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ADAPTIVE RISK AFFINITY IN MAMMALIAN EVOLUTION VIA INTEGER PARTITION AND FACTORIZATION MODELS

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Abstract. Many animal species, including humans, display risk-taking behaviors that are evolutionarily beneficial for survival, such as learning about predators or exploring high-stakes environments. However, the underlying evolutionary mechanisms driving this behavior, especially its link to reproductive dynamics, are not well understood. In this paper, we introduce a novel population dynamics model using Diophantine equations to reconstruct evolutionary pathways. Our approach leverages integer partition and factorization techniques to model discrete population changes in species exhibiting risk-taking and risk-averse behaviors. Unlike traditional continuous models, our framework allows multiple solutions, reflecting the diversity of evolutionary strategies. This model also demonstrates how environmental pressures shape adaptive risk affinity, influencing reproductive rates and survival. The implications for understanding species adaptively, particularly in mammals, are profound, offering new insights into evolutionary behavior beyond the constraints of classical game-theoretic approaches.

Keywords: population dynamics; diaphantine equations; integer partitions; factorizations.

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1. INTRODUCTION

Risk-taking behaviors are widespread across animal species, ranging from predator exploration in mammals to thrill-seeking activities in humans. These behaviors are traditionally explained by evolutionary advantages in learning, survival, and cognitive development (Rachman, 2004; Caro, 2005) in [5] and [2]. However, the precise evolutionary dynamics underlying risk affinity remain largely unexplored, particularly in species with low birth rates where survival hinges on adaptive learning and reproductive success. The interest in thrill-seeking activities, such as haunted houses or extreme sports, is thought to stem from evolutionary mechanisms that favor high-risk learning experiences (Zuckerman, 2007; Raghunathan and Corfman, 2006) in [9], [6]. These experiences enhance cognitive flexibility and survival skills, providing long-term evolutionary benefits. Additionally, altruistic risk-taking, such as defending kin or group members, poses another dimension of adaptive behavior linked to social cooperation and group survival (Nowak, 2006) in [4]. Another open question is the adaptivity of reproductive patterns in humans and the gender dynamics linked with it. Humans are in the primate lineage where many of human apes exhibit polygynous behavior, aiming for rapid reproduction. However, humans are able to support monogamous mating behavior (Buss and Schmitt, 1993) in [1]. This paper will shed light on why the flexibility in reproductive patterns in humans exist and give a glimpse of natural evolution of the sexes regarding risk-taking which is also examined extensively in research literature (van Vugt, 2018) [8].

Current models in evolutionary dynamics, including game-theoretic approaches, predominantly assume continuous payoffs or frequency-dependent selection strategies (Maynard Smith, 1982; McNamara and Houston, 2009) in [7] and [3]. While effective in explaining strategic interactions, these models struggle to capture the discrete, nonlinear dynamics inherent in low-birth-rate species, where population changes occur in integer steps.

This paper models a scenario where an animal species can take two different phenotypes: Risk-taking and risk averse. As a simplification, we assume that the phenotypic behavior in risk-taking is depicted by a two-zone environment where the risk-averse individuals reside in the low-stakes environment while high-stakes zone is populated by adventurous individuals. From generation to generation, individuals can freely migrate between these environments.

Evolutionary paths will be reconstructed by solving the integer-valued population dynamics equations with varying reproductive rates reversely, i.e. by starting with an end state of population numbers. Since these equations that need to be solved are Diophantine, multiple solutions are possible, as long as some positivity constraints that ensure survival of the population are satisfied. The connection between reproductive behavior and risk-taking will be explored, too.

