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STABILITY ANALYSIS AND ASYMPTOTIC BEHAVIOR OF A STOCHASTIC DELAYED SIRS EPIDEMIC MODEL WITH A CLASS OF NONLINEAR INCIDENCE RATES

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Abstract. This paper deals with a stochastic version of a deterministic delayed SIRS (Susceptible-Infective-Removed-Suceptible) epidemic model with nonlinear incidence rate. The stochastic model is obtained by taking into account random perturbations in the contact rate β due to environmental variations. We assume that the stochastic perturbation intensity is proportional to the number of infectious. Firstly, the existence of a unique global positive solution of the stochastic differential equations with delay describing the model is proved. Then, the stability of the disease-free equilibrium point is established under suitable conditions on the parameters of the model. We also study the asymptotic behaviour of the solution when the disease-free equilibrium (DFE) is unstable. Finally, numerical simulations are introduced to support our results.

Keywords: SIRS epidemic model; stochastic delay differential equations; stochastic stability; Lyapunov functionals technique.

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1. INTRODUCTION

In mathematical modelling, many systems used to model the evolution phenomena in science and industry do not only depend on the present state but also on the past ones. Stochastic delay differential equations (SDDEs) have been widely used to model such systems (see e.g. [1, 2, 3]). In particular, differential equations with delay have been used in epidemiology to model the incubation period in the disease transmission process (see e.g. [4, 5, 6]).
The SIRS epidemic model is one of the basic models introduced in [7] by Kermack and McKendrick to describe endemic infections, in which any infected individual may recover and become temporarily immune. Since its introduction in 1933 this model has been studied by many authors. It is based on the law of mass action which is not realistic. So there is a need to modify the classical linear incidence rate to capture the essential feature of the transmission process of communicable diseases. Nonlinear incidence rates have played an important role in ensuring that the model can give a reasonable quantitative description for disease dynamics such as saturation of the number of effective contact between infective individuals and susceptible individuals at high infective levels due to crowding of infective individuals or due to the protection measures by the susceptible individuals (see e.g. [8, 9, 10, 11, 12]). For example, in [6] a delayed SIRS epidemic model is considered. The authors deal with a class of nonlinear incidence rate of the form $\beta S(t)\int_0^h f(\tau)G(I(t-\tau))d\tau$, where $S(t)$, $I(t)$ and $R(t)$ denote respectively, the fractions of susceptible, infective and recovered individuals at time t and G is a nonlinear function checking some assumptions. This kind of incidence rate is used to model a disease spread in which transmission of the infection is through vectors which have an incubation time to become infectious. According to [13, 14], the vectors can be omitted from the equations by including a delay in the force of infection. More precisely, the model is described by the following ordinary differential system with distributed delays,

$$
(1)\qquad \begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu_1 S(t) - \beta S(t)\int_0^h f(\tau)G(I(t-\tau))d\tau + \nu R(t), \\ \frac{dI(t)}{dt} = \beta S(t)\int_0^h f(\tau)G(I(t-\tau))d\tau - (\mu_2+\gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - (\mu_3+\nu)R(t), \end{cases}
$$

with the initial condition,

$$(2)\qquad \begin{cases} S(\theta)=\phi_1(\theta), \quad I(\theta)=\phi_2(\theta), \quad R(\theta)=\phi_3(\theta); \\ \phi_i(0)>0, \quad \phi_i\in\mathscr{C}([-h,0];\mathbb{R}^+) \quad \text{for} \quad i=1,2,3. \end{cases}$$

The meaning of the parameters is:

- Λ represents the birth rate of the population and $\mu_i, i=1,2,3$ stand respectively for the death rate of susceptible "S", infected "I" and recovered "R" individuals with the natural assumption $\mu_1\leq\min\{\mu_2,\mu_3\}$.
- β is the contact rate between susceptible and infective individuals
- γ represents the recovery rate of infectives
- ν denotes the rate at which recovered individuals lose immunity and return to susceptible class
- h is a superior limit of incubation times in the vector population.

These parameters are assumed to be nonnegative.

The functions G and f satisfy the following conditions:

(3) (H_1) G is continuous and monotone increasing on $[0,\infty)$, with G(0)=0.

(H_2) $x/G(x)$ is monotone increasing on $[0,+\infty)$, with $\lim_{x\to 0}x/G(x)=1$.

(H_3) G is Lipschitz continuous on $[0,+\infty)$ and satisfies $0\leq G(x)\leq x$, $\forall x\geq 0$.

(H_4) f is a nonnegative continuous function with support $[0,h]$ such that $\int_0^h f(s)ds=1$.

The nonnegative continuous function f represents the incubation period distribution of the infection in the vector population, that is $f(\tau)$ denotes the fraction of vector population in which the time taken to become infectious is τ.

The reproduction number of the system (1) is

$$R_0=\frac{\beta\Lambda}{\mu_1(\mu_2+\gamma)}.$$

It is a threshold quantity which determines whether the disease will persist and prevail in a population or the disease simply dies out. It is proved in [6] that system (1) always has a disease-free equilibrium $E^0=(s_0,i_0,r_0)=(\frac{\Lambda}{\mu_1},0,0)$ and for $R_0>1$, the system (1) has a unique endemic equilibrium point $E^*=(s^*,i^*,r^*)$. By using a Lyapunov functional techniques, it is also proved that the disease-free equilibrium E^0 is globally asymptotically stable for $R_0\leq 1$, that is the disease dies out, and the endemic equilibrium E^* is globally asymptotically stable for $R_0 > 1$ and $\mu_1 s^* - \nu r^* \geq 0$, means that the disease persist in the population.

In this paper, we use the same argument of parameter perturbation used in [15] to derive from deterministic model (1), a stochastic model (see system (4)) which is a generalization of the stochastic *SIR* (Susceptible-Infective-Recovered) epidemic model studied in [5] where $G(x) = x$, $\mu_1 = \mu_2 = \mu_3 = \Lambda$, $\nu = 0$ and the noise intensity at each time is assumed to be a positive real σ. In this case it is proved that $0 < \beta < \min\{\gamma + \mu - \frac{1}{2}\sigma^2, 2\mu\}$ is a sufficient condition for the stochastic stability of the disease-free equilibrium. Assuming the above conditions on the death rates and function G but with $\nu > 0$, the stochastic stability of the disease-free equilibrium is proved in [16] provided that $2\mu_1 > \max\{\gamma - \nu, \beta + \nu\}$ and $\beta < \mu_2 + \gamma - \frac{1}{2}\sigma^2$. We note that in these cases the positivity of the model is not discussed. Recently many authors have studied a stochastic delayed epidemic models with perturbed parameter (see e.g. [4, 17, 18, 19, 20, 21, 22, 23, 24, 25]). But in these papers the random perturbations do not concern the contact rate β.

In this paper, first of all we show the existence of a unique global positive solution of our model. Then, by using a stochastic Lyapunov functional technique, we prove that if $2\mu_1 > \max\{\gamma - \nu, \beta - \nu\}$ and $\frac{\Lambda}{\mu_1}\beta < \mu_2 + \gamma - \frac{1}{2}\sigma^2\left(\frac{\Lambda}{\mu_1}\right)^4$, the disease-free equilibrium is almost surely exponentially stable for some class of initial conditions. In particular, if $G(x) < 1$ we can replace the previous second assumption with $\frac{\Lambda}{\mu_1}\beta < \mu_2 + \gamma - \frac{1}{2}\left(\sigma\frac{\Lambda}{\mu_1}\right)^2$. Also, by combining Lyapunov technique, the well-known variation of constants approach and stochastic analysis, we establish that the DFE is almost surely stable under the condition $\frac{\Lambda}{\mu_1}\beta < \gamma + \mu - \frac{1}{2}\sigma^2\left(\frac{\Lambda}{\mu_1}\right)^4$ and in the case where $G(x) < 1$ this condition can be replaced with $\frac{\Lambda}{\mu_1}\beta < \mu_2 + \gamma - \frac{1}{2}\left(\sigma\frac{\Lambda}{\mu_1}\right)^2$.

The rest of the paper is organized as follows. In section 2, we introduce the model and some preliminary definitions and results. In section 3, we investigate the existence of a unique global positive solution of our stochastic model. Section 4 deals with the stability of the unique DFE E^0. The asymptotic behaviour of the stochastic model around the endemic equilibrium point E^* of the deterministic model (1) when $R_0 > 1$ is considered in section 5. Finally, in section 6, we conclude and give some numerical simulations to support our theoretical results.

