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HUNTING COOPERATION AMONG PREDATORS EFFECTS ON THE PERSISTENCE AND BIFURCATION OF FOOD-WEB ECO-EPIDEMIOLOGICAL MODEL WITH ADDITIONAL FOOD TO PREDATORS

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Abstract: In this paper, the local bifurcation's occurrence conditions have been established for a food web eco-epidemiological model of four species, including first prey, second prey, susceptible predator, and infected predator with fear effect, and internal competition among prey populations, incorporating additional food resources and hunting cooperation among predators, and disease dynamics with treatment in the predator population with Holling type-II and Lotka-Volterra functional response. This model has fifteen equilibria, the saddle-node bifurcation have been shown close to the interior equilibrium points P_{13} and P_{14}, also at the first-second prey-free equilibrium point P_7, and the second prey-free equilibrium points P_9 and P_{10} a transcritical bifurcation occurred, while at the preys-infected predator-free equilibrium point P_3, the predators-free equilibrium point P_4, the second prey-infected predator-free equilibrium point P_5, the first prey-infected predator-free equilibrium point P_6, the infected predator-free equilibrium point P_8, first prey-free equilibrium points P_{11} and P_{12} have a transcritical and pitchfork bifurcations. Furthermore, conditions for Hopf bifurcation close to positive points P_{13} and P_{14} have also been examined. Numerical results for the set of hypothetical parameters support our analytical results regarding the persistence of this model and the occurrence of bifurcation using Mathematica.

Keywords: eco-epidemiological; hunting cooperation; additional food; bifurcation.
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1. INTRODUCTION

Mathematical modeling is a powerful tool used to understand complex phenomena and systems by representing them using mathematical concepts. as it allows us to study the behavior of complex systems and predict their outcomes under certain conditions, results of stability of equilibria, bifurcation and persistence, see [1-6] and the reference therein. Also hold great significance in eco-epidemiology, where the study of the dynamics of infectious diseases intersects with ecological and environmental factors.

Moreover, prey-predator model analysis is a key component in understanding the predation relationships between species in an ecosystem [7,8]. Since they need to eat to stay alive, predators usually try to get better at catching and killing prey because it will help them survive longer. Cooperative hunting is a common strategy used by several animals to improve their ability to capture and kill prey [9–11].

On the other hand, incorporating additional food sources in mathematical models introduces key ecological realism and flexibility into the model, these richer, more realistic models offer greater insights into ecological processes, allowing for better predictive power and providing valuable information for conservation efforts, ecosystem management, and understanding the resilience of ecosystems in the face of natural or anthropogenic disturbances. Many authors have merged eco-epidemiological prey-predator models with hunting cooperation, fear, additional food and treatment see [12-15].

Furthermore, eco-epidemiology looks at the interplay between host populations, infections, and ecosystems, and these mathematical concepts help explain how diseases spread, persist, or fade within populations, as well as the conditions under which these dynamics can change abruptly or stabilize [16,17].

Recent research also emphasizes bifurcation analysis as a powerful tool for exploring how small changes in parameters can lead to significant shifts in the system, which aligns with your interest in studying regime shifts in ecological and epidemiological systems, for example [18-20] and the reference therein.

The study of hunting cooperation among predators and its effects on food-web dynamics is an important topic in ecological modeling, especially in the context of eco-epidemiological models that also incorporate disease transmission and additional food sources. Recently, Shawka and Majeed [21] proposed and analyzed an eco-epidemiological model with fear, internal competition in the prey populations, and hunting cooperation, additional food, and a treatment in the predator's population where the predators feed on prey using two different types of functional responses. The purpose of this study is to test the analytic findings with numerical simulation and to establish the conditions of Hopf bifurcation near the positive equilibrium point and local bifurcation near the equilibrium points of a mathematical model given in [21].

2. MATHEMATICAL MODEL

A food web eco-epidemiological model comprising two preys, susceptible and infected predator with treatment, has been proposed and formulation in [21], as the following.

$$\left.\begin{aligned}
\frac{dL_1}{dT} &= \frac{a_1 L_1}{1+f_1 L_3} - b_1 L_1^2 - \frac{B_1 L_1 L_3}{C+\alpha\eta A+L_1},\\
\frac{dL_2}{dT} &= \frac{a_2 L_2}{1+f_2 L_3} - b_2 L_2^2 - (\rho + hL_3) L_2 L_3,\\
\frac{dL_3}{dT} &= a_3 L_3\left(1-\frac{L_3+L_4}{K}\right) + \frac{C_1(L_1+\eta A)L_3}{C+\alpha\eta A+L_1} + C_2(\rho+hL_3)L_2L_3 - (d+BL_4)L_3 + \frac{\gamma L_4}{\sigma+L_4},\\
\frac{dL_4}{dT} &= BL_3L_4 - \delta L_4 - \frac{\gamma L_4}{\sigma+L_4}.
\end{aligned}\right\} \qquad (1)$$

where $L_1(T), L_2(T), L_3(T), L_4(T)$ represents the total population density at time T of the first prey, second prey, susceptible predator, and infected predator, respectively, and the following assumptions have been assumed in order to construct the model:

1. It is assumed that a_1, a_2 are the intrinsic growth rates of the first and the second prey, respectively and there is an internal competition between their populations with b_1, b_2 rates, while f_1 and f_2 the fear rates of the first and second prey species from the susceptible predator, respectively.
2. The susceptible predator L_3 is capable of reproducing in logistic growth with carrying capacity K > 0, and intrinsic growth rate $a_3 > 0$, it is assumed that the susceptible predator consumes the first prey according to Holling type II functional response with attack rate $B_1 > 0$, half saturation rate $C \geq 1$, and with additional food A > 0 and the ratio of search rate for additional food is $\eta > 0$. It is assumed that the maximum growth rate of the predator when it consumes the prey and additional food is α.
3. The susceptible predator consumes the second prey with Lotka-Volterra type of functional response with attack rate ρ, hunting cooperation rate h, and the conversion rate constants $C_1 > 0$ and $0 < C_2 < 1$, respectively, Moreover, the susceptible predator faces the natural death at a rate $d > 0$.

4. Finally, it is assumed that the disease transmission between the predator population at a rate *B*, with the medical resource for treatment rate γ and the saturation factor that measure the effect of the delay in treatment for the infected with rate σ. Also the infected predator faces the natural death due to the effect of diseases at a rate $\delta > 0$.

3. LOCAL BIFURCATION ANALYSIS

In this section, it has been investigated how altering the parameter values affects the system's (1) dynamic behavior close to each equilibrium point (EP). Recall that the presence of the system's (1) non-hyperbolic equilibrium point is a prerequisite for bifurcation, but it is not a sufficient one. Consequently, an application of Sotomayor's theorem [22] is appropriate in the following theorems. Now, according to the Jacobian matrix $J(L_1, L_2, L_3, L_4)$ of system (1), which is given in [21].

$$J = [b_{ij}]_{4\times4}, \tag{2}$$

$$b_{11} = \frac{a_1}{1+f_1L_3} - 2b_1L_1 - \frac{(C+\alpha\eta A)B_1L_3}{(C+\alpha\eta A+L_1)^2}, \quad b_{12} = b_{14} = 0, \quad b_{13} = \frac{-a_1f_1L_1}{(1+f_1L_3)^2} - \frac{B_1L_1}{(C+\alpha\eta A+L_1)} < 0,$$

$$b_{21} = b_{24} = 0, \quad b_{22} = \frac{a_2}{1+f_2L_3} - 2b_2L_2 - (\rho + hL_3)L_3, \quad b_{23} = \frac{-a_2f_2L_2}{(1+f_1L_3)^2} - (\rho + 2hL_3)L_2,$$

$$b_{31} = \frac{C_1\big(C+\eta A(\alpha-1)\big)L_3}{(C+\alpha\eta A+L_1)^2}, \quad b_{32} = C_2(\rho + hL_3)L_3,$$

$$b_{33} = a_3 - \frac{a_3}{K}(2L_3 + L_4) + \frac{C_1(L_1+\eta A)}{(C+\alpha\eta A+L_1)} + C_2(\rho + 2hL_3)L_2 - (d + BL_4),$$

$$b_{34} = -\left(\frac{a_3}{K} + B\right)L_3 + \frac{\sigma\gamma}{(\sigma+L_4)^2}, \quad b_{41} = b_{42} = 0, \quad b_{43} = BL_4, \quad b_{44} = BL_3 - \delta - \frac{\sigma\gamma}{(\sigma+L_4)^2}.$$

It is obvious to confirm that for every vector that is nonzero $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ we have:

$$D^2F(X,\mu)(\psi, \psi) = [A_{i1}]_{4x1}, \tag{3}$$

$$A_{11} = 2\left[\left(-b_1 + \frac{B_1(C+\alpha\eta A)L_3}{(C+\alpha\eta A+L_1)^3}\right)(\psi_1)^2 - \left(\frac{a_1f_1}{(1+f_1L_3)^2} + \frac{B_1(C+\alpha\eta A)}{(C+\alpha\eta A+L_1)^2}\right)\psi_1\psi_3 + \left(\frac{a_1{f_1}^2L_1}{(1+f_1L_3)^3}\right)(\psi_3)^2\right],$$

$$A_{21} = -2\left[b_2(\psi_1)^2 + \left(\frac{a_2f_2}{(1+f_2L_3)^2} + (\rho + 2hL_3)\right)\psi_2\psi_3 + \left(h - \frac{a_2{f_2}^2}{(1+f_2L_3)^3}\right)L_2(\psi_3)^2\right],$$

$$A_{31} = 2\Bigg[\frac{-C_1\big(C+\eta A(\alpha-1)\big)L_3}{(C+\alpha\eta A+L_1)^3}(\psi_1)^2 + \frac{C_1\big(C+\eta A(\alpha-1)\big)}{(C+\alpha\eta A+L_1)^2}\psi_1\psi_3 + C_2(\rho + 2hL_3)\psi_2\psi_3 +$$
$$\left(C_2hL_2 - \frac{a_3}{K}\right)(\psi_3)^2 - \left(\frac{a_3}{K} + B\right)\psi_3\psi_4 - \left(\frac{\sigma\gamma}{(\sigma+L_4)^3}\right)(\psi_4)^2\Bigg],$$

$$A_{41} = 2\left[B\psi_3\psi_4 + \frac{\sigma\gamma}{(\sigma+L_4)^3}(\psi_4)^2\right].$$

And

$$D^3F(X,\mu)(\psi\,,\psi\,,\psi) = [M_{i1}]_{4x1}, \tag{4}$$

$$M_{11} = 6\left[\frac{-(C+\alpha\eta A)B_1L_3}{(C+\alpha\eta A+L_1)^4}(\psi_1)^3 + \frac{(C+\alpha\eta A)B_1}{(C+\alpha\eta A+L_1)^3}(\psi_1)^2\psi_3 + \frac{a_1{f_1}^2}{(1+f_1L_3)^3}\psi_1(\psi_3)^2 - \frac{a_1{f_1}^3}{(1+f_1L_3)^4}(\psi_3)^3\right],$$

$$M_{21} = 6\left[\left(\frac{a_2{f_2}^2}{(1+f_2L_3)^3} - h\right)\psi_2(\psi_3)^2 - \left(\frac{a_2{f_2}^3L_2}{(1+f_2L_3)^4}\right)(\psi_3)^3\right],$$

$$M_{31} = 6\left[\frac{C_1(C+\eta A(\alpha-1))L_3}{(C+\alpha\eta A+L_1)^4}(\psi_1)^3 - \frac{C_1(C+\eta A(\alpha-1))}{(C+\alpha\eta A+L_1)^3}(\psi_1)^2\psi_3 + C_2h\psi_2(\psi_3)^2 + \frac{\sigma\gamma}{(\sigma+L_4)^4}(\psi_4)^3\right],$$

$$M_{41} = -6\left[\frac{\sigma\gamma}{(\sigma+L_4)^4}(\psi_4)^3\right],$$

where $X = (L_1\,,L_2\,,L_3\,,L_4)^T$ and μ is any parameter.

Theorem 1. system (1) at the (EP) $P_3 = (0,\ 0,\ \bar{L}_3,\ 0)$ with the parameter $\bar{B} = \frac{\delta\sigma+\gamma}{\sigma\bar{L}_3}$ has a transcritical and pitchfork bifurcation if the following condition holds

$$\bar{B}\bar{W}_1 \neq \frac{\gamma}{\sigma^2} \tag{5}$$

Where $\bar{W}_1 = \frac{(C+\alpha\eta A)(a_3\bar{L}_3+\delta K)}{[(a_3-d)(C+\alpha\eta A)+C_1\eta A]}$. While saddle-node bifurcation cannot be occurs at $\bar{B}$.

Proof. From the Jacobian matrix J_3 which is given in Eq.(86) in [21] , system (1) at the (EP) P_3 has eigenvalue say (λ_{3L_4}) equal to zero at $B = \bar{B}$, then J_3 with $B = \bar{B}$ becomes $\bar{J}_3 = J_3(P_3\,,\bar{B}) = [\bar{\Omega}_{ij}]_{4\times 4}$,

where $\bar{\Omega}_{ij} = \Omega_{ij},\ \ i,j = 1,2,3,4$ which is given in Eq.(86) in [21] accept

$$\bar{\Omega}_{34} = -\left(\frac{a_3}{K}\bar{L}_3 + \delta\right) \text{ and } \bar{\Omega}_{44} = 0.$$

Now, let $\bar{\psi}^{[3]} = \left(\bar{\psi}_1^{[3]}\,,\bar{\psi}_2^{[3]}\,,\bar{\psi}_3^{[3]},\bar{\psi}_4^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalue $(\lambda_{3L_4}) = 0$.

Thus $(\bar{J}_3 - \lambda_{3L_4}I)\bar{\psi}^{[3]} = 0$, that gives $\bar{\psi}^{[3]} = \left(0\,,0\,,-\bar{W}_1\bar{\psi}_4^{[3]}\,,\bar{\psi}_4^{[3]}\right)^T$ where $\bar{W}_1$ given in the state of theorem and $\bar{\psi}_4^{[3]} \neq 0$ is any real number.

Let $\bar{\Omega}^{[3]} = \left(\bar{\Omega}_1^{[3]}\,,\bar{\Omega}_2^{[3]}\,,\bar{\Omega}_3^{[3]},\bar{\Omega}_4^{[3]}\right)^T$ be the eigenvector of $\bar{J}_3^T$ for $\lambda_{3L_4} = 0$.

Then we get $(\bar{J}_3^T - \lambda_{3L_4}I)\bar{\Omega}^{[3]} = 0$ then by solving this equation for $\bar{\Omega}^{[3]}$ we get

$$\bar{\Omega}^{[3]} = \left(0\,,0\,,0\,,\bar{\Omega}_4^{[3]}\right)^T, \quad \text{where } \bar{\Omega}_4^{[3]} \neq 0 \text{ is any real number.}$$

Now, consider $\frac{\partial f}{\partial B} = f_B(X\,,B) = \left(\frac{\partial f_1}{\partial B}, \frac{\partial f_2}{\partial B}, \frac{\partial f_3}{\partial B}, \frac{\partial f_4}{\partial B}\right)^T = (0\,,0\,,-L_3L_4\,,L_3L_4)^T$.

So, $f_{\mathrm{B}}(P_3\,,\bar{B}) = (0\,,0\,,0,0)^T$ and hence $\left(\bar{\Omega}^{[3]}\right)^T f_{\mathrm{B}}(P_3\,,\bar{B}) = 0$.

Therefore, using Sotomayor's theorem we get that the saddle-node bifurcation's conditions cannot be satisfied.

While the transcritical bifurcation's first condition is satisfied. Now, since

$$Df_B(X\,,B) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -L_4 & -L_3 \\ 0 & 0 & L_4 & L_3 \end{bmatrix},$$

where $Df_B(X\,,B)$ is the derivative of $f_B(X\,,B)$ with respect to $X = (L_1\,,L_2\,,L_3\,,L_4)^T$.

Further, it is observed that $$Df_B(P_3\,,\bar{B})\bar{\psi}^{[3]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{L}_3 \\ 0 & 0 & 0 & \bar{L}_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\bar{W}_1\bar{\psi}_4^{[3]} \\ \bar{\psi}_4^{[3]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{W}_1\bar{L}_3\bar{\psi}_4^{[3]} \\ \bar{L}_3\bar{\psi}_4^{[3]} \end{bmatrix},$$ so

$$\left(\bar{\Omega}^{[3]}\right)^T\left[Df_{\mathrm{B}}(P_3\,,\bar{B})\bar{\psi}^{[3]}\right] = \left(0\,,0\,,0\,,\bar{\Omega}_4^{[3]}\right)\left(0\,,0\,,\bar{W}_1\bar{L}_3\bar{\psi}_4^{[3]}\,,\bar{L}_3\bar{\psi}_4^{[3]}\right)^T = \bar{L}_3\bar{\Omega}_4^{[3]}\bar{\psi}_4^{[3]} \neq 0.$$

Moreover, by substituting $\bar{\psi}^{[3]}$ in (3) we get:

$$D^2 f_{\mathrm{B}}(P_3\,,\bar{B})\left(\bar{\psi}^{[3]}\,,\bar{\psi}^{[3]}\right) = \begin{bmatrix} 0 \\ 0 \\ -2\left[\frac{a_3}{K}\bar{W}_1{}^2 - \left(\frac{a_3}{K} + \bar{B}\right)\bar{W}_1 + \frac{\gamma}{\sigma^2}\right]\left(\bar{\psi}_4^{[3]}\right)^2 \\ 2\left[-\bar{B}\bar{W}_1 + \frac{\gamma}{\sigma^2}\right]\left(\bar{\psi}_4^{[3]}\right)^2 \end{bmatrix}$$

Hence, it obtains that $\left(\bar{\Omega}^{[3]}\right)^T\left[D^2 f_{\mathrm{B}}(P_3\,,\bar{B})\left(\bar{\psi}^{[3]}\,,\bar{\psi}^{[3]}\right)\right] = 2\left[-\bar{B}\bar{W}_1 + \frac{\gamma}{\sigma^2}\right]\bar{\Omega}_4^{[3]}\left(\bar{\psi}_4^{[3]}\right)^2 \neq 0$, under condition (5).

Thus, by using Sotomayor's theorem P_3 has a transcritical bifurcation at the parameter $\bar{B}$.

While if condition (5) not holds then there is no a transcritical bifurcation and by substituting

$\bar{\psi}^{[3]}$ in (4) we get $D^3 f_{\mathrm{B}}(P_3, \bar{B})(\bar{\psi}^{[3]}, \bar{\psi}^{[3]}, \bar{\psi}^{[3]}) = \begin{bmatrix} 6\frac{a_1 {f_1}^3}{(1+f_1 L_3)^4}\left(\bar{W}_1 \bar{\psi}_4^{[3]}\right)^3 \\ 0 \\ 6\frac{\gamma}{\sigma^3}\left(\bar{\psi}_4^{[3]}\right)^3 \\ -6\frac{\gamma}{\sigma^3}\left(\bar{\psi}_4^{[3]}\right)^3 \end{bmatrix}$, so

$$\left(\bar{\Omega}^{[3]}\right)^T \left[D^3 f_{\mathrm{B}}(P_3, \bar{B})(\bar{\psi}^{[3]}, \bar{\psi}^{[3]}, \bar{\psi}^{[3]})\right] = -6\frac{\gamma}{\sigma^3}\, \bar{\Omega}_4^{[3]} \left(\bar{\psi}_4^{[3]}\right)^3 \neq 0.$$

Hence again, by using Sotomayor's theorem system (1) at P_3 has pitchfork bifurcation with the parameter $\bar{B}$.

Theorem 2. Assume that condition (99), which is given in [21] and the following condition holds:

$$a_3 > C_2 h K \breve{L}_2 \tag{6}$$

Then system (1) at the (EP) $P_4 = (\breve{L}_1,\ \breve{L}_2,\ 0,\ 0)$ where $\breve{L}_1 = \frac{a_1}{b_1}$ and $\breve{L}_2 = \frac{a_2}{b_2}$ with the parameter $\breve{d} = a_3 + \frac{C_1(a_1 + \eta A b_1)}{a_1 + b_1(C + \alpha\eta A)} + \frac{a_2 C_2 \rho}{b_2}$ has a transcritical and pitchfork bifurcation, while saddle-node bifurcation cannot be occurs at $\breve{d}$.