2. MATHEMATICAL DETAILS

We consider two different types of populations in one species: One risk-averse (denoted by superscript α) and one with risky behavior (denoted by superscript r) and these will reside the corresponding environmental zones. Let the natural number t be the t-th generation and n is the integer that denotes number of individuals. Then for example $n_t^{(\alpha)}$ would denote the number of risk-averse individuals in the t-th generation. A simple population model with a completely risk-averse species would have the form

$$n_{t+1}^{(\alpha)} = n_t^{(\alpha)} + r_t^{(\alpha)} n_t^{(\alpha)} - c_t^{(\alpha)} n_t^{(\alpha)2} \quad (1)$$

where $r_t^{(\alpha)} \in \mathbb{Q}$ is the reproduction rate in t-th generation, $c_t^{(\alpha)} \in \mathbb{Q}$ is the competition constant that accounts e.g. for resource depletion and (1) holds for every $t \in [1, t_{\max}]$ with the most recent generation denoted by $t_{\max}$. For the reproduction rate we set $r_t^{(\alpha)} = \frac{p_t^{(\alpha)}}{A_t}$ for some integers $p_t^{(\alpha)}, A_t$ where we assume that A_t is a constant, we set $A_t = A$. In the same way we set $c_t^{(\alpha)} = \frac{q_t^{(\alpha)}}{A_t}$ for some integers $q_t^{(\alpha)}$. As a simplifying assumption we assume that the denominator A_t is the same for both parameters and that the integer competition $q_t^{(\alpha)}$ is also a constant. Multiplication of (1) with A leads to

$$An_{t+1}^{(\alpha)} = An_t^{(\alpha)} + p_t^{(\alpha)} n_t^{(\alpha)} - q_t^{(\alpha)} n_t^{(\alpha)} n_t^{(\alpha)}. \quad (2)$$

In this paper we are not interested on giving an initial condition and iterate the population dynamics forward, we will start with a given value $n_{t_{\max}}^{(\alpha)}$ at the most recent generation which is ecologically easy to determine by counting a population and then reverse-engineer the possible evolutionary path in the past that could have contributed to this population size. This means that the first equation we pose if $n_{t_{\max}}^{(\alpha)}$ is given has the form

$$An^{(\alpha)}_{t_{\max}} = \left(A + p^{(\alpha)}_{t_{\max}-1} - q^{(\alpha)}_{t_{\max}-1} n^{(\alpha)}_{t_{\max}-1}\right) n^{(\alpha)}_{t_{\max}-1}. \quad (3)$$

Equation (3) is a Diophantine equation which we can easily solve for $n^{(\alpha)}_{t_{\max}-1}$ by factoring the integer $An^{(\alpha)}_{t_{\max}}$. So one factor will be $n^{(\alpha)}_{t_{\max}-1}$ and another one $\left(A + p^{(\alpha)}_{t_{\max}-1} - q^{(\alpha)}_{t_{\max}-1} n^{(\alpha)}_{t_{\max}-1}\right)$. To ensure that (3) has a solution for every $An^{(\alpha)}_{t_{\max}}$ we must regard the reproductive parameter $p^{(\alpha)}_{t_{\max}-1}$ as a non-constant variable. For example, if $A = 5, n^{(\alpha)}_{t_{\max}} = 3, q^{(\alpha)}_{t_{\max}-1} = 1$, we would not have any solution if $q^{(\alpha)}_{t_{\max}-1}$ would have the constant value of 2. It is important that we have varying reproduction rates to admit a solution of (3). By recursive solution of (2) for all lower values of t we will gain a possible path of population sizes $n^{(\alpha)}_t$ and reproduction rates $p^{(\alpha)}_t$. Multiple solutions of above Diophantine equation can exist since the factorization of a number is not unique.

Even if $An^{(\alpha)}_{t_{\max}}$ is a prime number, we can assign one and the number itself in one order of factors and additionally in the other order. However, it must hold

$$A + p^{(\alpha)}_t - q^{(\alpha)}_t n^{(\alpha)}_t > 0 \quad (4)$$

for any valid solution, population sizes cannot be negative and if these are zero these would be permanently extinct. Equation (4) implies that the constraint $p^{(\alpha)}_t > q^{(\alpha)}_t n^{(\alpha)}_t - A$ must hold, in other words that sufficient reproduction must take place to ensure survival of the species. This condition will prune many reverse calculated population dynamics. Computational solution of (2) will require backtracking with pruning according to (4). We conclude the simple statement that a species that does not take risks has simple reproductive guidelines, namely bearing sufficiently many off-springs without major variability in reproductive behavior.

