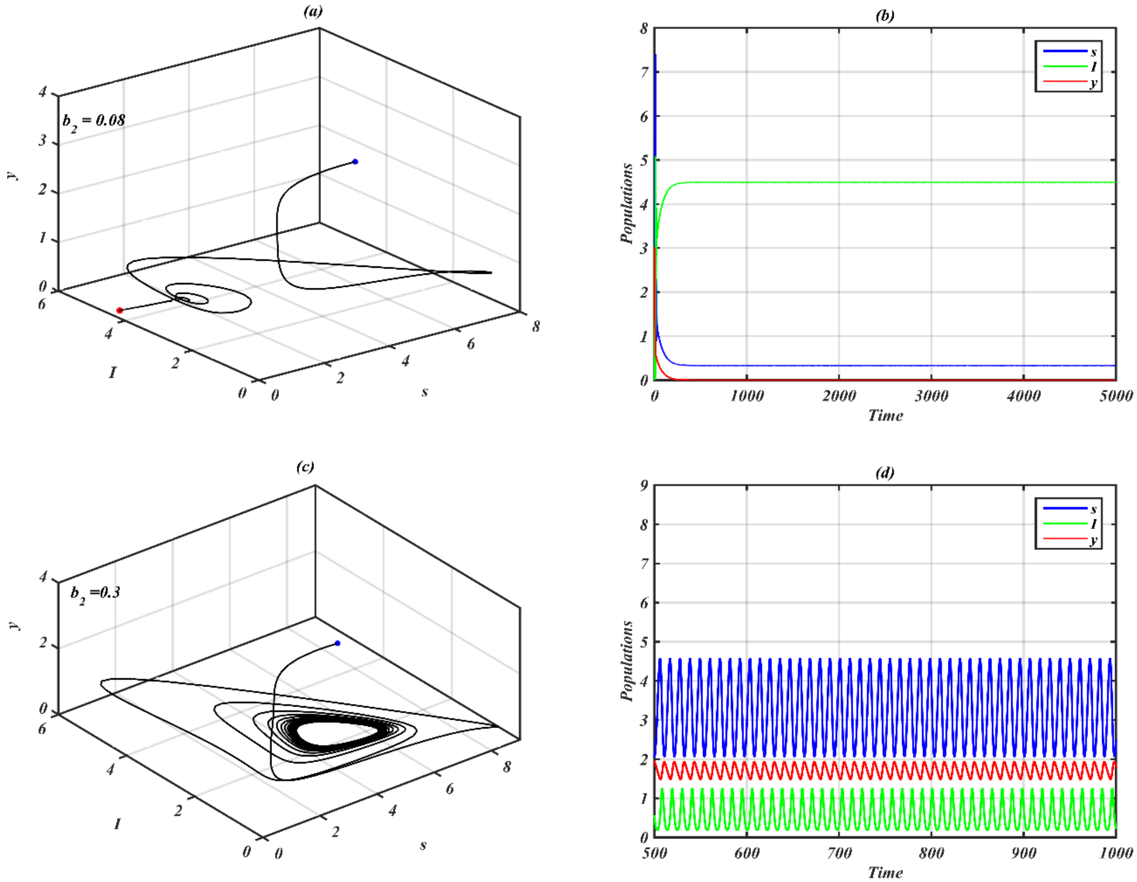

Figure 8. The system's trajectories (2) with Eq. (29) and different values of b_2. (a) Approaches asymptotically to $\bar{p}_2 = (3.56,4.49,0)$ for $b_2 = 0.08$. (d) Time series for $b_2 = 0.08$. (c) Approaches periodic attractor in $\mathcal{R}_+^3$ for $b_2 = 0.3$. (d) Time series for $b_2 = 0.3$.

Now, when $\mu \leq 0.18$, the solution of system (2) approaches $DFEP$. In addition, for $\mu \geq 0.23$ the system approaches a stable limit cycle, as illustration in Fig. 9. Otherwise, the $COEP$ of the system (2) is a GAS if $\mu \in [0.19,0.22]$ as illustrated in Fig. 1. Similar effect has been obtained, as that happened with varying μ when we varying the value h.