Proof. From the Jacobian matrix J_4 which is given in Eq.(91) in [21] , system (1) at the (EP) P_4 has eigenvalue say (λ_{4L_3}) equal to zero at $d = \breve{d}$, then J_4 with $d = \breve{d}$ becomes

$$\breve{J}_4 = J_4(P_4, \breve{d}) = \begin{bmatrix} -a_1 & 0 & -\left(\frac{{a_1}^2 f_1}{b_1} + \frac{a_1 B_1}{a_1 + b_1(C + \alpha\eta A)}\right) & 0 \\ 0 & -a_2 & \frac{-a_2(a_2 f_2 + \rho)}{b_2} & 0 \\ 0 & 0 & 0 & \frac{\gamma}{\sigma} \\ 0 & 0 & 0 & -\left(\delta + \frac{\gamma}{\sigma}\right) \end{bmatrix}.$$

Now, let $\breve{\psi}^{[4]} = \left(\breve{\psi}_1^{[4]}, \breve{\psi}_2^{[3]}, \breve{\psi}_3^{[4]}, \breve{\psi}_4^{[4]}\right)^T$ be the eigenvector corresponding to the eigenvalue $(\lambda_{4L_3}) = 0$.

Thus $(\breve{J}_4 - \lambda_{4L_3} I)\breve{\psi}^{[4]} = 0$, that gives $\breve{\psi}^{[4]} = \left(-\breve{W}_1 \breve{\psi}_3^{[4]}, -\breve{W}_2 \breve{\psi}_3^{[4]}, \breve{\psi}_3^{[4]}, 0\right)^T$ where $\breve{W}_1 = \left(\frac{a_1 f_1}{b_1} + \frac{B_1}{a_1 + b_1(C + \alpha\eta A)}\right)$, $\breve{W}_2 = \frac{(a_2 f_2 + \rho)}{b_2}$ and $\breve{\psi}_3^{[4]} \neq 0$ is any real number.

Let $\breve{\Omega}^{[4]} = \left(\breve{\Omega}_1^{[4]}, \breve{\Omega}_2^{[4]}, \breve{\Omega}_3^{[4]}, \breve{\Omega}_4^{[4]}\right)^T$ be the eigenvector of $\breve{J}_4^T$ for $\lambda_{4L_3} = 0$.

Then we get $\left(\breve{J}_4^T - \lambda_{4L_3} I\right)\breve{\Omega}^{[4]} = 0$ then by solving this equation for $\breve{\Omega}^{[4]}$ we get

$\breve{\Omega}^{[4]} = \left(0, 0, \breve{\Omega}_3^{[4]}, \frac{\gamma}{\sigma\delta+\gamma}\breve{\Omega}_3^{[4]}\right)^T$ where $\breve{\Omega}_3^{[4]} \neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial d} = f_d(X, d) = \left(\frac{\partial f_1}{\partial d}, \frac{\partial f_2}{\partial d}, \frac{\partial f_3}{\partial d}, \frac{\partial f_4}{\partial d}\right)^T = (0, 0, -L_3, 0)^T$.

So, $f_d\left(P_4, \breve{d}\right) = (0, 0, 0, 0)^T$ and hence $\left(\breve{\Omega}^{[4]}\right)^T f_d\left(P_4, \breve{d}\right) = 0$.

Therefore, using Sotomayor's theorem we get that the saddle-node bifurcation's conditions cannot be satisfied.

While the transcritical bifurcation's first condition is satisfied. Now, since

$$Df_d(X, d) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_d(X, d)$ is the derivative of $f_d(X, d)$ with respect to $X = (L_1, L_2, L_3, L_4)^T$.

Further, it is observed that $Df_d\left(P_4, \breve{d}\right)\breve{\psi}^{[4]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\breve{W}_1\breve{\psi}_3^{[4]} \\ -\breve{W}_2\breve{\psi}_3^{[4]} \\ \breve{\psi}_3^{[4]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\breve{\psi}_3^{[4]} \\ 0 \end{bmatrix}$, so

$$\left(\breve{\Omega}^{[4]}\right)^T\left[Df_d\left(P_4, \breve{d}\right)\breve{\psi}^{[4]}\right] = \left(0, 0, \breve{\Omega}_3^{[4]}, \frac{\gamma}{\sigma\delta+\gamma}\breve{\Omega}_3^{[4]}\right)\left(0, 0, -\breve{\psi}_3^{[4]}, 0\right)^T = -\breve{\Omega}_3^{[4]}\breve{\psi}_3^{[4]} \neq 0.$$

Moreover, by substituting $\breve{\psi}^{[4]}$ in (3) we get $D^2 f_d\left(P_4, \breve{d}\right)\left(\breve{\psi}^{[4]}, \breve{\psi}^{[4]}\right) = [\breve{A}_{i1}]_{4x1}$.

$$\breve{A}_{11} = 2\left[-b_1\breve{W}_1^{\,2} + \left(a_1 f_1 + \frac{B_1(C+\alpha\eta A)}{(C+\alpha\eta A+\breve{L}_1)^2}\right)\breve{W}_1 + a_1 f_1^2 \breve{L}_1\right]\left(\breve{\psi}_3^{[4]}\right)^2$$

$$\breve{A}_{21} = -2\left[b_2\breve{W}_2^{\,2} - (a_2 f_2 + \rho)\breve{W}_2 + (h - a_1 f_2^2)\breve{L}_2\right]\left(\breve{\psi}_3^{[4]}\right)^2$$

$$\breve{A}_{31} = -2\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)}{(C+\alpha\eta A+\breve{L}_1)^2}\breve{W}_1 + C_2\rho\breve{W}_2 - \left(\frac{C_2 h K\breve{L}_2 - a_3}{K}\right)\right]\left(\breve{\psi}_3^{[4]}\right)^2$$

$$\breve{A}_{41} = 0$$

Hence, it obtains that

$$\left(\breve{\Omega}^{[4]}\right)^T\left[D^2 f_d\left(P_4,\breve{d}\right)\left(\breve{\psi}^{[4]},\breve{\psi}^{[4]}\right)\right]=-2\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)}{(C+\alpha\eta A+\breve{L}_1)^2}\breve{W}_1+C_2\rho\breve{W}_2-\left(\frac{C_2hK\breve{L}_2-a_3}{K}\right)\right]\breve{\Omega}_3^{[4]}\left(\breve{\psi}_3^{[4]}\right)^2\neq 0,$$

under condition (99) which is given in [21], and condition (6). Thus, by using Sotomayor's theorem P_4 has a transcritical bifurcation at the parameter $\breve{d}$.

While if condition (6) not holds then there is no a transcritical bifurcation and by substituting $\breve{\psi}^{[4]}$ in (4) we get $D^3 f_d\left(P_4,\breve{d}\right)\left(\breve{\psi}^{[4]},\breve{\psi}^{[4]},\breve{\psi}^{[4]}\right)=[\breve{M}_{i1}]_{4x1}$.

$$\breve{M}_{11}=6\left[\frac{B_1(C+\alpha\eta A)}{(C+\alpha\eta A+\breve{L}_1)^3}\breve{W}_1{}^2-a_1f_1{}^2\breve{W}_1-a_1f_1{}^3\right]\left(\breve{\psi}_3^{[4]}\right)^3,$$

$$\breve{M}_{21}=-6\left[\left(a_2f_2{}^2-h\right)\breve{W}_2+a_2f_2^3\breve{L}_2\right]\left(\breve{\psi}_3^{[4]}\right)^3,$$

$$\breve{M}_{31}=-6\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)}{(C+\alpha\eta A+\breve{L}_1)^3}\breve{W}_1{}^2+C_2h\breve{W}_2\right]\left(\breve{\psi}_3^{[4]}\right)^3,$$

$$\breve{M}_{41}=0.$$

Then

$$\left(\breve{\Omega}^{[4]}\right)^T\left[D^3 f_d\left(P_4,\breve{d}\right)\left(\breve{\psi}^{[4]},\breve{\psi}^{[4]},\breve{\psi}^{[4]}\right)\right]=-6\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)}{(C+\alpha\eta A+\breve{L}_1)^4}\breve{W}_1{}^2+C_2h\breve{W}_2\right]\breve{\Omega}_3^{[4]}\left(\breve{\psi}_3^{[4]}\right)^3\neq 0,$$

under condition (99) which is given in [21].

Hence again , by using Sotomayor's theorem system (1) at P_4 has pitchfork bifurcation with the parameter $\breve{d}$.

Theorem 3. Assume that conditions (3, 97, 98, 99), which are given in [21] and the following condition holds

$$\bar{\bar{B}}\bar{\bar{W}}_2\neq\frac{\gamma}{\sigma^2}, \tag{7}$$

where

$$\bar{\bar{W}}_2=\frac{\bar{\bar{c}}_{11}\bar{\bar{c}}_{34}}{(-\bar{\bar{c}}_{31}\bar{\bar{c}}_{13}+\bar{\bar{c}}_{11}\bar{\bar{c}}_{33})}.$$

Then system (1) at the (EP) $P_5=\left(\bar{\bar{L}}_1,\ 0,\ \bar{\bar{L}}_3,\ 0\right)$ with the parameter $\bar{\bar{B}}=\frac{\delta\sigma+\gamma}{\sigma\bar{\bar{L}}_3}$ has (transcritical and pitchfork) bifurcation, while saddle-node bifurcation cannot be occur at $\bar{\bar{B}}$.

Proof. From the Jacobian matrix J_5 which is given in Eq.(93) in [21], system (1) at the (EP) P_5 has eigenvalue say $\left(\lambda_{5L_4}\right)$ equal to zero at $B=\bar{\bar{B}}$, then J_5 with $B=\bar{\bar{B}}$ becomes

$\bar{\bar{J}}_5=J_5\left(P_5,\bar{\bar{B}}\right)=\left[\bar{\bar{c}}_{ij}\right]_{4\times 4}$, where $\bar{\bar{c}}_{ij}=c_{ij}$, $i,j=1,2,3,4$ which is given in Eq.(93) in [21] accept $\bar{\bar{c}}_{34}=-\left(\frac{a_3}{K}\bar{\bar{L}}_3+\delta\right)<0$ and $\bar{\bar{c}}_{44}=0$.

Now, let $\bar{\bar{\psi}}^{[5]} = \left(\bar{\bar{\psi}}_1^{[5]}, \bar{\bar{\psi}}_2^{[5]}, \bar{\bar{\psi}}_3^{[5]}, \bar{\bar{\psi}}_4^{[5]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\left(\lambda_{5L_4}\right) = 0$.

Thus $\left(\bar{\bar{J}}_5 - \lambda_{5L_4} I\right)\bar{\bar{\psi}}^{[5]} = 0$, that gives $\bar{\bar{\psi}}^{[5]} = \left(\bar{\bar{W}}_1 \bar{\bar{\psi}}_4^{[5]}, 0, -\bar{\bar{W}}_2 \bar{\bar{\psi}}_4^{[5]}, \bar{\bar{\psi}}_4^{[5]}\right)^T$ where $\bar{\bar{W}}_1 = \frac{\bar{\bar{c}}_{13}\bar{\bar{c}}_{34}}{(-\bar{\bar{c}}_{13}\bar{\bar{c}}_{31} + \bar{\bar{c}}_{11}\bar{\bar{c}}_{33})}$, $\bar{\bar{W}}_2$ given in the state of theorem and $\bar{\bar{\psi}}_4^{[5]} \neq 0$ is any real number.

Let $\bar{\bar{\Omega}}^{[5]} = \left(\bar{\bar{\Omega}}_1^{[5]}, \bar{\bar{\Omega}}_2^{[5]}, \bar{\bar{\Omega}}_3^{[5]}, \bar{\bar{\Omega}}_4^{[5]}\right)^T$ be the eigenvector of $\bar{\bar{J}}_5^T$ for $\lambda_{5L_4} = 0$.

Then we get $\left(\bar{\bar{J}}_5^T - \lambda_{5L_4} I\right)\bar{\bar{\Omega}}^{[5]} = 0$ then by solving this equation for $\bar{\bar{\Omega}}^{[5]}$ we get

$\bar{\bar{\Omega}}^{[5]} = \left(0, 0, 0, \bar{\bar{\Omega}}_4^{[5]}\right)^T$ where $\bar{\bar{\Omega}}_4^{[5]} \neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial B} = f_B(X, B) = \left(\frac{\partial f_1}{\partial B}, \frac{\partial f_2}{\partial B}, \frac{\partial f_3}{\partial B}, \frac{\partial f_4}{\partial B}\right)^T = (0, 0, -L_3 L_4, L_3 L_4)^T$.

So, $f_B\left(P_5, \bar{\bar{B}}\right) = (0, 0, 0, 0)^T$ and hence $\left(\bar{\bar{\Omega}}^{[5]}\right)^T f_B\left(P_5, \bar{\bar{B}}\right) = 0$.

Therefore, using Sotomayor's theorem we get that the saddle-node bifurcation's conditions cannot be satisfied.

While the transcritical bifurcation's first condition is satisfied. Now, since

$$Df_B(X, B) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -L_4 & -L_3 \\ 0 & 0 & L_4 & L_3 \end{bmatrix},$$

where $Df_B(X, B)$ is the derivative of $f_B(X, B)$ with respect to $X = (L_1, L_2, L_3, L_4)^T$.

Further, it is observed that $Df_B\left(P_5, \bar{\bar{B}}\right)\bar{\bar{\psi}}^{[5]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\bar{L}}_3 \\ 0 & 0 & 0 & \bar{\bar{L}}_3 \end{bmatrix} \begin{bmatrix} \bar{\bar{W}}_1 \bar{\bar{\psi}}_4^{[5]} \\ 0 \\ -\bar{\bar{W}}_2 \bar{\bar{\psi}}_4^{[5]} \\ \bar{\bar{\psi}}_4^{[5]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{\bar{W}}_2 \bar{\bar{L}}_3 \bar{\bar{\psi}}_4^{[5]} \\ \bar{\bar{L}}_3 \bar{\bar{\psi}}_4^{[5]} \end{bmatrix}$, so

$\left(\bar{\bar{\Omega}}^{[5]}\right)^T \left[Df_B\left(P_5, \bar{\bar{B}}\right)\bar{\bar{\psi}}^{[5]}\right] = \left(0, 0, 0, \bar{\bar{\Omega}}_4^{[5]}\right)\left(0, 0, \bar{\bar{W}}_2 \bar{\bar{L}}_3 \bar{\bar{\psi}}_4^{[5]}, \bar{\bar{L}}_3 \bar{\bar{\psi}}_4^{[5]}\right)^T = \bar{\bar{L}}_3 \bar{\bar{\Omega}}_4^{[5]} \bar{\bar{\psi}}_4^{[5]} \neq 0$.

Moreover, by substituting $\bar{\bar{\psi}}^{[5]}$ in (3) we get: $D^2 f_B\left(P_5, \bar{\bar{B}}\right)\left(\bar{\bar{\psi}}^{[5]}, \bar{\bar{\psi}}^{[5]}\right) = \left[\bar{\bar{A}}_{i1}\right]_{4x1}$

$$\bar{\bar{A}}_{11} = 2\left[\left(-b_1 + \frac{B_1(C+\alpha\eta A)\bar{\bar{L}}_3}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^3}\right)\bar{\bar{W}}_1^{\ 2} + \left(\frac{a_1 f_1}{\left(1+f_1\bar{\bar{L}}_3\right)^2} + \frac{B_1(C+\alpha\eta A)}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^2}\right)\bar{\bar{W}}_1\bar{\bar{W}}_2 + \frac{a_1 f_1^2\bar{\bar{L}}_1}{\left(1+f_1\bar{\bar{L}}_3\right)^3}\bar{\bar{W}}_2\right]\left(\bar{\bar{\psi}}_4^{[5]}\right)^2$$

$$\bar{\bar{A}}_{21} = 0$$

$$\bar{\bar{A}}_{31} = -2\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)\bar{\bar{L}}_3}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^3}\bar{\bar{W}}_1^{\ 2} + \frac{C_1\left(C+\eta A(\alpha-1)\right)}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)}\bar{\bar{W}}_1\bar{\bar{W}}_2 + \frac{a_3}{K}\bar{\bar{W}}_2^{\ 2} - \left(\frac{a_3}{K} + B\right)\bar{\bar{W}}_2 + \frac{\gamma}{\sigma^2}\right]\left(\bar{\bar{\psi}}_4^{[5]}\right)^2$$

$$\bar{\bar{A}}_{41} = 2\left[-\bar{\bar{B}}\bar{\bar{W}}_2 + \frac{\gamma}{\sigma^2}\right]\left(\bar{\bar{\psi}}_4^{[5]}\right)^2.$$

Hence, it obtains that $\left(\bar{\bar{\Omega}}^{[5]}\right)^T\left[D^2 f_B\left(P_5, \bar{\bar{B}}\right)\left(\bar{\bar{\psi}}^{[5]}, \bar{\bar{\psi}}^{[5]}\right)\right] = 2\left[-\bar{\bar{B}}\bar{\bar{W}}_2 + \frac{\gamma}{\sigma^2}\right]\bar{\bar{\Omega}}_4^{[5]}\left(\bar{\bar{\psi}}_4^{[5]}\right)^2 \neq 0$, under condition $(3, 97, 98, 99)$ which are given in [21], and condition (7). Thus, by using Sotomayor's theorem P_5 has a transcritical bifurcation at the parameter $\bar{\bar{B}}$.

While if conditions (7) not holds then there is no a transcritical bifurcation and by substituting $\bar{\bar{\psi}}^{[5]}$ in (4) we get $D^3 f_B\left(P_5, \bar{\bar{B}}\right)\left(\bar{\bar{\psi}}^{[5]}, \bar{\bar{\psi}}^{[5]}, \bar{\bar{\psi}}^{[5]}\right) = \left[\bar{\bar{M}}_{i1}\right]_{4x1}$.

$$\bar{\bar{M}}_{11} = 6\left[\frac{-(C+\alpha\eta A)B_1\bar{\bar{L}}_3}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^4}\left(\bar{\bar{W}}_1\right)^3 - \frac{(C+\alpha\eta A)B_1}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^3}\left(\bar{\bar{W}}_1\right)^2\bar{\bar{W}}_2\psi_2 + \frac{a_1 f_1^{\ 2}}{\left(1+f_1\bar{\bar{L}}_3\right)^3}\bar{\bar{W}}_1\left(\bar{\bar{W}}_2\right)^2 - \frac{a_1 f_1^{\ 3}}{\left(1+f_1\bar{\bar{L}}_3\right)^4}\left(\bar{\bar{W}}_2\right)^3\right]\left(\bar{\bar{\psi}}_4^{[5]}\right)^3,$$

$$\bar{\bar{M}}_{21} = 0,$$

$$\bar{\bar{M}}_{31} = 6\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)L_3}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^4}\left(\bar{\bar{W}}_1\right)^3 + \frac{C_1\left(C+\eta A(\alpha-1)\right)}{\left(C+\alpha\eta A+\bar{\bar{L}}_1\right)^3}\left(\bar{\bar{W}}_1\right)^2\bar{\bar{W}}_2 + \frac{\gamma}{\sigma^3}(\psi_4)^3\right]\left(\bar{\bar{\psi}}_4^{[5]}\right)^3,$$

$$\bar{\bar{M}}_{41} = -6\frac{\gamma}{\sigma^3}\left(\bar{\bar{\psi}}_4^{[5]}\right)^3.$$

Then

$$\left(\bar{\bar{\Omega}}^{[5]}\right)^T\left[D^3 f_B\left(P_5, \bar{\bar{B}}\right)\left(\bar{\bar{\psi}}^{[5]}, \bar{\bar{\psi}}^{[5]}, \bar{\bar{\psi}}^{[5]}\right)\right] = -6\frac{\gamma}{\sigma^3}\bar{\bar{\Omega}}_4^{[5]}\left(\bar{\bar{\psi}}_4^{[5]}\right)^3 \neq 0.$$

Hence again , by using Sotomayor's theorem system (1) at P_5 has pitchfork bifurcation with the parameter $B = \bar{\bar{B}}$.

