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DYNAMICS OF A DENGUE MODEL WITH INCUBATION PERIODS AND VERTICAL TRANSMISSION IN A HETEROGENEOUS ENVIRONMENT

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Abstract. Dengue fever is one of the most common insect-borne infectious diseases in the world. In this paper, a dynamics model of dengue fever with incubation periods and vertical transmission in a heterogeneous environment is developed, taking into account the influence of incubation periods and vertical transmission on dengue virus transmission. Theoretically the existence of a global classical solution of the model is proved uniquely, and the threshold dynamics of the model is described using the basic reproduction number R_0. The global asymptotic stability of the disease-free equilibrium is proved by constructing the upper and lower solutions in combination with the properties of the basic reproduction number when $R_0 < 1$, and the disease-free equilibrium is unstable when $R_0 > 1$. There exists a global exponential attraction set in the system when $R_0 > 1$. Finally, numerical simulation and PRCC sensitivity analysis were combined to obtain that increasing the diffusion coefficients of both susceptible and infected populations exacerbates the spread of dengue virus. In reality, dengue virus transmission can be effectively controlled by reducing the frequency of crowd activities, good personal protection against mosquito bites, timely medical treatment, and effective vaccination, among which the reduction of mosquito bite rate has the most significant effect.

Keywords: incubation periods; vertical transmission; dengue fever model; basic reproduction number; sensitivity analysis.

2020 AMS Subject Classification: 35A02, 35B40, 37B25.

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1. INTRODUCTION

Dengue fever (DF) is a mosquito-borne tropical infectious disease caused by the dengue fever virus (DENV). The virus is transmitted primarily through the bites of infected female mosquitoes, mainly in our country to the Aedes aegypti and Aedes albopictus mosquitoes, mosquitoes infected with DENV can not only be carried for life, but also through the eggs of the virus that will be passed on to the offspring, and at the same time, people can also be transmitted through the mother and child, organ donation, etc[1, 2, 3]. Clinical studies have shown that the life cycle of mosquitoes is about 3 weeks and DENV generally has incubation periods of 5-8 days in mosquitoes and 3-14 days in humans[1]. The vast majority of people infected with DENV individuals do not have symptoms. Symptomatic of infected people are mainly fever, headache, muscle and joint pain. In severe cases, life-threatening dengue hemorrhagic fever (DHF), shock, or death[4]. According to WHO statistics, 80% of symptomatic DF infections have only mild symptoms, such as fever, while the remaining 20% of infected deteriorate further and develop severe symptoms[5].

During the 18th and 19th centuries, DF was prevalent mainly in the tropics, but the trend of globalization has led to a more rapid spread of DF, a wider range of epidemics, and the emergence of multiple DENV epidemics in different regions, resulting in most countries in the tropics and subtropics becoming high-risk areas. In China,it is more serious in Guangdong, Yunnan, Guangxi and Zhejiang[6]. There are four main serotypes of DENV, DENV-1, DENV-2, DENV-3 and DENV-4. When a person recovers from infection with one of these serotypes, he or she develops lifelong immunity to that type of serotype, but has only partial and short-lived cross-immunity to the other serotypes, and the lethality of subsequent infections with the other serotypes (secondary infections) can be as high as 5% to 8%[7]. During the last decade, the incidence rate of DF has increased dramatically around the world. From 2000 to 2019, the WHO reported cases have increased from 505,430 to 5.2 million, and the actual number of cases will be higher as many of them will be misdiagnosed as other febrile disease[1]. In the literature[8], it is known that the DENV is present in more than 110 counties around the world and there are 390 million cases of DENV infection each year, 96 million of which are clinically symptomatic, and based on prevalence estimates, a total of 3.9 billion people around the world are at risk of dengue infection, which kills about 25,000 people each year. In the first decade of the 21st century, it is estimated that approximately 3 million people were infected with the DENV and 6,000 died annually in 112 countries in South Asia[9]. In 2019, the domestic DF epidemic in the form of severe, the country reported a total of 22,599 cases of DF, the incidence of 1.63/100,000, the number of reported cases is second only to the big outbreak of DF in 2014[10]. Until now, in the control of DF, although a lot of research has been done on vaccine research, a truly effective vaccine has not yet been developed, so the prevention of DENV transmission is mainly dependent on avoiding the bite of mosquitoes, of which the main control strategies are spraying insecticides, genetic modification, insect technology sterilization and Wolbachia mosquito control technology. Due to the limited control strategies and the wide range of populations involved, DF has become one of the most researched insect-borne diseases in the world[11].

In existing research on DF, scholars use mathematical models to study, common are ordinary differential equations, stochastic differential equations and partial differential equations[12, 13, 14]. In 1970, Fischer and Halstead[15] first proposed a class of dynamics models reflecting the dynamics system of transmission of DF, with which they described the transmission of DHF due to successive infections with different viral types, and evaluated the number of cases and time intervals at which DHF occurs. In 1998, Esteva and Vargas[16] proposed the classical SIR-SI model of DF with standard incidence(1.1)

$$
\begin{cases}
\frac{\mathrm{d}S_h}{\mathrm{d}t} = \mu_h N_h - \frac{\beta_h b}{N_h+m} S_h I_v - \mu_h S_h, \\
\frac{\mathrm{d}I_h}{\mathrm{d}t} = \frac{\beta_h b}{N_h+m} S_h I_v - (\mu_h + \gamma_h) I_h \\
\frac{\mathrm{d}R_h}{\mathrm{d}t} = \gamma_h I_h - \mu_h R_h, \\
\frac{\mathrm{d}S_v}{\mathrm{d}t} = A - \frac{\beta_v b}{N_v+m} S_v I_h - \mu_v S_v, \\
\frac{\mathrm{d}I_v}{\mathrm{d}t} = \frac{\beta_v b}{N_v+m} S_v I_h - \mu_v I_v, \\
S_h(0), I_h(0), R_h(0), S_v(0), I_v(0) \geq 0.
\end{cases}
\tag{1.1}
$$

The model divides the population into into three compartments: susceptible populations S_h, infected population I_h and recovered population R_h, the mosquitoes are divided into two compartments: susceptible mosquito swarms S_v and infected mosquito swarms I_v. The model depicts the generalized process of the transmission of DF, and discusses the global asymptotic stability of the positive equilibrium point through the stability of the periodic solution, analyzes the effective mosquito control measures based on the threshold conditions to further prevent the transmission of DENV effectively.

Today, many scholars have further studied the vertical transmission of DF, cross-infection and the characteristics of DENV under different age structures on the basis of this classic model[17, 18, 19], among which in literature[17], based on hospitalization data of mothers infected with DENV during breastfeeding in the region of New Caledonia, it was found that there was a high risk of complications for mothers and infants, and that vertical transmission of DF was up to 90% of mothers' deliveries. In order to study the effects of climatic factors and cross-infection on DENV transmission, a dynamics model for the transmission of two strains of DENC between mosquitoes and populations with seasonal influences is proposed in literature[18]. Using the next-generation matrix method, the local asymptotic stability and global asymptotic stability are obtained by determining the disease-free periodic solution of the model. Subsequently following in 2017 in the literature[19], the authors investigated a class of Aedes aegypti mosquito population models with with a class age structure and obtained the threshold conditions controlling the growth and development of the stage structure of the Aedes aegypti mosquito population: when the basic reproduction number $R_0 < 1$, the local equilibrium state in the system is globally asymptotically stable; when the basic reproduction number $R_0 > 1$, the positive equilibrium states in the system are globally asymptotically stable. Through the analysis of various dengue dynamics models, the characteristics of DENV transmission and the transmission mechanism have been clarified. With the advent of the era of globalization and the increasing frequency of contacts around the world, infectious diseases can be transmitted from one area to another through the spread of the population or the migration of mosquito swarms, so the spread of DENV is not only related to time, but also related to spatial location, so some biomathematicians have used reaction-diffusion equations to construct a model of DF infectious diseases to describe the transmission mechanism of DENV spread in space and time. Taking into account spatial heterogeneity, in 2019, the problem of the free boundary of several types of reaction-diffusion systems is considered in the literature[20], the determining criterion of spreading and elimination of the disease is given, and the expansion capacity of the initial distribution region and the effect on the free boundary are analyzed by numerical simulation. In the following period of time, some scholars have considered the effects of spatio-temporal transmission, media coverage and time delay on DENV spread using reaction-diffusion equation in conjunction with the DENV spreading mechanism[21, 23, 22, 24], and using stability theory and optimal control theory criteria for the propagation and elimination of disease were given. Through existing research and real life, it is found that the spread of DENV is a complex process which is affected by various factors, including environment, human behavior, socio-economic factors, ect. Therefore, to effectively control and prevent DENV spread, it is necessary to consider various influencing factors in a comprehensive way and take the corresponding response measures. In previous research on DF, the influence of DENV incubation periods is often neglected in consideration of the convenience of model theory analysis, which will overestimate the risk of DENV transmission. In 2020 Zhou and Zheng[25] discussed a class of dengue thermodynamic models with latent time delays,and obtained the local asymptotic stability of the disease-free equilibrium and the endemic equilibrium using the linearization method. Furthermore, by constructing Lyapunov functional models, the criteria for determining the global asymptotic stability of the disease-free equilibrium and endemic equilibrium were obtained. In this paper, based on the previous research on DENV, optimize the classic SIR-SI model considering the effects of DENV incubation and vertical transmission, we use the next generation characteristic operator to calculate the basic reproduction number R_0. It is theoretically proven that when $R_0 < 1$, the global asymptotic stability of the disease-free equilibrium of the system is obtained; when $R_0 > 1$, the instability of the disease-free equilibrium of the system is demonstrated. The existence theorem of the global exponential attraction set of the system is proved by constructing upper and lower solutions. The accuracy of the relevant theory is verified through numerical simulation. Considering the sensitivity analysis of PRCC of each parameter to the basic reproduction number, some effective and feasible suggestions for preventing DF are given.

2. Model Formulation

Combining the DF dynamics model with time delays and vertical transmission in the literature[24], we established a SEIR-SI reaction-diffusion dynamics model with incubation and vertical transmission in a heterogeneous environment. The model flow diagram is as follows

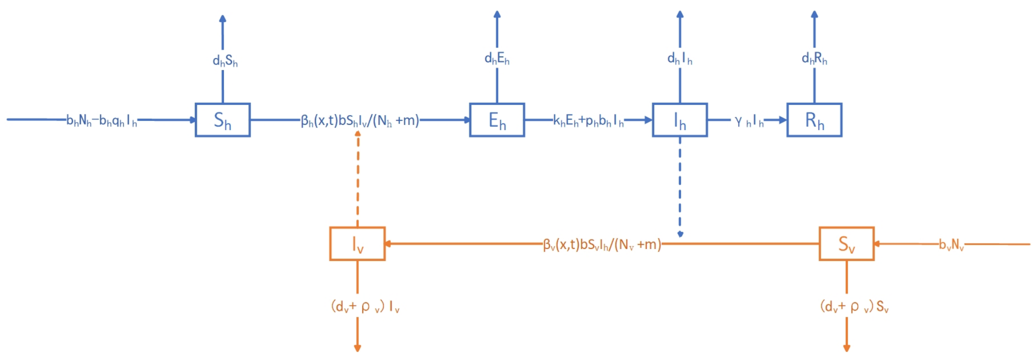


FIGURE 1. Flow diagram of DF SEIR-SI model

According to the figure 1, we construct the SEIR-SI model equation for DF with incubation periods and vertical transmission in a heterogeneous environment

$$
(2.1)\quad \begin{cases}
\frac{\partial S_h(x,t)}{\partial t} = \nabla\left(D_s(x,t)\nabla S_h\right) + b_hN_h - \frac{\beta_h(x,t)b}{N_h+m}S_hI_v - d_hS_h - p_hb_hI_h, & t>0, x\in\Omega,\\
\frac{\partial E_h(x,t)}{\partial t} = \nabla\left(D_e(x,t)\nabla E_h\right) + \frac{\beta_h(x,t)b}{N_h+m}S_hI_v - (d_h+k_h)E_h, & t>0, x\in\Omega,\\
\frac{\partial I_h(x,t)}{\partial t} = \nabla\left(D_i(x,t)\nabla I_h\right) + k_hE_h - (d_h+\gamma_h)I_h + p_hb_hI_h, & t>0, x\in\Omega,\\
\frac{\partial R_h(x,t)}{\partial t} = \nabla\left(D_r(x,t)\nabla R_h\right) + \gamma_hI_h - d_hR_h, & t>0, x\in\Omega,\\
\frac{\partial S_v(x,t)}{\partial t} = \nabla\left(D_v(x,t)\nabla S_v\right) + b_vN_v - \frac{\beta_v(x,t)b}{N_v+m}S_vI_h - (d_v+\rho_v)S_v, & t>0, x\in\Omega,\\
\frac{\partial I_v(x,t)}{\partial t} = \nabla\left(D_v(x,t)\nabla I_v\right) + \frac{\beta_v(x,t)b}{N_v+m}S_vI_h - (d_v+\rho_v)I_v, & t>0, x\in\Omega,\\
\frac{\partial S_h}{\partial n} = \frac{\partial E_h}{\partial n} = \frac{\partial I_h}{\partial n} = \frac{\partial R_h}{\partial n} = \frac{\partial S_v}{\partial n} = \frac{\partial I_v}{\partial n} = 0, & t>0, x\in\partial\Omega,\\
S_h(x,0), E_h(x,0), I_h(x,0), R_h(x,0), S_v(x,0), I_v(x,0) \ge 0, & x\in\Omega.
\end{cases}
$$

Where $S_h(x,t), E_h(x,t), I_h(x,t), R_h(x,t)$ represent the population densities of susceptible populations, exposed populations, infected populations and recovered populations at location x and time t, $S_v(x,t), I_v(x,t)$ represent the population densities of susceptible and infected mosquitoes at location x and time t, $D_s(x,t), D_e(x,t), D_i(x,t), D_r(x,t)$ represent the diffusion coefficients of susceptible populations, exposed populations, infected populations and recovered populations at location x and time t,$D_v(x,t)$ represents the diffusion coefficient of the mosquito populations at location x and time t. For convenience of discussion. To facilitate discussion, we assume that the activity range of the crowd will not cross the region Ω, which satisfies the Neumann boundary condition

$$\frac{\partial S_h}{\partial n} = \frac{\partial E_h}{\partial n} = \frac{\partial I_h}{\partial n} = \frac{\partial R_h}{\partial n} = \frac{\partial S_v}{\partial n} = \frac{\partial I_v}{\partial n} = 0, \qquad t > 0, x \in \partial\Omega,$$

where n is the out-of-unit normal vector of the boundary Ω, and the specific parameters represent the infectious disease significance as shown in table 1 here $p_h + q_h = 1$.