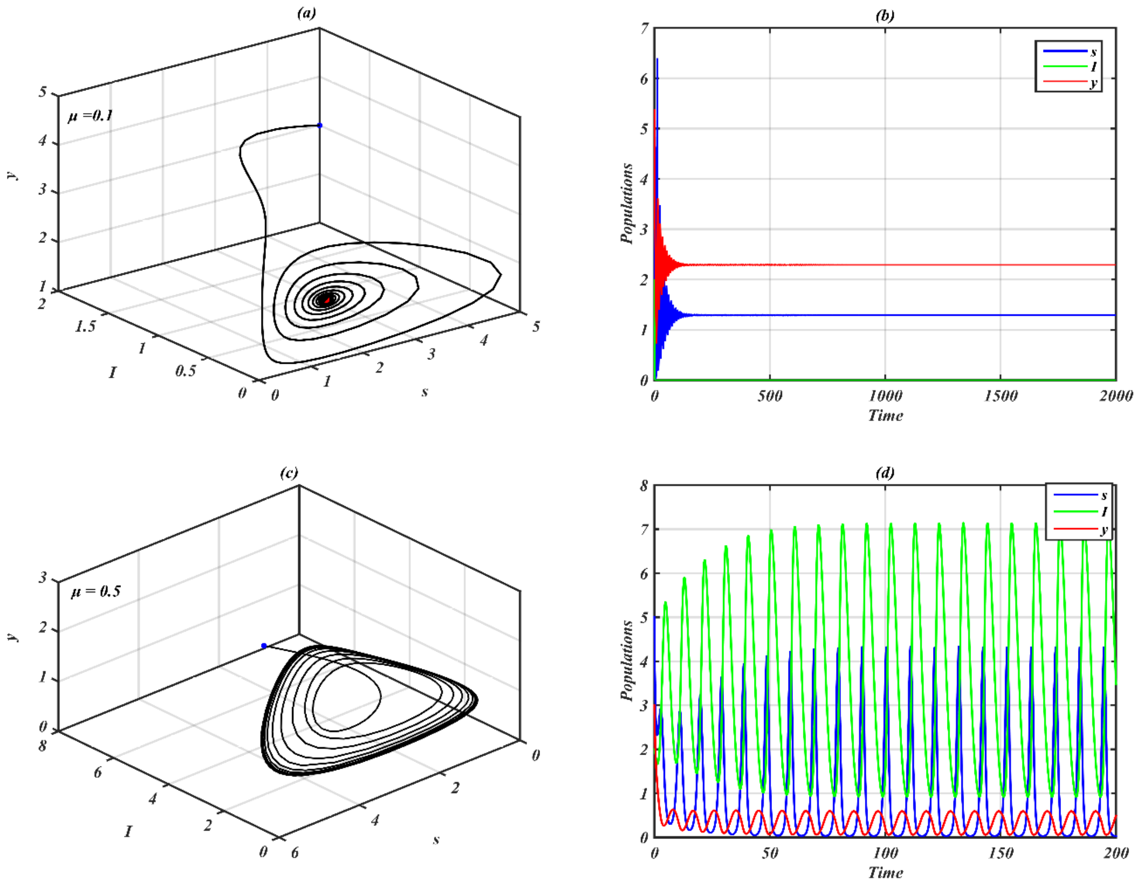

Figure 9. The system's trajectories (2) with Eq. (29) and different values of μ. (a) Approaches asymptotically to $\tilde{p}_3 = (1.29,0,2.29)$ for $\mu = 0.1$. (d) Time series for $\mu = 0.1$. (c) Approaches periodic attractor in $\mathcal{R}_+^3$ for $\mu = 0.5$. (d) Time series for $\mu = 0.5$.