The more interesting case is when the single Diophantine equation (1) is extended to a system of Diophantine equations

$$n^{(\alpha)}_{t+1} = n^{(\alpha)}_t + r^{(\alpha)}_t n^{(\alpha)}_t - c^{(\alpha)}_t n^{(\alpha)2}_t - I_t \quad (5a)$$

$$n^{(r)}_{t+1} = n^{(r)}_t + r^{(r)}_t n^{(r)}_t - c^{(r)}_t n^{(r)2}_t + I_t \quad (5b)$$

where I_t denotes the immigration rate into a high-stakes territory in t-th generation. In the two equations, the absolute value of I_t is the same, but with opposite sign due to continuity in the population during migration (that what is lost in one region is gained exactly in the same quantity in another region). In a similar fashion we define $r_t^{(r)} = \frac{p_t^{(r)}}{A_t}$ and $c_t^{(r)} = \frac{q_t^{(r)}}{A_t}$ where for simplicity all denominators A_t have the same value A, also the same as aforementioned denominator values. It makes sense, that in the two different environments $c_t^{(\alpha)} < c_t^{(r)}$ because the high-stakes environment has more confrontational threats than the other one. Thus, we can rewrite (5a, 5b) as

$$An_{t+1}^{(\alpha)} = \left(A + p_t^{(\alpha)} - q_t^{(\alpha)} n_t^{(\alpha)}\right) n_t^{(\alpha)} - AI_t \quad (6a)$$

$$An_{t+1}^{(r)} = \left(A + p_t^{(r)} - q_t^{(r)} n_t^{(r)}\right) n_t^{(r)} + AI_t \quad (6b)$$

To solve the system (6a, 6b) in case when we know the population sizes at most recent generation, namely $n_{t_{\max}}^{(\alpha)}$ and $n_{t_{\max}}^{(r)}$ we proceed as follows:

1. Reformulate equations (6a) and (6b) as a system of linear Diophantine equations

$$An_{t+1}^{(\alpha)} = x - AI_t, \quad An_{t+1}^{(r)} = y + AI_t \quad (7)$$

for some unknown integers x, y and solve (7) efficiently using the Extended Euclidean Algorithm, let the subscript 0 denote the solution found. Solutions always exist due to Bezout's Lemma and the fact that we have the greatest common divisor of the coefficients in (7) being always one.

2. Generate all solutions by using the identities

$$x = x_0 + A\,k \quad (8a)$$

$$y = y_0 - A\,k \quad (8b)$$

$$I_t = I_{t0} + k \quad (8c)$$

where $k \in \mathbb{Z}$ is an arbitrary integer. While I_t can both have positive and negative values (since net migration is positive when more individuals enter the high-stakes territory and negative when individuals leave the high-stakes territory), both x and y must be positive. To ensure positivity of left hand sides of (7) we must introduce the constraints

$$x > AI_t, \quad y > -AI_t \quad (9)$$

that will narrow the solution space and valid values of k which will be examined further.

3. For each of the solutions (8a) and (8b) for any valid k solve the factorization problem

$$x = \left(A + p_t^{(\alpha)} - q_t^{(\alpha)} n_t^{(\alpha)}\right) n_t^{(\alpha)}, \quad y = \left(A + p_t^{(r)} - q_t^{(r)} n_t^{(r)}\right) n_t^{(r)} \quad (10)$$

for $n_t^{(\alpha)}, n_t^{(r)}, p_t^{(\alpha)}, p_t^{(r)}$ and accept only solutions that satisfy the positivity constraint

$$A + p_t^{(\alpha)} - q_t^{(\alpha)} n_t^{(\alpha)} > 0, \quad A + p_t^{(r)} - q_t^{(r)} n_t^{(r)} > 0. \quad (11)$$

4. Repeat from Step 1 for next lower value of t until the desired depth of evolutionary history reconstruction is finished.