Parameters	Biological significance
$\beta_h(x,t)$	The transmission coefficient of DF from I_v to S_h
$\beta_v(x,t)$	The transmission coefficient of DF from I_h to S_v
γ_h	The recovery rate of patients infected with DENV
b	The biting rate of mosquito
k_h	The conversion rate from exposed population to infected population
ρ_v	Mosquito mortality from insecticides or other similar measures
m	The density of alternative hosts available as blood source
b_v	The breeding rate of mosquitoes
b_h	The birth rate of human population
d_v	The natural mortality of mosquitoes
d_h	The natural mortality rate of human population
p_h	The probability that an infected human will transmit DF to the next generation
q_h	The probability that an infected human won't transmit DF to the next generation

TABLE 1. The notation of SEIR-SI model (2.1)

3. SUITABILITY OF SOLUTIONS

In this section, we discuss the problem of the suitability of the solution of the system (2.1) in terms of the existence and uniqueness of the global classical solution and the domain of existence of the solution. Firstly, based on the content of Section 2, we make the following assumptions about the system (2.1).

(A1) Assume that the birth rate of the population is equal to the natural mortality rate of the population and that the mosquito breeding rate is equal to the sum of the natural mortality rate of mosquitoes and the mortality rate of mosquitoes due to other measures, that is $b_h = d_h, b_v = d_v + \rho_v$, when $S_h(x,0), E_h(x,0), I_h(x,0), R_h(x,0), S_v(x,0), I_v(x,0) \geq (\not\equiv) 0, x \in \Omega$, we define

$$\int_\Omega [S_h(x,t) + E_h(x,t) + I_h(x,t) + R_h(x,t)]\mathrm{d}x = M_h(t), \int_\Omega [S_v(x,t) + I_v(x,t)]\mathrm{d}x = M_v(t). \tag{3.1}$$

Next, we prove the global existence, uniqueness, boundednes and positivity of th0e classical solutions of the system (2.1), we take the vector $d = (d_1(\cdot,t),\ d_2(\cdot,t), d_3(\cdot,t), d_4(\cdot,t), d_5(\cdot,t), d_6(\cdot,t)) = (D_s(\cdot,t),\ D_e(\cdot,t), D_i(\cdot,t), D_r(\cdot,t), D_v(\cdot,t), D_v(\cdot,t))$, the operator $L = (L_1, L_2,\ L_3, L_4, L_5, L_6)$, in this paper we define the following differential operator

$$\begin{aligned} &L_i\phi := \nabla\left(d_i(\cdot)\nabla\phi\right), \\ &D\left(L_i\right) := \left\{\phi \in C^2(\Omega) \cap C^1(\overline{\Omega}) : L_i\phi \in C\left(\overline{\Omega}, \tfrac{\partial\phi}{\partial n} = 0, x \in \partial\Omega\right)\right\}. \end{aligned} \tag{3.2}$$

According to literature[14], we can get that L_i generates a C_0-semigroup $\{\mathbb{T}_i(t)\}_{t\geq 0}$, and at the same time, it makes $\omega_i(t) = \mathbb{T}_i(t)\phi$ is a solution of $\mathrm{d}\omega_i(t) = L_i\omega_i(t)\mathrm{d}t, t > 0$ and the solution satisfies $\mu_i(0) = \phi \in D\left(L_i\right)$, here

$$D\left(L_i\right) := \left\{\phi \in C(\overline{\Omega}) : \lim_{t\to 0^+} \frac{\left(\mathbb{T}_i(t) - \mathbb{I}_d\right)\phi}{t} \text{ existence } \right\},$$

where $\mathbb{I}_d$ denotes the unit operator. In order to be able to utilize the representation system (2.1) in the form of operators as above, we define the following nonlinear operators $F_i(i = 1, 2.....6)$

$$\begin{aligned} &F_1(\psi)(x) = b_h N_h - \tfrac{\beta_h b}{N_h+m}\psi_1\psi_6 - d_h\psi_1 - (1-q_h)\, b_h\psi_3, \\ &F_2(\psi)(x) = \tfrac{\beta_h b}{N_h+m}\psi_1\psi_6 - (d_h + k_h)\,\psi_2, \\ &F_3(\psi)(x) = k_h\psi_2 - (d_h + \gamma_h)\,\psi_3 + p_h b_h\psi_3, \\ &F_4(\psi)(x) = \gamma_h\psi_3 - d_h\psi_4, \\ &F_5(\psi)(x) = b_v N_v - \tfrac{\beta_v b}{N_v+m}\psi_5\psi_3 - (d_v + \rho_v)\,\psi_5, \\ &F_6(\psi)(x) = \tfrac{\beta_v b}{N_v+m}\psi_5\psi_3 - (d_v + \rho_v)\,\psi_6. \end{aligned} \tag{3.3}$$

Take $Y := C(\bar{\Omega}, R^6)$ to denote the state space with the upper bounding paradigm $\|\cdot\|_Y$, i.e.,

$$\|\omega\|_Y = \max\left\{\sup_{x\in\bar{\Omega}}|\omega_1(\cdot)|, \sup_{x\in\bar{\Omega}}|\omega_2(\cdot)|, \sup_{x\in\bar{\Omega}}|\omega_3(\cdot)|, \sup_{x\in\bar{\Omega}}|\omega_4(\cdot)|, \sup_{x\in\bar{\Omega}}|\omega_5(\cdot)|, \sup_{x\in\bar{\Omega}}|\omega_6(\cdot)|\right\}.$$

Let $Y^+ := C(\bar{\Omega}, R_+^6)$ denote the positive cone of Y, so (Y, Y^+) is a strongly ordered Banach space. Based on the operators defined in equation (3.2), the system (2.1) can be rewritten as the following equation

$$\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = L\omega(t) + F(\omega(t)), \omega(\cdot,0) = \psi \in D(L) \subset Y, \tag{3.4}$$

where $\omega(t) = (S_h(\cdot,t), E_h(\cdot,t), I_h(\cdot,t), R_h(\cdot,t), S_v(\cdot,t), I_v(\cdot,t),), F = (F_1, F_2, F_3, F_4, F_5, F_6)^T$. When $\psi_i = 0, F_i := 0, Y := C(\bar{\Omega})$. Since L is a contracted C_0-semigroup generated on Y and F satisfies the local Lipschitz condition with respect to $\omega(t)$, according to literature[14], it can be known there exists at least one saturated solution of the system (2.1) when the initial values satisfy $(S_h(x,0), E_h(x,0), I_h(x,0), R_h(x,0), S_v(x,0), I_v(x,0)) \in C(\Omega)$, $\omega(t) \in C^{2,1}(\Omega \times (0,t])$. The following theorem is given in order to prove the existence of uniqueness of the global classical solution.

Theorem 3.1. *For any initial data* $(S_h(x,0), E_h(x,0), I_h(x,0), R_h(x,0), S_v(x,0), I_v(x,0))$*, the solution of the system (2.1) satisfies the following condition*

$$\lim_{t\to\infty} \sup \int_\Omega (S_h(x,t) + E_h(x,t) + I_h(x,t) + R_h(x,t) + S_v(x,t) + I_v(x,t))\,\mathrm{d}x < \infty. \tag{3.5}$$

Proof. Take

$$M(t) = \int_\Omega [S_h(x,t) + E_h(x,t) + I_h(x,t) + R_h(x,t) + S_v(x,t) + I_v(x,t)]\mathrm{d}x = M_h(t) + M_v(t).$$

Considering the overall growth trend below, when the diffusion coefficient is only time-dependent, according to the system (2.1) we can obtain

$$\begin{aligned}
\frac{\partial M(t)}{\partial t} &= \int_\Omega \left(\frac{\partial S_h(x,t)}{\partial t} + \frac{\partial E_h(x,t)}{\partial t} + \frac{\partial I_h(x,t)}{\partial t} + \frac{\partial R_h(x,t)}{\partial t} + \frac{\partial S_v(x,t)}{\partial t} + \frac{\partial I_v(x,t)}{\partial t} \right) \mathrm{d}x \\
&= \int_\Omega (D_s(t) \triangle S_h + D_e(t) \triangle E_h + D_i(t) \triangle I_h + D_r(t) \triangle E_h + D_v(t) \triangle S_v + D_v(t) \triangle I_v)\,\mathrm{d}x \\
&\quad + \int_\Omega (b_h N_h + b_v N_v - d_h(S_h + E_h + I_h + R_h) - (d_v + \rho_v)(S_v + I_v))\,\mathrm{d}x.
\end{aligned}$$

Further by considering the boundary conditions, we can get

$$\begin{aligned}
\frac{\partial M(t)}{\partial t} \leq{} & D_s(t) \int_{\partial\Omega} \left(\frac{\partial S_h(x,t)}{\partial n} \right) \mathrm{d}x + D_e(t) \int_{\partial\Omega} \left(\frac{\partial E_h(x,t)}{\partial n} \right) \mathrm{d}x + D_i(t) \int_{\partial\Omega} \left(\frac{\partial I_h(x,t)}{\partial n} \right) \mathrm{d}x \\
& + D_r(t) \int_{\partial\Omega} \left(\frac{\partial R_h(x,t)}{\partial n} \right) \mathrm{d}x + D_v(t) \int_{\partial\Omega} \left(\frac{\partial S_v(x,t)}{\partial n} \right) \mathrm{d}x + D_v(t) \int_{\partial\Omega} \left(\frac{\partial I_v(x,t)}{\partial n} \right) \mathrm{d}x
\end{aligned}$$

$$+\int_{\Omega}\left(b_hN_h+b_vN_v-d_h(S_h+E_h+I_h+R_h)-(d_v+\rho_v)(S_v+I_v)\right)\mathrm{d}x$$
$$\leq\left(b_hN_h+b_vN_v\right)|\Omega|-BM(t),$$

where $B=min\{d_h,d_v+\rho_v\}$, so we can get

$$\lim_{t\to\infty}M(t)\leq\frac{\left(b_hN_h+b_vN_v\right)|\Omega|}{B}.$$

Therefore, the solution of the system (2.1) is bounded, i.e., equation (3.5) holds. The conclusion is proved. □

Next, we prove that the system (2.1) has a unique positive solution.

Theorem 3.2. *For any initial condition $\psi\in Y^+$ (Y^+ is the positive cone of Y). The system (2.1) has a unique saturated solution $\omega(\cdot,t;\psi)$ on $(0,T]$, and $\omega(\cdot,0;\psi)=\psi$. That is, if $t\in(0,T)$, then there is $\omega(\cdot,t;\psi)\in Y^+$ which is a solution to the system (2.1), where $T<\infty$.*

Proof. For any $\psi\in Y^+$ and $h\in[0,\infty)$, we can get

$$\psi+hF(\psi)=\begin{bmatrix}\psi_1+h\left(b_hN_h-\frac{\beta_h(x,t)b}{N_h+m}\psi_1\psi_6-(1-q_h)\,b_h\psi_3-d_h\psi_1\right)\\ \psi_2+h\left(\frac{\beta_h(x,t)b}{N_h+m}\psi_1\psi_6-(d_h+k_h)\,\psi_2\right)\\ \psi_3+h\left(k_h\psi_2-(d_h+\gamma_h)\,\psi_3+p_hb_h\psi_3\right)\\ \psi_4+h\left(\gamma_h\psi_3-d_h\psi_4\right)\\ \psi_5+h\left(b_vN_v-\frac{\beta_v(x,t)b}{N_v+m}\psi_5\psi_3-(d_v+\rho_v)\,\psi_5\right)\\ \psi_6+h\left(\frac{\beta_v(x,t)b}{N_v+m}\psi_5\psi_3-(d_v+\rho_v)\,\psi_6\right)\end{bmatrix}$$

$$\geq\begin{bmatrix}\psi_1\left[1-h\left(\frac{\tilde{\beta}_hb}{N_h+m}\psi_6+d_h\right)\right]\\ \psi_2\left[1-h\left(d_h+k_h\right)\right]\\ \psi_3\left[1-h\left(d_h+\gamma_h\right)\right]\\ \psi_4\left[1-hd_h\right]\\ \psi_5\left[1-h\left(\frac{\tilde{\beta}_vb}{N_v+m}\psi_3+d_v+\rho_v\right)\right]\\ \psi_6\left[1-h\left(d_v+\rho_v\right)\right],\end{bmatrix},$$

where

$$\tilde{\beta}_h=\max_{x\in\bar{\Omega},t\geq0}\beta_h(x,t),\tilde{\beta}_v=\max_{x\in\bar{\Omega},t\geq0}\beta_v(x,t),$$

so it can be deduced that

$$\lim_{h\to 0^+}\frac{1}{h}\operatorname{dist}\left(\psi+hF(\psi),Y^+\right)=0,\forall\psi\in Y^+,$$

where dist denotes the spatial distance formula. By Corollary 4 in literature[26], it is known that there exists a unique positive solution to the system (2.1) on $(0,T]$. The conclusion is valid. □

Combining boundedness of solutions in theorem 3.1 and the existence of uniqueness of positive solutions in theorem 3.2, it can be deduced that there exists a unique positive solution $\omega(\cdot,t;\psi)\in Y^+$ to the system (2.1) when $t\in[0,+\infty),\omega(\cdot,0;\psi)\in Y^+$.