It is observed that the influence of varying the values $d_1, \beta,$ and q_1 on the system (2) dynamics is only a quantitative effect when the parameter values (29) are used, and the system trajectory approaches asymptotically to $COEP$ with a different position.

The influence the value of d_2 of system (2) is studied. For $d_2 \leq 0.1$ the system approaches $DFEP$. While, for $d_2 \geq 0.7$ the system's approaches to $PFEP$, as illustration in Fig. 10. Otherwise, the $COEP$ of the system (2) is a GAS if $d_2 \in [0.2,0.6]$ as illustrated in Fig. 1.

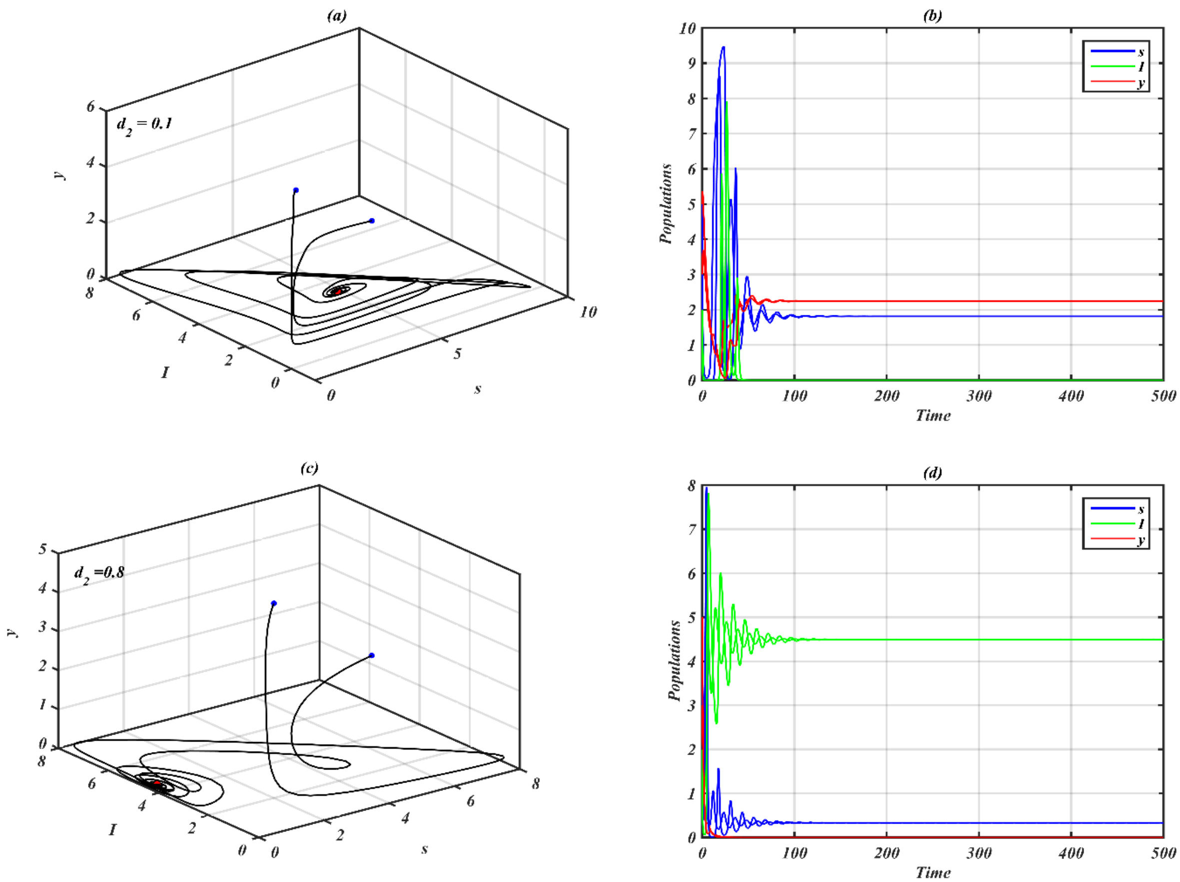

Figure 10. The trajectories of system (2) using Eq. (29) with different values of d_2. (a) The system approaches asymptotically to $\tilde{p}_3 = (1.81,0,2.24)$ for $d_2 = 0.1$. (b) Time series for $d_2 = 0.1$. (c) The system approaches to $\bar{p}_2 = (0.33,4.49,0)$ for $d_2 = 0.8$. (d) Time series for $d_2 = 0.8$.