2. Stochastic Model Derivation and Preliminary Results

2.1. Stochastic model derivation. Throughout this paper, we let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_t$ contains all P-null set and $\mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s$) on which are defined all the random variables considered. In the following, we consider the deterministic model described by the system (1) where we introduce in the contact rate β, an additive stochastic perturbation. Indeed, the contact process are inevitably affected by random perturbations due to stochastic environmental factors that can be modeled by a random variable $\tilde{\beta}$ with average value β and variance ϑ^2. In view of the well-known Central Limit Theorem, the potential infectious contacts $\tilde{\beta}dt$ made by each infected individual with each susceptible in the small time interval $[t, t+dt]$ is approximately given by (see e.g. [2, 15, 26])

$$\tilde{\beta}dt = \beta dt + \vartheta dW(t),$$

where W is a standard brownian motion. Moreover, in the real world the increase in the number of infectious occur with some spatial dispersion of these infectious that increases the variability in the contact process due to environmental variations. To take into account this situation, here we assume that the noise intensity at time t, depends on infectious population size $I(t)$. Therefore we replace βdt in system (1) with $\tilde{\beta}dt = \beta dt + \sigma I(t)dW(t)$, where σ is a positive real. Doing so, we obtain the stochastic model

$$(4) \qquad \begin{cases} dS(t) = \Big(\Lambda - \mu_1 S(t) - \beta S(t)\displaystyle\int_0^h f(\tau)G(I(t-\tau))d\tau + \nu R(t)\Big)dt \\ \qquad - \sigma S(t)I(t)\Big(\displaystyle\int_0^h f(\tau)G(I(t-\tau))d\tau\Big)dW(t), \\ dI(t) = \Big(\beta S(t)\displaystyle\int_0^h f(\tau)G(I(t-\tau))d\tau - (\mu_2+\gamma)I(t)\Big)dt \\ \qquad + \sigma S(t)I(t)\Big(\displaystyle\int_0^h f(\tau)G(I(t-\tau))d\tau\Big)dW(t), \\ dR(t) = \Big(\gamma I(t) - (\mu_3+\nu)R(t)\Big)dt. \end{cases}$$

The description of the parameters are the same as in the deterministic model (1) with the same assumptions.

The initial condition is given by

$$(5)\qquad \begin{cases} S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta), \quad \theta \in [-h, 0], \\ \varphi_i(0) > 0, \quad i = 1, 2, 3; \quad \varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{H}^3(I), \end{cases}$$

where $\mathscr{H}^n(I)$ is the set of $\mathscr{F}_0$-measurable $\mathscr{C}([-h,0];I)$-valued random variables, where I is a compact and connected subset of $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n : \backslash x_1 \geq 0, ..., x_n \geq 0\}$ such that $\mathbb{E}(||\varphi||^2) < \infty$, where $||\varphi|| = \sup_{\theta \in [-h,0]} |\varphi(\theta)|$ and $|\,.\,|$ stands for the Euclidean norm on $\mathbb{R}^n$.

2.2. Definitions and preliminary results. Let $a : \mathscr{C}([-h,0];I) \to \mathbb{R}^n$ and $b : \mathscr{C}([-h,0];I) \to \mathscr{M}_{n\times m}(\mathbb{R})$ be respectively a n-dimensional functional and $n \times m-$ matrix-valued functional. Let $B = (B(t))_{t\geq 0}$ be a m-dimensional Brownian motion process. Consider the following n-dimensional stochastic system with time delay:

$$(6)\qquad \begin{cases} dX(t) = a(X_t)dt + b(X_t)dB(t), \\ X_0 = \varphi \in \mathscr{H}^n(I), \end{cases}$$

where $X_t = \{X(t+\theta), \quad \theta \in [-h,0]\}$ is viewed as a $\mathscr{C}([-h,0];\mathbb{R}^n_+)-$valued stochastic process with $X_t(0) = X(t)$. Let $\{X(t,\varphi) : t \geq 0\}$ the solution of the stochastic system (6) with initial condition $X_0 = \varphi$.

Definition 1. Let $x^* \in \mathbb{R}^n$. A solution $X(t) = x^*$ of the stochastic system (6) with initial condition $X_0 = x^*$ is called a stationary solution if

$$a(x^*) = b(x^*) = 0.$$

The real number x^* is called a equilibrium point of the stochastic system (6). If $x^* = 0$, the stationary solution is said to be the trivial solution.

The stability analysis of a equilibrum point x^* of the stochastic system (6) can be reduced to the stability analysis of trivial solution of the stochastic system obtained from the transformation $Y(t,\varphi) = X(t,\varphi) - x^*$. For this reason, the following definitions and preliminary results are given with respect to the trivial solution.

Definition 2. Assume that for every initial value $\varphi \in \mathscr{H}^n(I)$, the system (6) admits a unique global solution which is denoted by $\{X(t,\varphi) : t \geq 0\}$. The trivial solution of system is said to be

***(i)*:** almost surely asymptotically stable in $\mathscr{H}^n(I)$ if for all $\varphi \in \mathscr{H}^n(I)$, $\lim_{t\to\infty} |X(t,\varphi)| = 0 \quad a.s.$

***(ii)*:** almost surely exponentially stable in $\mathscr{H}^n(I)$, if for all $\varphi \in \mathscr{H}^n(I)$, there exists $\alpha > 0$ such that $\limsup_{t\to\infty} \frac{1}{t} \ln |X(t,\varphi)| < -\alpha, \quad a.s.$

***(iii)*:** exponentially mean-square stable in $\mathscr{H}^n(I)$, if for all $\varphi \in \mathscr{H}^n(I)$, there exists $\alpha > 0$ such that $\lim_{t\to+\infty} \frac{1}{t} \ln E(|X(t,\varphi)|^2) < -\alpha, \quad a.s.$

One of the important properties in population dynamics is the persistence which means every species will never become extinct. More often, in many stochastic epidemic models obtained from a deterministic model by adding white noise to the dynamics, do not have an endemic equilibrium. Analyzing the persistence of this disease in the population helps in particular to determine the conditions under which a disease introduced into a community becomes endemic.

Definition 3. The solution of the system (6) is said to be persistent with probability one if, for every initial value $\varphi \in \mathscr{H}^n(I)$, the solution $\{X(t,\varphi) : t \geq 0\}$ has the property that

$$\liminf_{t\to\infty} X_i(t,\varphi) > 0 \quad a.s \text{ for all } 1 \leq i \leq n.$$

Let $V : [0,\infty) \times \mathscr{H}^n(I) \to \mathbb{R}$ be a functional. The generating operator $\mathscr{L}$ of the system (6) is defined (see e.g. [2, 3, 27]) by the formula

$$\mathscr{L}V(t,\varphi) = \lim_{\Delta\to 0} \frac{\mathbb{E}(V(t+\Delta, X_{t+\Delta})|X_t = \varphi) - V(t,\varphi)}{\Delta}.$$

Suppose that the functional V can be written in the form

$$V(t,\varphi) = V^*(t,\varphi(0),\varphi)$$

where V^* is a $\mathbb{R}$-valued functional defined on $[0,\infty) \times \mathbb{R}^n \times \mathscr{H}^n(I)$.

For any $(t,x) \in [-\tau,\infty) \times \mathbb{R}^n$ and any $\varphi \in \mathscr{H}^n(I)$ such that $\varphi(0) = x$, we put

$$V_\varphi(t,x) = V^*(t,x,\varphi)$$

where $x = \varphi(0) = X(t)$ and $\varphi = X_t$.

Let D be the class of all functionals V for which function $V_\varphi(t,x)$ has two continuous derivatives with respect to x and one bounded derivative with respect to t for almost all $t \geq 0$. For functionals in D, the generating operator $\mathscr{L}$ of system (6) becomes

(7)
$$\mathscr{L}V(t,\varphi) = \mathscr{L}V_\varphi(t,X(t)) = \frac{\partial V_\varphi(t,\varphi(0))}{\partial t} + \nabla V_\varphi^T(t,\varphi(0))a(\varphi) + \frac{1}{2}trace[b^T(\varphi)\nabla^2 V_\varphi(t,\varphi(0))b(\varphi)]$$

where $\nabla V_\varphi(t,x) = \left(\frac{\partial V_\varphi(t,x)}{\partial x_1}, \ldots, \frac{\partial V_\varphi(t,x)}{\partial x_n}\right)$, $\quad \nabla^2 V_\varphi(t,x) = \left(\frac{\partial^2 V_\varphi(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}$.

The following theorem which is a corollary of Theorems 3 in [28], gives a sufficient condition for the stability of the trivial solution.