4. Basic Reproduction Number

Combining the assumptions **(A1)**, we can know that the system (2.1) has a disease-free equilibrium $E_0=(S_{h0}(x),0,\ 0,0,S_{v0}(x),0)$, where $S_{h0}=N_h$, $S_{v0}=N_v$. Then, for the convenience of later discussion, linearizing the second, third, fourth, and sixth equations of the system (2.1) at the disease-free equilibrium yields

$$\begin{cases}\frac{\partial E_h}{\partial t}=\nabla(D_e(x,t)\nabla E_h(x,t))+\frac{\beta_h(x,t)b}{N_h+m}S_{h0}I_v-(d_h+k_h)E_h,\\ \frac{\partial I_h}{\partial t}=\nabla(D_i(x,t)\nabla I_h(x,t))+k_hE_h-(d_h+\gamma_h)I_h+p_hb_hI_h,\\ \frac{\partial R_h}{\partial t}=\nabla(D_r(x,t)\nabla R_h(x,t))+\gamma_hI_h-d_hR_h,\\ \frac{\partial I_v}{\partial t}=\nabla(D_v(x,t)\nabla I_v(x,t))+\frac{\beta_v(x,t)b}{N_v+m}S_{v0}I_h-(d_v+\rho_v)I_v.\end{cases}\tag{4.1}$$

Inspired by literature[27], we give the basic reproduction number R_0 for the system (2.1) by the spectral radius of the next-generation infection operator. We divide the source of the population inside the silo into three components: the newly infected population $\mathscr{F}$, the population moving in and out of the silo by other means $\mathscr{V}$ and the population moving in to the silo by diffusion $\mathscr{D}$, then the system (4.1) is equivalent to

$$\frac{\partial\mu}{\partial t}=\mathscr{D}-\mathscr{V}+\mathscr{F},x\in\Omega,t>0,$$

where

$$\mu=\begin{bmatrix}E_h\\ I_h\\ R_h\\ I_v\end{bmatrix},\mathscr{D}=\begin{bmatrix}\nabla(D_e(x,t)\nabla E_h(x,t))\\ \nabla(D_i(x,t)\nabla I_h(x,t))\\ \nabla(D_r(x,t)\nabla R_h(x,t))\\ \nabla(D_v(x,t)\nabla I_v(x,t))\end{bmatrix},\mathscr{V}=\begin{bmatrix}(d_h+k_h)E_h\\ (d_h+\gamma_h)I_h-k_hE_h-p_hb_hI_h\\ d_hR_h-\gamma_hI_h\\ (d_v+\rho_v)I_v\end{bmatrix},$$

and

$$\mathscr{F} = \begin{bmatrix} \frac{\beta_h(x,t)b}{N_h+m} S_{h0} I_v \\ 0 \\ 0 \\ \frac{\beta_v(x,t)b}{N_v+m} S_{v0} I_h \end{bmatrix}.$$

Using its Taylor expansion at the origin, we make the following definition

$$F = \begin{bmatrix} 0 & 0 & 0 & \frac{\beta_h(x,t)b}{N_h+m} S_{h0} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_v(x,t)b}{N_v+m} S_{v0} & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} d_h + k_h & 0 & 0 & 0 \\ -k_h & (d_h+\gamma_h) - p_h b_h & 0 & 0 \\ 0 & -\gamma_h & d_h & 0 \\ 0 & 0 & 0 & d_v + \rho_v \end{bmatrix},$$

and

$$D = \begin{bmatrix} \nabla(D_e(x,t)\nabla) & 0 & 0 & 0 \\ 0 & \nabla(D_i(x,t)\nabla) & 0 & 0 \\ 0 & 0 & \nabla(D_r(x,t)\nabla) & 0 \\ 0 & 0 & 0 & \nabla(D_v(x,t)\nabla) \end{bmatrix}.$$

So the system (4.1) is equivalent to

$$\frac{\partial \mu}{\partial t} = D\mu - V\mu + F\mu, x \in \Omega, t > 0.$$

Let $T(t)$ be the solution semigroup of the following linear equation

$$\frac{\partial \mu}{\partial t} = D\mu - V\mu, x \in \Omega, t > 0. \tag{4.2}$$

According to the definition of the next-generation infection operator, let the initial state distribution be ϕ, and take the differential operator

$$\mathscr{L}(\phi)(x) := F(x) \int_0^\infty T(t)\phi \, \mathrm{d}t.$$

Then the basic reproduction number R_0 can be defined as

$$R_0 := r(\mathscr{L}).$$

Based on the above conclusion, in combination with[27], taking $B = D - V$ we can obtain the following lemma and theorem.

Lemma 4.1. *The sign of $\lambda^* := s(B+F)$ is the same as the sign of $R_0 - 1$.*

Proof. Obviously, B is an infinitesimal generator of the $T(t)$. Note that $T(t)$ is a semigroup in the sense of a positive direction, i.e., it holds for all $t \geq 0$. And B is the 4×4 matrix corresponding to the bounded linear operators of $\Omega \times (0, +\infty) \longrightarrow R^4$ and

$$(\lambda I - B)^{-1}\phi = \int_0^\infty e^{-\lambda t} T(t)\phi \mathrm{d}t, \quad \forall \lambda > s(B), \phi \in X_1. \tag{4.3}$$

From the derivation of the C_0-semigroup properties in[14], it is known that $s(B) < 0$. Let $\lambda = 0$ in equation (4.3), we get

$$-B^{-1}\phi = \int_0^\infty T(t)\phi \mathrm{d}t, \quad \forall \phi \in \bar{\Omega} \times (0, \infty).$$

Thus, $\mathscr{L} = -FB^{-1}$. Define the linear operator $A := B + F$. By the equivalence system, A generates a positive C_0-semigroup. Using the characteristic operator, we know that $s(A)$ has the same sign as $r\left(-FB^{-1}\right) - 1 = R_0 - 1$. □

Lemma 4.2. *Suppose $D_e(x,t), D_i(x,t), D_r(x,t), D_v(x,t) \geq 0$, if the elliptic eigenvalue problem*

$$\begin{cases} -D\phi + V\phi = \lambda_0 F\phi, & x \in \Omega, \\ \frac{\partial \phi}{\partial n} = 0, & x \in \partial\Omega., \end{cases} \tag{4.4}$$

exists a unique positive eigenvalue λ_0 with a positive eigenfunction, then the basic reproduction number is $R_0 = r\left(-FB^{-1}\right) = r\left(-B^{-1}F\right) = 1/\lambda_0$.

Proof. Let

$$F_\varepsilon(x) = F(x) + \varepsilon E, \quad V_\varepsilon(x) = V(x) - \varepsilon E,$$

where $\varepsilon > 0$ is a constant and E is a 4×4 matrix whose elements are all 1. Consider the following system of equations

$$\begin{cases} \frac{\partial u}{\partial t} = Du - V_\varepsilon(x)u, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{4.5}$$

Denote $T_\varepsilon(t)\phi$ as a solution of the system (4.5) and satisfying $T_\varepsilon(0)\phi = \phi$. The following definitions are made

$$L_\varepsilon(\phi)(x) := F_\varepsilon(x) \int_0^\infty T_\varepsilon(t)\phi(x)\mathrm{d}t.$$

Obviously, L_ε is a strongly positive and compact operator. Therefore, it has a positive spectral radius denoted by $R_0(\varepsilon)$ and is an eigenvalue of algebraic multiplicity 1 and a positive eigenvector ϕ_ε. Thus, by the definition of the basic reproduction number we have

$$F_\varepsilon(x)\int_0^\infty T_\varepsilon(t)\phi_\varepsilon \mathrm{d}t = R_0(\varepsilon)\phi_\varepsilon.$$

Let B_ε be the generator of the continuous semigroupan $T_\varepsilon(t)$, then, we have

$$B_\varepsilon\phi = D\triangle\phi - V_\varepsilon(x)\phi.$$

Since $T_\varepsilon(t)$ is a positive semigroup, taking into account the presolution operator $(\lambda I - B_\varepsilon)^{-1}$ for the infinitesimal generator B_ε yields

$$(\lambda I - B_\varepsilon)^{-1}\phi = \int_0^\infty e^{-\lambda t}T_\varepsilon(t)\phi \mathrm{d}t, \quad \forall\lambda > s(B_\varepsilon), \phi\in X_1.$$

Due to the boundedness and continuity of the parameters, it is possible to restrict ε to be small enough such that $s(B_\varepsilon) < 0$. Let $\lambda = 0$, then for all ϕ it holds that $-B_\varepsilon^{-1}\phi = \int_0^\infty T_\varepsilon(t)\phi \mathrm{d}t$. Therefore, we obtain

$$-F_\varepsilon(x)B_\varepsilon^{-1}\phi_\varepsilon = R_0(\varepsilon)\phi_\varepsilon.$$

Set

$$\psi_\varepsilon := -B_\varepsilon^{-1}\phi_\varepsilon.$$

It follows that ψ_ε is positive, and bylemma (4.1) that

$$F_\varepsilon(x)\psi_\varepsilon = -R_0(\varepsilon)B_\varepsilon\psi_\varepsilon,$$

Therefore, combining weakly coupled elliptic system eigenvalue problem leads to the following system

$$\begin{cases} -D\phi_\varepsilon + V_\varepsilon(x)\phi_\varepsilon = \frac{1}{R_0(\varepsilon)}F_\varepsilon(x)\phi_\varepsilon, & x\in\Omega, \\ \frac{\partial\phi_\varepsilon}{\partial n} = 0, & x\in\partial\Omega. \end{cases}$$

There exists a unique positive eigenvalue $\lambda_\varepsilon = \frac{1}{R_0(\varepsilon)}$ and the corresponding eigenfunction is positive. Utilizing the regress theory of linear operators and letting $\varepsilon\to 0$, we can obtain $R_0^{-1} = \lambda_0$.

□

Combining the above two lemmas, we consider the following principal eigenvalue problem for the system (4.1) of equivalent equations

$$\begin{cases} -\nabla(D_e(x,t)\nabla\phi_1)+(d_h+k_h)\phi_1=\lambda\frac{\beta_h(x,t)b}{N_h+m}S_{h0}\phi_4, \\ -\nabla(D_i(x,t)\nabla\phi_2)-k_h\phi_1+(d_h+\gamma_h)\phi_2-p_hb_h\phi_2=0, \\ -\nabla(D_r(x,t)\nabla\phi_3)-\gamma_h\phi_2+d_h\phi_3=0, \\ -\nabla(D_v(x,t)\nabla\phi_4)+(d_v+\rho_v)\phi_4=\lambda\frac{\beta_v(x,t)b}{N_v+m}S_{v0}\phi_2. \end{cases} \tag{4.6}$$

Where $R_0=\frac{1}{\lambda}$. To further understand the nature of the basic reproduction number, the following theorem can be obtained by combining the lemma 4.1 and Conclusion.