Finally, for the parameter $q_2 \leq 0.02$ the system approaches a stable limit cycle, while for $q_2 \geq 0.5$ the system approaches $PFEP$, as illustration in Fig. 11. Otherwise, the $COEP$ of the system (2) is a GAS if $q_2 \in [0.03,0.4]$ as illustrated in Fig. 1.

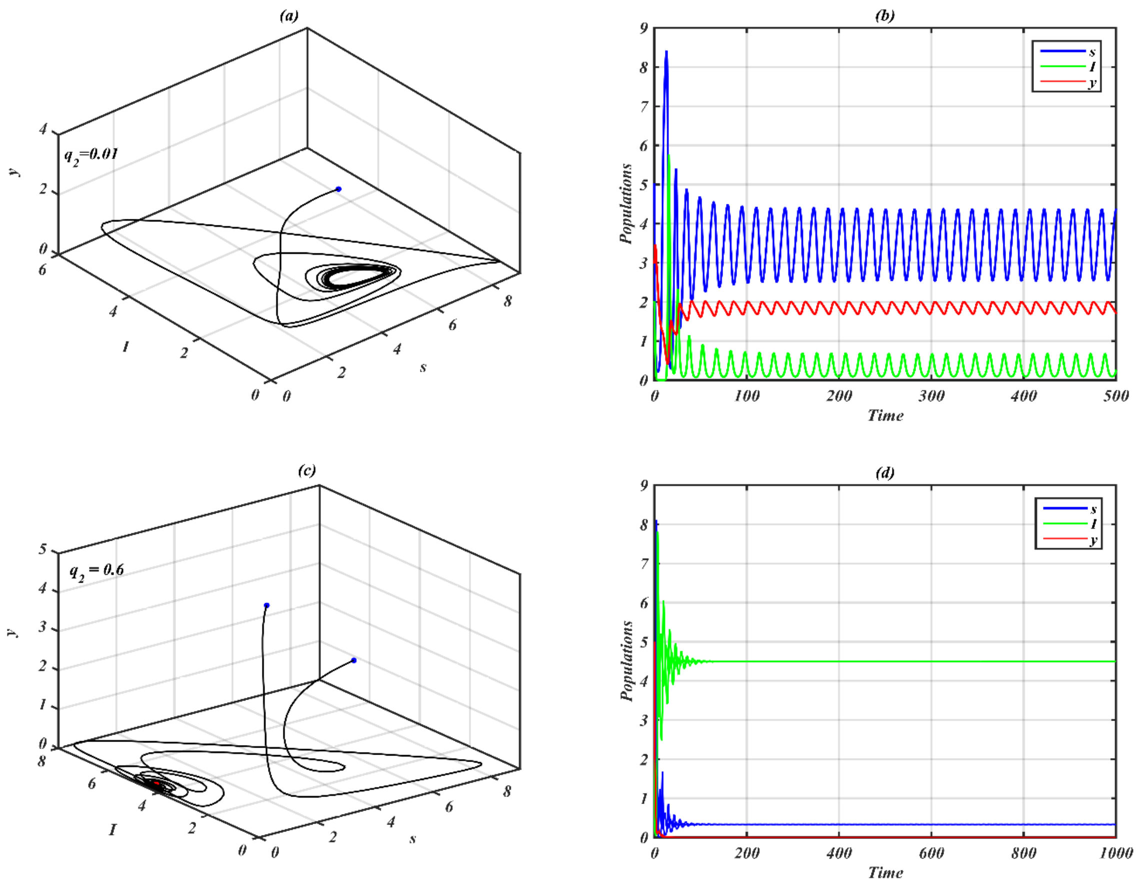

Figure 11. The system's trajectories (2) with Eq. (29) and different values of q_2. (a) Approaches periodic attractor in $\mathcal{R}_+^3$ for $q_2 = 0.01$. (b) Time series for $q_2 = 0.01$. (c) The system approaches asymptotically $\bar{p}_2 = (0.33,4.49,0)$ for $q_2 = 0.6$. (d) Time series for $q_2 = 0.6$.