Theorem 4. Assume that conditions $(3, 99, 104, 105)$, which are given in $[21]$ and the following conditions hold

$$\frac{\hat{L}_3}{(C+\alpha\eta A)} \leq \widehat{W}_2, \tag{8}$$

$$\left(b_1 + \frac{\hat{a}_1 f_1\widehat{W}_2}{(1+f_1\hat{L}_3)^2}\right) = \frac{B_1}{(C+\alpha\eta A)}\left(\frac{\hat{L}_3}{(C+\alpha\eta A)} - \widehat{W}_2\right), \tag{9}$$

$$1 < \frac{f_1\widehat{W}_2}{(1+f_1\hat{L}_3)} \tag{10}$$

where

$$\widehat{W}_2 = \frac{\hat{d}_{22}\hat{d}_{31}}{\hat{d}_{23}\hat{d}_{32}-\hat{d}_{22}\hat{d}_{33}}.$$

Then system (1) at the (EP) $P_6 = \left(0,\ \hat{L}_2,\ \hat{L}_3,\ 0\right)$ with the parameter $\hat{a}_1 = \frac{(1+f_1\hat{L}_3)B_1\hat{L}_3}{(C+\alpha\eta A)}$ has (transcritical and pitchfork) bifurcation, while saddle-node bifurcation cannot be occur at $\hat{a}_1$.

Proof. From the Jacobian matrix J_6 which is given in Eq.(100) in [21] , system (1) at the (EP) P_6, has eigenvalue say $\left(\lambda_{6L_1}\right)$ equal to zero at $a_1 = \hat{a}_1$, then J_6 with $a_1 = \hat{a}_1$ becomes $\hat{J}_6 = J_6\,(P_6\,, \hat{a}_1) = \left[\hat{d}_{ij}\right]_{4\times4}$, where $\hat{d}_{ij} = d_{ij}$; $i, j = 1,2,3,4$ which are given in Eq.(100) in [21] accept $\hat{d}_{11} = 0$.

Now, let $\hat{\psi}^{[6]} = \left(\hat{\psi}_1^{[6]}, \hat{\psi}_2^{[6]}, \hat{\psi}_3^{[6]}, \hat{\psi}_4^{[6]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\left(\lambda_{6L_1}\right) = 0$.

Thus $\left(\hat{J}_6 - \lambda_{6L_1}I\right)\hat{\psi}^{[6]} = 0$, that gives $\hat{\psi}^{[6]} = \left(\hat{\psi}_1^{[6]}, -\widehat{W}_1\hat{\psi}_1^{[6]}, \widehat{W}_2\hat{\psi}_1^{[6]}, 0\right)^T$ where $\hat{\psi}_1^{[6]} \neq 0$ is any real number, $\widehat{W}_1 = \frac{\hat{d}_{23}\hat{d}_{31}}{\hat{d}_{23}\hat{d}_{32}-\hat{d}_{22}\hat{d}_{33}}$ and $\widehat{W}_2$ given in the state of theorem.

Let $\widehat{\Omega}^{[6]} = \left(\widehat{\Omega}_1^{[6]}, \widehat{\Omega}_2^{[6]}, \widehat{\Omega}_3^{[6]}, \widehat{\Omega}_4^{[6]}\right)^T$ be the eigenvector of $\hat{J}_6^T$ for $\left(\lambda_{6L_1}\right) = 0$.

Then we get $\left(\hat{J}_6^T - \lambda_{6L_1}I\right)\widehat{\Omega}^{[6]} = 0$ then by solving this equation for $\widehat{\Omega}^{[6]}$ we get

$\widehat{\Omega}^{[6]} = \left(\widehat{\Omega}_1^{[6]}, 0, 0, 0\right)^T$, where $\widehat{\Omega}_1^{[6]} \neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial a_1} = f_{a_1}(X\,, a_1) = \left(\frac{\partial f_1}{\partial a_1}, \frac{\partial f_2}{\partial a_1}, \frac{\partial f_3}{\partial a_1}, \frac{\partial f_4}{\partial a_1}\right)^T = \left(\frac{L_1}{1+f_1L_3}, 0, 0, 0\right)^T$.

So, $f_{a_1}(P_6\,, \hat{a}_1) = (0\,, 0\,, 0\,, 0)^T$ and hence $\left(\widehat{\Omega}^{[6]}\right)^T f_{a_1}(P_6\,, \hat{a}_1) = 0$.

Therefore, using Sotomayor's theorem we get that the saddle-node bifurcation's conditions cannot be satisfied.

While the transcritical bifurcation's first condition is satisfied. Now, since

$$Df_{a_1}(X\,, a_1) = \begin{bmatrix} \frac{1}{1+f_1L_3} & 0 & \frac{-f_1L_1}{(1+f_1L_3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_{a_1}(X, a_1)$ is the derivative of $f_{a_1}(X, a_1)$ with respect to $X = (L_1, L_2, L_3, L_4)^T$.

Further, it is observed that $Df_{a_1}(P_6, \hat{a}_1)\hat{\psi}^{[6]} = \begin{bmatrix} \frac{1}{1+f_1\hat{L}_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1^{[6]} \\ -\widehat{W}_1\hat{\psi}_1^{[6]} \\ \widehat{W}_2\hat{\psi}_1^{[6]} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+f_1\hat{L}_3}\hat{\psi}_1^{[6]} \\ 0 \\ 0 \\ 0 \end{bmatrix}$, so

$$\left(\hat{\Omega}^{[6]}\right)^T \left[Df_{a_1}(P_6, \hat{a}_1)\hat{\psi}^{[6]}\right] = \left(\hat{\Omega}_1^{[6]}, 0, 0, 0\right)\left(\frac{1}{1+f_1\hat{L}_3}\hat{\psi}_1^{[6]}, 0, 0, 0\right)^T = \frac{1}{1+f_1\hat{L}_3}\hat{\Omega}_1^{[6]}\hat{\psi}_1^{[6]} \neq 0$$

Moreover, by substituting $\hat{\psi}^{[6]}$ in (3) we get: $D^2 f_{a_1}(P_6, \hat{a}_1)\left(\hat{\psi}^{[6]}, \hat{\psi}^{[6]}\right) = [\hat{A}_{i1}]_{4x1}$

$$\hat{A}_{11} = 2\left[-\left(b_1 + \frac{\hat{a}_1 f_1 \widehat{W}_2}{(1+f_1\hat{L}_3)^2}\right) + \frac{B_1}{(C+\alpha\eta A)}\left(\frac{\hat{L}_3}{(C+\alpha\eta A)} - \widehat{W}_2\right)\right]\left(\hat{\psi}_1^{[6]}\right)^2$$

$$\hat{A}_{21} = -2\left[b_2\widehat{W}_1^{\,2} - \left(\frac{a_2 f_2}{(1+f_2\hat{L}_3)^2} + \left(\rho + 2h\hat{L}_3\right)\right)\widehat{W}_1\widehat{W}_2 + \left(h - \frac{a_2 f_2^{\,2}}{(1+f_2\hat{L}_3)^3}\right)\hat{L}_2\widehat{W}_2^{\,2}\right]\left(\hat{\psi}_1^{[6]}\right)^2$$

$$\hat{A}_{31} = 2\left[\frac{-C_1\left(C+\eta A(\alpha-1)\right)\hat{L}_3}{(C+\alpha\eta A)^3} + \frac{C_1\left(C+\eta A(\alpha-1)\right)}{(C+\alpha\eta A)^2}\widehat{W}_2 - C_2\left(\rho + h\hat{L}_3\right)\widehat{W}_1\widehat{W}_2 + \left(C_2 h\hat{L}_2 - \frac{a_3}{K}\right)\widehat{W}_2^{\,2}\right]\left(\hat{\psi}_1^{[6]}\right)^2$$

$$\hat{A}_{41} = 0.$$

Hence, it obtains that $\left(\hat{\Omega}^{[6]}\right)^T\left[D^2 f_{a_1}(P_6, \hat{a}_1)\left(\hat{\psi}^{[6]}, \hat{\psi}^{[6]}\right)\right] = 2\left[-\left(b_1 + \frac{\hat{a}_1 f_1\widehat{W}_2}{(1+f_1\hat{L}_3)^2}\right) + \frac{B_1}{(C+\alpha\eta A)}\left(\frac{\hat{L}_3}{(C+\alpha\eta A)} - \widehat{W}_2\right)\right]\hat{\Omega}_1^{[6]}\left(\hat{\psi}_1^{[6]}\right)^2 \neq 0$, under conditions $(3, 99, 104, 105)$, which are given in [21], and condition (8). Thus, by using Sotomayor's theorem P_6 has a transcritical bifurcation at the parameter $\hat{a}_1$.

While if condition (8) not holds and according to condition (9), then there is no a transcritical bifurcation and by substituting $\hat{\psi}^{[6]}$ in (4) we get $D^3 f_{a_1}(P_6, \hat{a}_1)\left(\hat{\psi}^{[6]}, \hat{\psi}^{[6]}, \hat{\psi}^{[6]}\right) = [\widehat{M}_{i1}]_{4x1}$.

$$\widehat{M}_{11} = 6\left[\frac{B_1}{(C+\alpha\eta A)^2}\left(\widehat{W}_2 - \frac{\hat{L}_3}{(C+\alpha\eta A)}\right) + \frac{\hat{a}_1 f_1^{\,2}\widehat{W}_2^{\,2}}{(1+f_1\hat{L}_3)^3}\left(1 - \frac{f_1\widehat{W}_2}{(1+f_1\hat{L}_3)}\right)\right]\left(\hat{\psi}_1^{[6]}\right)^3$$

$$\widehat{M}_{21} = -6\left[\frac{a_2 f_2^2\widehat{W}_2}{(1+f_2\hat{L}_3)^3}\left(\widehat{W}_1\widehat{W}_2 + \frac{f_2\hat{L}_2}{(1+f_2\hat{L}_3)}\right) + h\widehat{W}_1\widehat{W}_2^{\,2}\right]\left(\hat{\psi}_1^{[6]}\right)^3$$

$$\widehat{M}_{31} = 6\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)}{(C+\alpha\eta A)^3}\left(\frac{\hat{L}_3}{(C+\alpha\eta A)} - \widehat{W}_2\right) - C_2 h\widehat{W}_1\widehat{W}_2^{\,2}\right]\left(\hat{\psi}_1^{[6]}\right)^3$$

$$\widehat{M}_{41} = 0$$

Then

$$\left(\widehat{\Omega}^{[6]}\right)^T\left[D^3 f_{a_1}(P_6,\hat{a}_1)\left(\hat{\psi}^{[6]},\hat{\psi}^{[6]},\hat{\psi}^{[6]}\right)\right]=6\left[\frac{B_1}{(C+\alpha\eta A)^2}\left(\widehat{W}_2-\frac{\hat{L}_3}{(C+\alpha\eta A)}\right)+\frac{\hat{a}_1 {f_1}^2 {\widehat{W}_2}^2}{(1+f_1\hat{L}_3)^3}\left(1-\frac{f_1\widehat{W}_2}{(1+f_1\hat{L}_3)}\right)\right]\widehat{\Omega}^{[6]}\left(\hat{\psi}_1^{[6]}\right)^3\neq 0,$$

under conditions $(3, 99, 104, 105)$, which are given in $[21]$, and condition (10). Hence again , by using Sotomayor's theorem system (1) at P_6 has pitchfork bifurcation with the parameter $\hat{a}_1$.

Theorem 5. Assume that conditions $(3, 22, 110, 111)$, which are given in $[21]$, then system (1) at the (EP) $P_7=\left(0,\ 0,\ \check{L}_3,\ \check{L}_4\right)$ with the parameter $\check{a}_2=\left(1+f_1\check{L}_3\right)\left(\rho+h\check{L}_3\right)\check{L}_3$ has a transcritical bifurcation, while neither $(\text{saddle–node or pitchfork})$ bifurcation can be occur at $\check{a}_2$.

Proof. From the Jacobian matrix J_7 which is given in Eq.(106) in $[21]$, system (1) at the (EP) P_7 has eigenvalue say $\left(\lambda_{7L_2}\right)$ equal to zero at $a_2=\check{a}_2$, then J_7 with $a_2=\check{a}_2$ becomes $\check{J}_7=J_7(P_7\,,\check{a}_2)=\left[\check{e}_{ij}\right]_{4\times4}$, where $\check{e}_{ij}=e_{ij}$; $i,j=1,2,3,4$ which are given in Eq.(106) in $[21]$ accept $\check{e}_{22}=0$.

Now, let $\check{\psi}^{[7]}=\left(\check{\psi}_1^{[7]},\check{\psi}_2^{[7]},\check{\psi}_3^{[7]},\check{\psi}_4^{[7]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\left(\lambda_{7L_2}\right)=0$.

Thus $\left(\check{J}_7-\lambda_{7L_2}I\right)\check{\psi}^{[7]}=0$, that gives $\check{\psi}^{[7]}=\left(0\,,\check{\psi}_2^{[7]}\,,-\widecheck{W}_1\check{\psi}_2^{[7]}\,,\widecheck{W}_2\check{\psi}_2^{[7]}\right)^T$ where $\check{\psi}_2^{[7]}\neq 0$ is any real number , $\widecheck{W}_1=\frac{\check{e}_{32}\check{e}_{44}}{\check{e}_{33}\check{e}_{44}-\check{e}_{34}\check{e}_{43}}$ and $\widecheck{W}_2=\frac{\check{e}_{32}\check{e}_{43}}{\check{e}_{33}\check{e}_{44}-\check{e}_{34}\check{e}_{43}}$.

Let $\widecheck{\Omega}^{[7]}=\left(\widecheck{\Omega}_1^{[7]},\widecheck{\Omega}_2^{[7]},\widecheck{\Omega}_3^{[7]},\widecheck{\Omega}_4^{[7]}\right)^T$ be the eigenvector of $\check{J}_7^T$ for $\left(\lambda_{7L_2}\right)=0$.

Then we get $\left(\check{J}_7^T-\lambda_{7L_2}I\right)\widecheck{\Omega}^{[7]}=0$ then by solving this equation for $\widecheck{\Omega}^{[7]}$ we get

$$\widecheck{\Omega}^{[7]}=\left(0\,,\widecheck{\Omega}_2^{[7]}\,,0\,,0\right)^T,\text{ where } \widecheck{\Omega}_2^{[7]}\neq 0 \text{ is any real number.}$$

Now, consider $\frac{\partial f}{\partial a_2}=f_{a_2}(X\,,a_2)=\left(\frac{\partial f_1}{\partial a_2},\frac{\partial f_2}{\partial a_2},\frac{\partial f_3}{\partial a_2},\frac{\partial f_4}{\partial a_2}\right)^T=\left(0\,,\frac{L_2}{1+f_2L_3}\,,0\,,0\right)^T$.

So, $f_{a_2}(P_7\,,\check{a}_2)=(0\,,0\,,0,0)^T$ and hence $\left(\widecheck{\Omega}^{[7]}\right)^T f_{a_2}(P_7\,,\check{a}_2)=0$.

Therefore, using Sotomayor's theorem we get that the saddle-node bifurcation's conditions cannot be satisfied.

While the transcritical bifurcation's first condition is satisfied. Now, since

$$Df_{a_2}(X\,,a_2)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \frac{1}{1+f_2L_3}-\frac{f_2L_2}{(1+f_2L_3)^2} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

where $Df_{a_2}(X\,,a_2)$ is the derivative of $f_{a_2}(X\,,a_2)$ with respect to $X=(L_1\,,L_2\,,L_3\,,L_4)^T$.

Further, it is observed that $Df_{a_2}(P_7,\breve{a}_2)\breve{\psi}^{[7]}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \frac{1}{1+f_2\breve{L}_3} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\ \breve{\psi}_2^{[7]}\\ -\breve{W}_1\breve{\psi}_2^{[7]}\\ \breve{W}_2\breve{\psi}_2^{[7]}\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{1+f_2\breve{L}_3}\breve{\psi}_2^{[7]}\\ 0\\ 0\end{bmatrix}$, so

$$\left(\breve{\Omega}^{[7]}\right)^T\left[Df_{a_2}(P_7\,,\breve{a}_2)\breve{\psi}^{[7]}\right]=\left(0\,,\breve{\Omega}_2^{[7]}\,,0\,,0\right)\left(0\,,\frac{1}{1+f_2\breve{L}_3}\breve{\psi}_2^{[7]}\,,0\,,0\right)^T=\frac{1}{1+f_2\breve{L}_3}\breve{\Omega}_2^{[7]}\breve{\psi}_2^{[7]}\neq 0$$

Moreover, by substituting $\breve{\psi}^{[7]}$ in (3) we get: $D^2f_{a_2}(P_7\,,\breve{a}_2)\left(\breve{\psi}^{[7]}\,,\breve{\psi}^{[7]}\right)=[\breve{A}_{i1}]_{4x1}$

$\breve{A}_{11}=0,$

$\breve{A}_{21}=-2\left[b_2-\left(\frac{\breve{a}_2f_2}{(1+f_2\breve{L}_3)^3}+\left(\rho+2h\breve{L}_3\right)\right)\breve{W}_1\right]\left(\breve{\psi}_2^{[7]}\right)^2,$

$\breve{A}_{31}=-2\left[C_2\left(\rho+2h\breve{L}_3\right)\breve{W}_1+\frac{a_3}{K}\breve{W}_1{}^2-\left(\frac{a_3}{K}+B\right)\breve{W}_1\breve{W}_2+\frac{\gamma\sigma}{(\sigma+\breve{L}_4)^3}\breve{W}_2{}^2\right]\left(\breve{\psi}_2^{[7]}\right)^2,$

$\breve{A}_{41}=2\left[-B\breve{W}_1+\frac{\gamma\sigma}{(\sigma+\breve{L}_4)^3}\breve{W}_2\right]\breve{W}_2\left(\breve{\psi}_2^{[7]}\right)^2.$

Hence, it obtains that

$$\left(\breve{\Omega}^{[7]}\right)^T\left[D^2f_{a_2}(P_7,\breve{a}_2)\left(\breve{\psi}^{[7]},\breve{\psi}^{[7]}\right)\right]=-2\left[b_2-\left(\frac{\breve{a}_2f_2}{(1+f_2\breve{L}_3)^3}+\left(\rho+2h\breve{L}_3\right)\right)\breve{W}_1\right]\breve{\Omega}_2^{[7]}\left(\breve{\psi}_2^{[7]}\right)^2\neq 0,$$

under conditions $(3, 22, 110, 111)$, which is given in [21]. Thus, by using Sotomayor's theorem P_7 has a transcritical bifurcation at the parameter $\breve{a}_2$.

Theorem 6. Assume that conditions $(3, 99, 115, 116, 117)$, which are given in [21] and the following condition holds

$$-\tilde{B}\widetilde{W}_3\neq\frac{\gamma}{\sigma^2}, \tag{11}$$

where

$$\widetilde{W}_3 = \frac{\tilde{f}_{11}\tilde{f}_{22}\tilde{f}_{34}}{\left(\tilde{f}_{22}\tilde{f}_{13}\tilde{f}_{31}+\tilde{f}_{11}(\tilde{f}_{23}\tilde{f}_{32}-\tilde{f}_{22}\tilde{f}_{33})\right)}.$$

Then system (1) at the (EP) $P_8 = (\tilde{L}_1,\ \tilde{L}_2,\ \tilde{L}_3, 0)$ with the parameter $\tilde{B} = \frac{\delta\sigma+\gamma}{\sigma\tilde{L}_3}$ has (transcritical and pitchfork) bifurcation, while saddel-node bifurcation cannot be occurs at $\tilde{B}$.