Theorem 1. *Assume that both a and b satisfy the local Lipschitz condition and suppose that there exists a functional* $V(t,\varphi) \in D$ *such that*

$$c_1|\varphi(0)|^2 \leq V(t,\varphi) \leq c_2||\varphi||^2 \quad and \quad \mathscr{L}V(\varphi,t) \leq -\alpha|\varphi(0)|^2$$

where c_1, c_2, *and* α *are all positive constants. Then for all* $\varphi \in \mathscr{H}^n(I)$, *there exists a positive constant q such that the solution of system* (6) *satisfies*

$$\lim_{t\to\infty} \frac{1}{t}\ln(|X(t)|) < -q \quad a.s.$$

That is the trivial solution of the system (6) *is almost surely exponentially stable.*

Remark 1. The stochastic model (4) with the initial condition (5) can be written in the form of (6), with $B(t) = (W(t), W_1(t), W_2(t))$ a three-dimensional Brownian motion process and for any $\phi \in \mathscr{C}([-h,0];\mathbb{R}^3_+)$

$$a(\phi) = \begin{pmatrix} \Lambda - \mu_1\phi_1(0) - \beta\phi_1(0)\int_{-h}^{0} f(-\theta)G(\phi_2(\theta))d\theta - \nu\phi_3(0) \\ \beta\phi_1(0)\int_{-h}^{0} f(-\theta)G(\phi_2(\theta))d\theta - (\mu_2+\gamma)\phi_2(0) \\ \gamma\phi_2(0) - (\mu_3+\nu)\phi_3(0) \end{pmatrix},$$

$$b(\phi) = \begin{pmatrix} -\sigma\phi_1(0)\varphi_2(0)\int_{-h}^{0} f(-\theta)G(\phi_2(\theta))d\theta & 0 & 0 \\ \sigma\phi_1(0)\phi_2(0)\int_{-h}^{0} f(-\theta)G(\phi_2(\theta))d\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that E^0 is also the disease-free equilibrium position of the stochastic model (4).

3. Global Existence and Positivity

Under assumptions (H_3) and (H_4), the coefficients of the system (4) are locally Lipschitz continuous, but do not satisfy the linear growth condition. So we only have the existence and uniqueness of a local solution, that is for any initial condition $\phi \in \mathscr{H}^3(I)$ there is a unique local solution $(S(t), I(t), R(t))$ of the system (4) on $t \in [0, \tau_e)$, where τ_e denotes the explosion time (see e.g. [2] Theorem 2.8 page 154). In order to prove that this solution is global we will establish the positivity and the no-explosion in a finite time of the solution for any initial condition.

Let us set $\Delta = \{(x,y,z) \in \mathbb{R}^3 / \quad x > 0, y > 0, z > 0, \quad x+y+z < \frac{\Lambda}{\mu_1}\}$ and $\mathscr{H}^3(\Delta)$ be the class of $\mathscr{F}_0$-measurable $\mathscr{C}([-h,0];\Delta)$-valued random variables.

Let $N(t) = S(t) + I(t) + R(t)$ denotes the whole population size at time $t \in [-h, \tau_e)$.

We consider the following stopping times:

$$\begin{aligned}
\tau^{\{-\}} &= \inf\{t \in [0,\infty), \min\{S(t), I(t), R(t)\} \notin (0,\infty)\}, \\
\tau_0 &= \inf\{t \in [0,\infty), \min\{S(t), I(t), R(t)\} \notin (0, \tfrac{\Lambda}{\mu_1})\}, \\
\tau_k &= \inf\{t \in [0,\infty), \min\{S(t), I(t), R(t)\} \notin (\tfrac{1}{k}, \tfrac{\Lambda}{\mu_1})\}, \text{ for all integers } k \geq k_0,
\end{aligned}$$

where $k_0 \in \mathbb{N}^*$ is such that $\min\{S(0), I(0), R(0)\} \in (\frac{1}{k_0}, \frac{\Lambda}{\mu_1})$, with the convention $\inf \emptyset = \infty$.

Lemma 1. *Let us assume that the initial condition φ of (4) belongs to $\mathscr{H}^3(\Delta)$. Then*

***(i)*:** $\tau^{\{-\}} \geq \tau_e \quad a.s.$

***(ii)*:** $\sup_{t \in [-h, \tau_e)} N(t) \leq \frac{\Lambda}{\mu_1} \quad$ *a.s. Moreover* $\tau_0 = \tau^{\{-\}} = \tau_e \quad a.s.$

***(iii)*:** *The sequence of stopping times $(\tau_k)_{k \geq k_0}$ converges τ_0 a.s.*

Proof. In view of Itô's formula for all $t \in [-h, \tau^{\{-\}})$ we have

$$\begin{aligned}
&\ln[S(t)I(t))R(t)] - \ln[\varphi_1(0)\varphi_2(0)\varphi_3(0)] \\
&= \int_0^t \left[\frac{\Lambda}{S(s)} - \mu_1 + \nu \frac{R(s)}{S(s,)} - \beta \int_0^h f(\tau) G(I(s-\tau)) d\tau \right] ds \\
&+ \int_0^t \left[\beta \frac{S(s)}{I(s)} \int_0^h f(\tau) G(I(s-\tau)) d\tau - (\mu_2 + \gamma) - \frac{\sigma^2}{2} \left(S(s) \int_0^h f(\tau) G(I(s-\tau)) d\tau \right)^2 \right] ds \\
&+ \int_0^t \left[\gamma \frac{I(s)}{R(s)} - (\mu_3 + \nu) - \frac{\sigma^2}{2} \left(I(s) \int_0^h f(\tau) G(I(s-\tau)) d\tau \right)^2 \right] ds
\end{aligned}$$

$$-\sigma\int_0^t I(s)\Big(\int_0^h f(\tau)G(I(s-\tau))d\tau\Big)dW(s)+\sigma\int_0^t S(s)\Big(\int_0^h f(\tau)G(I(s-\tau))d\tau\Big)dW(s)$$

Obviously, for all $t\in[-h,\tau^{\{-\}})$, $\min\{S(t),I(t),R(t)\}>0$. It follows that

$$\begin{aligned}
&\ln[S(t)I(t))R(t)]-\ln[\varphi_1(0)\varphi_2(0)\varphi_3(0)]\\
&\geq\int_0^{t\wedge\tau^{\{-\}}}\Big[-(\mu_1+\mu_2+\gamma+\mu_3+\nu)-\beta\int_0^h f(\tau)G(I(s-\tau))d\tau\Big]ds\\
&-\frac{\sigma^2}{2}\int_0^{t\wedge\tau^{\{-\}}}\Big(I^2(s)+S^2(s)\Big)\Big(\int_0^h f(\tau)G(I(s-\tau))d\tau\Big)^2ds\\
&+\sigma\int_0^{t\wedge\tau^{\{-\}}}(S(s)+I(s))\Big(\int_0^h f(\tau)G(I(s-\tau))d\tau\Big)dW(s)=J(t).
\end{aligned}\tag{8}$$

Assume that $P(\{\tau^{\{-\}}<\tau_e\})>0$. By continuity of the solution of the system (4), we have on the event $\{\tau^{\{-\}}<\tau_e\}$

$$S(\tau^{\{-\}})I(\tau^{\{-\}})R(\tau^{\{-\}})=0.$$

Hence

$$\lim_{t\to\tau^{\{-\}}}\ln[S(t)I(t)R(t)]=-\infty.\tag{9}$$

Combining (8) and (9), we have on the event $\{\tau^{\{-\}}<\tau_e\}$ that $-\infty\geq J(\tau^{\{-\}})$.

Therefore

$$\{\tau^{\{-\}}<\tau_e\}\subset\{-\infty\geq J(\tau^{\{-\}})\}.$$

Since $J(\tau^{\{-\}})$ is finite on $\{\tau^{\{-\}}<\tau_e\}$, we have a contradiction. So necessarily $P(\{\tau^{\{-\}}<\tau_e\})=0$ and (i) is proved.

For any initial condition $\varphi\in\mathscr{H}^3(\Delta)$, the total size of the population $N(t)$ at time $t\in[-h,\tau_e)$ is described by the equation

$$\begin{cases}dN(t)=(\Lambda-\mu_1S(t)-\mu_2I(t)-\mu_3R(t))dt,\\ N(0)=\phi_1(0)+\phi_2(0)+\phi_3(0).\end{cases}$$

In view of **(i)**, for any $t\in[-h,\tau_e)$, we see that $\min\{S(t),I(t),R(t)\}>0$ a.s.

Since $\mu_1\leq\min\{\mu_2,\mu_3\}$, we get

$$\begin{aligned}
dN(t)&=[\Lambda-(\mu_2-\mu_1)I(t)-(\mu_3-\mu_1)R(t)-\mu_1N(t)]dt\\
&=[-\Phi(t)+(\Lambda-\mu_1N(t))]dt,
\end{aligned}$$

where $\Phi(t) = (\mu_2 - \mu_1)I(t) + (\mu_3 - \mu_1)R(t) \geq 0$.