8. CONCLUSION

An investigation of the influence of hunting cooperation and anti-predator behavior induced by the fear of predation for a Holling-II prey-predator model has been carried out analytically and numerically. The analytical and numerical findings show that cooperation leads the maximal number of individuals to escape predation through fear. The existence and stability of equilibria of the model are given and the existence of bifurcation is investigated. In addition, the influence of the fear effect on the population dynamics of the model is investigated, and find that the fear effect can not only reduce the population density of both predator and prey but also prevent the occurrence of limit cycle oscillation and increase the stability of the system.

Appendix A:

The coefficients of the equation (5) can be determined as follows:

$$L_0 = ch^2Km\mu^3 - c^2h^2K\alpha\mu^3 + h^2Km^2r\mu^2b_1 + 2hm^2r\mu b_1^2 - 2h^2m^2r\mu b_1d_2 - h^2Km^2\mu^2b_1q_1 + ch^2Km\alpha\mu^2b_1q_1 - 2h^2m^2r\mu b_1q_2$$

$$L_1 = hKm\mu^3 - 2chK\alpha\mu^3 - 2chKm\mu^2b_1 + 2c^2hK\alpha\mu^2b_1 - 2hKm^2r\mu b_1^2 - m^2rb_1^3 + 3ch^2Km\mu^2d_2 - 3c^2h^2K\alpha\mu^2d_2 + 2h^2Km^2r\mu b_1d_2 + 2hm^2rb_1^2d_2 - h^2m^2rb_1d_2^2 + hKm\alpha\mu^2b_1q_1 + 2hKm^2\mu b_1^2q_1 - 2chKm\alpha\mu b_1^2q_1 - 2h^2Km^2\mu b_1d_2q_1 + 2ch^2Km\alpha\mu b_1d_2q_1 + 3ch^2Km\mu^2q_2 - 3c^2h^2K\alpha\mu^2q_2 + 2h^2Km^2r\mu b_1q_2 + 2hm^2rb_1^2q_2 - 2h^2m^2rb_1d_2q_2 - 2h^2Km^2\mu b_1q_1q_2 + 2ch^2Km\alpha\mu b_1q_1q_2 - h^2m^2rb_1q_2^2$$

$$L_2 = -K\alpha\mu^3 - Km\mu^2b_1 + 2cK\alpha\mu^2b_1 + cKm\mu b_1^2 - c^2K\alpha\mu b_1^2 + Km^2rb_1^3 + 3hKm\mu^2d_2 - 6chK\alpha\mu^2d_2 - 4chKm\mu b_1d_2 + 4c^2hK\alpha\mu b_1d_2 - 2hKm^2rb_1^2d_2 + 3ch^2Km\mu d_2^2 - 3c^2h^2K\alpha\mu d_2^2 + h^2Km^2rb_1d_2^2 - Km\alpha\mu b_1^2q_1 - Km^2b_1^3q_1 + cKm\alpha b_1^3q_1 + 2hKm\alpha\mu b_1d_2q_1 + 2hKm^2b_1^2d_2q_1 - 2chKm\alpha b_1^2d_2q_1 - h^2Km^2b_1d_2^2q_1 + ch^2Km\alpha b_1d_2^2q_1 + 3hKm\mu^2q_2 - 6chK\alpha\mu^2q_2 - 4chKm\mu b_1q_2 + 4c^2hK\alpha\mu b_1q_2 - 2hKm^2rb_1^2q_2 + 6ch^2Km\mu d_2q_2 - 6c^2h^2K\alpha\mu d_2q_2 + 2h^2Km^2rb_1d_2q_2 + 2hKm\alpha\mu b_1q_1q_2 + 2hKm^2b_1^2q_1q_2 - 2chKm\alpha b_1^2q_1q_2 - 2h^2Km^2b_1d_2q_1q_2 + 2ch^2Km\alpha b_1d_2q_1q_2 + 3ch^2Km\mu q_2^2 - 3c^2h^2K\alpha\mu q_2^2 + h^2Km^2rb_1q_2^2 - h^2Km^2b_1q_1q_2^2 + ch^2Km\alpha b_1q_1q_2^2$$