Theorem 4.1. *When the diffusion coefficient is independent of spatial location, i.e., $D_e(x,t)=\tilde{D}_e, D_i(x,t)=\tilde{D}_i$, $D_r(x,t)=\tilde{D}_r, D_v(x,t)=\tilde{D}_v$, the basic reproduction number R_0 of the system (2.1) satisfies $sign(1-R_0)=sign(\lambda_1)$. Where $(\lambda_1;\phi_1,\phi_2,\phi_3,\phi_4)$ is the principal feature pair of the following characterization problem*

$$\begin{cases} -\tilde{D}_e\Delta\phi_1+(d_h+k_h)\phi_1=\frac{\beta_h(x,t)b}{N_h+m}S_{h0}\phi_4+\lambda_1\phi_1, & x\in\Omega, \\ -\tilde{D}_i\Delta\phi_2-k_h\phi_1+(d_h+\gamma_h)\phi_2-p_hb_h\phi_2=\lambda_1\phi_2, & x\in\Omega, \\ -\tilde{D}_r\Delta\phi_3-\gamma_h\phi_2+d_h\phi_3=\lambda_1\phi_3, & x\in\Omega, \\ -\tilde{D}_v\Delta\phi_4+(d_v+\rho_v)\phi_4=\frac{\beta_v(x,t)b}{N_v+m}S_{v0}\phi_2+\lambda_1\phi_4, & x\in\Omega. \end{cases} \tag{4.7}$$

Remark 1: According to the literature[28] and the literature[29], it can be obtained that the basic reproduction number is positively correlated with the coefficients of the transmission of DENV $\beta_h(x,t),\beta_v(x,t)$, i.e., the higher the coefficient of the transmission of DENV, the greater the basic reproduction number R_0, we give the following estimation of the range of basic reproduction number R_0

$$\sqrt{\frac{\frac{\beta_h^m b}{N_h+m}k_h\frac{\beta_v^m b}{N_v+m}N_vN_h}{(d_h+k_h)(d_h+\gamma_h-p_hb_h)(d_v+\rho_v)}}\le R_0\le\sqrt{\frac{\frac{\beta_h^M b}{N_h+m}k_h\frac{\beta_v^M b}{N_v+m}N_vN_h}{(d_h+k_h)(d_h+\gamma_h-p_hb_h)(d_v+\rho_v)}},$$

where

$$\beta_h^m=\min_{x\in\bar{\Omega},t\ge0}\beta_h(x,t)\quad \beta_v^m=\min_{x\in\bar{\Omega},t\ge0}\beta_v(x,t),$$
$$\beta_h^M=\max_{x\in\bar{\Omega},t\ge0}\beta_h(x,t)\quad \beta_v^M=\max_{x\in\bar{\Omega},t\ge0}\beta_v(x,t).$$

Remark 2: When all parameters are constants independent of spatial position and time, i.e., $\beta_h(x,t)=\beta_h^*, \beta_v(x,t)=\beta_v^*$, the basic reproduction number R_0^* of the system (2.1) is

$$R_0^* = \sqrt{\frac{\frac{\beta_h^* b}{N_h+m} k_h \frac{\beta_v^* b}{N_v+m} N_v N_h}{(d_h+k_h)(d_h+\gamma_h-p_h b_h)(d_v+\rho_v)}}. \tag{4.8}$$

According to equation (4.8), we can get that the basic reproduction number R_0^* is positively related to the transmission coefficient and vertical transmission rate of DENV, and inversely related to the recovery rate. Therefore, the vertical transmission of DENV increases the transmission risk of DENV. When the vertical transmission probability of DF is higher, the transmission coefficient is higher, the recovery rate is lower the larger the value of the basic reproduction number R_0^* is, the higher the risk of DENV transmission is.

5. Threshold Dynamics

In this section, we focus on the influence of global dynamics of the system (2.1). To prove the global asymptotic stability of the disease-free equilibrium point of the system (2.1), inspired by literature[31, 27], let X be a decomposable Banach space, $J(t)_{t\geq 0}$ is a continuous semigroup on X, where the space X is made as follows

$$X = X_1 \oplus X_2, dim(X) < \infty.$$

Define $M: X \to X_1, (I-M): X \to X_2$ to be the orthogonal projection. According to literature[30], the following lemma can be obtained

Lemma 5.1. *For any bounded set B in X, there exists positive numbers t_b, C, α and any $\varepsilon > 0$ such that $\|MJ(t)B\|_{t\geq t_b}$ is bounded on a finite dimensional subspace X_1 of X, and the following conclusion holds*

$$\|(I-M)J(t)B\| \leq Ce^{-\alpha t} + \varepsilon, t \geq t_b,$$

where $M: X \to X_1$ is a bounded projection.

Let $\mathrm{H} = \mathrm{L}^2(\Omega) \cap C^{2,1}(\Omega)$ and $\mathrm{H}_1 = \mathrm{H}_0^1(\Omega) \cap C^{2,1}(\Omega)$, denote $\mathrm{H}^6 = \mathrm{H}\times\mathrm{H}\times\mathrm{H}\times\mathrm{H}\times\mathrm{H}\times\mathrm{H}$ and $\mathrm{H}_1^6 = \mathrm{H}_1\times\mathrm{H}_1\times\mathrm{H}_1\times\mathrm{H}_1\times\mathrm{H}_1\times\mathrm{H}_1$ to be Banach spaces with the following paradigms

$$\|(S_h,E_h,I_h,R_h,S_v,I_v)\|_{\mathrm{H}^6} = \max\left\{\|S_h\|_{\mathrm{H}}, \|E_h\|_{\mathrm{H}}, \|I_h\|_{\mathrm{H}}, \|R_h\|_{\mathrm{H}}, \|S_v\|_{\mathrm{H}}, \|I_v\|_{\mathrm{H}}\right\},$$

we define the number of paradigms on the space H to be $\|f(x,t)\|_{\mathrm{H}} = \max_{x\in \mathrm{H}} \|f(x,t)\|$, and

$$\|(S_h, E_h, I_h, R_h, S_v, I_v)\|_{\mathrm{H}_1^6} := \max\left\{\|S_h\|_{\mathrm{H}_1}, \|E_h\|_{\mathrm{H}_1}, \|I_h\|_{\mathrm{H}_1}, \|R_h\|_{\mathrm{H}_1}, \|S_v\|_{\mathrm{H}_1}, \|I_v\|_{\mathrm{H}_1}\right\}.$$

Based on the above definitions, the following theorem is obtained for the system (2.1) having a global exponential attractor set.

Theorem 5.1. *If the system (2.1) has an attraction set $\mathbb{B}_{\bar{R}} \subset \mathrm{H}^6$ which satisfies lemma 5.1, then the system (2.1) has a global exponential attraction set $\mathbb{Q}$, which can attract to any bounded set* H^6.

Proof. By theorem 3.1 and theorem 3.2 we obtain that if the initial value of the system (2.1) satisfies

$$\psi = w(0) = (S_h(x,0), E_h(x,0), I_h(x,0), R_h(x,0), S_v(x,0), I_v(x,0))^T \in \mathrm{H}^6,$$

then the system (2.1) has the global classical solution

$$w(t) = (S_h(\cdot,t), E_h(\cdot,t), I_h(\cdot,t), R_h(\cdot,t), S_v(\cdot,t), I_v(\cdot,t))^T \in C^0\left(\mathrm{H}^6, [0,\infty)\right).$$

Based on the definition of operators in Section 3, we transform the system (2.1) into the following operator problem

$$\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = L\omega(t) + F(\omega(t)), \omega(\cdot,0) = \psi,$$

where the map $L+F : \mathrm{H}_1 \to \mathrm{H}$ is called the gradient type operator. We can obtain that the system (2.1) with the following semigroup of operators

$$J(t) = (J_1(t), J_2(t), J_3(t), J_4(t), J_5(t), J_6(t))^T,$$

and $J(t)\psi = \omega(t;\psi)$. Next, we show that the operator semigroup $J(t)$ has an attractor set $\mathbb{B}_{\bar{R}} \subset \mathrm{H}^6$. When the diffusion coefficients are independent of the spatial position, we obtain the inner product of the first equation of the system (2.1) with S_h

$$\begin{aligned}(5.1)\quad &\left\langle D_s\Delta S_h(x,t) + b_hN_h - \frac{\beta_h(x,t)b}{N_h+m}S_h(x,t)I_v(x,t) - d_hS_h - p_hb_hI_h, S_h\right\rangle_{\mathrm{H}} \\ = \ &\int_\Omega D_s\Delta S_h(x,t)\cdot S_h(x,t)\mathrm{d}x + \int_\Omega b_hN_h\cdot S_h(x,t)\mathrm{d}x - \int_\Omega \frac{\beta_h(x,t)b}{N_h+m}S_h(x,t)I_v(x,t)\cdot S_h(x,t)\mathrm{d}x \\ &-\int_\Omega d_hS_h(x,t)\cdot S_h(x,t)\mathrm{d}x - \int_\Omega p_hb_hI_h(x,t)\cdot S_h(x,t)\mathrm{d}x\end{aligned}$$

$$\leq \quad -D_s \|S_h\|^2_{\mathrm{H}_{\frac{1}{2}}} + \int_\Omega b_h N_h S_h \mathrm{d}x,$$

where $\mathrm{H}_{\frac{1}{2}}$ is the subspace of fractional curtains generated by the sector operator L. By lemma 4.1, it follows that S_h is bounded.Hence the following inequality can be obtained

$$\int_\Omega b_h N_h \cdot S_h \mathrm{d}x \leq K_1,$$

since $\mathrm{H}_{\frac{1}{2}} \to \mathrm{H}$, there exists $K > 0$ such that

$$\|S_h\|_{\mathrm{H}_{\frac{1}{2}}} \geq K \|S_h\|_{\mathrm{H}}, \forall S_h \in \mathrm{H}_{\frac{1}{2}},$$

so the equation (5.1) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|S_h\|^2_{\mathrm{H}} \leq -D_s K^2 \|S_h\|^2_{\mathrm{H}} + K_1,$$

based on the above derivation, we can obtain

$$\|S_h\|^2_{\mathrm{H}} \leq e^{-2D_sK^2t} \|S_h(x,0)\|^2_{\mathrm{H}} + \frac{K_1}{D_sK^2}\left(1 - e^{-2D_sK^2t}\right).$$

Next, using the same method, the second equation of the system (2.1) can be inner product with E_h in the space H

$$\left\langle D_e \Delta E_h(x,t) + \frac{\beta_h(x,t)b}{N_h+m} S_h(x,t) I_v(x,t) - (d_h + k_h) E_h(x,t), E_h \right\rangle_{\mathrm{H}}$$

$$\begin{aligned} &= \int_\Omega D_e \Delta E_h(x,t) \cdot E_h(x,t)\mathrm{d}x + \int_\Omega \frac{\beta_h(x,t)b}{N_h+m} S_h(x,t) I_v(x,t) \cdot E_h(x,t)\mathrm{d}x \\ &\quad - \int_\Omega (d_h + k_h) E_h(x,t) \cdot E_h(x,t)\mathrm{d}x \\ &\leq -D_e \|E_h\|^2_{\mathrm{H}_{\frac{1}{2}}} + \int_\Omega \frac{\beta_h(x,t)b}{N_h+m} S_h I_v \cdot E_h \mathrm{d}x. \end{aligned} \tag{5.2}$$

From theorem 3.1 and theorem 3.2, we get

$$\int_\Omega \frac{\beta_h(x,t)b}{N_h+m} S_h I_v \cdot E_h \mathrm{d}x \leq K_2.$$

Similarly since $\mathrm{H}_{\frac{1}{2}} \to \mathrm{H}$, there exists $K > 0$ such that

$$\|E_h\|_{\mathrm{H}_{\frac{1}{2}}} \geq K \|E_h\|_{\mathrm{H}}, \forall E_h \in \mathrm{H}_{\frac{1}{2}},$$

the equation (5.2) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_h\|_{\mathrm{H}}^2 \leq -D_eK^2\|E_h\|_{\mathrm{H}}^2 + K_2,$$

therefore,

$$\|E_h\|_{\mathrm{H}}^2 \leq e^{-2D_eK^2t}\|E_h(x,0)\|_{\mathrm{H}}^2 + \frac{K_2}{D_eK^2}\left(1-e^{-2D_eK^2t}\right).$$

Similarly the inner product of the third, fourth, fifth and sixth equations of the system (2.1) with I_h, R_h, S_v and I_v respectively

$$\begin{bmatrix} \langle D_i\Delta I_h(x,t)+k_hE_h(x,t)-(d_h+\gamma_h)I_h(x,t)+p_hb_hI_h(x,t),I_h\rangle_{\mathrm{H}} \\ \langle D_r\Delta R_h(x,t)+\gamma_hI_h(x,t)-d_hR_h(x,t),R_h\rangle_{\mathrm{H}} \\ \left\langle D_v\Delta S_v(x,t)+b_vN_v-\frac{\beta_v(x,t)b}{N_v+m}S_v(x,t)I_h(x,t)-(d_v+\rho_v)S_v(x,t),S_v\right\rangle_{\mathrm{H}} \\ \left\langle D_v\Delta I_v(x,t)+\frac{\beta_v(x,t)b}{N_v+m}S_v(x,t)I_h(x,t)-(d_v+\rho_v)I_v(x,t),I_v\right\rangle_{\mathrm{H}} \end{bmatrix}$$

$$\leq \begin{bmatrix} -D_i\|I_h\|_{\mathrm{H}_{\frac{1}{2}}}^2+\int_\Omega(k_hE_h+p_hb_hI_h)\cdot I_h\mathrm{d}x \\ -D_r\|R_h\|_{\mathrm{H}_{\frac{1}{2}}}^2+\int_\Omega\gamma_hI_h\cdot R_h\mathrm{d}x \\ -D_v\|S_v\|_{\mathrm{H}_{\frac{1}{2}}}^2+\int_\Omega b_vN_v\cdot S_v\mathrm{d}x \\ -D_v\|I_v\|_{\mathrm{H}_{\frac{1}{2}}}^2+\int_\Omega\frac{\beta_v(x,t)b}{N_v+m}S_vI_h\cdot I_v\mathrm{d}x \end{bmatrix}. \tag{5.3}$$

According to theorem 3.1, theorem 3.2 and similar proofs of equation (5.1) and (5.2), we can obtain that there exists a positive number K_3,K_4,K_5,K_6 for which the following conclusions holds

$$\begin{aligned} \|I_h\|_{\mathrm{H}}^2 &\leq e^{-2D_iK^2t}\|I_h(x,0)\|_{\mathrm{H}}^2+\tfrac{K_3}{D_iK^2}\left(1-e^{-2D_iK^2t}\right), \\ \|R_h\|_{\mathrm{H}}^2 &\leq e^{-2D_rK^2t}\|R_h(x,0)\|_{\mathrm{H}}^2+\tfrac{K_4}{D_rK^2}\left(1-e^{-2D_rK^2t}\right), \\ \|S_v\|_{\mathrm{H}}^2 &\leq e^{-2D_vK^2t}\|S_v(x,0)\|_{\mathrm{H}}^2+\tfrac{K_5}{D_vK^2}\left(1-e^{-2D_vK^2t}\right), \\ \|I_v\|_{\mathrm{H}}^2 &\leq e^{-2D_vK^2t}\|I_v(x,0)\|_{\mathrm{H}}^2+\tfrac{K_6}{D_vK^2}\left(1-e^{-2D_vK^2t}\right). \end{aligned} \tag{5.4}$$

If $\bar{R}^2 > \max\left\{\frac{K_1}{D_sK^2},\frac{K_2}{D_eK^2},\frac{K_3}{D_iK^2},\frac{K_4}{D_rK^2},\frac{K_5}{D_vK^2},\frac{K_6}{D_vK^2}\right\}$, then there exists $t^* > 0$, for any $t \geq t^*$, satisfying $\omega(t;\psi) \subset \mathbb{B}_{\bar{R}}$. Therefore, $\mathbb{B}_{\bar{R}} \subset \mathrm{H}^6$ is an attractor set. Next, we prove that the system (2.1) has a global exponential attractor.