Therefore, by virtue of a comparison theorem, we obtain that for any $t \in [0, \tau_e)$,

$$N(t) \quad \leq \quad (N(0) - \frac{\Lambda}{\mu_1})e^{-\mu_1 t} + \frac{\Lambda}{\mu_1} \quad a.s. \quad \text{where } N(0) \in \left(0, \frac{\Lambda}{\mu_1}\right).$$

It follows that $\sup_{t\in[0,\tau_e)} N(t) \leq \dfrac{\Lambda}{\mu_1}$ which leads to $\tau^{\{-\}} \leq \tau_e$ $a.s.$ since $\max\{S(t), I(t), R(t)\} < \dfrac{\Lambda}{\mu_1}$ $a.s.$

Hence the solution might explode only toward $-\infty$ which implies that $\tau_0 = \tau^{\{-\}} = \tau_e$ a.s. Therefore we have **(ii)**.

For $k \geq k_0$, let set $A_k = (\frac{1}{k}, \frac{\Lambda}{\mu_1})$. $(A_k)_{k\geq k_0}$ is increasing and converges to $\bigcup_{k\geq k_0} A_k = (0, \frac{\Lambda}{\mu_1})$. Therefore the sequence of stopping time $(\tau_k)_{k\geq k_0}$ is increasing and there exists $\tau_\infty \in [0, \infty)$ such that $\lim_{k\to\infty} \tau_k = \tau_\infty$. Since for any $k \geq k_0$, $A_k \subset (0, \frac{\Lambda}{\mu_1})$, we have $\tau_\infty \leq \tau_0$. So, if $\tau_\infty = \infty$ then $\tau_\infty = \tau_0 = \infty$.

Now, let us assume that $\tau_\infty < \infty$ and put $Y(t) = \min\{S(t), I(t), R(t)\}$, for any $t \in [0, \tau_e)$. Since for all $k \geq k_0$, $Y(\tau_k) \notin A_k$ and $Y(\tau_k) \to Y(\tau_\infty)$. It follows that $Y(\tau_\infty) \notin (0, \frac{\Lambda}{\mu_1})$. Hence $\tau_\infty = \tau_0$ wich gives **(iii)**. □

Theorem 2. *For any initial condition* $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{H}^3(\Delta)$, *the system* (4) *admits a unique solution* $(S(t), I(t), R(t))$ *on* $t \geq 0$, *and this solution remains in* Δ *with probability* 1.

Proof. In view of the Lemma 1, for any initial condition $\varphi \in \mathscr{H}^3(\Delta)$ the system (4) has a unique local positive solution $(S(t), I(t), R(t))$ on $t \in [0, \tau_e)$ and $\lim_{k\to\infty} \tau_k = \tau_e$. In order to establish the existence and uniqueness of a global positive solution it is enough to prove that $\lim_{k\to\infty} \tau_k = \infty$. Let $k_0 \in \mathbb{N}^*$ such that $\min\{\varphi_1(0), \varphi_2(0), \varphi_3(0)\} > \frac{1}{k_0}$. Consider the function V_0 defined for any vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ by

$$V_0(x) = -\ln\left(\frac{\mu_1 x_1}{\Lambda}\right) - \ln\left(\frac{\mu_1 x_2}{\Lambda}\right) - \ln\left(\frac{\mu_1 x_3}{\Lambda}\right).$$

By virtue of Itô's formula, we get that for any $t \in [-h, \tau_e)$ and $X(t) = (S(t), I(t), R(t))$

$$\begin{aligned} & dV_0(X(t)) \\ = \quad & \left[-\frac{\Lambda}{S(t)} + \mu_1 - \nu\frac{R(t)}{S(t)} + \beta\int_0^h f(\tau)G(I(t-\tau))d\tau + \frac{\sigma^2}{2}\left(I(t)\int_0^h f(\tau)G(I(t-\tau))d\tau\right)^2\right]dt \end{aligned}$$

$$+\Big[-\beta\frac{S(t)}{I(t)}\int_0^h f(\tau)G(I(t-\tau))d\tau+(\mu_2+\gamma)+\frac{\sigma^2}{2}\Big(S(t)\int_0^h f(\tau)G(I(t-\tau))d\tau\Big)^2\Big]dt$$
$$+\Big[-\gamma\frac{I(t)}{R(t)}+(\mu_3+\nu)\Big]dt+\sigma S(t)\Big(\int_0^h f(\tau)G(I(t-\tau))d\tau\Big)dW_t$$
$$-\sigma I(t)\Big(\int_0^h f(\tau)G(I(t-\tau))d\tau\Big)dW(t).$$

Since for any $s\in[-h,t\wedge\tau_k)$, we see that $S(s),I(s),R(s)\in\left(\frac{1}{k},\frac{\Lambda}{\mu_1}\right)\quad a.s.$

For any $k\geq k_0$, we obtain that

$$\begin{aligned}V_0(X(t\wedge\tau_k)) &\leq V_0(X(0))+\int_0^{t\wedge\tau_k}\Big[\mu_1+\mu_2+\gamma+\mu_3+\nu+\beta\int_0^h f(\tau)G(I(s-\tau))d\tau\Big]ds\\ &\quad+\frac{\sigma^2}{2}\int_0^{t\wedge\tau_k}(I(s)^2+S(s)^2)\left(\int_0^h f(\tau)G(I(s-\tau))d\tau\right)^2 ds\\ &\quad+\sigma\int_0^{t\wedge\tau_k}(S(s)-I(s))\left(\int_0^h f(\tau)G(I(s-\tau))d\tau\right)dW(s).\end{aligned}$$

Let us put $K(z)=f(s-z)$ for all $z\in[s-h,s]$. Then in view of (H_3) and (H_4), for any $s\in[0,t\wedge\tau_k)$ we have

$$\int_0^h f(\tau)G(I(s-\tau))d\tau=\int_{s-h}^s f(s-z)G(I(z))dz\leq\frac{\Lambda}{\mu_1}\int_{s-h}^s K(z)dz=\frac{\Lambda}{\mu_1}\quad a.s. \tag{10}$$

So, we get

$$V_0(X(t\wedge\tau_k))\leq V_0(X(0))+C_0t\wedge\tau_k+\sigma\int_0^{t\wedge\tau_k}(S(s)-I(s))\int_0^h f(\tau)G(I(s-\tau))d\tau dW(s), \tag{11}$$

where $C_0=3\bar{\mu}+\gamma+\nu+\beta\frac{\Lambda}{\mu_1}+\sigma^2\left(\frac{\Lambda}{\mu_1}\right)^4$ and $\bar{\mu}=\max\{\mu_2,\mu_3\}$.

On the other hand we have

$$\mathbb{E}\Big[\int_0^{t\wedge\tau_k}(S(s)-I(s))\Big(\int_0^h f(\tau)G(I(s-\tau))d\tau\Big)dW(s)\Big]=0.$$

Since, one sees from (10) that for every $s\in[0,t\wedge\tau_k)$,

$$(S(s)-I(s))\int_0^h f(\tau)G(I(s-\tau))d\tau\leq\Big(\frac{\Lambda}{\mu_1}\Big)^2\quad \text{a.s.}$$

It follows that for any $t\geq 0$

$$\mathbb{E}[V_0(X(t\wedge\tau_k))]\leq C_0t\wedge\tau_k+V_0(X(0))\leq C_0t+V_0(X(0)). \tag{12}$$

Now, since $S(t\wedge\tau_k), I(t\wedge\tau_k), R(t\wedge\tau_k)$ are in $(\frac{1}{k}, \frac{\Lambda}{\mu_1})$, we have $V_0(X(t\wedge\tau_k)) > 0$.