$$L_3 = -3K\alpha\mu^2d_2 - 2Km\mu b_1d_2 + 4cK\alpha\mu b_1d_2 + cKmb_1^2d_2 - c^2K\alpha b_1^2d_2 + 3hKm\mu d_2^2 - 6chK\alpha\mu d_2^2 - 2chKmb_1d_2^2 + 2c^2hK\alpha b_1d_2^2 + ch^2Kmd_2^3 - c^2h^2K\alpha d_2^3 - Km\alpha b_1^2d_2q_1 + hKm\alpha b_1d_2^2q_1 - 3K\alpha\mu^2q_2 - 2Km\mu b_1q_2 + 4cK\alpha\mu b_1q_2 + cKmb_1^2q_2 - c^2K\alpha b_1^2q_2 + 6hKm\mu d_2q_2 - 12chK\alpha\mu d_2q_2 - 4chKmb_1d_2q_2 + 4c^2hK\alpha b_1d_2q_2 + 3ch^2Kmd_2^2q_2 - 3c^2h^2K\alpha d_2^2q_2 - Km\alpha b_1^2q_1q_2 + 2hKm\alpha b_1d_2q_1q_2 + 3hKm\mu q_2^2 - 6chK\alpha\mu q_2^2 - 2chKmb_1q_2^2 + 2c^2hK\alpha b_1q_2^2 + 3ch^2Kmd_2q_2^2 - 3c^2h^2K\alpha d_2q_2^2 + hKm\alpha b_1q_1q_2^2 + ch^2Kmq_2^3 - c^2h^2K\alpha q_2^3$$

$$L_4 = -3K\alpha\mu d_2^2 - Kmb_1d_2^2 + 2cK\alpha b_1d_2^2 + hKmd_2^3 - 2chK\alpha d_2^3 - 6K\alpha\mu d_2q_2 - 2Kmb_1d_2q_2 + 4cK\alpha b_1d_2q_2 + 3hKmd_2^2q_2 - 6chK\alpha d_2^2q_2 - 3K\alpha\mu q_2^2 - Kmb_1q_2^2 + 2cK\alpha b_1q_2^2 + 3hKmd_2q_2^2 - 6chK\alpha d_2q_2^2 + hKmq_2^3 - 2chK\alpha q_2^3$$

$$L_5 = -h^2m^2rx^6\mu^2b_1 - K\alpha d_2^3 - 3K\alpha d_2^2q_2 - 3K\alpha d_2q_2^2 - K\alpha q_2^3.$$

Appendix B:

Reformulating the system (2) in the vector form as follows

$$\frac{dX}{dt} = F(X),$$

where $X = (s, I, y)^T$ and $F = (sf_1, If_2, yf_3)^T$. Then direct computation to obtain the second derivative of F can be written as