Proof. From the Jacobian matrix J_8 which is given in Eq.(112) in [21] , system (1) at the (EP) P_8 has eigenvalue say (λ_{8L_4}) equal to zero at $B = \tilde{B}$, then J_8 with $B = \tilde{B}$ becomes

$\tilde{J}_8 = J_8(P_8 , \tilde{B}) = [\tilde{f}_{ij}]_{4\times4}$, where $\tilde{f}_{ij} = f_{ij}$; $i, j = 1,2,3,4$ which are given in Eq.(112) in [21] accept $\tilde{f}_{34} = -\left(\frac{a_3}{K}\tilde{L}_3 + \delta\right)$ and $\tilde{f}_{44} = 0$.

Now, let $\tilde{\psi}^{[8]} = \left(\tilde{\psi}_1^{[8]} , \tilde{\psi}_2^{[8]} , \tilde{\psi}_3^{[8]}, \tilde{\psi}_4^{[8]}\right)^T$ be the eigenvector corresponding to the eigenvalue $(\lambda_{8L_4}) = 0$.

Thus $(\tilde{J}_8 - \lambda_{8L_4} I)\tilde{\psi}^{[8]} = 0$, that gives $\tilde{\psi}^{[8]} = \left(-\widetilde{W}_1\tilde{\psi}_4^{[8]} , -\widetilde{W}_2\tilde{\psi}_4^{[8]} , \widetilde{W}_3\tilde{\psi}_4^{[8]} , \tilde{\psi}_4^{[8]}\right)^T$ where $\widetilde{W}_1 = \frac{\tilde{f}_{13}\tilde{f}_{22}\tilde{f}_{34}}{\left(\tilde{f}_{22}\tilde{f}_{13}\tilde{f}_{31}+\tilde{f}_{11}(\tilde{f}_{23}\tilde{f}_{32}-\tilde{f}_{22}\tilde{f}_{33})\right)}$, $\widetilde{W}_2 = \frac{\tilde{f}_{11}\tilde{f}_{23}\tilde{f}_{34}}{\left(\tilde{f}_{22}\tilde{f}_{13}\tilde{f}_{31}+\tilde{f}_{11}(\tilde{f}_{23}\tilde{f}_{32}-\tilde{f}_{22}\tilde{f}_{33})\right)}$, $\widetilde{W}_3$ given in the state of theorem, and $\bar{\psi}_4^{[3]} \neq 0$ is any real number.

Let $\widetilde{\Omega}^{[8]} = \left(\widetilde{\Omega}_1^{[8]} , \widetilde{\Omega}_2^{[8]} , \widetilde{\Omega}_3^{[8]}, \widetilde{\Omega}_4^{[8]}\right)^T$ be the eigenvector of $\tilde{J}_8^T$ for $(\lambda_{8L_4}) = 0$.

Then we get $(\tilde{J}_8^T - \lambda_{8L_4} I)\widetilde{\Omega}^{[8]} = 0$ then by solving this equation for $\widetilde{\Omega}^{[8]}$ we get

$\widetilde{\Omega}^{[8]} = \left(0 , 0 , 0 , \widetilde{\Omega}_4^{[8]}\right)^T$, where $\widetilde{\Omega}_4^{[8]} \neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial B} = f_B(X , B) = \left(\frac{\partial f_1}{\partial B}, \frac{\partial f_2}{\partial B}, \frac{\partial f_3}{\partial B}, \frac{\partial f_4}{\partial B}\right)^T = (0 , 0 , -L_3L_4 , L_3L_4)^T$.

So, $f_B(P_8 , \tilde{B}) = (0 , 0 , 0, 0)^T$ and hence $\left(\widetilde{\Omega}^{[8]}\right)^T f_B(P_8 , \tilde{B}) = 0$.

Therefore, using Sotomayor’s theorem we get that the saddle-node bifurcation's conditions cannot be satisfied.

While the transcritical bifurcation's first condition is satisfied. Now, since

$$Df_B(X , B) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -L_4 & -L_3 \\ 0 & 0 & L_4 & L_3 \end{bmatrix}$$

where $Df_B(X\,,B)$ is the derivative of $f_B(X\,,B)$ with respect to $X=(L_1\,,L_2\,,L_3\,,L_4)^T$.

Further, it is observed that $Df_{\mathrm{B}}\big(P_8\,,\tilde{B}\big)\tilde{\psi}^{[8]}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-\tilde{L}_3\\0&0&0&\tilde{L}_3\end{bmatrix}\begin{bmatrix}-\widetilde{W}_1\tilde{\psi}_4^{[8]}\\-\widetilde{W}_2\tilde{\psi}_4^{[8]}\\\widetilde{W}_3\tilde{\psi}_4^{[8]}\\\tilde{\psi}_4^{[8]}\end{bmatrix}=\begin{bmatrix}0\\0\\-\tilde{L}_3\widetilde{W}_3\tilde{\psi}_4^{[8]}\\\tilde{L}_3\tilde{\psi}_4^{[8]}\end{bmatrix}$, so

$\big(\tilde{\Omega}^{[8]}\big)^T\big[Df_{\mathrm{B}}\big(P_8\,,\tilde{B}\big)\tilde{\psi}^{[8]}\big]=\big(0\,,0\,,0\,,\tilde{\Omega}_4^{[8]}\big)\big(0\,,0\,,-\tilde{L}_3\widetilde{W}_3\tilde{\psi}_4^{[8]}\,,\tilde{L}_3\tilde{\psi}_4^{[8]}\big)^T=\tilde{L}_3\tilde{\Omega}_4^{[8]}\tilde{\psi}_4^{[8]}\neq 0.$

Moreover, by substituting $\tilde{\psi}^{[8]}$ in (3) we get $D^2f_{\mathrm{B}}\big(P_8\,,\tilde{B}\big)\big(\tilde{\psi}^{[8]}\,,\tilde{\psi}^{[8]}\big)=[\tilde{A}_{i1}]_{4x1}$

$$\tilde{A}_{11}=2\left[\left(-b_1+\frac{B_1(C+\alpha\eta A)\tilde{L}_3}{(C+\alpha\eta A+\tilde{L}_1)^3}\right)\widetilde{W}_1^{\,2}+\left(\frac{a_1f_1}{(1+f_1\tilde{L}_3)^2}+\frac{B_1(C+\alpha\eta A)}{(C+\alpha\eta A+\tilde{L}_1)^2}\right)\widetilde{W}_1\widetilde{W}_3+\frac{a_1f_1^2\tilde{L}_1}{(1+f_1\tilde{L}_3)^3}\widetilde{W}_3^{\,2}\right]\left(\tilde{\psi}_4^{[8]}\right)^2,$$

$$\tilde{A}_{21}=-2\left[b_2\widetilde{W}_2^{\,2}-\left(\frac{a_2f_2}{(1+f_2\tilde{L}_3)^2}+\left(\rho+2h\tilde{L}_3\right)\right)\widetilde{W}_2\widetilde{W}_3+\left(h-\frac{a_2f_2^{\,2}}{(1+f_2\tilde{L}_3)^3}\right)\tilde{L}_2\widetilde{W}_3^{\,2}\right]\left(\tilde{\psi}_4^{[8]}\right)^2,$$

$$\tilde{A}_{31}=-2\left[\frac{C_1\big(C+\eta A(\alpha-1)\big)\widetilde{W}_1}{(C+\alpha\eta A+\tilde{L}_1)^2}\left(\widetilde{W}_3+\frac{\tilde{L}_3\widetilde{W}_1}{(C+\alpha\eta A+\tilde{L}_1)}\right)+C_2\big(\rho+2h\tilde{L}_3\big)\widetilde{W}_2\widetilde{W}_3-\left(C_2h\tilde{L}_2-\frac{a_3}{K}\right)\widetilde{W}_3^{\,2}+\left(\frac{a_3}{K}+\tilde{B}\right)\widetilde{W}_3+\frac{\gamma}{\sigma^2}\right]\left(\tilde{\psi}_4^{[8]}\right)^2,$$

$$\tilde{A}_{41}=2\left[\tilde{B}\widetilde{W}_3+\frac{\gamma}{\sigma^2}\right]\left(\tilde{\psi}_4^{[8]}\right)^2.$$

Hence, it obtains that $\big(\tilde{\Omega}^{[8]}\big)^T\big[D^2f_{\mathrm{B}}\big(P_8\,,\tilde{B}\big)\big(\tilde{\psi}^{[8]}\,,\tilde{\psi}^{[8]}\big)\big]=2\left[\tilde{B}\widetilde{W}_3+\frac{\gamma}{\sigma^2}\right]\tilde{\Omega}^{[8]}\left(\tilde{\psi}_4^{[8]}\right)^2\neq 0,$ under conditions $(3, 99{,}115{,}116{,}117)$ which are given in $[21]$, and condition (11). Thus, by using Sotomayor's theorem P_8 has a transcritical bifurcation at the parameter $\tilde{B}$.

While if condition (11) not holds then there is no a transcritical bifurcation and by substituting $\tilde{\psi}^{[8]}$ in (4) we get $D^3f_B\big(P_8\,,\tilde{B}\big)\big(\tilde{\psi}^{[8]}\,,\tilde{\psi}^{[8]},\tilde{\psi}^{[8]}\big)=[\widetilde{M}_{i1}]_{4x1}$.

$$\widetilde{M}_{11}=6\left[\frac{B_1(C+\alpha\eta A)\tilde{L}_3\widetilde{W}_1^{\,2}}{(C+\alpha\eta A+\tilde{L}_1)^3}\left(\widetilde{W}_3+\frac{\tilde{L}_3\widetilde{W}_1}{(C+\alpha\eta A+\tilde{L}_1)}\right)-\frac{a_1f_1^2\widetilde{W}_3^{\,2}}{(1+f_1\tilde{L}_3)^3}\left(\widetilde{W}_1+\frac{f_1\widetilde{W}_3}{(1+f_1\tilde{L}_3)}\right)\right]\left(\tilde{\psi}_4^{[8]}\right)^3,$$

$$\widetilde{M}_{21}=-6\left[\left(\frac{a_2f_2^{\,2}}{(1+f_1\tilde{L}_3)^3}-h\right)\widetilde{W}_2\widetilde{W}_3^{\,2}+\frac{a_2f_2^3}{(1+f_1\tilde{L}_3)^4}\widetilde{W}_3^{\,3}\right]\left(\tilde{\psi}_4^{[8]}\right)^3,$$

$$\widetilde{M}_{31}=-6\left[\frac{C_1\big(C+\eta A(\alpha-1)\big)\widetilde{W}_1}{(C+\alpha\eta A+\tilde{L}_1)^4}\left(\frac{\widetilde{W}_1^{\,2}}{(C+\alpha\eta A+\tilde{L}_1)}+\widetilde{W}_3^{\,2}\right)+C_2h\widetilde{W}_2\widetilde{W}_3^{\,2}-\frac{\gamma}{\sigma^3}\right]\left(\tilde{\psi}_4^{[8]}\right)^3,$$

$$\widetilde{M}_{41}=-6\left[\frac{\gamma}{\sigma^3}\right]\left(\tilde{\psi}_4^{[8]}\right)^3.$$

Then

$$\left(\tilde{\Omega}^{[8]}\right)^T\left[D^3 f_{\mathrm{B}}\left(P_8\,,\tilde{B}\right)\left(\tilde{\psi}^{[8]}\,,\tilde{\psi}^{[8]},\tilde{\psi}^{[8]}\right)\right]=-6\frac{\gamma}{\sigma^3}\tilde{\Omega}^{[8]}\left(\tilde{\psi}_4^{[8]}\right)^3\neq 0.$$

Hence again , by using Sotomayor's theorem system (1) at P_8 has pitchfork bifurcation with the parameter $\tilde{B}$.

Theorem 7. Assume that conditions $(3, 99, 121, 122, 123)$, which are given in $[21]$, then system (1) at the (EP) $P_9=\left(\tilde{\tilde{L}}_1, 0, \tilde{\tilde{L}}_3, \tilde{\tilde{L}}_4\right)$ with the parameter $\tilde{\tilde{a}}_2=\left(1+f_1\tilde{\tilde{L}}_3\right)\left(\rho+h\tilde{\tilde{L}}_3\right)\tilde{\tilde{L}}_3$ has a transcritical bifurcation, while neither $(\text{saddle– node or pitchfork})$ bifurcation can be occur.

Proof. From the Jacobian matrix J_9 which is given in Eq.(118) in $[21]$, system (1) at the (EP) P_9 has eigenvalue say $\left(\lambda_{9L_2}\right)$ equal to zero at $a_2=\tilde{\tilde{a}}_2$, then J_9 with $a_2=\tilde{\tilde{a}}_2$ becomes $\tilde{\tilde{J}}_9=J_9\left(P_9\,,\tilde{\tilde{a}}_2\right)=\left[\tilde{\tilde{g}}_{ij}\right]_{4\times 4}$, where $\tilde{\tilde{g}}_{ij}=g_{ij}$; $i,j=1{,}2{,}3{,}4$ which are given in Eq.(118) in $[21]$ accept $\tilde{\tilde{g}}_{22}=0$.

Now, let $\tilde{\tilde{\psi}}^{[9]}=\left(\tilde{\tilde{\psi}}_1^{[9]}\,,\tilde{\tilde{\psi}}_2^{[9]}\,,\tilde{\tilde{\psi}}_3^{[9]}\,,\tilde{\tilde{\psi}}_4^{[9]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\left(\lambda_{9L_2}\right)=0$.

Thus $\left(\tilde{\tilde{J}}_9-\lambda_{9L_2}I\right)\tilde{\tilde{\psi}}^{[9]}=0$, that gives $\tilde{\tilde{\psi}}^{[9]}=\left(-\tilde{\tilde{W}}_1\tilde{\tilde{\psi}}_2^{[9]}\,,\tilde{\tilde{\psi}}_2^{[9]}\,,\tilde{\tilde{W}}_2\tilde{\tilde{\psi}}_2^{[9]}\,,-\tilde{\tilde{W}}_3\tilde{\tilde{\psi}}_2^{[9]}\right)^T$ where

$$\tilde{\tilde{W}}_1=\frac{\tilde{\tilde{g}}_{13}\tilde{\tilde{g}}_{32}\tilde{\tilde{g}}_{44}}{\left(\tilde{\tilde{g}}_{13}\tilde{\tilde{g}}_{31}\tilde{\tilde{g}}_{44}-\tilde{\tilde{g}}_{11}\left(\tilde{\tilde{g}}_{33}\tilde{\tilde{g}}_{44}-\tilde{\tilde{g}}_{34}\tilde{\tilde{g}}_{43}\right)\right)},\ \tilde{\tilde{W}}_2=\frac{\tilde{\tilde{g}}_{11}\tilde{\tilde{g}}_{32}\tilde{\tilde{g}}_{44}}{\left(\tilde{\tilde{g}}_{13}\tilde{\tilde{g}}_{31}\tilde{\tilde{g}}_{44}-\tilde{\tilde{g}}_{11}\left(\tilde{\tilde{g}}_{33}\tilde{\tilde{g}}_{44}-\tilde{\tilde{g}}_{34}\tilde{\tilde{g}}_{43}\right)\right)},$$

$$\tilde{\tilde{W}}_3=\frac{\tilde{\tilde{g}}_{11}\tilde{\tilde{g}}_{32}\tilde{\tilde{g}}_{43}}{\left(\tilde{\tilde{g}}_{13}\tilde{\tilde{g}}_{31}\tilde{\tilde{g}}_{44}-\tilde{\tilde{g}}_{11}\left(\tilde{\tilde{g}}_{33}\tilde{\tilde{g}}_{44}-\tilde{\tilde{g}}_{34}\tilde{\tilde{g}}_{43}\right)\right)},\ \text{ and } \tilde{\tilde{\psi}}_2^{[9]}\neq 0 \text{ is any real number.}$$

Let $\tilde{\tilde{\Omega}}^{[9]}=\left(\tilde{\tilde{\Omega}}_1^{[9]}\,,\tilde{\tilde{\Omega}}_2^{[9]}\,,\tilde{\tilde{\Omega}}_3^{[9]},\tilde{\tilde{\Omega}}_4^{[9]}\right)^T$ be the eigenvector of $\tilde{\tilde{J}}_9^T$ for $\left(\lambda_{9L_2}\right)=0$.

Then we get $\left(\tilde{\tilde{J}}_9^T-\lambda_{9L_2}I\right)\tilde{\tilde{\Omega}}^{[9]}=0$ then by solving this equation for $\tilde{\tilde{\Omega}}^{[9]}$ we get

$$\tilde{\tilde{\Omega}}^{[9]}=\left(0\,,\tilde{\tilde{\Omega}}_2^{[9]}\,,0\,,0\right)^T, \text{where } \tilde{\tilde{\Omega}}_2^{[9]}\neq 0 \text{ is any real number.}$$

Now, consider $\frac{\partial f}{\partial a_2}=f_{a_2}(X\,,a_2)=\left(\frac{\partial f_1}{\partial a_2},\frac{\partial f_2}{\partial a_2},\frac{\partial f_3}{\partial a_2},\frac{\partial f_4}{\partial a_2}\right)^T=\left(0\,,\frac{L_2}{1+f_2L_3}\,,0\,,0\right)^T$.

So, $f_{a_2}\left(P_9\,,\tilde{\tilde{a}}_2\right)=(0\,,0\,,0,0)^T$ and hence $\left(\tilde{\tilde{\Omega}}_2^{[9]}\right)^T f_{a_2}\left(P_9\,,\tilde{\tilde{a}}_2\right)=0$.

Therefore, using Sotomayor's theorem we get that the saddle-node bifurcation's conditions cannot be satisfied.

While the transcritical bifurcation's first condition is satisfied. Now, since

$$Df_{a_2}(X\,,a_2)=\begin{bmatrix}0&0&0&0\\0&\frac{1}{1+f_2L_3}-\frac{f_2L_2}{(1+f_2L_3)^2}&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

where $Df_{a_2}(X\,,a_2)$ is the derivative of $f_{a_2}(X\,,a_2)$ with respect to $X=(L_1\,,L_2\,,L_3\,,L_4)^T$.