Let $L_1 := D_s\Delta : \mathrm{H}_1 \to \mathrm{H}$ be a symmetric operator, hence the eigenvectors e_i belonging to the eigenvalue $\lambda_{S_h,i}$ (the ith eigenvalue with respect to S_h) are complete orthogonal bases in H. That is, for any $S_h \in \mathrm{H}$

$$S_h = \sum_{j=1}^{\infty} x_j e_j, \|S_h\|_{\mathrm{H}}^2 = \sum_{j=1}^{\infty} x_j^2.$$

Further, we get $\forall N_{S_h} > 0, \exists C_{S_h} \geq 1$ (the constant associated with S_h) which satisfies $-N_{S_h} \geq \lambda_{S_h,i}, \forall i \geq C_{S_h}+1$

$$\mathrm{H}_1^{C_{S_h}} = \mathrm{span}\left\{e_1, e_2, \cdots, e_{C_{S_h}}\right\}, \mathrm{H}_2^{C_{S_h}} = \left(\mathrm{H}_1^{C_{S_h}}\right)^{\perp}.$$

For any $S_h \in \mathrm{H}$ can be decomposed as

$$S_h = MS_h + (I-M)S_h := S_{h,1} + S_{h,2},$$
$$S_{h,1} = \sum_{k=1}^{C_{S_h}} x_k e_k \in \mathrm{H}_1^{C_{S_h}}, S_{h,2} = \sum_{i=C_{S_h+1}}^{\infty} x_i e_i \in \mathrm{H}_2^{C_{S_h}},$$

where $M : \mathrm{H} \to \mathrm{H}_1^{C_{S_h}}$ is an orthogonal projection. E_h, I_h, R_h, S_v and I_v have similar decomposition forms. Since $J(t)$ is a bounded attractor, for any bounded set $\mathbb{B}_{\bar{R}} \subset \mathrm{H}^6$, there exists a positive t_0. Suppose $t_0 > t^*$ such that $(\omega_1(t; S_h(x,0)), \omega_2(t;\ \ E_h(x,0)),\ \ \omega_3(t; I_h(x,0)), \omega_4(t; R_h(x,0)), \omega_5(t; S_v(x,0)), \omega_6(t; I_v(x,0))) \subset \mathbb{B}_{\bar{R}}$, where

$$\|\omega_1(t; S_h(x,0))\|_{\mathrm{H}}^2 = \|J_1(t)S_h(x,0)\|_{\mathrm{H}}^2 \leq \bar{R}^2,$$
$$\|\omega_2(t; E_h(x,0))\|_{\mathrm{H}}^2 = \|J_2(t)E_h(x,0)\|_{\mathrm{H}}^2 \leq \bar{R}^2,$$
$$\|\omega_3(t; I_h(x,0))\|_{H}^2 = \|J_3(t)I_h(x,0)\|_{\mathrm{H}}^2 \leq \bar{R}^2,$$
$$\|\omega_4(t; R_h(x,0))\|_{\mathrm{H}}^2 = \|J_4(t)R_h(x,0)\|_{\mathrm{H}}^2 \leq \bar{R}^2,$$
$$\|\omega_5(t; S_v(x,0))\|_{\mathrm{H}}^2 = \|J_5(t)S_v(x,0)\|_{\mathrm{H}}^2 \leq \bar{R}^2,$$
$$\|\omega_6(t; I_v(x,0))\|_{\mathrm{H}}^2 = \|J_6(t)I_v(x,0)\|_{\mathrm{H}}^2 \leq \bar{R}^2,$$

so

$$\|MJ(t)\psi\|_{\mathrm{H}^6} \leq \bar{R}, \forall t \geq t_0. \tag{5.5}$$

Making an inner product of the first equation of the system (2.1) with $S_{h,2}$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle S_h, S_{h,2}\right\rangle_{\mathrm{H}} \\
&=\left\langle D_s\Delta S_h(x,t)+b_hN_h-\frac{\beta_h(x,t)b}{N_h+m}S_h(x,t)I_v(x,t)-d_hS_h(x,t)-p_hb_hI_h, S_{h,2}\right\rangle_{\mathrm{H}} \\
&=\left\langle D_sS_h(x,t), S_{h,2}(x,t)\right\rangle_{\mathrm{H}}+\left\langle b_hN_h-d_hS_h(x,t)-p_hb_hI_h-\frac{\beta_h(x,t)b}{N_h+m}S_h(x,t)I_v(x,t), S_{h,2}\right\rangle_{\mathrm{H}} \\
&=\left\langle D_sS_{h,1}(x,t)+D_sS_{h,2}(x,t), S_{h,2}(x,t)\right\rangle_{\mathrm{H}}+\langle b_hN_h-d_hS_h(x,t)-p_hb_hI_h \\
&\quad -\frac{\beta_h(x,t)b}{N_h+m}S_h(x,t)I_v(x,t), S_{h,2}\right\rangle_{\mathrm{H}} \\
&\le\left\langle D_sS_{h,2}(x,t), S_{h,2}(x,t)\right\rangle_{\mathrm{H}}+\left\langle b_hN_h, S_{h,2}\right\rangle_{\mathrm{H}} \\
&\le\left\langle D_sS_{h,2}(x,t), S_{h,2}(x,t)\right\rangle_{\mathrm{H}}+b_hN_h\bar{R},
\end{aligned}$$

where

$$\begin{aligned}
\left\langle D_sS_{h,2}(x,t), S_{h,2}(x,t)\right\rangle_{\mathrm{H}} &= -D_s\left\|S_{h,2}\right\|^2_{\mathrm{H}_{\frac{1}{2}}} \\
&= D_s\sum_{i=C_{S_h}+1}^{\infty}x_i^2\lambda_{S_h,i} \\
&\le -D_sN_{S_h}\sum_{i=C_{S_h}+1}^{\infty}x_i^2 = -D_sN_{S_h}\left\|S_{h,2}\right\|^2_{\mathrm{H}},
\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|S_{h,2}\right\|^2_{\mathrm{H}} \le -D_sN_{S_h}\left\|S_{h,2}\right\|^2_{\mathrm{H}}+b_hN_h\bar{R},$$

further it is possible to obtain

$$\left\|S_{h,2}\right\|^2_{\mathrm{H}} \le e^{-2D_sN_{S_h}(t-t_0)}\left\|S_{h,2}(t_0)\right\|^2_{\mathrm{H}}+\frac{b_hN_h\bar{R}}{D_sN_{S_h}}\left(1-e^{D_sN_{S_h}(t-t_0)}\right), \quad \forall t>t_0.$$

According to equation (5.3), similar to the above derivation we can get

$$\begin{aligned}
&\left\|E_{h,2}\right\|^2_{\mathrm{H}} \le e^{-2D_eN_{E_h}(t-t_0)}\left\|E_{h,2}(t_0)\right\|^2_{\mathrm{H}}+\tfrac{\beta_h(x,t)b\bar{R}}{D_eN_{S_h}(N_h+m)}\left(1-e^{-2D_eN_{E_h}(t-t_0)}\right), && \forall t>t_0, \\
&\left\|I_{h,2}\right\|^2_{\mathrm{H}} \le e^{-2D_iN_{I_h}(t-t_0)}\left\|I_{h,2}(t_0)\right\|^2_{\mathrm{H}}+\tfrac{p_hb_h\bar{R}^2+k_h\bar{R}}{D_iN_{S_h}}\left(1-e^{-2D_iN_{I_h}(t-t_0)}\right), && \forall t>t_0, \\
&\left\|R_{h,2}\right\|^2_{\mathrm{H}} \le e^{-2D_rN_{R_h}(t-t_0)}\left\|R_h(t_0)\right\|^2_{\mathrm{H}}+\tfrac{\gamma_h\bar{R}}{D_rN_{S_h}}\left(1-e^{-2D_rN_{R_h}(t-t_0)}\right), && \forall t>t_0, \\
&\left\|S_{v,2}\right\|^2_{\mathrm{H}} \le e^{-2D_vN_{S_v}(t-t_0)}\left\|S_v(t_0)\right\|^2_{H}+\tfrac{b_vN_v}{D_vN_{S_v}}\left(1-e^{-2D_vN_{S_v}(t-t_0)}\right), && \forall t>t_0, \\
&\left\|I_{v,2}\right\|^2_{\mathrm{H}} \le e^{-2D_vN_{I_v}(t-t_0)}\left\|I_v(t_0)\right\|^2_{\mathrm{H}}+\tfrac{\beta_v(x,t)b\bar{R}^2}{D_vN_{S_v}(N_v+m)}\left(1-e^{-2D_vN_{I_v}(t-t_0)}\right), && \forall t>t_0.
\end{aligned}$$

lemma 5.1 holds and we can obtain that the system (2.1) has a global exponential attractor set $\mathbb{Q}$. The theorem holds. □

In conjunction with the above theorem, we discuss the global stability of the disease-free equilibrium of the system (2.1).

Theorem 5.2. *(1) When $R_0 < 1$, the disease-free equilibrium of the system (2.1) is globally asymptotically stable, i.e.,*

$$\begin{aligned}&\lim_{t\to\infty} S_h(x,t) = S^*_{h0}, \quad \lim_{t\to\infty} E_h(x,t) = 0, \quad \lim_{t\to\infty} I_h(x,t) = 0, \quad \lim_{t\to\infty} R_h(x,t) = 0,\\ &\lim_{t\to\infty} S_v(x,t) = S^*_{v0}, \quad \lim_{t\to\infty} I_v(x,t) = 0,\end{aligned}$$

where $S^*_{h0} = N_h, S^*_{v0} = N_v$.

(2) When $R_0 > 1$, if there exists a function $\Gamma(x)$ such that the following inequality holds

$$\begin{aligned}&\lim_{t\to\infty} S_h(x,t) \geq \Gamma(x), \quad \lim_{t\to\infty} E_h(x,t) \geq \Gamma(x), \quad \lim_{t\to\infty} I_h(x,t) \geq \Gamma(x), \quad \lim_{t\to\infty} R_h(x,t) \geq \Gamma(x)\\ &\lim_{t\to\infty} S_v(x,t) \geq \Gamma(x), \quad \lim_{t\to\infty} I_v(x,t) \geq \Gamma(x),\end{aligned}$$

Then DF will form an endemic epidemic.