$$D^2F(X) = (m_{i1})_{3\times1},$$

where

$$\begin{aligned}
m_{11} = {} & \frac{2}{k(1+\alpha y)}\left(-r + \frac{hky(c+my)^2(1+\alpha y)(1+h(c+my)\beta I)}{(1+h(c+my)(s+\beta I))^3}\right) v_1^2 \\
& +2v_1\left[\left(-\frac{r}{k(1+\alpha y)} + \frac{hy(c+my)^2\beta(1+h(c+my)(-s+\beta I))}{(1+h(c+my)(s+\beta I))^3} - \delta\right) v_2\right. \\
& \left.+\left(\frac{r(-k+2s+I)\alpha}{k(1+\alpha y)^2} + \frac{2hmsy(c+my)}{(1+h(c+my)(s+\beta I))^3} + \frac{-my+hs(c+my)^2}{(1+h(c+my)(s+\beta I))^2} - \frac{(c+my)}{(1+h(c+my)(s+\beta I))}\right) v_3\right] \\
& +s\left[-\frac{2h^2y(c+my)^3\beta^2v_2^2}{(1+h(c+my)(s+\beta I))^3} + 2\left(\frac{r\alpha}{k(1+\alpha y)^2}\right.\right. \\
& \left.+\frac{h(c+my)\beta(c+c^2h(s+\beta I)+2chmy(s+\beta I)+my(3+hmy(s+\beta I)))}{(1+h(c+my)(s+\beta I))^3}\right) v_2v_3 \\
& \left.+2\left(\frac{r(k-s-I)\alpha^2}{k(1+\alpha y)^3} - \frac{m(1+ch(s+\beta I))}{(1+h(c+my)(s+\beta I))^3}\right) v_3^2\right]
\end{aligned}$$

$$\begin{aligned}
m_{21} = {} & \frac{2}{(1+h(c+my)(s+\beta I))^3}[-h^2Iy(c+my)^3v_1^2 + hy(c+my)^2(1+hs(c+my))\beta v_2^2 \\
& -(c + c^3h^2s(s+\beta I) + my(2+hmsy(3+hmy(s+\beta I))) \\
& +c^2h(\beta I + s(2+3hmy(s+\beta I))) + chmy(\beta I + s(5+3hmy(s+\beta I))))v_2v_3 \\
& -mI(1+ch(s+\beta I))v_3^2 + v_1((hy(c+my)^2(1+h(c+my)(s+\beta I)) \\
& +(1+h(c+my)(s+\beta I))^3\delta)v_2 + hI(c+my)(c+c^2h(s+\beta I) \\
& +2chmy(s+\beta I) + my(3+hmy(s+\beta I)))v_3)]
\end{aligned}$$

$$\begin{aligned}
m_{31} = {} & \frac{1}{(1+h(c+my)(s+\beta I))^3}[-2(1+h(c+my)(s+\beta I))^3\mu v_1v_3 \\
& +2b_2(h^2Iy(c+my)^3v_1^2 - hy(c+my)^2(1+hs(c+my))\beta v_2^2 \\
& +(c + c^3h^2s(s+\beta I) + my(2+hmsy(3+hmy(s+\beta I))) \\
& +c^2h(\beta I + s(2+3hmy(s+\beta I))) + chmy(\beta I + s(5+3hmy(s+\beta I))))v_2v_3 \\
& +mI(1+ch(s+\beta I))v_3^2 + h(c+my)v_1(-y(c+my)(1+h(c+my)(s+\beta I))v_2 \\
& -I(c+c^2h(s+\beta I) + 2chmy(s+\beta I) + my(3+hmy(s+\beta I)))v_3)) \\
& -2b_1(hy(c+my)^2(1+hI(c+my)\beta)v_1^2 + v_1(hy(c+my)^2\beta(1-h(c+my)(s+\beta I))v_2 \\
& -(c+2my+chs(c+my) + hI(c+my)(c^2hs + 2c(1+hmsy) \\
& +my(3+hmsy))\beta + h^2I^2(c+my)^3\beta^2)v_3) \\
& +s(-h^2y(c+my)^3\beta^2v_2^2 + h(c+my)\beta(c+c^2h(s+\beta I) + 2chmy(s+\beta I) \\
& +my(3+hmy(s+\beta I)))v_2v_3 - m(1+ch(s+\beta I))v_3^2))]
\end{aligned}$$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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