Further, it is observed that $Df_{a_2}(P_9,\tilde{\tilde{a}}_2)\tilde{\tilde{\psi}}^{[9]}=\begin{bmatrix}0&0&0&0\\0&\frac{1}{1+f_2\tilde{\tilde{L}}_3}&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}-\tilde{\tilde{W}}_1\tilde{\tilde{\psi}}_2^{[9]}\\ \tilde{\tilde{\psi}}_2^{[9]}\\ \tilde{\tilde{W}}_2\tilde{\tilde{\psi}}_2^{[9]}\\ -\tilde{\tilde{W}}_3\tilde{\tilde{\psi}}_2^{[9]}\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{1+f_2\tilde{\tilde{L}}_3}\tilde{\tilde{\psi}}_2^{[9]}\\0\\0\end{bmatrix}$,

then

$$\left(\tilde{\tilde{\Omega}}^{[9]}\right)^T\left[Df_{a_2}(P_9,\tilde{\tilde{a}}_2)\tilde{\tilde{\psi}}^{[9]}\right]=\left(0\,,\tilde{\tilde{\Omega}}_2^{[9]}\,,0\,,0\right)\left(0\,,\frac{1}{1+f_2\tilde{\tilde{L}}_3}\tilde{\tilde{\psi}}_2^{[9]}\,,0\,,0\right)^T=\frac{1}{1+f_2\tilde{\tilde{L}}_3}\tilde{\tilde{\Omega}}_2^{[9]}\tilde{\tilde{\psi}}_2^{[9]}\neq 0$$

Moreover, by substituting $\tilde{\tilde{\psi}}^{[9]}$ in (3) we get $D^2f_{a_2}(P_9\,,\tilde{\tilde{a}}_2)\left(\tilde{\tilde{\psi}}^{[9]}\,,\tilde{\tilde{\psi}}^{[9]}\right)=\left[\tilde{\tilde{A}}_{i1}\right]_{4x1}$.

$$\tilde{\tilde{A}}_{11}=2\left[\left(-b_1+\frac{B_1(C+\alpha\eta A)\tilde{\tilde{L}}_3}{\left(C+\alpha\eta A+\tilde{\tilde{L}}_1\right)^3}\right)\tilde{\tilde{W}}_1^{\,2}+\left(\frac{a_1f_1}{\left(1+f_1\tilde{\tilde{L}}_3\right)^2}+\frac{B_1(C+\alpha\eta A)}{\left(C+\alpha\eta A+\tilde{\tilde{L}}_1\right)^2}\right)\tilde{\tilde{W}}_1\tilde{\tilde{W}}_2+\frac{a_1f_1^2\tilde{\tilde{L}}_1}{\left(1+f_1\tilde{\tilde{L}}_3\right)^3}\tilde{\tilde{W}}_2^{\,2}\right]\left(\tilde{\tilde{\psi}}_2^{[9]}\right)^2,$$

$$\tilde{\tilde{A}}_{21}=-2\left[b_2+\left(\frac{\tilde{\tilde{a}}_2f_2}{\left(1+f_2\tilde{\tilde{L}}_3\right)^3}+\left(\rho+2h\tilde{\tilde{L}}_3\right)\right)\tilde{\tilde{W}}_2\right]\left(\tilde{\tilde{\psi}}_2^{[9]}\right)^2,$$

$$\tilde{\tilde{A}}_{31}=2\left[\frac{-C_1(C+\eta A(\alpha-1))\tilde{\tilde{W}}_1}{\left(C+\alpha\eta A+\tilde{\tilde{L}}_1\right)^2}\left(\tilde{\tilde{W}}_2+\frac{\tilde{\tilde{L}}_3\tilde{\tilde{W}}_1}{\left(C+\alpha\eta A+\tilde{\tilde{L}}_1\right)}\right)+C_2\left(\rho+2h\tilde{\tilde{L}}_3\right)\tilde{\tilde{W}}_2-\frac{a_3}{K}\tilde{\tilde{W}}_2^{\,2}+\right.$$

$$\left.\left(\frac{a_3}{K}+B\right)\tilde{\tilde{W}}_2\tilde{\tilde{W}}_3+\frac{\gamma\sigma}{\left(\sigma+\tilde{\tilde{L}}_4\right)^3}\tilde{\tilde{W}}_3^{\,2}\right]\left(\tilde{\tilde{\psi}}_2^{[9]}\right)^2,$$

$$\tilde{\tilde{A}}_{41}=-2\left[B\tilde{\tilde{W}}_2-\frac{\gamma\sigma}{\left(\sigma+\tilde{\tilde{L}}_4\right)^3}\tilde{\tilde{W}}_3\right]\tilde{\tilde{W}}_3\left(\tilde{\tilde{\psi}}_2^{[9]}\right)^2.$$

Hence, it obtains that

$$\left(\tilde{\tilde{\Omega}}^{[9]}\right)^T\left[D^2f_{a_2}(P_9,\tilde{\tilde{a}}_2)\left(\tilde{\tilde{\psi}}^{[9]},\tilde{\tilde{\psi}}^{[9]}\right)\right]=-2\left[b_2+\left(\frac{\tilde{\tilde{a}}_2f_2}{\left(1+f_2\tilde{\tilde{L}}_3\right)^3}+\rho+2h\tilde{\tilde{L}}_3\right)\tilde{\tilde{W}}_2\right]\tilde{\tilde{\Omega}}_2^{[9]}\left(\tilde{\tilde{\psi}}_2^{[9]}\right)^2\neq 0,$$

under conditions $(3, 99, 121, 122, 123)$, which are given in $[21]$. Thus, by using Sotomayor's theorem P_9 has a transcritical bifurcation at the parameter $\tilde{\tilde{a}}_2$.

Similarly for (EP) $P_{10}=\left(\tilde{\tilde{\tilde{L}}}_1, 0,\ \tilde{\tilde{\tilde{L}}}_3,\ \tilde{\tilde{\tilde{L}}}_4\right)$.

Theorem 8. Assume that conditions $(3, 99, 127, 128, 129)$, which is given in $[21]$ and the following conditions hold

$$\frac{\dot{L_3}}{(C+\alpha\eta A)}\leq \dot{W}_2, \tag{12}$$

$$\left(b_1+\frac{\dot{a}_1 f_1 \dot{W}_2}{(1+f_1\dot{L_3})^2}\right)=\frac{B_1}{(C+\alpha\eta A)}\left(\frac{\dot{L_3}}{(C+\alpha\eta A)}-\dot{W}_2\right), \tag{13}$$

$$1<\frac{f_1\dot{W}_2}{(1+f_1\dot{L_3})}, \tag{14}$$

where

$$\dot{W}_2=\frac{\dot{n}_{22}\dot{n}_{31}\dot{n}_{44}}{\dot{n}_{23}\dot{n}_{32}\dot{n}_{44}+\dot{n}_{22}(\dot{n}_{34}\dot{n}_{43}-\dot{n}_{33}\dot{n}_{44})}.$$

Then system (1) at the (EP) $P_{11}=\left(0,\ \dot{L}_2,\ \dot{L_3},\ \dot{L}_4\right)$ with the parameter $\dot{a}_1=\frac{(1+f_1\dot{L_3})B_1\dot{L_3}}{(C+\alpha\eta A)}$ has $(\text{transcritical and pitchfork})$ bifurcation, while saddel-node bifurcation cannot be occur at $\dot{a}_1$.

Proof. From the Jacobian matrix J_{11} which is given in Eq.(124) in $[21]$, system (1) at the (EP) P_{11} has eigenvalue say $\left(\lambda_{11L_1}\right)$ equal to zero at $a_1=\dot{a}_1$, then J_{11} with $a_1=\dot{a}_1$ becomes $\dot{J}_{11}=J_{11}(P_{11}, \dot{a}_1)=\left[\dot{n}_{ij}\right]_{4\times 4}$, where $\dot{n}_{ij}=n_{ij}$; $i,j=1,2,3,4$ which are given in Eq.(124) in $[21]$ accept $\dot{n}_{11}=0$.

Now, let $\dot{\psi}^{[11]}=\left(\dot{\psi}_1^{[11]}, \dot{\psi}_2^{[11]}, \dot{\psi}_3^{[11]}, \dot{\psi}_4^{[11]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\left(\lambda_{11L_1}\right)=0$.

Thus $\left(\dot{J}_{11}-\lambda_{11L_1}I\right)\dot{\psi}^{[11]}=0$, that gives $\dot{\psi}^{[11]}=\left(\dot{\psi}_1^{[11]}, -\dot{W}_1\dot{\psi}_1^{[11]}, \dot{W}_2\dot{\psi}_1^{[11]}, -\dot{W}_3\dot{\psi}_1^{[11]}\right)^T$ where $\dot{W}_1=\frac{\dot{n}_{23}\dot{n}_{31}\dot{n}_{44}}{\dot{n}_{23}\dot{n}_{32}\dot{n}_{44}+\dot{n}_{22}(\dot{n}_{34}\dot{n}_{43}-\dot{n}_{33}\dot{n}_{44})}$, $\dot{W}_3=\frac{\dot{n}_{22}\dot{n}_{31}\dot{n}_{43}}{\dot{n}_{23}\dot{n}_{32}\dot{n}_{44}+\dot{n}_{22}(\dot{n}_{34}\dot{n}_{43}-\dot{n}_{33}\dot{n}_{44})}$, $\dot{W}_2$ given in the state of theorem, and $\dot{\psi}_1^{[11]}\neq 0$ is any real number.

Let $\dot{\Omega}^{[11]}=\left(\dot{\Omega}_1^{[11]}, \dot{\Omega}_2^{[11]}, \dot{\Omega}_3^{[11]}, \dot{\Omega}_4^{[11]}\right)^T$ be the eigenvector of $\dot{J}_{11}^T$ for $\left(\lambda_{11L_1}\right)=0$.

Then we get $\left(\dot{J}_{11}^T-\lambda_{11L_1}I\right)\dot{\Omega}^{[11]}=0$ then by solving this equation for $\dot{\Omega}^{[11]}$ we get

$\dot{\Omega}^{[11]}=\left(\dot{\Omega}_1^{[11]}, 0, 0, 0\right)^T$, where $\dot{\Omega}_1^{[11]}\neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial a_1} = f_{a_1}(X, a_1) = \left(\frac{\partial f_1}{\partial a_1}, \frac{\partial f_2}{\partial a_1}, \frac{\partial f_3}{\partial a_1}, \frac{\partial f_4}{\partial a_1}\right)^T = \left(\frac{L_1}{1+f_1L_3}, 0, 0, 0\right)^T$.

So, $f_{a_1}(P_{11}, \dot{a}_1) = (0, 0, 0, 0)^T$ and hence $\left(\dot{\Omega}^{[11]}\right)^T f_{a_1}(P_{11}, \dot{a}_1) = 0$.

Therefore, using Sotomayor's theorem we get that the saddle-node bifurcation's conditions cannot be satisfied.

While the transcritical bifurcation's first condition is satisfied. Now, since

$$Df_{a_1}(X, a_1) = \begin{bmatrix} \frac{1}{1+f_1L_3} & 0 & \frac{-f_1L_1}{(1+f_1L_3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_{a_1}(X, a_1)$ is the derivative of $f_{a_1}(X, a_1)$ with respect to $X = (L_1, L_2, L_3, L_4)^T$.

Further, it is observed that $Df_{a_1}(P_{11}, \dot{a}_1)\dot{\psi}^{[11]} = \begin{bmatrix} \frac{1}{1+f_1\dot{L_3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi}_1^{[11]} \\ -\dot{W}_1\dot{\psi}_1^{[11]} \\ \dot{W}_2\dot{\psi}_1^{[11]} \\ -\dot{W}_3\dot{\psi}_1^{[11]} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+f_1\dot{L_3}}\dot{\psi}_1^{[11]} \\ 0 \\ 0 \\ 0 \end{bmatrix}$, so

$$\left(\dot{\Omega}^{[11]}\right)^T \left[Df_{a_1}(P_{11}, \dot{a}_1)\dot{\psi}^{[11]}\right] = \left(\dot{\Omega}_1^{[11]}, 0, 0, 0\right)\left(\frac{1}{1+f_1\dot{L_3}}\dot{\psi}_1^{[11]}, 0, 0, 0\right)^T = \frac{1}{1+f_1\dot{L_3}}\dot{\Omega}_1^{[11]}\dot{\psi}_1^{[11]} \neq 0$$

Moreover, by substituting $\dot{\psi}_1^{[11]}$ in (3) we get $D^2 f_{a_1}(P_{11}, \dot{a}_1)\left(\dot{\psi}^{[11]}, \dot{\psi}^{[11]}\right) = [\dot{A}_{i1}]_{4x1}$.

$$\dot{A}_{11} = 2\left[-\left(b_1 + \frac{\dot{a}_1 f_1 \dot{W}_2}{(1+f_1\dot{L_3})^2}\right) + \frac{B_1}{(C+\alpha\eta A)}\left(\frac{\dot{L_3}}{(C+\alpha\eta A)} - \dot{W}_2\right)\right]\left(\dot{\psi}_1^{[11]}\right)^2,$$

$$\dot{A}_{21} = -2\left[b_2\dot{W}_1^{\,2} - \left(\frac{a_2 f_2}{(1+f_2\dot{L_3})^2} + \left(\rho + 2h\dot{L_3}\right)\right)\dot{W}_1\dot{W}_2 + \left(h - \frac{a_2 f_2^{\,2}}{(1+f_2\dot{L_3})^3}\right)\dot{L_2}\dot{W}_2\right]\left(\dot{\psi}_1^{[11]}\right)^2,$$

$$\dot{A}_{31} = -2\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)}{(C+\alpha\eta A)^2}\left(\frac{\dot{L_3}}{(C+\alpha\eta A)} - \dot{W}_2\right) + C_2(\rho + 2h\dot{L_3})\dot{W}_1\dot{W}_2 - \left(C_2 h\dot{L_2} - \frac{a_3}{K}\right)\dot{W}_2^{\,2} - \right.$$
$$\left.\left(\frac{a_3}{K} + B\right)\dot{W}_2\dot{W}_3 + \frac{\gamma\sigma}{(\sigma+\dot{L_4})^3}\dot{W}_3\right]\left(\dot{\psi}_1^{[11]}\right)^2,$$

$$\dot{A}_{41} = 2\left[-B\dot{W}_2 + \frac{\gamma\sigma}{(\sigma+\dot{L_4})^3}\dot{W}_3\right]\dot{W}_3\left(\dot{\psi}_1^{[11]}\right)^2.$$

Hence, it obtains that $\left(\dot{\Omega}^{[11]}\right)^T\left[D^2 f_{a_1}(P_{11}, \dot{a}_1)\left(\dot{\psi}^{[11]}, \dot{\psi}^{[11]}\right)\right]$

$$= 2\left[-\left(b_1 + \frac{\dot{a}_1 f_1 \dot{W}_2}{(1+f_1\dot{L_3})^2}\right) + \frac{B_1}{(C+\alpha\eta A)}\left(\frac{\dot{L_3}}{(C+\alpha\eta A)} - \dot{W}_2\right)\right]\dot{\Omega}_1^{[11]}\left(\dot{\psi}_1^{[11]}\right)^2 \neq 0,$$

under conditions $(3, 99, 127, 128, 129)$, which are given in [21] and condition (12). Thus, by using Sotomayor's theorem P_{11} has a transcritical bifurcation at the parameter $\dot{a}_1$.

While if condition (12) not holds and according to condition (13) then there is no a transcritical bifurcation and by substituting $\dot{\psi}^{[11]}$ in (4) we get $D^3 f_{a_1}(P_{11}, \dot{a}_1)\left(\dot{\psi}^{[11]}, \dot{\psi}^{[11]}, \dot{\psi}^{[11]}\right) = [\dot{M}_{i1}]_{4x1}$.

$$\dot{M}_{11} = 6\left[\frac{\dot{a}_1 {f_1}^2 {\dot{W}_2}^2}{(1+f_1\dot{L_3})^3}\left(1 - \frac{f_1\dot{W}_2}{(1+f_1\dot{L_3})}\right) - \frac{B_1}{(C+\alpha\eta A)^2}\left(\frac{\dot{L_3}}{(C+\alpha\eta A)} - \dot{W}_2\right)\right]\left(\dot{\psi}_1^{[11]}\right)^3,$$

$$\dot{M}_{21} = 6\left[\frac{a_2 f_2^2 {\dot{W}_2}^2}{(1+f_2\dot{L_3})^3}\left(\frac{f_2\dot{L_2}}{(1+f_2\dot{L_3})} - \dot{W}_1\right) + h\dot{W}_1\right]\left(\dot{\psi}_1^{[11]}\right)^3,$$

$$\dot{M}_{31} = 6\left[\frac{C_1\left(C+\eta A(\alpha-1)\right)}{(C+\alpha\eta A)^3}\left(\frac{\dot{L}_3}{(C+\alpha\eta A)} - \dot{W}_2\right) - C_2 h\dot{W}_1{\dot{W}_2}^2 - \frac{\gamma\sigma}{(\sigma+\dot{L_4})^3}{\dot{W}_3}^3\right]\left(\dot{\psi}_1^{[11]}\right)^3,$$

$$\dot{M}_{41} = 6\frac{\gamma\sigma}{(\sigma+\dot{L_4})^3}{\dot{W}_3}^3\left(\dot{\psi}_1^{[11]}\right)^3.$$

Then $\left(\dot{\Omega}^{[11]}\right)^T\left[D^3 f_{a_1}(P_{11}, \dot{a}_1)\left(\dot{\psi}^{[11]}, \dot{\psi}^{[11]}, \dot{\psi}^{[11]}\right)\right]$

$$= 6\left[\frac{\dot{a}_1 {f_1}^2 {\dot{W}_2}^2}{(1+f_1\dot{L_3})^3}\left(1 - \frac{f_1\dot{W}_2}{(1+f_1\dot{L_3})}\right) - \frac{B_1}{(C+\alpha\eta A)^2}\left(\frac{\dot{L_3}}{(C+\alpha\eta A)} - \dot{W}_2\right)\right]\dot{\Omega}^{[11]}\left(\dot{\psi}_1^{[11]}\right)^3 \neq 0,$$

under conditions $(3, 99, 127, 128, 129)$, which is given in $[21]$ and condition (14). Hence again , by using Sotomayor's theorem system (1) at P_{11} has pitchfork bifurcation with the parameter $\dot{a}_1$.

Similarly for (EP) $P_{12} = \left(0,\ \ddot{L}_2,\ \ddot{L}_3,\ \ddot{L}_4\right)$.

Theorem 9. Assume that conditions $(99,\ 132,\ 133,\ 135)$, which are given in $[21]$ and the following conditions hold

$$a_3 + \frac{C_1(L_1^* + \eta A)}{(C+\alpha\eta A + L_1^*)} + C_2(\rho + 2hL_3^*)L_2^* > 2\frac{a_3}{K}L_3^* + \frac{a_3}{K}L_4^* + BL_4^* + \frac{\chi_{44}^*(\chi_{11}^*\chi_{23}^*\chi_{32}^* + \chi_{22}^*\chi_{13}^*\chi_{31}^*) + \chi_{11}^*\chi_{22}^*\chi_{34}^*\chi_{43}^*}{\chi_{11}^*\chi_{22}^*\chi_{44}^*}, \tag{15}$$

$$W_7^* \neq W_8^*. \tag{16}$$

Where

$$W_7^* = \Bigg(\left[\frac{(C+\alpha\eta A)B_1 L_3^*}{(C+\alpha\eta A + L_1^*)^3}{W_1^*}^2 + \left(\frac{a_1 f_1}{\left(1+f_1 L_3^*\right)^2} + \frac{(C+\alpha\eta A)B_1}{(C+\alpha\eta A + L_1^*)^2}\right)W_1^* + \left(\frac{a_1 {f_1}^2 L_1^*}{\left(1+f_1 L_3^*\right)^3}\right)\right]W_4^* + \left[\left(\frac{a_2 f_2}{\left(1+f_2 L_3^*\right)^2} + (\rho + 2hL_3^*)\right)W_2^* + \frac{a_2 {f_2}^2 L_2^*}{\left(1+f_2 L_3^*\right)^3}\right]W_5^* - \left[C_2 h L_2^* + \left(\frac{a_3}{K} + B\right)W_3^*\right] + [BW_3^*]W_6^*\Bigg),$$

$$W_8^* = \Big(\big[b_1 {W_1^*}^2\big] W_4^* + \big[b_2 {W_2^*}^2 + hL_2^*\big] W_5^* - \Big[\frac{C_1\big(C+\eta A(\alpha-1)\big)W_1^*}{(C+\alpha\eta A+L_1^*)^2}\Big(1+\frac{L_3^* W_1^*}{(C+\alpha\eta A+L_1^*)}\Big) + C_2(\rho+2hL_3^*)W_2^* + \frac{a_3}{K} + \Big(\frac{\sigma\gamma}{(\sigma+L_4^*)^3}\Big){W_3^*}^2\Big] + \Big[\frac{\sigma\gamma}{(\sigma+L_4)^3}{W_3^*}^2\Big] W_6^*\Big).$$

With

$$W_1^* = \frac{\chi_{13}^*}{\chi_{11}^*},\ \ W_2^* = \frac{\chi_{23}^*}{\chi_{22}^*},\ \ W_3^* = \frac{\chi_{43}^*}{\chi_{44}^*},\ \ W_4^* = \frac{\chi_{31}^*}{\chi_{11}^*},\ \ W_5^* = \frac{\chi_{32}^*}{\chi_{22}^*},\ \ W_6^* = \frac{\chi_{34}^*}{\chi_{44}^*}.$$

Then system (1) at the (EP) $P_{13} = (L_1^*, L_2^*, L_3^*, L_4^*)$ with the parameter $d^* = a_3 - 2\frac{a_3}{K}L_3^* - \frac{a_3}{K}L_4^* + \frac{C_1(L_1^*+\eta A)}{(C+\alpha\eta A+L_1^*)} + C_2(\rho+2hL_3^*)L_2^* - BL_4^* - \frac{\chi_{44}^*(\chi_{11}^*\chi_{23}^*\chi_{32}^*+\chi_{22}^*\chi_{13}^*\chi_{31}^*)+\chi_{11}^*\chi_{22}^*\chi_{34}^*\chi_{43}^*}{\chi_{11}^*\chi_{22}^*\chi_{44}^*}$ has a saddle-node bifurcation, while neither a transcritical nor pitchfork bifurcation can be occur at $d = d^*$.