Proof. (1) When $R_0 < 1$, according to lemma 4.1 and lemma 4.2 we can obtain that there exists $\varepsilon > 0$ such that $\lambda_\varepsilon(S^*_h + \varepsilon, S^*_v + \varepsilon) < 0$. From the first and fifth equations of the system (2.1) the following inequality can be obtained

$$\begin{cases} \frac{\partial S_h}{\partial t} \leq \nabla(D_s(x,t)\nabla S_h) + b_h N_h - d_h S_h(x,t), & x \in \Omega, t > 0,\\ \frac{\partial S_v}{\partial t} \leq \nabla(D_v(x,t)\nabla S_v) + b_v N_v - (d_v + \rho_v) S_v(x,t), & x \in \Omega, t > 0,\\ \frac{\partial S_h}{\partial n} = \frac{\partial S_v}{\partial n} = 0, & x \in \partial\Omega, t > 0.\end{cases}$$

So there exists $t_1 > 0$, when $t > t_1, x \in \bar{\Omega}$, $S_h < S^*_{h0} + \varepsilon$ and $S_v < S^*_{v0} + \varepsilon$ hold. Comparing theorems we get

$$\begin{cases} \frac{\partial E_h}{\partial t} \leq \nabla(D_e(x,t)\nabla E_h) + \frac{\beta_h(x,t)b}{N_h+m}(S^*_{h0} + \varepsilon) I_v - (d_h + k_h) E_h, & x \in \Omega, t > t_1,\\ \frac{\partial I_h}{\partial t} \leq \nabla(D_i(x,t)\nabla I_h) + k_h E_h - (d_h + \gamma_h) I_h + p_h b_h I_h, & x \in \Omega, t > t_1,\\ \frac{\partial R_h}{\partial t} \leq \nabla(D_r(x,t)\nabla R_h) + \gamma_h I_h - d_h R_h, & x \in \Omega, t > t_1,\\ \frac{\partial I_v}{\partial t} \leq \nabla(D_v(x,t)\nabla I_v) + \frac{\beta_v(x,t)b}{N_v+m}(S^*_{v0} + \varepsilon) I_h - (d_v + \rho_v) I_v, & x \in \Omega, t > t_1.\end{cases}$$

Take $(\bar{E}_h(x,t),\bar{I}_h(x,t),\bar{R}_h(x,t),\bar{I}_v(x,t),)=\left(Me^{\bar{\lambda}t}\bar{\phi}_1(x),Me^{\bar{\lambda}t}\bar{\phi}_2(x),Me^{\bar{\lambda}t}\bar{\phi}_3(x),Me^{\bar{\lambda}t}\bar{\phi}_4(x)\right)$ to be a solution that satisfies the following system of equations

$$\begin{cases} \frac{\partial \bar{E}_h}{\partial t}=\nabla(D_e(x,t)\nabla \bar{E}_h)+\frac{\beta_h(x,t)b}{N_h+m}(S_{h0}^*+\varepsilon)\bar{I}_v-(d_h+k_h)\bar{E}_h, \\ \frac{\partial \bar{I}_h}{\partial t}=\nabla(D_i(x,t)\nabla \bar{I}_h)+k_h\bar{E}_h-(d_h+\gamma_h)\bar{I}_h+p_hb_h\bar{I}_h, \\ \frac{\partial \bar{R}_h}{\partial t}=\nabla(D_r(x,t)\nabla \bar{R}_h)+\gamma_h\bar{I}_h-d_h\bar{R}_h, \\ \frac{\partial \bar{I}_v}{\partial t}=\nabla(D_v(x,t)\nabla \bar{I}_v)+\frac{\beta_v(x,t)b}{N_v+m}(S_{v0}^*+\varepsilon)\bar{I}_h-(d_v+\rho_v)\bar{I}_v. \end{cases}$$

According to lemma 4.1 we can get that $\bar{\phi}_1(x),\bar{\phi}_2(x),\bar{\phi}_3(x),\bar{\phi}_4(x)$ are the eigenfunctions under $\bar{\lambda}<0$. According to the comparison theorem we can get that when $x\in\Omega, t>t_1$ we have

$$E_h(x,t)\le \bar{E}_h(x,t), I_h(x,t)\le \bar{I}_h(x,t), R_h(x,t)\le \bar{R}_h(x,t), I_v(x,t)\le \bar{I}_v(x,t).$$

So we can obtain

$$E_h\le Me\bar{\lambda}(t)\bar{\phi}_1(x), I_h\le Me\bar{\lambda}(t)\bar{\phi}_2(x), R_h\le Me\bar{\lambda}(t)\bar{\phi}_3(x), I_v\le Me\bar{\lambda}(t)\bar{\phi}_4(x),$$

since $\bar{\lambda}<0$, we take the limit on both sides when the time $t\to\infty$

$$\lim_{t\to\infty}E_h=0,\ \lim_{t\to\infty}I_h=0,\ \lim_{t\to\infty}R_h=0,\ \lim_{t\to\infty}I_v=0. \tag{5.6}$$

Next we show that $\lim_{t\to\infty}S_h=N_h, \lim_{t\to\infty}S_v=N_v$. According to equation (5.6), we get that when $t\ge t_1, x\in\bar{\Omega}$, there exists $\kappa>0$ such that $0<E_h<\kappa, 0<I_h<\kappa, 0<R_h<\kappa, 0<I_v<\kappa$ hold. By the first equation of the system (2.1) we can obtain that there exists a $t_2>0$ and that $S_h(x,t)$ is an upper solution of the following equation

$$\begin{cases} \frac{\partial \mu(x,t)}{\partial t}=\nabla(D_s(x,t)\nabla\mu)+b_hN_h-\frac{\beta_h(x,t)b}{N_h+m}\mu(x,t)\kappa-d_h\mu(x,t)-p_hb_h\kappa, & x\in\Omega, t>t_2, \\ \frac{\partial\mu}{\partial n}=0, & x\in\partial\Omega, t>t_2, \\ \mu(x,t_2)=S_h(x,t_2), \end{cases} \tag{5.7}$$

and is the lower solution to the following problem

$$\begin{cases} \frac{\partial v(x,t)}{\partial t}=\nabla(D_s(x,t)\nabla v)+b_hN_h-d_hv(x,t), & x\in\Omega, t>t_2, \\ \frac{\partial v}{\partial n}=0, & x\in\partial\Omega, t>t_2, \\ v(x,t_2)=S_h(x,t_2). \end{cases} \tag{5.8}$$

According to the comparison theorem, we can get that the following conclusions hold when $t \geq t_2, x \in \Omega$

$$\mu(x,t) \leq S_h(x,t) \leq \nu(x,t).$$

Similar to the proof of theorem 5.1, we can obtain that when the diffusion coefficient is independent of spatial location there is a global exponential attractor set in the system (5.7) and in the system (5.8). Similar to the proof of literature[31], we can get

$$\lim_{t\to\infty}\mu(x,t) = S_{h,0}(\kappa,x)^-, \lim_{t\to\infty}\nu(x,t) = S_{h,0}(\kappa,x)^+,$$

where $S_{h,0}(\kappa,x)^-$ and $S_{h,0}(\kappa,x)^+$ represent the steady states of the system (5.7) and (5.8) respectively. Thus when we take $\kappa \to 0$, we have

$$S_{h,0}(\kappa,x)^-, S_{h,0}(\kappa,x)^+ \to S_{h0} = N_h, t \to \infty.$$

Similarly we can get $\lim_{t\to\infty} S_v = N_v$. The first half of the theorem is proved, i.e., the disease-free equilibrium of the system (2.1) is globally asymptotically stable.

(2) When $R_0 > 1$, by lemma 4.1 then there exists $\varepsilon > 0$ such that $\lambda\left(S_h^* - \varepsilon, S_v^* - \varepsilon\right) > 0$. This means that there exists a $\tilde{t}_1 > 0$ satisfying $S_h(x,t) > S_{h0}^* - \varepsilon$ and $S_v(x,t) > S_{v0}^* - \varepsilon$. When $t \geq \tilde{t}_1$ and $x \in \overline{\Omega}$, by the principle of comparison, we can get

$$\begin{cases} \frac{\partial E_h}{\partial t} \geq \nabla(D_e(x,t)\nabla E_h) + \frac{\beta_h(x,t)b}{N_h+m}(S_{h0}^* - \varepsilon)I_v - (d_h + k_h)E_h, & x \in \Omega, t > t_1, \\ \frac{\partial I_h}{\partial t} \geq \nabla(D_i(x,t)\nabla I_h) + k_h E_h - (d_h + \gamma_h)I_h + p_h b_h I_h, & x \in \Omega, t > t_1, \\ \frac{\partial R_h}{\partial t} \geq \nabla(D_r(x,t)\nabla R_h) + \gamma_h I_h - d_h R_h, & x \in \Omega, t > t_1, \\ \frac{\partial I_v}{\partial t} \geq \nabla(D_v(x,t)\nabla I_v) + \frac{\beta_v(x,t)b}{N_v+m}(S_{v0}^* - \varepsilon)I_h - (d_v + \rho_v)I_v, & x \in \Omega, t > t_1. \end{cases}$$

For $x \in \Omega, t > \tilde{t}_1$, we denote

$$\left(\tilde{E}_h(x,t), \tilde{I}_h(x,t), \tilde{R}_h(x,t), \tilde{I}_v(x,t),\right) = \left(Pe^{\tilde{\lambda}t}\tilde{\varphi}_1(x), Pe^{\tilde{\lambda}t}\tilde{\varphi}_2(x), Pe^{\tilde{\lambda}t}\tilde{\varphi}_3(x), Pe^{\tilde{\lambda}t}\tilde{\varphi}_4(x)\right),$$

where $\left(\tilde{E}_h(x,t), \tilde{I}_h(x,t), \widetilde{R}_h(x,t), \tilde{I}_v(x,t),\right)$ satisfies the following equation

$$\begin{cases} \frac{\partial E_h}{\partial t} = \nabla(D_e(x,t)\nabla E_h) + \frac{\beta_h(x,t)b}{N_h+m}(S_{h0}^* - \varepsilon)I_v - (d_h + k_h)E_h, \\ \frac{\partial I_h}{\partial t} = \nabla(D_i(x,t)\nabla I_h) + k_h E_h - (d_h + \gamma_h)I_h + p_h b_h I_h, \\ \frac{\partial R_h}{\partial t} = \nabla(D_r(x,t)\nabla R_h) + \gamma_h I_h - d_h R_h, \\ \frac{\partial I_v}{\partial t} = \nabla(D_v(x,t)\nabla I_v) + \frac{\beta_v(x,t)b}{N_v+m}(S_{v0}^* - \varepsilon)I_h - (d_v + \rho_v)I_v, \end{cases}$$

where $(\tilde{\varphi}_1(x), \tilde{\varphi}_2(x), \tilde{\varphi}_3(x), \tilde{\varphi}_4(x))$ are the eigenfunctions with respect to $\tilde{\lambda} > 0$. According to the comparison theorem, when $x \in \Omega, t > \tilde{t}_1$ we have

$$E_h(x,t) \geq \tilde{E}_h(x,t), I_h(x,t) \geq \tilde{I}_h(x,t), R_h(x,t) \geq \tilde{R}_h(x,t), I_v(x,t) \geq \tilde{I}_v(x,t),$$

then we obtain

$$E_h(x,t) \geq Pe^{\tilde{\lambda}t}\tilde{\varphi}_1(x), I_h(x,t) \geq Pe^{\tilde{\lambda}t}\tilde{\varphi}_2(x), R_h(x,t) \geq Pe^{\tilde{\lambda}t}\tilde{\varphi}_3(x), I_v(x,t) \geq Pe^{\tilde{\lambda}t}\tilde{\varphi}_4(x).$$

Taking the limit on both sides, we can get

$$\lim_{t\to\infty}\inf\left(E_h(x,t), I_h(x,t), R_h(x,t), I_v(x,t)\right) \geq \left(P\tilde{\varphi}_1(x), P\tilde{\varphi}_2(x), P\tilde{\varphi}_3(x), \tilde{\varphi}_4(x)\right).$$

According to the boundedness of the system (2.1), we can obtain that there exists constants $K > 0, \tilde{t_2} > 0$, when $x \in \Omega, t \geq \tilde{t_2}$, we can get

$$\left(E_h(x,t), I_h(x,t), R_h(x,t), I_v(x,t)\right) \leq (K,K,K,K).$$

Based on the above conclusions we can obtain that S_h and S_v satisfy the following equations

$$\begin{cases} \frac{\partial S_h}{\partial t} \geq \nabla(D_s(x,t)\nabla S_h) + b_h N_h - \left(d_h + p_h b_h K + \frac{\beta_h(x,t)bK}{N_h+m}\right) S_h(x,t), & x \in \Omega, t > 0, \\ \frac{\partial S_v}{\partial t} \geq \nabla(D_v(x,t)\nabla S_v) + b_v N_v - \left(d_v + \rho_v + \frac{\beta_v(x,t)bK}{N_v+m}\right) S_v(x,t), & x \in \Omega, t > 0, \\ \frac{\partial S_h}{\partial n} = \frac{\partial S_v}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases}$$

So

$$\lim_{t\to\infty}\inf S_h(x,t) \geq b_h N_h / \left(d_h + p_h b_h K + \frac{\beta_h(x,t)bK}{N_h+m}\right),$$
$$\lim_{t\to\infty}\inf S_v(x,t) \geq b_v N_v / \left(d_v + \rho_v + \frac{\beta_v(x,t)bK}{N_v+m}\right).$$

In summary, the endemic equilibrium of the system (2.1) is uniformly persistent when $\Gamma(x) := \min\left\{b_h N_h / \left(d_h + p_h b_h K + \frac{\beta_h(x,t)bK}{N_h+m}\right), b_v N_v / \left(d_v + \rho_v + \frac{\beta_v(x,t)bK}{N_v+m}\right), P\tilde{\varphi}_1(x), P\tilde{\varphi}_2(x), P\tilde{\varphi}_3(x), P\tilde{\varphi}_4(x)\right\}$ and $x \in \Omega$ are taken. Thus the theorem is proved. □

6. NUMERICAL SIMULATION

In this section, we verify the above conclusions by numerical simulation, then we obtain the parameters that have a greater influence on the basic reproduction number R_0 by performing PRCC sensitivity analysis on the basic reproduction number. Let the space be $\Omega = [0, \pi]$. We take $b_h = d_h = 0.5, b_v = 1, \rho_v = d_v = 0.5$, and the rest of the parameters take values in table 2.