Therefore, we get

$$\begin{aligned}\mathbb{E}[V_0(X(t\wedge\tau_k))] &= \mathbb{E}[V_0(X(t\wedge\tau_k))1_{\{\tau_k\le t\}}] + \mathbb{E}[V_0(X(t\wedge\tau_k))1_{\{\tau_k> t\}}]\\ &\ge \mathbb{E}[U(X(t\wedge\tau_k))1_{\{\tau_k\le t\}}].\end{aligned}$$

In view of Lemma 1, for all $\phi\in\mathscr{H}^3(\Delta)$, we see that $\min\{S(t\wedge\tau_k), I(t\wedge\tau_k), R(t\wedge\tau_k)\} < \frac{\Lambda}{\mu_1}$.
It follows that $\min\{S(\tau_k), I(\tau_k), R(\tau_k)\} = \frac{1}{k}$ on $\{\tau_k\le t\}$ which implies that
$U(X(\tau_k)) \ge -\ln\left(\frac{\mu_1}{\Lambda k}\right)$.
Hence

$$\mathbb{E}[V_0(X(t\wedge\tau_k))] \ge \mathbb{E}[V_0(X(t\wedge\tau_k))1_{\{\tau_k\le t\}}] \ge -\ln\left(\frac{\mu_1}{\Lambda k}\right)\mathbb{P}(\tau_k\le t). \tag{13}$$

Combining (12) and (13), for any $t\ge 0$,we get that

$$\mathbb{P}(\tau_k\le t) \le \frac{C_0 t + V_0(X(0))}{\ln\left(\frac{\Lambda k}{\mu_1}\right)}.$$

By letting $k\to\infty$, we obtain for any $t\ge 0$, $P(\tau_0\le t) = 0$. Consequently $P(\tau_0=\infty)=1$. Now, since $\tau_e=\tau_0$ a.s., we obtain that $\tau_e=\infty$ a.s. □

4. STABILITY ANALYSIS AND ASYMPTOTIC BEHAVIOUR

In this section, we study the stability of the disease-free equilibrium $E^0 = (\frac{\Lambda}{\mu_1}, 0, 0)$ of the model (4) which can also be reduced to the stability analysis of the trivial solution $y_0 = (0,0,0)$ of a new system. Indeed, let put $y_1(t) = S(t) - s_0$, $y_2(t) = I(t)$, $y_3(t) = R(t)$, $\forall t\ge 0$ with $s_0 = \frac{\Lambda}{\mu_1}$. Then by virtue of Itô's formula, we get the following system

$$\left\{\begin{aligned} dy_1(t) &= \Big[-\mu_1 y_1(t) - \beta y_1(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau - \beta s_0\int_0^h f(\tau)G(y_2(t-\tau))d\tau + \nu y_3(t)\Big]dt\\ &\quad -\sigma y_2(t)(y_1(t)+s^0)\Big(\int_0^h f(\tau)G(y_2(t-\tau))d\tau\Big)dW(t),\\ dy_2(t) &= \Big[\beta y_1(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau + \beta s_0\int_0^h f(\tau)G(y_2(t-\tau))d\tau - (\mu_2+\gamma)y_2(t)\Big]dt\\ &\quad +\sigma y_2(t)(y_1(t)+s_0)\Big(\int_0^h f(\tau)G(y_2(t-\tau))d\tau\Big)dW(t),\\ dy_3(t) &= \Big[\gamma y_2(t) - (\mu_3+\nu)y_3(t)\Big]dt,\end{aligned}\right. \tag{14}$$

with initial condition

$$\left\{\begin{aligned} &y_1(\theta)=\psi_1(\theta),\quad y_2(\theta)=\psi_2(\theta),\quad y_3(\theta)=\psi_3(\theta),\\ &\psi_i(0)>0,\quad i=1,2,3;\quad \psi=(\psi_1,\psi_2,\psi_3)\in\mathscr{H}^3(\Gamma),\end{aligned}\right.$$

where $\mathscr{H}^3(\Gamma)$ is the $\mathscr{F}_0$-measurable $\mathscr{C}([-h,0];\Gamma)$-valued random variable such that,

$$\forall\psi \in \mathscr{H}^3(\Gamma), \quad \mathbb{E}(||\psi||^2) < \infty$$

$$\text{where} \quad \Gamma = \Big\{(x,y,z) \in \mathbb{R}^3 / -\frac{\Lambda}{\mu_1} < x \leq 0, \quad 0 < y, \quad 0 < z, \text{ and } x+y+z \leq 0\Big\}.$$

Remark 2. The assertion $\varphi \in \mathscr{H}^3(\Delta)$ is equivalent to $\psi = \varphi - E^0 \in \mathscr{H}^3(\Gamma)$ and the assertion $X(t) = (S(t), I(t), R(t)) \in \Delta$ is equivalent to $y(t) = X(t) - E^0 \in \Gamma, t \geq 0$. So by virtue of Theorem 2, for any initial condition $\psi = (\psi_1, \psi_2, \psi_3) \in \mathscr{H}^3(\Gamma)$, the system (14) admits a unique solution $\{(y_1(t), y_2(t), y_3(t)) : t \geq 0\}$, and this solution remains in Γ with probability one.

Theorem 3. *For any initial condition $\psi = (\psi_1, \psi_2, \psi_3) \in \mathscr{H}^3(\Gamma)$, the trivial solution $y_0 = (0,0,0)$ of the system* (14) *is almost surely exponentially stable in $\mathscr{H}^3(\Gamma)$ under the assumptions*

$$(i) \quad 2\mu_1 > \max\{\gamma - \nu, s_0\beta + \nu\}, \qquad (ii) \quad s_0\beta < \mu_2 + \gamma - \frac{1}{2}\sigma^2 s_0^4. \tag{15}$$

In particular, if $G(x) < 1$ for all $x \in [0,\infty)$, the trivial solution $y_0 = (0,0,0)$ remains almost surely exponentially stable in $\mathscr{H}^3(\Gamma)$ by replacing the assumption (ii) with $s_0\beta < \mu_2 + \gamma - \frac{1}{2}(s_0\sigma)^2$.

Proof. In view of Remark 2, for any $\psi = (\psi_1, \psi_2, \psi_3) \in \mathscr{H}^3(\Gamma)$, we see that

$$-\frac{\Lambda}{\mu_1} < \psi_1(\theta) \leq 0, \quad 0 < \psi_2(\theta) \leq \frac{\Lambda}{\mu_1} \quad \text{and} \quad 0 < \psi_3(\theta) \leq \frac{\Lambda}{\mu_1}, \quad \text{for all } \theta \in [-h,0]. \tag{16}$$

Let us consider the Lyapunov functional $V \in D$, defined by

$$V : [0,\infty) \times \mathscr{H}^3(\Gamma) \to \mathbb{R}_+, \quad (t,\psi) \mapsto V(t,\psi) = V_1(\psi) + V_2(t,\psi) \text{ with}$$

$$\begin{aligned} V_1(\psi) &= \psi_1(0)^2 + \lambda_1\psi_2(0)^2 + \psi_3(0)^2, \\ V_2(t,\psi) &= \lambda_2 \int_0^h f(\tau) \int_{-\tau}^0 G^2(\psi_2(s)) d\tau ds, \end{aligned}$$

where λ_1 and λ_2 are tow positive constants and $\psi(\theta) = y_t(\theta) = y(t+\theta), \theta \in [-h,0]$.

Otherwise $\quad V_1(y_t) = y_1(t)^2 + \lambda_1 y_2(t)^2 + y_3(t)^2, \quad V_2(t,y_t) = \lambda_2 \int_0^h f(\tau) \int_{t-\tau}^t G^2(y_2(u)) d\tau du.$

By using (7), we get

$$\mathscr{L}V_1(\psi)$$

$$\begin{aligned}
&= -2\beta s_0 y_1(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau - 2\beta y_1^2(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau - 2\mu_1 y_1^2(t)\\
&\quad +2\nu y_1(t)y_3(t) + 2\lambda_1\beta s_0 y_2(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau + 2\lambda_1\beta y_1(t)y_2(t)\int_0^h f(\tau)G(z_2(t-\tau))d\tau\\
&\quad -2\lambda_1(\mu_2+\gamma)y_2^2(t) + 2\gamma y_2(t)y_3(t) - 2(\mu_3+\nu)y_3^2(t)\\
&\quad +(1+\lambda_1)\sigma^2 y_2^2(t)(y_1(t)+s_0)^2\Big(\int_0^h f(\tau)G(y_2(t-\tau))d\tau\Big)^2.
\end{aligned}$$

By virtue of (H_4) and Hölder's inequality, we have

$$\begin{aligned}
\mathscr{L}V_1(\psi) &\leq \beta s_0 y_1^2(t) + \beta s_0\int_0^h f(\tau)G^2(y_2(t-\tau))d\tau - 2\mu_1 y_1^2(t) + \nu y_1^2(t) + \nu y_3^2(t) + \lambda_1\beta s_0 y_2^2(t)\\
&\quad +\lambda_1\beta s_0\int_0^h f(\tau)G^2(y_2(t-\tau))d\tau - 2\lambda_1(\mu_2+\gamma)y_2^2(t) + \gamma y_2^2(t) + \gamma y_3^2(t) - 2(\mu_3+\nu)y_3^2(t)\\
&\quad -2\beta y_1^2(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau + 2\lambda_1\beta y_1(t)y_2(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau\\
&\quad +(1+\lambda_1)\sigma^2 y_2^2(t)(y_1(t)+s_0)^2\int_0^h f(\tau)G^2(y_2(t-\tau))d\tau.
\end{aligned}$$