Proof. From the characteristic equation $\lambda^4 + E_1\lambda^3 + E_2\lambda^2 + E_3\lambda + E_4 = 0$ of Jacobian matrix J_{13} which is given in Eq.(131) in [21], system (1) at the (EP) P_{13} has eigenvalue equal to zero say $\big(\lambda_{13L_3} = 0\big)$ if and only if $E_4 = 0$ then P_{13} will be non-hyperbolic equilibrium point then J_{13} with $d = d^*$ becomes $J_{13}^* = J_{13}(P_{13}, d^*) = \big[\chi_{ij}^*\big]_{4\times 4}$, where $\chi_{ij}^* = \chi_{ij}$; $i,j = 1,2,3,4$ which are given in Eq.(130) in [21] accept $\chi_{33}^* = \frac{\chi_{44}^*(\chi_{11}^*\chi_{23}^*\chi_{32}^*+\chi_{22}^*\chi_{13}^*\chi_{31}^*)+\chi_{11}^*\chi_{22}^*\chi_{34}^*\chi_{43}^*}{\chi_{11}^*\chi_{22}^*\chi_{44}^*}$.

Now, let $\psi^{[13]} = \Big(\psi_1^{[13]}, \psi_2^{[13]}, \psi_3^{[13]}, \psi_4^{[13]}\Big)^T$ be the eigenvector corresponding to the eigenvalue $\big(\lambda_{13L_3}\big) = 0$.

Thus $\big(J_{13}^* - \lambda_{13L_3} I\big)\psi^{[13]} = 0$, that gives $\psi^{[13]} = \Big(-W_1^*\psi_3^{[13]}, -W_2^*\psi_3^{[13]}, \psi_3^{[13]}, -W_3^*\psi_3^{[13]}\Big)^T$ where W_1^*, W_2^*, W_3^* given in the state of theorem, and $\psi_3^{[13]} \neq 0$ is any real number.

Let $\Omega^{[13]} = \Big(\Omega_1^{[13]}, \Omega_2^{[13]}, \Omega_3^{[13]}, \Omega_4^{[13]}\Big)^T$ be the eigenvector of J_{13}^{*T} for $\lambda_{13L_3} = 0$.

Then we get $\big(J_{13}^{*T} - \lambda_{13L_3} I\big)\Omega^{[13]} = 0$ then by solving this equation for $\Omega^{[13]}$ we get

$\Omega^{[13]} = \Big(-W_4^*\Omega_3^{[13]}, -W_5^*\Omega_3^{[13]}, \Omega_3^{[13]}, -W_6^*\Omega_3^{[13]}\Big)^T$, where W_4^*, W_5^*, W_6^* given in the state of theorem, and $\Omega_1^{[13]} \neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial d} = f_d(X, d) = \Big(\frac{\partial f_1}{\partial d}, \frac{\partial f_2}{\partial d}, \frac{\partial f_3}{\partial d}, \frac{\partial f_4}{\partial d}\Big)^T = (0, 0, -L_3, 0)^T$.

So, $f_d(P_{13}, d^*) = (0, 0, -L_3^*, 0)^T$ and hence $\left(\Omega^{[13]}\right)^T f_d(P_{13}, d^*) = -L_3^*\Omega_3^{[13]} \neq 0$.

Moreover, by substituting $\psi^{[13]}$ in (3) we get $D^2 f(P_{13}, d^*)\left(\psi^{[13]}, \psi^{[13]}\right) = [A_{i1}^*]_{4\times 1}$.

$$A_{11}^* = 2\left[\left(-b_1 + \frac{(C+\alpha\eta A)B_1 L_3^*}{(C+\alpha\eta A+L_1^*)^3}\right) W_1^{*2} + \left(\frac{a_1 f_1}{\left(1+f_1 L_3^*\right)^2} + \frac{(C+\alpha\eta A)B_1}{(C+\alpha\eta A+L_1^*)^2}\right) W_1^* + \left(\frac{a_1 {f_1}^2 L_1^*}{\left(1+f_1 L_3^*\right)^3}\right)\right]\left(\psi_3^{[13]}\right)^2,$$

$$A_{21}^* = -2\left[b_2 W_2^{*2} - \left(\frac{a_2 f_2}{\left(1+f_2 L_3^*\right)^2} + (\rho + 2hL_3^*)\right) W_2^* + \left(h - \frac{a_2 {f_2}^2}{\left(1+f_2 L_3^*\right)^3}\right) L_2^*\right]\left(\psi_3^{[13]}\right)^2,$$

$$A_{31}^* = 2\left[\frac{-C_1\left(C+\eta A(\alpha-1)\right)W_1^*}{(C+\alpha\eta A+L_1^*)^2}\left(1 + \frac{L_3^* W_1^*}{(C+\alpha\eta A+L_1^*)}\right) - C_2(\rho + 2hL_3^*)W_2^* + \left(C_2 h L_2^* - \frac{a_3}{K}\right) + \left(\frac{a_3}{K} + B\right) W_3^* - \left(\frac{\sigma\gamma}{(\sigma+L_4^*)^3}\right) W_3^{*2}\right]\left(\psi_3^{[13]}\right)^2,$$

$$A_{41}^* = -2\left[B - \frac{\sigma\gamma}{(\sigma+L_4)^3} W_3^*\right] W_3^* \left(\psi_3^{[13]}\right)^2.$$

Hence, it obtains that

$$\left(\Omega^{[13]}\right)^T\left[D^2 f(P_{13}, d^*)\left(\psi^{[13]}, \psi^{[13]}\right)\right] = 2\left[-\left(\left[\left(\frac{(C+\alpha\eta A)B_1 L_3^*}{(C+\alpha\eta A+L_1^*)^3} W_1^{*2} + \left(\frac{a_1 f_1}{(1+f_1 L_3^*)^2} + \frac{(C+\alpha\eta A)B_1}{(C+\alpha\eta A+L_1^*)^2}\right) W_1^* + \left(\frac{a_1 {f_1}^2 L_1^*}{(1+f_1 L_3^*)^3}\right)\right] W_4^* + \left[\left(\frac{a_2 f_2}{(1+f_2 L_3^*)^2} + (\rho + 2hL_3^*)\right) W_2^* + \frac{a_2 {f_2}^2 L_2^*}{(1+f_2 L_3^*)^3}\right] W_5^* - \left[C_2 h L_2^* + \left(\frac{a_3}{K} + B\right) W_3^*\right] + [BW_3^*]W_6^*\right) + \left(\left[b_1 W_1^{*2}\right]W_4^* + \left[b_2 W_2^{*2} + hL_2^*\right]W_5^* - \left[\frac{C_1\left(C+\eta A(\alpha-1)\right)W_1^*}{(C+\alpha\eta A+L_1^*)^2}\left(1 + \frac{L_3^* W_1^*}{(C+\alpha\eta A+L_1^*)}\right) + C_2(\rho + 2hL_3^*)W_2^* + \frac{a_3}{K} + \left(\frac{\sigma\gamma}{(\sigma+L_4^*)^3}\right) W_3^{*2}\right] + \left[\frac{\sigma\gamma}{(\sigma+L_4)^3} W_3^{*2}\right] W_6^*\right)\right]\Omega_3^{[13]}\left(\psi_3^{[13]}\right)^2$$

$$\left(\Omega^{[13]}\right)^T\left[D^2 f(P_{13}, d^*)\left(\psi^{[13]}, \psi^{[13]}\right)\right] = 2[-W_7^* + W_8^*]\Omega_3^{[13]}\left(\psi_3^{[13]}\right)^2 \neq 0,$$

under conditions $(99, 132, 133, 135)$, which are given in $[21]$ and condition $(15, 16)$ are hold. Thus, by using Sotomayor's theorem P_{13} has a saddle-node bifurcation at the parameter d^* but a transcritical and pitchfork bifurcation cannot be occur.

Similarly for (EP) $P_{14} = (L_1^{**}, L_2^{**}, L_3^{**}, L_4^{**})$.

4.HOPF BIFURCATION ANALYSIS

In this section, the following theorem shows that when the possibility of a Hopf bifurcation (HB) happening near the positive (EP) P_{13} of the system (1), an application to the Hopf bifurcation [23] for local bifurcation is appropriate.

Theorem 10. Assume that conditions $(99, 132-135)$ which are given in $[21]$ and the following conditions hold:

$$\Delta_1 < \frac{E_1^3}{4} \tag{17}$$

$$\chi_{22}\chi_{44} + \chi_{11}(\chi_{22} + \chi_{44}) > 1 \tag{18}$$

$$a_3 + \frac{C_1(L_1^* + \eta A)}{(\mathrm{C} + \alpha\eta A + L_1^*)} + C_2(\rho + 2hL_3^*)L_2^* > \left(2\frac{a_3}{K}L_3^* + \left(\frac{a_3}{K} + B\right)L_4^*\right) \tag{19}$$

$$\frac{E_3(d^\star)}{E_1(d^\star)}\left(-(\chi_{11} + \chi_{22} + \chi_{44}) + \chi_{11}\chi_{22} + \chi_{44}(\chi_{11} + \chi_{22}) + 2E_3(d^\star)\right) \neq$$

$$(\chi_{11}\chi_{22}\chi_{33} +)E_3(d^\star) - \left(\frac{E_3(d^\star)}{E_1(d^\star)}\right)^2\left(2\left(\chi_{11}\chi_{22} + \chi_{44}(\chi_{11} + \chi_{22}) + E_3(d^\star)\right)\right) \tag{20}$$

then at the parameter value $d = d^\star$ system (1) has a (HB) near the positive point P_{13}.

Proof. Consider the characteristic equation which is given in Eq.(131) in [21] of system (1) at P_{13}, we must choose a parameter let's say $(d^\star)$ using the (HB) theorem for $n = 4$, to confirm that the required and sufficient conditions for (HB) to occur satisfy that

$E_{\mathrm{i}}(d^\star) > 0$; i = 1,3,4, $\Delta_1(d^\star) = E_1E_2 - E_3 > 0$, $E_1^3 - 4\Delta_1 > 0$ and

$\Delta_2(d^\star) = (E_1E_2 - E_3)E_3 - E_2^2E_4 = 0$.

Then $E_{\mathrm{i}}(d^\star) > 0$; i = 1,3,4, $\Delta_1(d^\star) > 0$ and $E_1^3 - 4\Delta_1 > 0$ under conditions $(99, 132 - 135)$ which are given in [21], with condition (17) are holds.

On the other hand, it is observed that $\Delta_2 = 0$ gives

$$-\mathcal{R}_1 d^{\star 3} + \mathcal{R}_2 d^{\star 2} - \mathcal{R}_3 d^\star + \mathcal{R}_4 = 0, \tag{21}$$

where

$\mathcal{R}_1 = [(\chi_{11} + \chi_{22} + \chi_{44})(-1 + \chi_{22}\chi_{44} + \chi_{11}(\chi_{22} + \chi_{44}))]$,

$\mathcal{R}_2 = 3\mathcal{R}_1\mathcal{H}_4 + \mathcal{H}_1$,

$\mathcal{R}_3 = (3\mathcal{R}_1\mathcal{H}_4 + 2\mathcal{H}_1)\mathcal{H}_4 + \mathcal{H}_2$,

$\mathcal{R}_4 = (\mathcal{R}_1\mathcal{H}_4^2 + \mathcal{H}_1\mathcal{H}_4 + \mathcal{H}_2)\mathcal{H}_4 + \mathcal{H}_3$.

With

$\mathcal{H}_1 = \chi_{13}\chi_{31} + \chi_{23}\chi_{32} + \chi_{43}\chi_{34} + {\chi_{22}}^3\chi_{44} - (2 + \chi_{13}\chi_{31} + \chi_{23}\chi_{32}){\chi_{44}}^2 + {\chi_{11}}^3(\chi_{22} + \chi_{44}) - {\chi_{22}}^2(2 + \chi_{13}\chi_{31} + \chi_{43}\chi_{34} - 2{\chi_{44}}^2) + \chi_{22}\chi_{44}(-5 - 3\chi_{13}\chi_{31} - 2\chi_{23}\chi_{32} - 2\chi_{43}\chi_{34} + {\chi_{44}}^2) + \chi_{11}\left({\chi_{22}}^3 + 6{\chi_{22}}^2\chi_{44} - (5 + 2\chi_{13}\chi_{31} + 3\chi_{23}\chi_{32} + 2\chi_{43}\chi_{34})\chi_{44} + {\chi_{44}}^3 - \chi_{22}(5 + 2(\chi_{13}\chi_{31} + \chi_{23}\chi_{32}) + 3\chi_{43}\chi_{34} - 6{\chi_{44}}^2)\right) + {\chi_{11}}^2\left(-2 - \chi_{23}\chi_{32} - \chi_{43}\chi_{34} + 2({\chi_{22}}^2 + 3\chi_{22}\chi_{44} + {\chi_{44}}^2)\right)$,

$\mathcal{H}_2 = [\chi_{22}(2+\chi_{13}\chi_{31}+\chi_{43}\chi_{34})(\chi_{13}\chi_{31}+\chi_{23}\chi_{32}+\chi_{43}\chi_{34}) + (2+\chi_{13}\chi_{31}+\chi_{23}\chi_{32})(\chi_{13}\chi_{31}+\chi_{23}\chi_{32}+\chi_{43}\chi_{34})\chi_{44} - \chi_{22}(5+3\chi_{13}\chi_{31}+2\chi_{23}\chi_{32}+2\chi_{43}\chi_{34}){\chi_{44}}^2 - (1+\chi_{13}\chi_{31}+\chi_{23}\chi_{32}){\chi_{44}}^3 - {\chi_{22}}^3(1+\chi_{13}\chi_{31}+\chi_{43}\chi_{34}-{\chi_{44}}^2) + {\chi_{22}}^2\chi_{44}(-5-3\chi_{13}\chi_{31}-2\chi_{23}\chi_{32}-2\chi_{43}\chi_{34}+{\chi_{44}}^2) + {\chi_{11}}^3(-1+{\chi_{22}}^2-\chi_{23}\chi_{32}-\chi_{43}\chi_{34}+3\chi_{22}\chi_{44}+{\chi_{44}}^2) + {\chi_{11}}^2({\chi_{22}}^3+6{\chi_{22}}^2\chi_{44}-(5+2\chi_{13}\chi_{31}+3\chi_{23}\chi_{32}+2\chi_{43}\chi_{34})\chi_{44}+{\chi_{44}}^3-\chi_{22}(5+2\chi_{13}\chi_{31}+2\chi_{23}\chi_{32}+3\chi_{43}\chi_{34}-6{\chi_{44}}^2)) + \chi_{11}((2+\chi_{23}\chi_{32}+\chi_{43}\chi_{34})(\chi_{13}\chi_{31}+\chi_{23}\chi_{32}+\chi_{43}\chi_{34}) + 3{\chi_{22}}^3\chi_{44} - (5+2\chi_{13}\chi_{31}+3\chi_{23}\chi_{32}+2\chi_{43}\chi_{34}){\chi_{44}}^2 - {\chi_{22}}^2(5+2\chi_{13}\chi_{31}+2\chi_{23}\chi_{32}+3\chi_{43}\chi_{34}-6{\chi_{44}}^2) + 3\chi_{22}\chi_{44}(-2(2+\chi_{13}\chi_{31}+\chi_{23}\chi_{32}+\chi_{43}\chi_{34})+{\chi_{44}}^2))],$

$$\begin{aligned}\mathcal{H}_3 = [&{\chi_{22}}^2(\chi_{13}(\chi_{31}+\chi_{23}\chi_{31}\chi_{32})+\chi_{43}\chi_{34}+\chi_{23}(\chi_{32}+\chi_{32}\chi_{43}\chi_{34})) + \chi_{22}(\chi_{23}\chi_{32}(2+\chi_{23}\chi_{32})\\
&+\chi_{43}\chi_{34}(2+\chi_{43}\chi_{34}) - {\chi_{22}}^2(1+\chi_{13}\chi_{31}+\chi_{43}\chi_{34}) + \chi_{13}\chi_{31}(2+\chi_{23}\chi_{32}\\
&+\chi_{43}\chi_{34}))\chi_{44} + (\chi_{13}\chi_{31}+\chi_{23}\chi_{32}+(1+\chi_{13}\chi_{31}+\chi_{23}\chi_{32})\chi_{43}\chi_{34} - {\chi_{22}}^2(2\\
&+2\chi_{13}\chi_{31}+\chi_{23}\chi_{32}+\chi_{43}\chi_{34})){\chi_{44}}^2 - \chi_{22}(1+\chi_{13}\chi_{31}+\chi_{23}\chi_{32}){\chi_{44}}^3 + {\chi_{11}}^3(\chi_{22}\\
&+\chi_{44})(-1-\chi_{23}\chi_{32}-\chi_{43}\chi_{34}+\chi_{22}\chi_{44}) + \chi_{11}(\chi_{22}({\chi_{13}}^2{\chi_{31}}^2+\chi_{13}\chi_{31}(2\\
&+\chi_{43}\chi_{34}) - {\chi_{22}}^2(1+\chi_{13}\chi_{31}+\chi_{43}\chi_{34}) + (2+\chi_{23}\chi_{32})(\chi_{23}\chi_{32}+\chi_{43}\chi_{34}))\\
&+({\chi_{13}}^2{\chi_{31}}^2+\chi_{13}\chi_{31}(2+\chi_{23}\chi_{32})+(2+\chi_{43}\chi_{34})(\chi_{23}\chi_{32}+\chi_{43}\chi_{34}) - {\chi_{22}}^2(5\\
&+3\chi_{13}\chi_{31}+3\chi_{23}\chi_{32}+3\chi_{43}\chi_{34}))\chi_{44} + \chi_{22}(-5+{\chi_{22}}^2-3\chi_{13}\chi_{31}-3\chi_{23}\chi_{32}\\
&-3\chi_{43}\chi_{34}){\chi_{44}}^2 + (-1+{\chi_{22}}^2-\chi_{13}\chi_{31}-\chi_{23}\chi_{32}){\chi_{44}}^3) + {\chi_{11}}^2(\chi_{13}\chi_{31}+\chi_{23}\chi_{32}\\
&+\chi_{23}\chi_{32}\chi_{13}\chi_{31}+\chi_{43}\chi_{34}+\chi_{13}\chi_{31}\chi_{43}\chi_{34}+{\chi_{22}}^3\chi_{44}-(2+\chi_{13}\chi_{31}+2\chi_{23}\chi_{32}\\
&+\chi_{43}\chi_{34}){\chi_{44}}^2 - {\chi_{22}}^2(2+\chi_{13}\chi_{31}+\chi_{23}\chi_{32}+2\chi_{43}\chi_{34}-2{\chi_{44}}^2) + \chi_{22}\chi_{44}(-5\\
&-3\chi_{13}\chi_{31}-3\chi_{23}\chi_{32}-3\chi_{43}\chi_{34}+{\chi_{44}}^2))],\end{aligned}$$

$\mathcal{H}_4 = a_3 + \frac{C_1(L_1^*+\eta A)}{(\mathrm{C}+\alpha\eta A+L_1^*)} + C_2(\rho+2hL_3^*)L_2^* - \left(2\frac{a_3}{K}L_3^* + \left(\frac{a_3}{K}+B\right)L_4^*\right).$

Then by using Descartes rule of sign, Eq.(21) has a unique positive root $(d^\star)$ if in addition to conditions $(99, 132-135)$ which are given in $[21]$ with conditions $(18, 19)$ and one of the following sets of conditions hold

$\mathcal{R}_2 < 0, \mathcal{R}_3 < 0, \mathcal{R}_4 < 0$ $\quad OR \quad$ $\mathcal{R}_2 < 0, \mathcal{R}_3 > 0, \mathcal{R}_4 < 0$ $\quad OR$

$\mathcal{R}_2 > 0, \mathcal{R}_3 < 0, \mathcal{R}_4 < 0$ $\quad OR \quad$ $\mathcal{R}_2 > 0, \mathcal{R}_3 > 0, \mathcal{R}_4 < 0$

Now, at $d = d^\star$ the characteristic Eq.(131) which is given in $[21]$ can be written as

$$\left(\lambda_{13}^2 + \frac{E_3}{E_1}\right)\left(\lambda_{13}^2 + E_1\lambda_{13} + \frac{\Delta_1}{E_1}\right) = 0,$$

which has four roots $\lambda_{13L_1,L_2} = \pm\mathrm{i}\sqrt{\frac{E_3}{E_1}}$ and $\lambda_{13L_3,L_4} = \frac{1}{2}\left(-E_1 \pm \sqrt{E_1^2 - 4\frac{\Delta_1}{E_1}}\right).$

Clearly, at $d = d^\star$ there are two eigenvalues $(\lambda_{13L_1}$ and $\lambda_{13L_2})$ are pure imaginary and the other two eigenvalues are real and negative $(\lambda_{13L_3}$ and $\lambda_{13L_4})$. Now for all values of d in the neighborhood of $d^\star$ the roots in general of the following form

$$\lambda_{13L_1,L_2} = \varsigma_1 \pm i\varsigma_2,\ \ \lambda_{13L_3,L_4} = \frac{1}{2}\left(-E_1 \pm \sqrt{E_1^2 - 4\frac{\Delta_1}{E_1}}\right).$$

Clearly, $Re\left(\lambda_{13L_1,L_2}(d)\right)\big|_{d=d^\star} = \varsigma_1(d^\star) = 0$, this indicates that at $d = d^\star$, the first of the necessary and sufficient requirements for Hopf bifurcation is met.