Parameters	Value	Sources	Parameters	Value	Sources
p_h	0.8	[24]	N_v	18	[24]
k_h	0.2778	[17]	D_s	0.45	[33]
b	5.1	[31]	D_e	0.45	[33]
N_h	7	[24]	D_r	0.45	[33]
m	3	[24]	D_v	0.375	[31]
γ_h	0.1429	[32]	D_i	0.45	[31]

TABLE 2. Specific values for each parameter in the system (2.1)

6.1. The Impact of DENV Transmission Coefficients.

According to the literature[24] on the value of the infection coefficient, we discuss the various population density change curves of the system (2.1) for $R_0 < 1$ and $R_0 > 1$ in the following two cases, respectively.

Case 1: Let $\beta_h(x,t) = 0.015(1+0.3(\sin x+\sin t)), \beta_v(x,t) = 0.01(1+0.2(\sin x+\cos t))$, at this time $0.04 \leq R_0 \leq 0.09$, we assume $(S_h, E_h, I_h, R_h, S_v, I_v) = (1+\cos x, 2+\cos x, 3+\cos x, 1+\cos x, 6+5\cos x, 18-S_v)$. Then we can get the following statistical graphs of changes in density of various populations

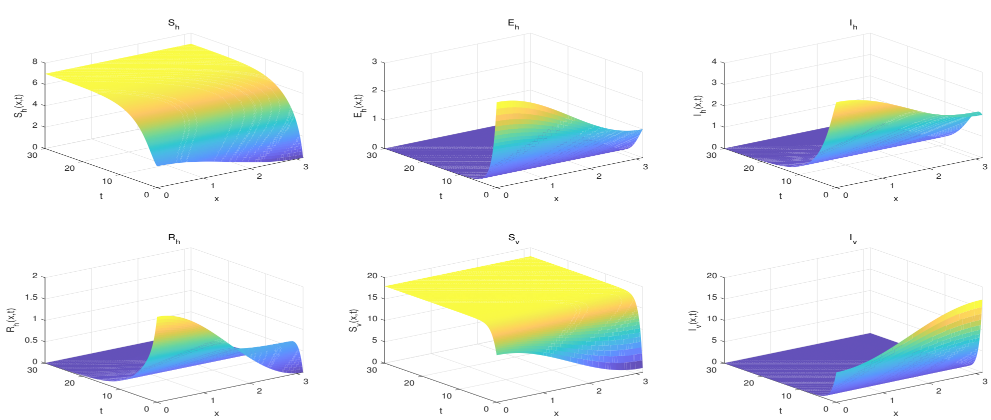


FIGURE 2. Surface plot of density variation in each population compartment of the system (2.1) when $R_0 < 1$

According to figure 2 it can be seen that when the basic reproduction number $R_0 < 1$, the disease-free equilibrium of the system (2.1) is globally asymptotically stabilized over time, which is the same as the conclusion of the theorem in this paper. To further observe the trend of the change in population density over time, the change in density of various populations is plotted at different locations when $R_0 < 1$, as shown in figure 3.

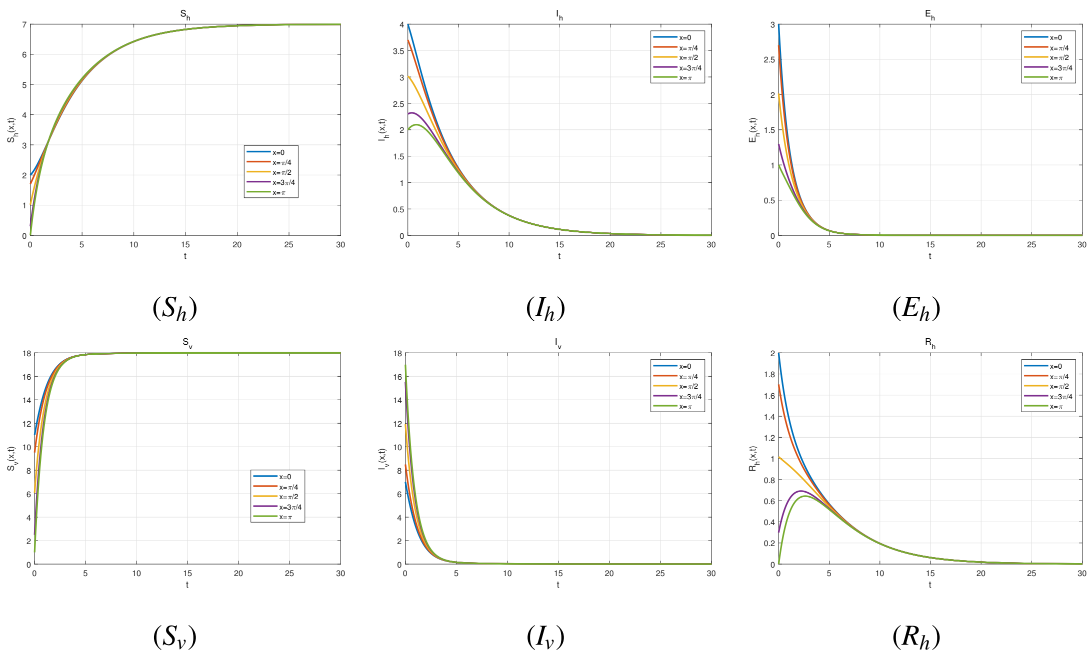


FIGURE 3. The change curve of population density of each compartment in the system (2.1) when $R_0 < 1$

From figure 3, we know that when $R_0 < 1$, the system (2.1) eventually converges to the disease-free equilibrium, and the mosquitoes can reach the disease-free equilibrium faster. Since the density of the recovered populations is small at the positions $x = \pi$ and $x = \frac{3\pi}{4}$, there is a process of rising and then falling. To observe the epidemiological trend of DF when the basic reproduction number $R_0 > 1$, we discuss the second case.

Case 2: Let $\beta_h(x,t) = 0.4(1+0.3(\sin x + \sin t)), \beta_v(x,t) = 0.3(1+0.2(\sin x + \cos t))$,at this time $1.24 \leq R_0 \leq 2.48$, we assume $(S_h, E_h, I_h, R_h, S_v,\ I_v) = (1+\cos x, 2+\cos x, 3+\cos x, 1+\cos x, 6+5\cos x, 18-S_v)$. Then we can get the following statistical graphs of changes in the density of various populations. From figure 4, we know that DF will form an endemic disease over time when $R_0 > 1$. To observe the trend of population density over time, we plotted the density change of various populations at different positions when $R_0 > 1$, as shown in figure 5.

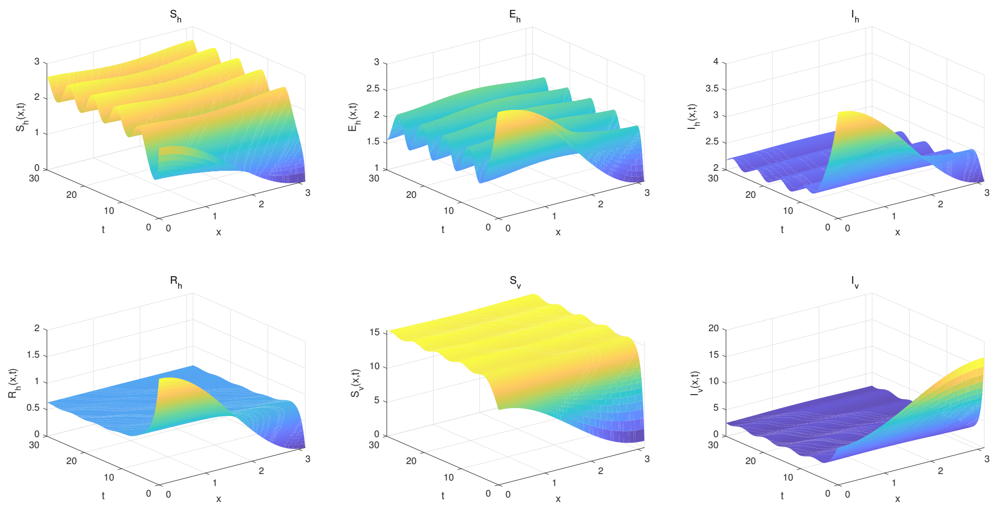
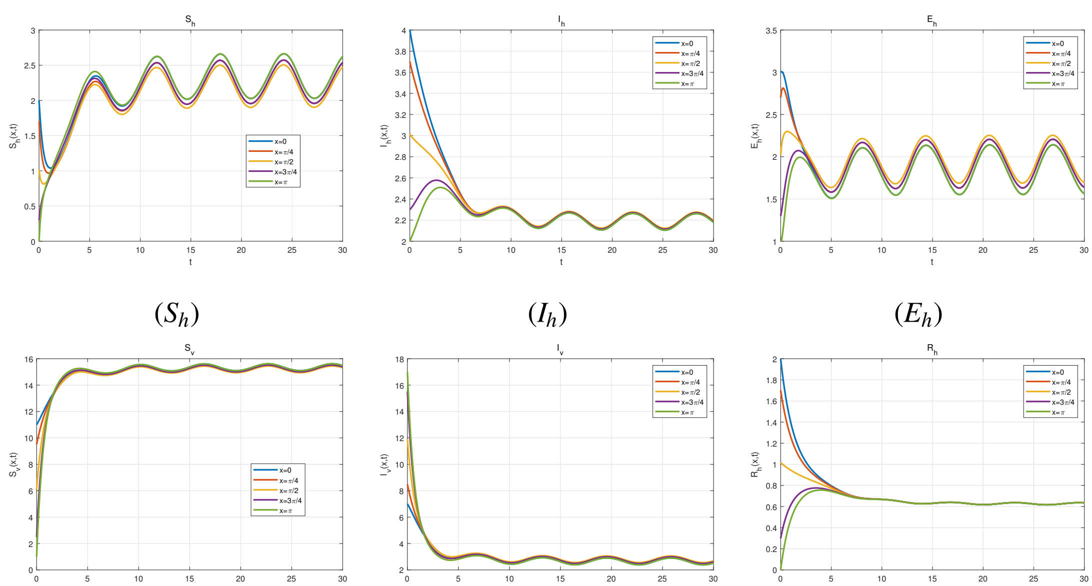


FIGURE 4. Surface plot of density variation in each population compartment of the system (2.1) when $R_0 > 1$



(S_h) (I_h) (E_h)

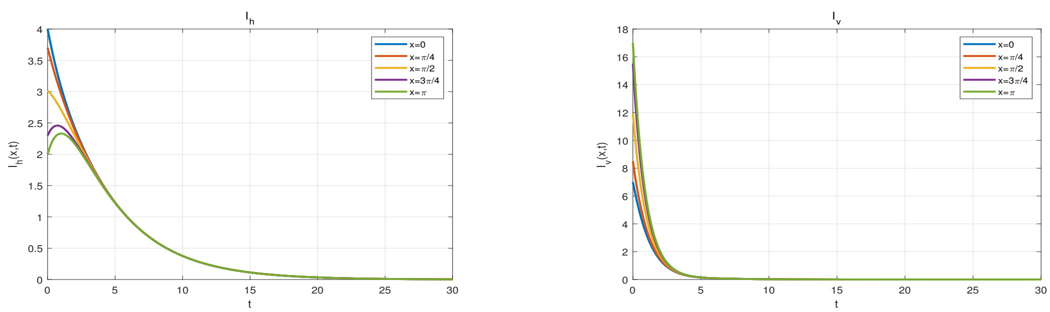
(S_v) (I_v) (R_h)

FIGURE 5. The change curve of population density in system (2.1) when $R_0 > 1$

From figure 5, we know that in the system (2.1), DF tends to stabilize periodically when $R_0 > 1$. It indicates that when the basic reproduction number $R_0 > 1$, the system (2.1) has a periodic solution.

6.2. The Influence of Diffusion Coefficient on the Transmission of DENV. In this section, we focus on how changing the diffusion coefficient of susceptible and infected populations affects the transmission of DENV. We discuss the following four cases.

Case 1: Let $\beta_h(x,t) = 0.015(1+0.3(\sin x+\sin t))$, $\beta_v(x,t) = 0.01(1+0.2(\sin x+\cos t))$, $D_s = D_e = D_r = 0.45, D_i = 1, D_v = 0.375$, we assume $(S_h, E_h, I_h, R_h, S_v, I_v) = (1+\cos x, 2+\cos x, 3+\cos x, 1+\cos x, 6+5\cos x, 18-S_v)$. Then we have plotted the changes in the infected populations and infected mosquitoes as shown in figure 6.