By virtue of (16), (H_3) and (H_4) we have

$$\begin{aligned}
y_1(t)y_2(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau &= \psi_1(0)\psi_2(0)\int_{-h}^0 f(-\theta)G(\psi_2(\theta))d\theta \leq 0\\
-2\beta y_1^2(t)\int_0^h f(\tau)G(y_2(t-\tau))d\tau &= -2\beta\psi_1(0)^2\int_{-h}^0 f(-\theta)G(\psi_2(\theta))d\theta \leq 0\\
y_2^2(t)(y_1(t)+s_0)^2 &= \psi_2^2(0)(\psi_1(0)+s_0)^2 \leq s_0^4.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{L}V_1(\psi) &\leq (\beta s_0 - 2\mu_1 + \nu)y_1^2(t) + ((\beta s_0 - 2\mu_2 - 2\gamma)\lambda_1 + \gamma)y_2^2(t) - (2\mu_3 - \gamma + \nu)y_3^2(t)\\
&\quad + s_0(\beta + s_0^3\sigma^2)(1+\lambda_1)\int_0^h f(\tau)G^2(y_2(t-\tau))d\tau. \qquad (17)
\end{aligned}$$

Let us put $\lambda_2 = (1+\lambda_1)s_0(\beta+\sigma^2 s_0^3)$ and $\lambda_1 = \frac{4\gamma+2\mu_2+\sigma^2 s_0^4}{2(2\gamma+2\mu_2-\sigma^2 s_0^4-2\beta s_0)}$. We have

$$\begin{aligned}
\mathscr{L}V_2(t,\psi) &= (1+\lambda_1)s_0(\beta+\sigma^2 s_0^3)\Big[G^2(y_2(t)) - \int_0^h f(\tau)G^2(y_2(t-\tau))d\tau\Big]\\
&\leq (1+\lambda_1)s_0(\beta+\sigma^2 s_0^3)\Big[y_2^2(t) - \int_0^h f(\tau)G^2(y_2(t-\tau))d\tau\Big]. \qquad (18)
\end{aligned}$$

Now, it follows from (17) and (18) that

$$\mathscr{L}V(t,\psi) \leq -\Big(2\mu_1 - \beta s_0 - \nu\Big)y_1^2(t) + \Big[(2\beta s_0 - 2\mu_2 - 2\gamma + \sigma^2 s_0^4)\lambda_1 + \gamma + \beta s_0 + \sigma^2 s_0^4\Big]y_2^2(t)$$

$$
\begin{aligned}
& -\Big(2\mu_3-\gamma+\nu\Big)y_3^2(t)\\
\leq\;& -\Big(2\mu_1-s_0\beta-\nu\Big)y_1^2(t)-\Big(\gamma+\mu_2-\frac{1}{2}\sigma^2 s_0^4-s_0\beta\Big)y_2^2(t)-\Big(2\mu_3-\gamma+\nu\Big)y_3^2(t).
\end{aligned}
$$

Finally, we have

$$\mathscr{L}V(t,\psi)\leq -C_1|\psi(0)|^2,$$

where

$$C_1=\min\{2\mu_1-\nu-s_0\beta,\quad \mu_2+\gamma-\frac{1}{2}\sigma^2 s_0^4-s_0\beta,\quad 2\mu_3-\gamma+\nu\}.$$

On the other hand

$$c_1|\varphi(0)|^2\leq V(t,\varphi)\leq c_2||\varphi||^2,$$

with $c_1=\min\{1,\quad \lambda_1,\quad s_0(\beta+\sigma^2 s_0^3)\}$ and $c_2=\max\{1,\quad \lambda_1,\quad s_0(\beta+\sigma^2 s_0^3)(1+\lambda_1)\}$.

Now, if (15) is satisfied, then by Theorem 1, the trivial solution $y_0=(0,0,0)$ of the system (14) is almost surely exponentially stable in $\mathscr{H}^3(\Gamma)$.

In particular, if $G(x)<1$ for all $x\in[0,\infty)$, then in view of (16) we have

$$(1+\lambda_1)\sigma^2 y_2^2(t)(y_1(t)+s_0)^2\int_0^h f(\tau)G^2(y_2(t-\tau))d\tau\leq(1+\lambda_1)s_0^2\sigma^2 y_2^2(t).$$

Therefore, in this case the relation (17) becomes

$$
\begin{aligned}
\mathscr{L}V_1(\psi)\;\leq\;& -\Big(2\mu_1-s_0\beta-\nu\Big)y_1^2(t)+\Big((s_0\beta-2\mu_2-2\gamma+s^2\sigma^2)\lambda_1+\gamma+s_0^2\sigma^2\Big)y_2^2(t)\\
& -\Big(2\mu_3-\gamma+\nu\Big)y_3^2(t)+(1+\lambda_1)s_0\beta\int_0^h f(\tau)G^2(y_2(t-\tau))d\tau.
\end{aligned}\tag{19}
$$

Now, by taking $\lambda_2=(1+\lambda_1)s_0\beta$, we obtain that

$$
\begin{aligned}
\mathscr{L}V_2(t,\psi)\;=\;& (1+\lambda_1)s_0\beta\Big[G^2(y_2(t))-\int_0^h f(\tau)G^2(y_2(t-\tau))d\tau\Big]\\
\leq\;& (1+\lambda_1)s_0\beta\Big[y_2^2(t)-\int_0^h f(\tau)G^2(y_2(t-\tau))d\tau\Big].
\end{aligned}\tag{20}
$$

By combining (19) and (20), we get

$$
\begin{aligned}
\mathscr{L}V(t,\psi)\;\leq\;& -\Big(2\mu_1-\beta s_0-\nu\Big)y_1^2(t)+\Big((2\beta s_0-2\mu_2-2\gamma+s_0^2\sigma^2)\lambda_1+\gamma+s_0\beta+s_0^2\sigma^2\Big)y_2^2(t)\\
& -\Big(2\mu_3-\gamma+\nu\Big)y_3^2(t)\\
\leq\;& -\Big(2\mu_1-s_0\beta-\nu\Big)y_1^2(t)-\Big(\gamma+\mu_2-\frac{1}{2}(s_0\sigma)^2-s_0\beta\Big)y_2^2(t)-\Big(2\mu_3-\gamma+\nu\Big)y_3^2(t).
\end{aligned}
$$

Therefore

$$\mathscr{L}V(t,\psi) \leq -C_2|\psi(0)|^2,$$

where

$$C_2 = \min\{2\mu_1 - \nu - s_0\beta, \quad \mu_2 + \gamma - \frac{1}{2}(s_0\sigma)^2 - s_0\beta, \quad 2\mu_3 - \gamma + \nu\}.$$

Hence, if assumptions (15) with condition (ii) replaced with $s_0\beta < \mu_2 + \gamma - \frac{1}{2}(s_0\sigma)^2$ is satisfied, then the almost sure exponential stability in $\mathscr{H}^3(\Gamma)$ of the trivial solution $y_0 = (0,0,0)$ of the system (14) follows from Theorem 1 □

Since the stability of the disease-free equilibrium $E^0 = (\frac{\Lambda}{\mu_1}, 0, 0)$ of the model (4) is equivalent to the stability of the trivial solution $y_0 = (0,0,0)$ of the system (14), we get the following corollary.

Corollary 1. *Under the assumptions* (15) *of Theorem 3, the disease-free equilibrium* E^0 *of the model* (4) *is almost surely exponentially stable for any initial condition* $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{H}^3(\Delta)$. *In particular, if* $G(x) < 1$ *for all* $x \in [0,\infty)$, *the trivial solution* $y_0 = (0,0,0)$ *remains almost surely exponentially stable in* $\mathscr{H}^3(\Gamma)$ *by replacing the assumption* (ii) *with* $s_0\beta < \mu_2 + \gamma - \frac{1}{2}(s_0\sigma)^2$.

Remark 3. In Figure 2, without condition (i) of Theorem 3, we draw three sample paths of the stochastic model (4) for three different values of $\sigma(0.03, 0.044, 0.052)$. This figure suggest that only the assumption (ii) of Theorem 3 is sufficient to ensure the asymptotic stability of the disease-free equilibrium E^0 of the stochastic model (4).

Now, we establish a stability result for the disease-free equilibrium E^0 of the model (4) in accordance with the observation of Remark 2. This is done by combining a stochastic Lyapunov technique, variation of constants approach and martingale convergence theory (see [29, 30]).

Theorem 4. *Let* $s_0\beta < \mu_2 + \gamma - \frac{1}{2}s_0^4\sigma^2$, *then the disease-free equilibrium* $E^0 = (\frac{\Lambda}{\mu_1}, 0, 0)$ *of model (4) is globally asymptotically almost surely stable for any initial condition* $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{H}^3(\Delta)$. *Moreover, if* $G(x) < 1$ *for all* $x \in [0,\infty)$, *the disease-free equilibrium of the system* (4) *is globally asymptotically almost surely stable under the condition* $s_0\beta < \mu_2 + \gamma - \frac{1}{2}(s_0\sigma)^2$.