In order to confirm the transversality criterion, we need to demonstrate that

$$\Theta^\star(d^\star)\,\Psi^\star(d^\star) + \Gamma^\star(d^\star)\Phi^\star(d^\star) \neq 0,$$

Note that for $d = d^\star$ we have $\varsigma_1 = 0$ and $\varsigma_2 = \sqrt{\frac{E_3}{E_1}}$, substituting the value of ς_2 gives the following simplifications:

$$\Psi^\star(d^\star) = -2\,E_3(d^\star),$$

$$\Phi^\star(d^\star) = 2\frac{\varsigma_2(d^\star)}{E_1(d^\star)}\left(E_1(d^\star)E_2(d^\star) - 2E_3(d^\star)\right),$$

$$\Theta^\star(d^\star) = E_4'(d^\star) - E_2'(d^\star)\frac{E_3(d^\star)}{E_1(d^\star)},$$

$$\Gamma^\star(d^\star) = \sqrt{\frac{\mathrm{E}_3}{E_1}}(d^\star)\left(E_3'(d^\star) - E_1'(d^\star)\frac{E_3(d^\star)}{E_1(d^\star)}\right),$$

where

$$E_1' = \frac{\partial E_1}{\partial d}\Big|_{d=d^\star} = 1,$$

$$E_2' = \frac{\partial E_2}{\partial d}\Big|_{d=d^\star} = -(\chi_{11} + \chi_{22} + \chi_{44}),$$

$$E_3' = \frac{\partial E_3}{\partial d}\Big|_{d=d^\star} = \chi_{11}\chi_{22} + \chi_{44}(\chi_{11} + \chi_{22}),$$

$$E_4' = \frac{\partial E_4}{\partial d}\Big|_{d=d^\star} = -\chi_{11}\chi_{22}\chi_{33}.$$

$$\Theta^\star(d^\star)\Psi^\star(d^\star) + \Gamma^\star(d^\star)\Phi^\star(d^\star) = 2\left[\frac{E_3(d^\star)}{E_1(d^\star)}\left(-(\chi_{11} + \chi_{22} + \chi_{44}) + \chi_{11}\chi_{22} + \chi_{44}(\chi_{11} + \chi_{22}) + 2E_3(d^\star) - \frac{E_3(d^\star)}{E_1(d^\star)}\left(2\left(\chi_{11}\chi_{22} + \chi_{44}(\chi_{11} + \chi_{22}) + E_3(d^\star)\right)\right)\right) + (\chi_{11}\chi_{22}\chi_{33})E_3(d^\star)\right] \neq 0,$$

under conditions $(99, 132 - 135)$ which are given in $[21]$ and condition (20), so we obtain that the (HB) happens at the parameter $d = d^\star$ around the (EP) P_{13}.

Similarly for (EP) $P_{14} = (L_1^{**}, L_2^{**}, L_3^{**}, L_4^{**})$.

5. NUMERICAL ANALYSES

In order to confirm our analytical findings and investigate the effects of varying the values of each parameter on the dynamic behaviour of the system, the dynamic behaviour of system (1) is investigated numerically using the Mathematica program. As shown in Figure (1), system (1) has a (GAS) positive equilibrium point, with the following hypothetical parameter setup that meets the stability conditions for the positive equilibrium point.

$$\left.\begin{array}{l} a_1 = 0.05,\ a_2 = 0.08,\ a_3 = 0.07,\ f_1 = 0.05,\ f_2 = 0.06,\ b_1 = 0.01,\ b_2 = 0.015, \\ B_1 = 0.07,\ C = 1,\ \alpha = 1.9,\ \eta = 0.06,\ A = 0.05,\ \rho = 0.05,\ h = 0.03,\ K = 3, \\ C_1 = 0.06,\ C_2 = 0.02,\ d = 0.04,\ B = 0.04,\ \gamma = 0.008,\ \sigma = 0.009,\ \delta = 0.005 \end{array}\right\} \quad (22)$$

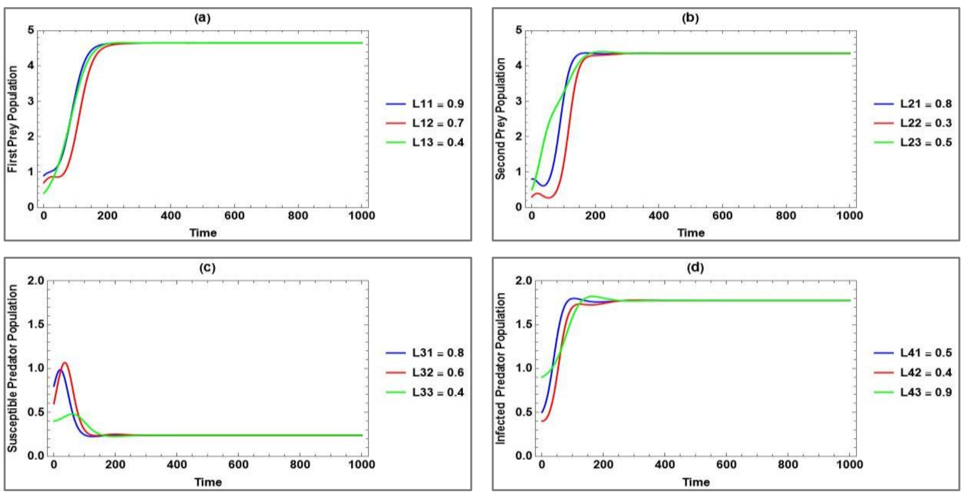

Figure -1 The time series of system (1) that started from three different initial points $(0.9, 0.8, 0.8, 0.5)$, $(0.7, 0.3, 0.6, 0.4)$ and $(0.4, 0.5, 0.4, 0.9)$ for the data given in (22). (a) the trajectory of L_1 as a function of time, (b) trajectory of L_2 as a function of time, (c) trajectory of L_3 as a function of time, (d) the trajectory of L_4 as a function of time, approaches to $P_{13} = (4.647, 4.355, 0.237, 1.774)$.

Now, to examine how the values of the parameters affect the system's dynamic behavior, we varied one parameter at each time with the rest parameters given in (22), and the result is displayed in Table 2.

Table 2: System (1)'s Dynamical Behavior

Range of Parameter	Stable Point	Bifurcation	Persistence
$0.001 \leq a_1 < 0.01985$ $0.01985 \leq a_1 \leq 1$	P_{11} P_{13}	 0.01985	*Not Persists* *Persists*
$0.001 \leq a_2 < 0.0143$ $0.0143 \leq a_2 \leq 1$	P_9 P_{13}	 0.0143	*Not Persists* *Persists*
$0.001 \leq a_3 \leq 5$	P_{13}		*Persists*
$0.001 \leq f_1 < 4.5657$ $4.5657 \leq f_1 \leq 10$	P_{13} P_{11}	 4.5657	*Persists* *Not Persists*
$0.001 \leq f_2 < 20.13$ $20.13 \leq f_2 < 30$	P_{13} P_9	 20.13	*Persists* *Not Persists*
$0.001 \leq b_1 < 2.12$ $2.12 \leq b_1 \leq 5$	P_{13} P_{11}	 2.12	*Persists* *Not Persists*
$0.001 \leq b_2 < 6.45$ $6.45 \leq b_2 < 10$	P_{13} P_9	 6.45	*Persists* *Not Persists*
$0.061 \leq B_1 < 0.162$ $0.162 \leq B_1 \leq 1$	P_{13} P_{11}	 0.162	*Persists* *Not Persists*
$0.01 \leq \alpha \leq 10$	P_{13}		*Persists*
$0.001 \leq \eta \leq 3$	P_{13}		*Persists*
$0.001 \leq A \leq 10$	P_{13}		*Persists*
$0.001 \leq K < 0.015$ $0.015 \leq K < 0.895$ $0.895 \leq K < 0.91207$ $0.91207 \leq K \leq 10$	P_4 P_8 P_5 P_{13}	 0.015 0.895 0.91207	*Not Persists* *Not Persists* *Not Persists* *Persists*
$0.001 \leq h <0.7696$	P_{13}		*Persists*
$1 \leq C \leq 10$	P_{13}		*Persists*
$0.001 \leq C_1 \leq 0.069$	P_{13}		*Persists*
$0.0001 \leq C_2 < 1$	P_{13}		*Persists*
$0.001 \leq \rho < 0.31$ $0.31 \leq \rho \leq 5$	P_{13} P_9	 0.31	*Persists* *Not Persists*
$0.001 \leq d < 0.09301$ $0.09301 \leq d < 0.255$ $0.255 \leq d < 1$	P_{13} P_8 P_4	 0.09301 0.255	*Persists* *Not Persists* *Not Persists*
$0.0001 \leq B < 0.0187321$ $0.0187321 \leq B \leq 1$	P_3 P_{13}	 0.0187321	*Not Persists* *Persists*
$0.001 \leq \delta < 0.0266248$ $0.0266248 \leq \delta < 1$	P_{13} P_3	 0.0266248	*Persists* *Not Persists*
$0.0001 \leq \gamma < 0.0217155$ $0.0217155 \leq \gamma \leq 1$	P_{13} P_3	 0.0217155	*Persists* *Not Persists*
$0.01 \leq \sigma \leq 1$	P_{13}		*Persists*

The effect of varying the intrinsic growth rate of the first prey in the range $0.001 \leq a_1 < 0.01985$ was studied, it is observed that the solution of system (1) approach to P_{11} , however increasing this parameter further $0.01985 \leq a_1 \leq 1$ the solution approach to P_{13} as shown in Figure (2).

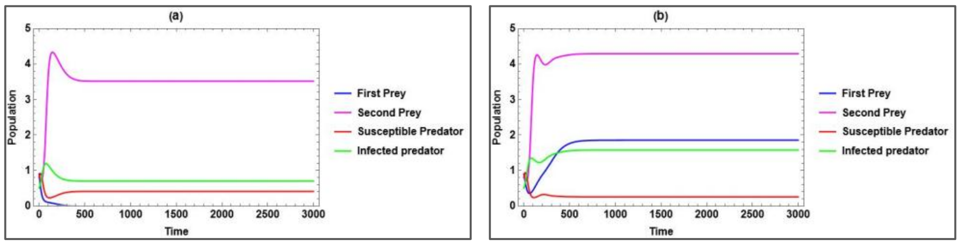

Figure -2 The trajectories of system (1) using data given in (22) with different values of a_1. (a) the trajectory approaches to $P_{11} = (0\,, 3.515\,, 0.407, 0.698)$ when a_1= 0.01, (b) the trajectory approaches to $P_{13} = (1.852, 4.289\,, 0.251\,, 1.573)$ when $a_1 = 0.025$.

The effect of varying the intrinsic growth rate of the second prey in the range $0.001 \leq a_2 < 0.0143$ was studied, it is observed that the solution of system (1) approach to P_9 , however increasing this parameter further $0.0143 \leq a_2 \leq 1$ the solution approach to P_{13} as shown in Figure (3).

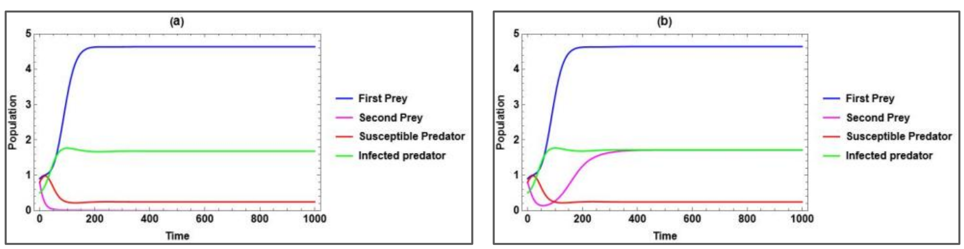

Figure -3 The trajectories of system (1) using data given in (22) with different values of a_2. (a) the trajectory approaches to $P_9 = (4.637\,, 0\,, 0.243, 1.679)$ when a_2= 0.01, (b) the trajectory approaches to $P_{13} = (4.640,\ 1.048\,, 0.241\,, 1.702)$ when $a_2 = 0.03$.

The effect of varying the fear rate of the first prey from susceptible predator in the range $0.001 \leq f_1 < 4.5657$ was studied, it is observed that the solution of system (1) still approaches to P_{13}, however increasing this parameter further $4.5657 \leq f_1 \leq 10$ the solution approaches to P_{11} as shown in Figure (4).

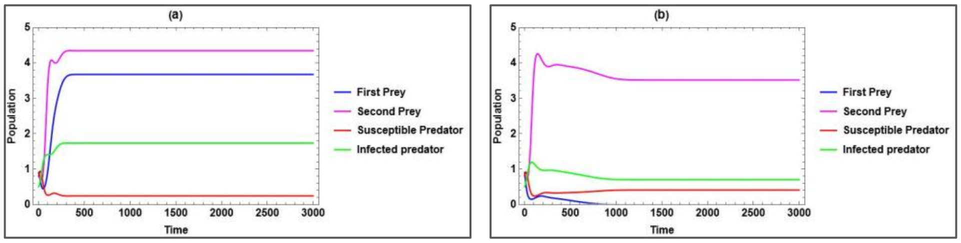

Figure -4 The trajectories of system (1) using data given in (22) with different values of f_1. (a) the trajectory approaches to $P_{13} = (4.673\,, 4.343\,, 0.239, 1.732)$ when $f_1 = 1$, (b) the trajectory approaches to $P_{11} = (0, 3.515\,, 0.407\,, 0.698)$ when $f_1 = 5$.

The effect of varying the fear rate of the second prey from susceptible predator in the range $0.001 \leq f_2 < 20.13$ was studied, it is observed that the solution of system (1) still approaches to P_{13} , however increasing this parameter further $20.13 \leq f_2 < 30$ the solution approach to P_9 as shown in Figure (5).

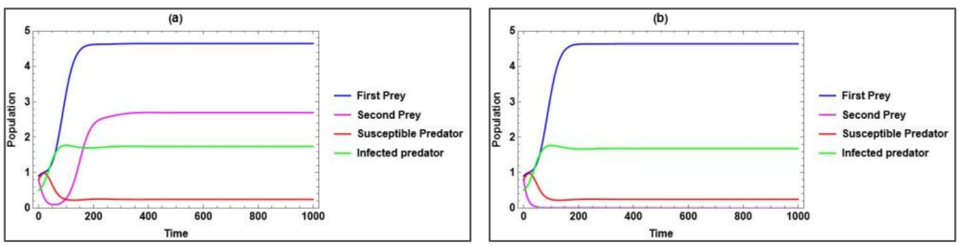

Figure -5 The trajectories of system (1) using data given in (22) with different values of f_2. (a) the trajectory approaches to $P_{13} = (4.643\,, 2.693\,, 0.239, 1.738)$ when $f_2 = 2$, (b) the trajectory approaches to $P_9 = (4.637, 0\,, 0.243\,, 1.679)$ when $f_2 = 21$.

The effect of varying the internal competition rate of first prey in the range $0.001 \leq b_1 < 2.12$ was studied, it is observed that system (1) still approach to P_{13} , however increasing this parameter further $2.12 \leq b_1 \leq 5$ the system approach to P_{11} as shown in Figure (6).

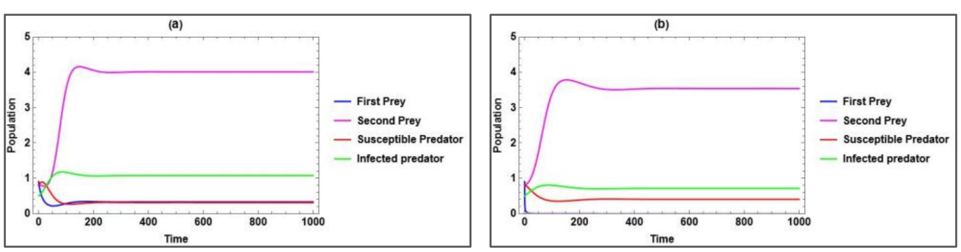

Figure -6 The trajectories of system (1) using data given in (22) with different values of b_1. (a) the trajectory approaches to $P_{13} = (0.329, 4.012, 0.309, 1.074)$ when $b_1 = 0.1$, (b) the trajectory approaches to $P_{11} = (0, 3.537, 0.403, 0.709)$ when $b_1 = 3.5$.

The effect of varying the internal competition rate of second prey in the range $0.001 \leq b_2 < 6.45$ was studied, it is observed that the solution of system (1) still approach to P_{13}, however increasing this parameter further $6.45 \leq b_2 < 10$ the solution approach to P_9 as shown in Figure (7).

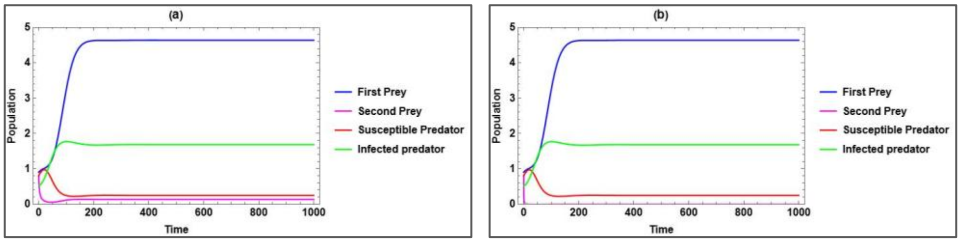

Figure -7 The trajectories of system (1) using data given in (22) with different values of b_2. (a) the trajectory approaches to $P_{13} = (4.637, 0.129, 0.243, 1.682)$ when $b_2 = 0.5$, (b) the trajectory approaches to $P_9 = (4.637, 0, 0.243, 1.679)$ when $b_2 = 8$.

The effect of varying the maximum rate of predation in the range $0.061 \leq B_1 < 0.162$ was studied, it is observed that the solution of system (1) still approach to P_{13}, however increasing this parameter further $0.162 \leq B_1 \leq 1$ the solution approach to P_{11} as shown in Figure (8).

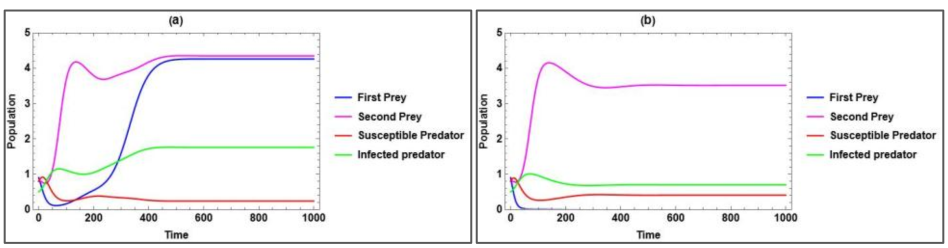

Figure -8 The trajectories of system (1) using data given in (22) with different values of B_1. (a) the trajectory approaches to $P_{13} = (4.262, 4.351, 0.238, 1.759)$ when $B_1 = 0.15$, (b) the trajectory approaches to $P_{11} = (0, 3.515, 0.407, 0.698)$ when $B_1 = 0.2$.