As a remark, to solve the system of Diophantine equations (6a) and (6b) more efficiently, we could restrict the maximum of $|I_t|$ (the bars denote absolute value) to some value which would narrow the admissible solutions for $n_t^{(\alpha)}, n_t^{(r)}, p_t^{(\alpha)}, p_t^{(r)}$ even further. This makes sense particularly when simulating animal species that tend to be more cautious. If $|I_t|$ takes high values, possible reproductive rates are more likely to show more variability. Factorization problems lead to more variability in solution if dealing with composite numbers (i.e. numbers that are not prime). From basic number theory it is known that the density of prime numbers decreases as the number gets larger. Thus, the higher the number range, the more different factorizations we can achieve in (10). While minimum reproductive conditions stated by (11) prune several factorizations, the likelihood that we obtain different valid reproduction rate histories with different values will increase if $|I_t|$ takes larger values. So, species that are more adventurous may show greater variability in mating and reproduction behavior. This is particularly important when reconstructing evolutionary paths of humans.

3. EFFECTIVE EVOLUTIONARY PARAMETERS

Especially for high risk-taking species, a plenty of solutions of the population dynamical equation can be expected. Therefore, effective statistical measures should be computed to estimate expected behavior and variance in risk-taking or reproductive behavior of the species. Using above algorithm we define for any integer-valued quantity X_i, where the index i denotes the i-th valid solution of (6a) and (6b) the mean value

$$\mu_X = \frac{1}{m} \sum_{i=1}^{m} X_i \quad (12)$$

and the standard deviation

$$\sigma_X = \sqrt{\frac{1}{m}\sum_{i=1}^{m} X_i^2 - \mu_X^2} \quad (13)$$

Where m is the total number of valid solutions. Higher-order statistical moments are possible, too, but mean (12) and standard deviation (13) are most important, particularly if many solutions exist. This is due to the law of large numbers in statistics which states that when having a very large number of samples, the samples will follow a Gaussian distribution that is uniquely characterized by its mean and its standard deviation. Another metric that can be interesting is the ratio

$$d = \frac{\mu_X}{\sigma_X} \quad (14)$$

which measures average reality with the ability to change the pattern. High risk-affinity causes σ_X to increase, leading to lower ratio d when assuming similar values of μ_X. Thus, species where high-volatility exposure is more common, e.g. humans, will blur behavioral patterns way more than other species. Traditionally, females are assumed to be way more risk averse than males. This is resembled due to the fact that females are the only group that allow to bear off-springs and according to constraints (9) and (11), a minimum amount of off-springs must be produced. However, even if this gender disparity is significant, the abundance of solution in Diophantine equation solution paths will create more variability, allowing for females also adapting risky behavior for certain cultures like participation in large-game hunting or even warfare.

4. CONCLUSION

We have examined reverse population dynamics of species that take risks by a simplified model. This reverse population dynamics is computed by recursively solving a system of Diophantine equations. While the simplified single-equation model (3) showed very basic tendency to survival of the species, the more general model (6a, 6b) for adventurous species showed much richer dynamics in possible evolutionary paths including reproductive behavior. The latter is important especially for humans, since they are extremely adaptive to the environment and are able to enter high-stakes regions more easily than most other animal species e.g. by risking the life to help strangers out of a dangerous situation. We have found that reproduction rates are varying the more adventurous a species is. This applies clearly for humans

that could aim for very fast reproduction in some cultures and very slow in other cultures. Humans are unique in the variability of reproductive strategies compared to other primates, they can adapt polygynous mating as well as monogamic mating. On the other hand, gender roles are much more fluid than in other animal species and from above considerations it can be assumed that despite the existence of sex differences in risk-taking or aggression shaped by natural selection, the variability induced by the ability to take great risks e.g. for learning from predators will enhance the variability in gender dynamics. These facts could have lead our world towards more gender equality within the recent decades.

On the other hand, more thrill-seeking activities surged around the world from haunted houses to bungee jumping emerged during the past decades, where the former activity can appear in form of Extreme Haunted Houses where a high-volatility situation is simulated far more realistic e.g. by incorporating sensation of pain than in normal haunted houses. This trend could also resemble the link between reproductive and gender dynamics and general risk-taking behavior.

Further investigations of this model will be performed in the future.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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