The proof of this Theorem requires the useful nonnegative semimartingale convergence result established in Lipster and Shiryayev ([31] Theorem 7, p.139).

Lemma 2. *Let A_1 and A_2 be two continuous adapted increasing processes on $t \geq 0$ with $A_1(0) = A_2(0) = 0$ a.s. Let M be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let Z be a nonnegative measurable random variable such that $\mathbb{E}(Z) < \infty$. Define*

$$X(t) = Z + A_1(t) - A_2(t) + M(t) \quad \text{for } t \geq 0.$$

If X is nonnegative, then

$$\left\{ \lim_{t\to\infty} A_1(t) < \infty \right\} \subset \left\{ \lim_{t\to\infty} X(t) < \infty \right\} \cap \left\{ \lim_{t\to\infty} A_2(t) < \infty \right\} \quad a.s.,$$

where $E \subset F$ a.s., means $P(E \cap F^c) = 0$.

In particular, if $\lim_{t\to\infty} A_1(t) < \infty$ a.s., then

$$\lim_{t\to\infty} X(t) < \infty, \lim_{t\to\infty} A_2(t) < \infty \quad \text{and} \quad \lim_{t\to\infty} \mid M(t) \mid < \infty \quad a.s$$

That is, all of the processes X, A_2, and M converge to finite random variables.

Proof of Theorem 4. We will first prove separately the asymptotic stability for every component of the solution $(I(t), R(t), S(t)), t \geq 0$ of the system (4) and then conclude.
For any $(x_1, x_2, x_3) \in \mathbb{R}^3$, let us put $Pr_2(x_1, x_2, x_3) = x_2$. So, $Pr_2 \circ \varphi = \varphi_2$. Since we can rewrite $S(t) = N(t) - R(t) - I(t) = H(t, I(t))$ as a continuous function of t and $I(t)$ which is bounded by $s_0 = \frac{\Lambda}{\mu_1}$ according to Theorem 2. It follows that the infectious size I(t) of the model (4) is described by the following equation

$$\begin{aligned} dI(t) &= \Big(\beta H(t, I(t)) \int_0^h f(\tau) G(I(t-\tau)) d\tau - (\mu_2 + \gamma) I(t)\Big) dt \\ &\quad + \sigma H(t, I(t)) I(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau dW(t). \end{aligned} \tag{21}$$

with initial condition $Pr_2 \circ \varphi = \varphi_2$ for any $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{H}^3(\Delta)$.
Let us consider the functional $\bar{V}(Pr_2 \circ \varphi) = \bar{V}_1(Pr_2 \circ \varphi) + \bar{V}_2(t, Pr_2 \circ \varphi)$, for all $\varphi \in \mathscr{H}^3(\Delta)$, where $\bar{V}_1(\varphi_2) = |\varphi_2(0)|^2$, $\bar{V}_2(t, \varphi_2) = \lambda \int_{-h}^0 f(-\theta) G(\varphi_2(\theta)) d\theta$ and $\varphi_2(\theta) = I(t+\theta)$.

In view of Theorem 2, for any φ_2 bounded by $\frac{\Lambda}{\mu_1}$ on $[-h,0]$, we get that

$$\begin{aligned}\mathscr{L}\bar{V}_1(\varphi_2) &= 2\beta H(t,I(t))I(t)\int_0^h f(\tau)G(I(t-\tau))d\tau - 2(\mu_2+\gamma)I(t)^2 \\ &\quad +\sigma^2\Big(H(t,I(t))I(t)\int_0^h f(\tau)G(I(t-\tau))d\tau\Big)^2.\end{aligned}$$

By virtue of (H_4) and Hölder's inequality, we obtain

$$\begin{aligned}\mathscr{L}\bar{V}_1(\varphi_2) &\leq 2\beta s_0 I(t)\int_0^h f(\tau)G(I(t-\tau))d\tau - 2(\mu_2+\gamma)I(t)^2 + s_0^4\sigma^2\Big(\int_0^h f(\tau)G(I(t-\tau))d\tau\Big)^2, \\ &\leq \beta s_0 I(t)^2 + \beta s_0\int_0^h f(\tau)G^2(I(t-\tau))d\tau - 2(\mu_2+\gamma)I(t)^2 + s_0^4\sigma^2\int_0^h f(\tau)G^2(I(t-\tau))d\tau \\ &\leq \Big(\beta s_0 - 2(\mu_2+\gamma)\Big)I(t)^2 + \Big(\beta s_0 + s_0^4\sigma^2\Big)\int_0^h f(\tau)G^2(I(t-\tau))d\tau.\end{aligned}$$

Let us put $\lambda = s_0(\beta+\sigma^2 s_0^3)$. In view of (H_3), we have

$$\begin{aligned}\mathscr{L}\bar{V}_2(t,\varphi_2) &= s_0(\beta+\sigma^2 s_0^3)\Big[G^2(I(t)) - \int_0^h f(\tau)G^2(I(t-\tau))d\tau\Big] \\ &\leq s_0(\beta+\sigma^2 s_0^3)\Big[I(t)^2 - \int_0^h f(\tau)G^2(I(t-\tau))d\tau\Big].\end{aligned}$$

Finally, we get

$$\begin{aligned}\mathscr{L}\bar{V}(t,\varphi_2) &\leq \Big(2\beta s_0 - 2(\mu_2+\gamma) + s_0^4\sigma^2\Big)I(t)^2 \\ &\leq -2\Big((\mu_2+\gamma) - \frac{1}{2}\sigma^2 s_0^4 - \beta s_0\Big)|\varphi_2(0)|^2.\end{aligned}$$

Moreover, if $G(x) < 1$ for all $x \in [0,\infty)$, by using the same approach as in the proof of Theorem 3, we get

$$\mathscr{L}\bar{V}(t,\varphi_2) \leq -2\Big((\mu_2+\gamma) - \frac{1}{2}(s_0\sigma)^2 - \beta s_0\Big)|\varphi_2(0)|^2.$$

Then, by virtue of Theorem 1, when $s_0\beta < \mu_2+\gamma-\frac{1}{2}\sigma^2 s_0^4$ or $s_0\beta < \mu_2+\gamma-\frac{1}{2}(s_0\sigma)^2$ and if $G(x) < 1$ for any $x \in [0,\infty)$, we have $\lim_{t\to\infty}\frac{1}{t}\ln(I(t)) < -p$, where p is a positive constant. That is, there exists two positive constants p_1 and p_2 such that

$$I(t) < p_1\exp(-p_2 t) \text{ for any } t \geq 0. \tag{22}$$

Now, consider the third equation of the model (4). From the well-know variation of constants approach, we obtain

$$R(t) = R(0)e^{-(\mu_3+\nu)t} + \gamma \int_0^t I(s) e^{(\mu_3+\nu)(s-t)} ds.$$

In view of (22), we get for all $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that for any $t > T(\varepsilon)$, $I(t) < \varepsilon \frac{\mu_3+\nu}{\gamma}$

$$\begin{aligned} R(t) &= R(0)e^{-(\mu_3+\nu)t} + \gamma \int_0^t I(s) e^{(\mu_3+\nu)(s-t)} ds \\ &\leq (R(0)-\varepsilon)e^{-(\mu_3+\nu)t} + \varepsilon, \end{aligned} \tag{23}$$

By letting $\varepsilon \to 0$, we have $\lim_{t\to\infty} R(t) \leq \lim_{t\to\infty} R(0)e^{-(\mu_3+\nu)t}$.