The effect of varying the maximum rate of predation in the range $0.001 \leq \rho < 0.31$ was studied, it is observed that the solution of system (1) still approach to P_9, however increasing this parameter further $0.31 \leq \rho \leq 5$ the solution approach to P_{13} as shown in Figure (9).

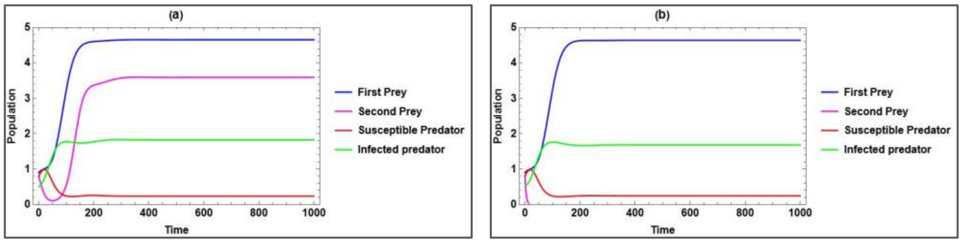

Figure -9 The trajectories of system (1) using data given in (22) with different values of ρ. (a) the trajectory approaches to $P_9 = (4.637, 0, 0.243, 1.679)$ when ρ= 0.32, (b) the trajectory approaches to $P_{13} = (4.652, 3.589, 0.234, 1.825)$ when $\rho = 0.1$.

The effect of varying the carrying capacity of the susceptible predator in the range $0.001 \leq K < 0.015$ was studied, it is observed that system (1) approach to P_4, however increasing this parameter further $0.015 \leq K < 0.895$ the system approach to P_8, moreover increasing this parameter in the range $0.895 \leq K < 0.91207$ the system approach to P_5, again increasing this parameter further $0.91207 \leq K \leq 10$ the system approach to P_{13} as shown in Figure (10).

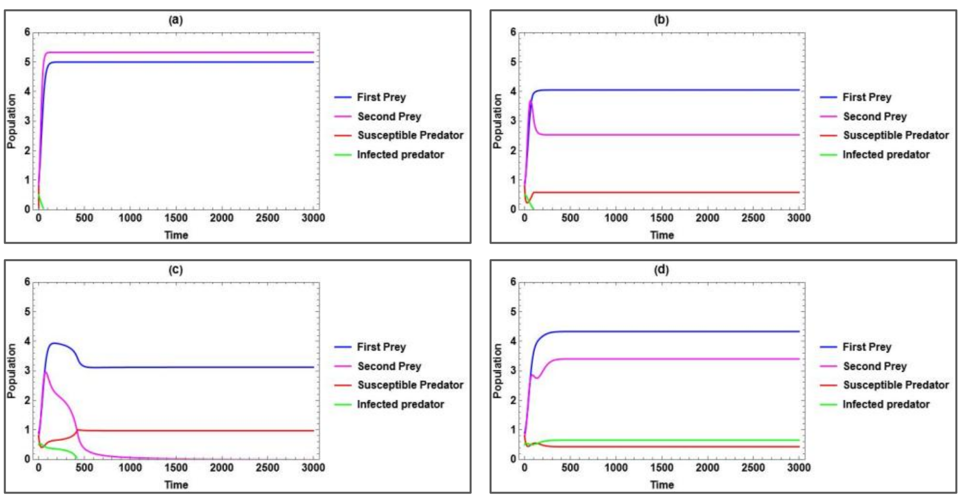

Figure -10 The trajectories of system (1) using data given in (22) with different values of K. (a) the trajectory approaches to $P_4 = (4.998, 5.328, 0, 0)$ when K= 0.001, (b) the trajectory approaches to $P_8 = (\ 4.051,\ 2.533,\ 0.582, 0\ )$when$K = 0.5$, (c) the trajectory approaches to $P_5 = (3.121, 0, 0.970, 0)$ when $K = 0.9$, (d) the trajectory approaches to $P_{13} = (4.332, 3.401, 0.429, 0.649)$ when $K = 1$.

The effect of varying the natural death rate of susceptible predator in the range $0.001 \leq d < 0.09301$ was studied, it is observed that the solution of system (1) still approach to P_{13} , however increasing this parameter further $0.09301 \leq d < 0.255$ the solution approach to P_8, moreover increasing this parameter in the range $0.255 \leq d < 1$ the solution approach to P_4 as shown in Figure (11).

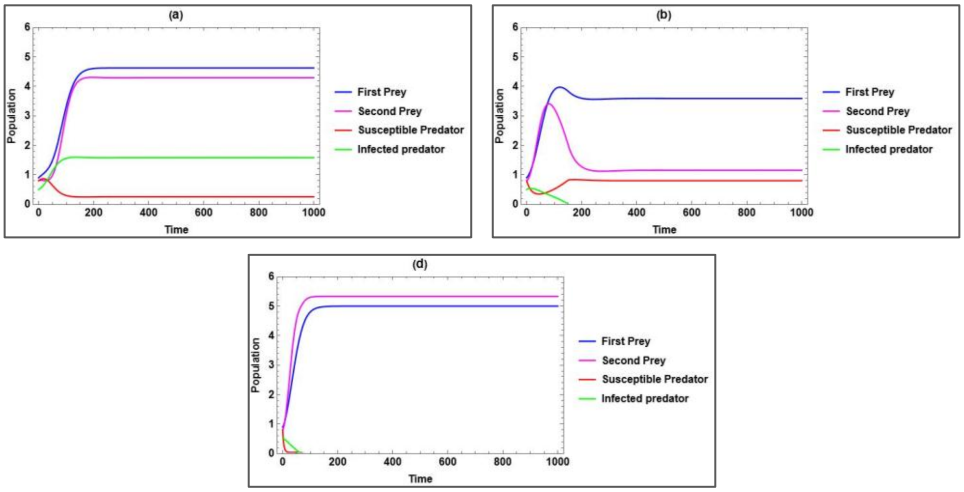

Figure -11 The trajectories of system (1) using data given in (22) with different values of d. (a) the trajectory approaches to $P_{13} = (4.626, 4.292, 0.250, 1.581)$ when $d = 0.05$, (b) the trajectory approaches to $P_8 = (3.590, 1.152, 0.798, 0)$ when $d = 0.1$, (c) the trajectory approaches to $P_4 = (5, 5.333, 0, 0)$ when $d = 0.3$.

The effect of varying the disease transmission rate in the range $0.0001 \leq B < 0.0187321$ was studied, it is observed that the solution of system (1) approach to P_3, however increasing this parameter further $0.0187321 \leq B \leq 1$ the solution approach to P_{13} as shown in Figure (12).

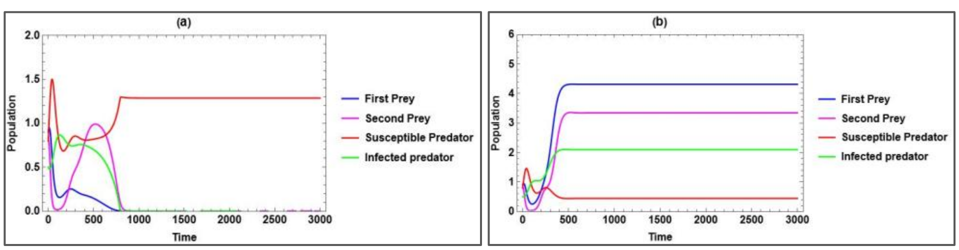

Figure -12 The trajectories of system (1) using data given in (22) with different values of B. (a) the trajectory approaches to $P_3 = (0, 0, 1.286, 0)$ when $B = 0.0187$, (b) the trajectory approaches to $P_{13} = (4.312, 3.341, 0.440, 2.094)$ when $B = 0.02$.

The effect of varying the death rates of infected predator in the range $0.001 \leq \delta < 0.0266248$ was studied, it is observed that the solution of system (1) still approach to P_{13} , however increasing this parameter further $0.174 \leq \delta < 1$ the solution approach to P_3 as shown in Figure (13).

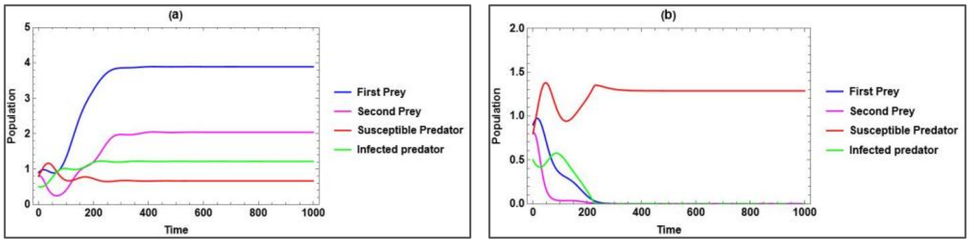

Figure -13 The trajectories of system (1) using data given in (22) with different values of δ. (a) the trajectory approaches to $P_{13} = (3.890, 2.039, 0.663, 1.2171)$ when $\delta = 0.02$, (b) the trajectory approaches to $P_3 = (0, 0, 1.286, 0)$ when $\delta = 0.03$.

The effect of varying the maximum medical resource supplied for treatment in the range $0.0001 \leq \gamma < 0.0217155$ was studied, it is observed that the solution of system (1) still approach to P_{13} , however increasing this parameter further $0.0217155 \leq \gamma \leq 1$ the solution approach to P_3 as shown in Figure (14).

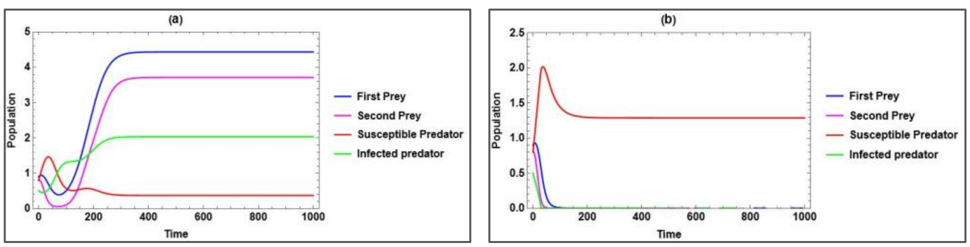

Figure -14 The trajectories of system (1) using data given in (22) with different values of γ. (a) the trajectory approaches to $P_{13} = (4.432, 3.710, 0.370, 2.031)$ when $\gamma = 0.02$, (b) the trajectory approaches to $P_3 = (0, 0, 1.286, 0)$ when $\gamma = 0.03$.

The effect of varying the carrying capacity and the medical resource for treatment rate was studied, it is found that solution of system (1) will approach to the (EP) $P_6 = (0,\ 0.541,\ 0.886,\ 0)$ as it shows in Figure (15).

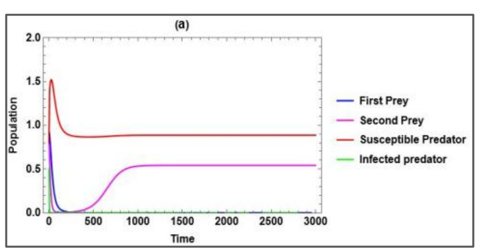

Figure -15 The trajectory of system (1) using data given in (22) approaches to $P_6 = (0,\ 0.541,\ 0.886,\ 0)$ when $K = 2$, $\gamma = 0.08$.

The effect of varying the intrinsic growth rates of the first and the second prey was studied, it is found that solution of system (1) will approach to the (EP) $P_7 = (0,\ 0,\ 0.473,\ 0.564)$ as it shows in Figure (16).

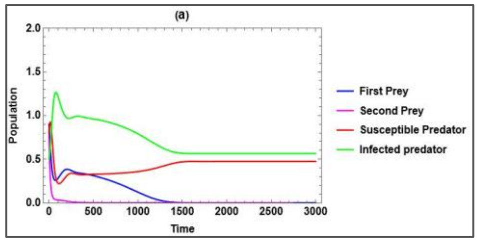

Figure -16 The trajectory of system (1) using data given in (22) approaches to $P_7 = (0,\ 0,\ 0.473,\ 0.564)$ when $a_1 = 0.02$ and $a_2 = 0.01$.

6. THE CONCLUSIONS AND DISCUSSIONS

In this paper, the conditions of the local bifurcation have been established for a food web eco-epidemiological model with fear and internal competition effects in the first and second prey populations, the susceptible predator feeds on preys using two different types of functional responses, as well as with additional food and hunting cooperation, on the other hand treatment is presumed to be being administered to the infected predator, and it is observed that near the equilibrium points:

- At P_7 , P_9 and P_{10} system (1) possesses a transcritical bifurcation only at the intrinsic growth rate of the second prey $(\breve{a}_2,\ \tilde{\tilde{a}}_2)$ respectively while P_3 , P_4 , P_5 , P_6 , P_8 , P_{11} and P_{12} have a transcritical and pitchfork bifurcations at the disease transmission rate $(\bar{B})$, the natural death rate of susceptible predator $(\breve{d})$, the disease transmission rate $(\bar{\bar{B}})$, the intrinsic growth rate of the first prey $(\hat{a}_1)$, the disease transmission rate $(\tilde{B})$, and the intrinsic growth rate of the first prey $(\dot{a}_1)$ respectively.
- At P_{13} and P_{14} system (1) possesses a saddle-node bifurcation at the natural death rate of susceptible predator (d^*).

Furthermore, investigations for (HB) at the natural death rate of susceptible predator $(d^\star)$ near P_{13} and P_{14} are carried out. On the other hand, numerical simulations were conducted using the Mathematica program for three distinct initial points and one hypothetical set of data given in (22). The results showed that:

1. The parameters that are most efficient in managing the stability of system (1) are a_i, f_i, $b_i, B_1, \rho, K, d, B, \gamma,$ and δ ; $i = 1,2$.
2. The stability of system (1), where the solutions are still approaching the positive equilibrium point, is not affected by the parameters $a_3, C,$ $\alpha,$ $\eta,$ A, h, C_j and σ; $j = 1,2$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A.A. Majeed, The Impact of Fear and Anti-Predator Behavior on the Dynamics of Stage-Structure Prey–Predator Model with a Harvesting, Iraqi J. Sci. 65 (2024), 7089–7101. https://doi.org/10.24996/ijs.2024.65.12.23.

[2] E. Dahy, A.M. Elaiw, A.A. Raezah, et al. Global Properties of Cytokine-Enhanced HIV-1 Dynamics Model with Adaptive Immunity and Distributed Delays, Computation 11 (2023), 217. https://doi.org/10.3390/computation11110217.

[3] D.S. Al-Jaf, The Role of Linear Type of Harvesting on Two Competitive Species Interaction, Commun. Math. Biol. Neurosci. 2024 (2024), 27. https://doi.org/10.28919/cmbn/8426.

[4] Z. Zhu, Y. Chen, Z. Li, F. Chen, Stability and Bifurcation in a Leslie–Gower Predator–Prey Model with Allee Effect, Int. J. Bifurcat. Chaos 32 (2022), 2250040. https://doi.org/10.1142/S0218127422500407.

[5] A.A. Majeed, A.J. Kadhim, The Bifurcation Analysis and Persistence of the Food Chain Ecological Model with Toxicant, J. Phys.: Conf. Ser. 1818 (2021), 012191. https://doi.org/10.1088/1742-6596/1818/1/012191.

[6] M. Agarwal, R. Pathak, Persistence and Optimal Harvesting of Prey-Predator Model with Holling Type III Functional Response, Int. J. Eng. Sci. Technol. 4 (2013), 78–96. https://doi.org/10.4314/ijest.v4i3.6.

[7] R.N. Shalan, Local Stability of Prey-Predator with Holling type I Functional Response, Glob. J. Pure Appl. Math. 13 (2017), 967-980.

[8] A.G. Frahan, On the Mathematical Model of Two- Prey and Two-Predator Species, Iraqi J. Sci. 61 (2020), 608–619. https://doi.org/10.24996/ijs.2020.61.3.17.

[9] D.P. Hector, Cooperative Hunting and Its Relationship to Foraging Success and Prey Size in an Avian Predator, Ethology 73 (1986), 247–257. https://doi.org/10.1111/j.1439-0310.1986.tb00915.x.

[10] P.E. Stander, Cooperative Hunting in Lions: The Role of the Individual, Behav. Ecol. Sociobiol. 29 (1992), 445–454. https://www.jstor.org/stable/4600646.

[11] S. Creel, N.M. Creel, Communal Hunting and Pack Size in African Wild Dogs, Lycaon Pictus, Anim. Behav. 50 (1995), 1325–1339. https://doi.org/10.1016/0003-3472(95)80048-4.

[12] M. Hossain, N. Pal, S. Samanta, Impact of Fear on an Eco-Epidemiological Model, Chaos Solitons Fractals 134 (2020), 109718. https://doi.org/10.1016/j.chaos.2020.109718.

[13] I. Bashkirtseva, T. Perevalova, L. Ryashko, Stochastic Phenomena in the Eco-Epidemiological Model with Fear-Induced Prey Refuge and Treatment of Infected Predator, SSRN, (2024). https://doi.org/10.2139/ssrn.4977897.

[14] J. Liu, B. Liu, P. Lv, T. Zhang, An Eco-Epidemiological Model with Fear Effect and Hunting Cooperation, Chaos Solitons Fractals 142 (2021), 110494. https://doi.org/10.1016/j.chaos.2020.110494.

[15] U. Ghosh, A.A. Thirthar, B. Mondal, P. Majumdar, Effect of Fear, Treatment, and Hunting Cooperation on an Eco-Epidemiological Model: Memory Effect in Terms of Fractional Derivative, Iran. J. Sci. Technol. Trans. A: Sci. 46 (2022), 1541–1554. https://doi.org/10.1007/s40995-022-01371-w.

[16] R.N. Shalan, The Local Stability of an Eco-Epidemiological Model Involving a Harvesting on Predator Population, Int. J. Adv. Sci. Technol. 29 (2020), 946- 955.

[17] T. Gaber, R. Herdiana, Widowati, Dynamical Analysis of an Eco-Epidemiological Model Experiencing the Crowding Effect of Infected Prey, Commun. Math. Biol. Neurosci. 2024 (2024), 3. https://doi.org/10.28919/cmbn/8353.

[18] Dina Sultan Al-Jaf, The Effect of Competing Predators in an Ecosystem, Commun. Math. Biol. Neurosci. 2024 (2024), 106. https://doi.org/10.28919/cmbn/8851.

[19] Alaa Khadim Mohammed1, Salam Jasim Majeed, Bifurcation Analysis of an Eco-Epidemiological Model Involving Prey Refuge, Fear Impact and Hunting Cooperation, J. Educ. Pure Sci.-Univ. Thi-Qar 14 (2024), 132-148. https://doi.org/10.32792/jeps.v14i2.438.

[20] C. Zhang, S. Liu, J. Huang, W. Wang, Stability and Hopf Bifurcation in an Eco-Epidemiological System with the Cost of Anti-Predator Behaviors, Math. Biosci. Eng. 20 (2023), 8146–8161. https://doi.org/10.3934/mbe.2023354.

[21] I.I. Shawka, A.A. Majeed, Hunting Cooperation Among Predators Effects on the Dynamics of Food-Web Eco-Epidemiological Model with Additional Food to Predators, Commun. Math. Biol. Neurosci. 2025 (2025), 23. https://doi.org/10.28919/cmbn/9063.

[22] L. Perko, Differential Equations and Dynamical Systems, Springer, New York, 2013. https://doi.org/10.1007/978-1-4613-0003-8.

[23] M. Haque, E. Venturino, Increase of the Prey May Decrease the Healthy Predator Population in Presence of Disease in the Predator, HERMIS, 7 (2006), 38–59. https://hdl.handle.net/2318/10329.