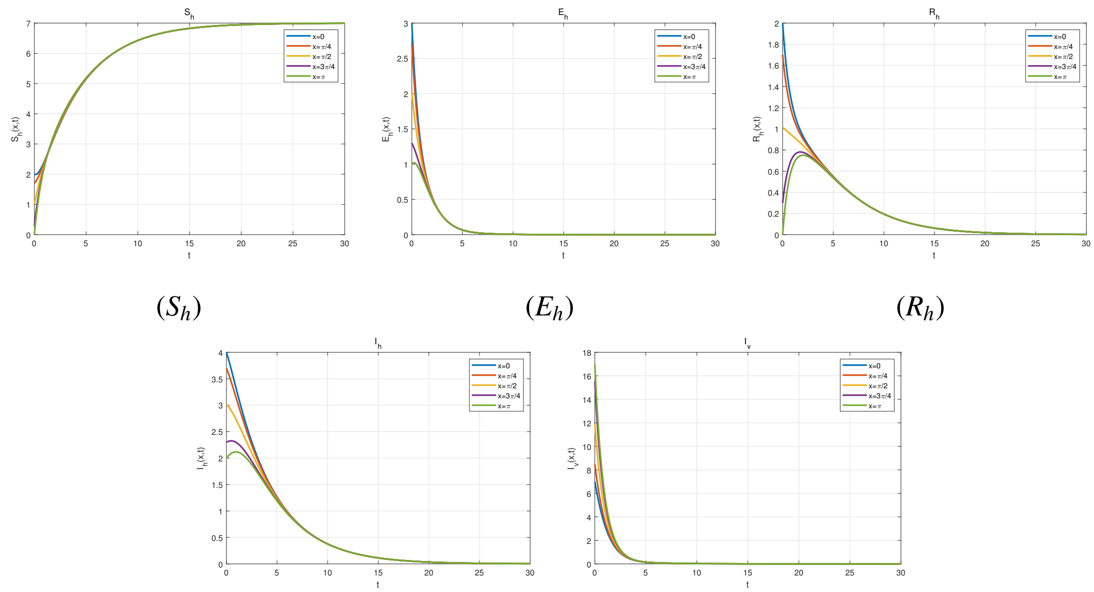


(I_h) (I_v)

FIGURE 6. Plot of changes in infected populations and infected mosquitoes densities in system (2.1) when $R_0 < 1, D_i = 1$

Compared with figure 6 and figure 3, it can be seen that increasing the diffusion coefficient of the infected populations causes the infected populations in different locations to reach the same value of density faster. To observe the impact of changing the diffusion coefficients of susceptible populations, exposed populations, and convalescent populations on the epidemiological trend of DF when $R_0 < 1$, we discuss the following case.

Case 2: Let $\beta_h(x,t) = 0.015(1+0.3(\sin x+\sin t))$, $\beta_v(x,t) = 0.01(1+0.2(\sin x+\cos t))$, $D_s = D_e = D_r = 1, D_i = 0.45, D_v = 0.375$, we assume $(S_h, E_h, I_h, R_h, S_v, I_v) = (1+\cos x, 2+\cos x, 3+\cos x, 1+\cos x, 6+5\cos x, 18-S_v)$. We have drawn the following population density change curves for each compartment, as shown in figure 7.



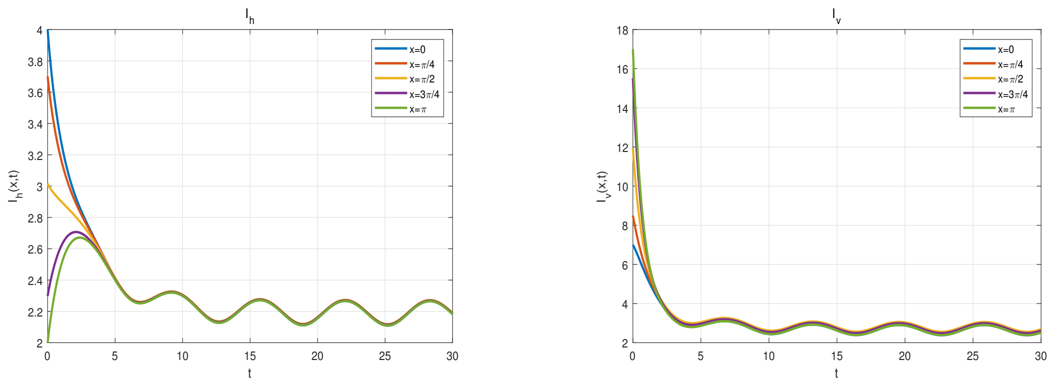
(S_h) (E_h) (R_h)

(I_h) (I_v)

FIGURE 7. Plot of the variation of population density in each compartment of the system (2.1) when $R_0 < 1, D_s = 1$

Compared with figure 7 and figure 3, it can be seen that when $R_0 < 1$ increasing the diffusion coefficient of the susceptible populations results in the susceptible population, the latent patient population, and the recovering population reaching the same density value faster, and does not have any significant delaying or advancing effect on the final extinction time of DF. Next, we discuss the impact of changing the diffusion coefficient of infected populations on the trend of DF epidemic when $R_0 > 1$. In this paper, we discuss the following case.

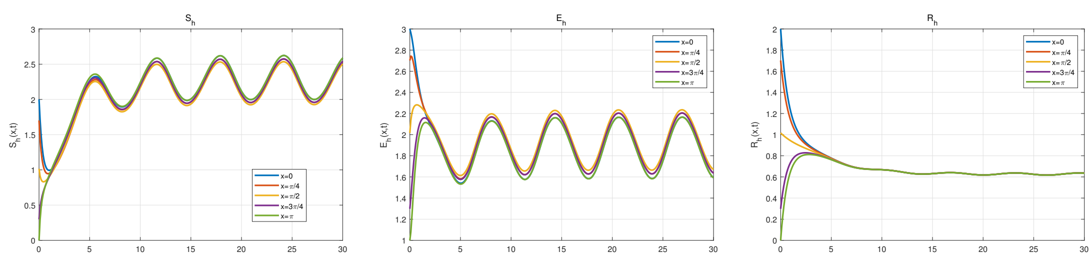
Case 3: Let $\beta_h(x,t) = 0.4(1+0.3(\sin x + \sin t)), \beta_v(x,t) = 0.3(1+0.2(\sin x + \cos t)), D_s = D_e = D_r = 0.45, D_i = 1, D_v = 0.375$, we assume $(S_h, E_h, I_h, R_h, S_v, I_v) = (1+\cos x, 2+\cos x, 3+\cos x, 1+\cos x, 6+5\cos x, 18-S_v)$. Then we have plotted the change curve of the infected populations and mosquitoes. Compared with figure 8 and figure 5, it can be seen that increasing the diffusion coefficient of the infected populations will accelerate the transmission of DENV when $R_0 > 1$, and the densities of the infected populations at different locations are closer during the DF epidemic. Next, we discuss the impact of changing the diffusion coefficients of susceptible populations, exposed populations, and convalescent populations on the epidemiological trend of DF when $R_0 > 1$.



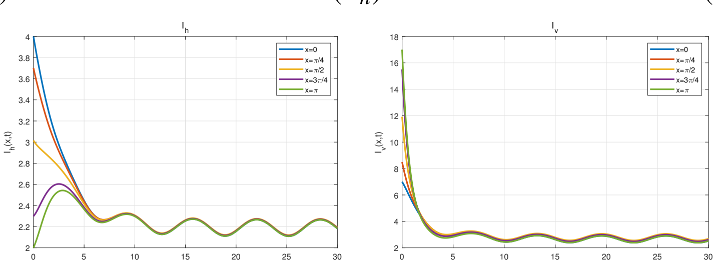
(I_h) (I_v)

FIGURE 8. Plot of changes in infected populations and infected mosquitoes densities in system (2.1) when $R_0 > 1, D_i = 1$

Case 4: Let $\beta_h(x,t) = 0.4(1+0.3(\sin x + \sin t)), \beta_v(x,t) = 0.3(1+0.4(\sin x + \cos t)), D_s = D_e = D_r = 1, D_i = 0.45, D_v = 0.375$, we assume $(S_h, E_h, I_h, R_h, S_v, I_v) = (1+\cos x, 2+\cos x, 3+\cos x, 1+\cos x, 6+5\cos x, 18-S_v)$. Then we have drawn the following population density change curves for each compartment, as shown in figure 9.



(S_h) (E_h) (R_h)



(I_h) (I_v)

FIGURE 9. Plot of the variation of population density in each compartment of the system (2.1) when $R_0 > 1, D_s = 1$

Compared with figure 9 and figure 5, it can be seen that increasing the diffusion coefficient of susceptible populations will not only cause DF to form endemic diseases earlier, but also lead to closer density values of susceptible, expose, and recovering populations when DF forms endemic diseases. Therefore, in reality, reducing the frequency of activities of susceptible and infected populations is conducive to the prevention and treatment of DF.

6.3. Sensitivity Analysis. In this subsection, we performed PRCC sensitivity analysis of the basic reproduction number R_0 according to equation (4.8),and plotted the histogram of the PRCC sensitivity analysis of each parameter to the basic reproduction number R_0 when the parameters are constant, as shown in figure 10

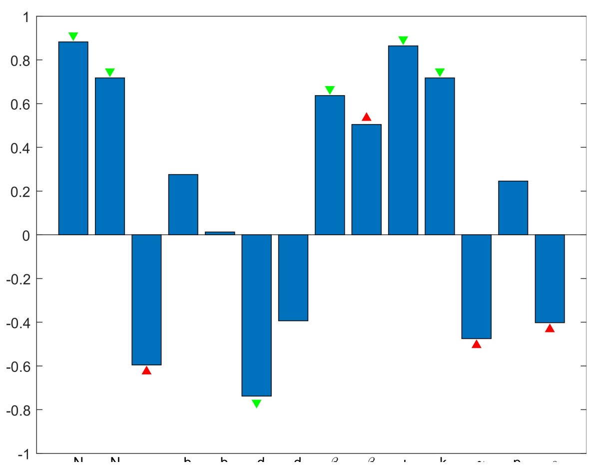


FIGURE 10. PRCC sensitivity analysis of each parameter to the basic reproduction number R_0

According to the PRCC sensitivity analysis, it can be obtained that the total population density N_h, the total population density of mosquitoes N_v, the birth rate b_h, the transmission coefficient of DENV β_h, β_v, the vertical transmission rate p_h, the conversion rate of the exposed population to the infected population k_h and the biting rate b were positively correlated with the basic reproduction number R_0, while other population densities m, the natural mortality rates of the population and mosquitoes d_h, d_v, the recovery rates γ_h and the mortality rates of mosquitoes due to other measures ρ_v were negatively correlated with the basic reproduction number R_0. When the PRCC sensitivity coefficient exceeds 0.6, there is a significant correlation; when the PRCC sensitivity coefficient is greater than 0.4, there is a more obvious correlation, then we can obtain the total density of the population N_h, the total density of mosquitoes N_v, the natural mortality rate of the population d_h, the DENV transmission coefficient from mosquitoes to population β_h, The conversion rate k_h from the population of exposed to the infected populations and the mosquito bite rate b have a significant effect on the basic reproduction number R_0. The population recovery rate γ_h, the mosquito mortality rate due to various other measures ρ_v, transmission coefficient of DENV from population to mosquito swarms β_v and other population densities m have more significant correlations on the basic reproduction number R_0. Combining positive and negative correlations with the real-life situation, the risk of DENV transmission can be reduced by taking personal protection measures to reduce the number of mosquito bites, using efficient mosquito killing measures, developing effective vaccines, and seeking medical treatment as early as possible. Since the mosquito bite rate b has a significant effect on the basic reproduction number R_0, the prevention and control of DF should focus on reducing the mosquito bite rate in real life.

7. Conclusion

In this paper, we incorporate the factors of incubation periods and vertical transmission of DENV into the spread of DF, and develop a class of dengue dynamics models in a heterogeneous environment with the effects of incubation periods and vertical transmission. Theoretically we prove the existence and uniqueness of global classical solutions for the system (2.1); and we obtain the expression of the basic reproduction number R_0 and the related properties by using the next-generation matrix operator. We show the threshold-type dynamics in terms of the basic reproduction number R_0: the disease-free equilibrium is global asymptotic stable when $R_0 < 1$, the disease-free equilibrium is unstable when $R_0 > 1$; and the existence of a global exponential attractor set for the system (2.1) is proved analytically by constructing upper and lower solutions.In the end,combined with numerical simulation, we verify the correctness of the theory, and find that shortening the incubation periods of DF can reduce the risk of dengue virus transmission through PRCC sensitivity analysis, and the vertical transmission of DENV in the population can increase the risk of DF transmission, but the effect is not obvious. In reality, the risk of DENV transmission can be reduced by reducing the frequency of crowd activities, taking good personal protection measures to reduce the rate of mosquito bites, seeking medical treatment as early as possible, and developing an effective vaccine, among which the reduction of the rate of mosquito bites has the most significant effect on the prevention and control of DENV.

In this paper, although the basic reproduction number can be utilized to describe the related threshold dynamics, the explicit expression of the basic regeneration number is not given in this paper, and only an approximate range of values of the basic reproduction number can be obtained. Therefore, finding a new method to calculate the exact value of the basic reproduction number is a key task in the future. DF transmission is a complex process, which is affected by a variety of factors such as weather variations and human activities, and the 2014 Guangdong DF outbreak was the most serious outbreak to date, Yi Jing, Xia Wang[34] evaluated the effects of a variety of factors on DF, including weather variables and human activities, with respect to the current outbreak of the DF, and the results showed that there was a significant correlation between the density of adult mosquitoes and the increasing number of cases, and that the weather variables might lead to the increase of complexity in spread of DENV. Combined with the latest research[35], it is found that the DF incidence in children is much higher than that in adults, and the development of a vaccine is difficult because the pathogenesis of DF has not yet been clarified. Therefore, incorporating the effects of weather variations and anthropogenic characteristics on DF in the model is our next research focus.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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