Hence

$$\lim_{t\to\infty} R(t) = 0 \quad a.s$$

Let us now prove that $\lim_{t\to\infty}\Big(\frac{\Lambda}{\mu_1} - S(t)\Big) = 0$. From the first equation of the model (4), we get

$$\begin{aligned} \frac{\Lambda}{\mu_1} - S(t) &= \frac{\Lambda}{\mu_1} - S(0) + \beta \int_0^t S(s) \int_0^h f(\tau) G(I(s-\tau)) d\tau ds - \int_0^t \Big[\mu_1\Big(\frac{\Lambda}{\mu_1} - S(s)\Big) + \nu R(s)\Big] ds \\ &\quad + \sigma \int_0^t S(s) I(s) \Big(\int_0^h f(\tau) G(I(s-\tau)) d\tau\Big) dW(s). \end{aligned}$$

In view of Theorem 2, Hölder inequality and (22), we obtain

$$\begin{aligned} \lim_{t\to\infty} \beta \int_0^t S(s) \int_0^h f(\tau) G(I(s-\tau)) d\tau ds &\leq \lim_{t\to\infty} \beta \int_0^t S(s) \int_{s-h}^s f(s-u) I(u) du ds \\ &\leq \lim_{t\to\infty} h\beta \int_0^t S(s) \times \sup_{u\in[s-h,s]} I(u) ds \\ &\leq h p_1 \beta s_0 \exp(p_2 h) \Big(\lim_{t\to\infty} \int_0^t \exp(-C_2 s) ds\Big) < \infty. \end{aligned}$$

Therefore, by virtue of Lemma 2, we get

$$\lim_{t\to\infty}\Big(\frac{\Lambda}{\mu_1} - S(t)\Big) < \infty \quad \text{and} \quad \lim_{t\to\infty} \int_0^t \Big[\mu_1\Big(\frac{\Lambda}{\mu_1} - S(s)\Big) + \nu R(s)\Big] ds < \infty \quad a.s.$$

In accordance with Theorem 2, $R(t)$ and $\frac{\Lambda}{\mu_1} - S(s)$ are positives forll all $t \geq 0$, we get

$$\lim_{t\to\infty} \int_0^t \Big(\frac{\Lambda}{\mu_1} - S(s)\Big) ds = \int_0^\infty \Big(\frac{\Lambda}{\mu_1} - S(s)\Big) ds < \infty. \tag{24}$$

Assume that $\frac{\Lambda}{\mu_1} - S(s)$ does not converge almost surely to 0. Then there is a set $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that for all $\omega \in \Omega_1$,

$$\liminf_{t\to\infty}\left(\tfrac{\Lambda}{\mu_1}-S(t,\omega)\right)=\tau(\omega)>0.$$

Then, there exists a $T>0$ such that $\frac{\Lambda}{\mu_1}-S(t,\omega)>\frac{1}{2}\tau(\omega)$ for all $t\geq T$. It follows that

$$\begin{aligned}\lim_{t\to\infty}\int_0^t\left(\frac{\Lambda}{\mu_1}-S(s,\omega)\right)ds &= \int_0^T\left(\frac{\Lambda}{\mu_1}-S(s,\omega)\right)ds+\int_T^\infty\left(\frac{\Lambda}{\mu_1}-S(s,\omega)\right)ds\\ &\geq \int_T^\infty\left(\frac{\Lambda}{\mu_1}-S(s,\omega)\right)ds=\infty.\end{aligned}$$

Therefore, $\Omega_1\subset\Omega_2$, where $\Omega_2=\left\{\omega,\int_T^\infty\left(\frac{\Lambda}{\mu_1}-S(s,\omega)\right)ds=\infty\right\}$. Hence $P(\Omega_2)>0$, which contradicts (24). So, we have

$$\lim_{t\to\infty}\left(\frac{\Lambda}{\mu_1}-S(t)\right)=0\quad a.s.$$

Finally, we have proved that, when $t\to\infty$, $(S(t),I(t),R(t))\to(\frac{\Lambda}{\mu_1},0,0)\quad a.s.$ □

5. Persistence and Asymptotic Behaviour When $R_0>1$

The stochastic model (4) obtained from the deterministic system (1) has a single equilibrium position which is the disease-free equilibrium E^0. However, even if the endemic equilibrium E^* of the deterministic model is not an equilibrium state of the stochastic model, it is interesting to study the asymptotic behaviour of the solution of the stochastic model around this point. In the following results, we discuss the persistence of the epidemic and the asymptotic behaviour of the solution of the stochastic model (4) around the endemic equilibrium E^* of the deterministic model (1) when $R_0>1$.

Theorem 5. *If the disease-free equilibrium E^0 of the model* (4) *is unstable in $\mathscr{H}^3(\Delta)$, then for any initial condition $\varphi\in\mathscr{H}^3(\Delta)$, the solution of the system* (6) *is persistent with probability one, that is there exists a constant $\xi\in(0,\frac{\Lambda}{\mu_1})$ such that,*

$$\liminf_{t\to\infty}S(t)\geq\xi,\quad \liminf_{t\to\infty}I(t)\geq\xi,\quad \liminf_{t\to\infty}R(t)\geq\xi.$$

Proof. Le us assume that for any initiale condition $\varphi\in\mathscr{H}^3(\Delta)$ the DFE E^0 of the system (4) is unstable and the trivial solution of the equation (21) describing the infectious size I(t) with initiale condition $\varphi_2=Pr_2(\varphi)$ is stable.

It follows that

$\forall \varepsilon > 0$, $\exists \eta(\varepsilon) > 0$ and $\exists T(\varepsilon) > 0$ such that $\|I(t)\| < \varepsilon \dfrac{\mu_3+\nu}{\gamma}$, $\forall t \geq T(\varepsilon)$ and $\| Pr_2(\varphi) \| < \eta(\varepsilon)$.

From (23) we obtain

$$R(t) \leq (R(0)-\varepsilon)e^{-(\mu_3+\nu)t} + \varepsilon,$$

By letting $\varepsilon \to 0$, for any $t \geq 0$, we have $\lim_{t\to\infty} R(t) \leq \lim_{t\to\infty} R(0)e^{-(\mu_3+\nu)t}$.

Hence

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} I(t) = 0 \quad a.s. \tag{25}$$

Since $\mu_1 \leq \min\{\mu_2, \mu_3\}$, the size of the whole population $N(t)$ in the model (4) verifies

$$N(t) = e^{-\mu_1 t}\Big(N(0) + \int_0^t e^{\mu_1 s}(\Lambda - \alpha_1 I(s) + \alpha_2 R(s))ds\Big), \quad \text{where } \alpha_1 = \mu_2 - \mu_1 \text{ and } \alpha_2 = \mu_3 - \mu_1.$$

In view of (25) for any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that for any $t > T(\varepsilon)$, $I(t) < \varepsilon$ and $R(t) < \varepsilon$, we have

$$N(t) \geq N(0)e^{-\mu_1 t} + \frac{\Lambda - (\alpha_1+\alpha_2)\varepsilon}{\mu_1}\Big(1 - e^{-\mu_1 t}\Big).$$

By letting $\varepsilon \to 0$, we obtain

$$\liminf_{t\to\infty} N(t) \geq \frac{\Lambda}{\mu_1} \quad a.s.\ \forall t > T(\varepsilon)$$

Since $N(t) = S(t) + I(t) + R(t)$, by virtue of Lemma 1 and (25), we have

$$\lim_{t\to\infty} N(t) = \frac{\Lambda}{\mu_1} \quad \text{and} \quad \lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu_1} \quad a.s.$$

So, the disease-free equilibrium E^0 is stable, which is a contradiction since by hypothesis the disease-free equilibrium E^0 is assumed to be unstable. Therefore, the trivial solution of the equation (21) describing the infectious size I(t) is unstable. Finally, there exists a constant $\xi > 0$ such that

$$\liminf_{t\to\infty} I(t) > \xi, \quad \liminf_{t\to\infty} R(t) > \xi \text{ and } \liminf_{t\to\infty} S(t) > \xi.$$

□

Combining with the Theorem 2, the Lemma 3 means that there exists three positives constants ξ_s, ξ_i, ξ_r, such that $\inf_{u\geq 0} S(u) = \xi_s$, $\inf_{u\geq 0} I(u) = \xi_i$, $\inf_{u\geq 0} R(u) = \xi_r$ and for all $t \geq 0$,

$$S^2(t) \leq (s_0 - \xi_i - \xi_r)^2 = \rho_1,\ I^2(t) \leq (s_0 - \xi_s - \xi_r)^2 = \rho_2 \text{ and}$$

$$I^2(t) + S^2(t) \leq (s_0 - \xi_r)^2 - 2\xi_i\xi_s = \rho_3. \tag{26}$$

Theorem 6. *Let $E^* = (s^*, i^*, r^*)$ be the endemic equilibrium of the deterministic model* (1)*. If $R_0 > 1$ and $\mu_1 s^* - \nu r^* \geq 0$, then for any initial condition $\varphi \in \mathscr{H}^3(\Delta)$, the solution of the model* (4) *under the assumption of the Theorem 5 has the property*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \Big[(S(s)-s^*)^2 + (I(s)-i^*)^2 + (x_3 - r^*)^2\Big] ds \leq \frac{\varpi\sigma^2 M^*}{\zeta}, \tag{27}$$

where

$$\begin{aligned} \varpi &> \max\Big\{\frac{\mu_1+\mu_2+\gamma+\frac{\mu_1^2}{\mu_2+\gamma}}{A}, \frac{2\nu}{(\mu_2+\gamma)B}, \frac{2\nu^2}{\mu_1 B}\Big\}, \\ \zeta &= \min\Big\{\varpi A - \Big(\mu_1+\mu_2+\gamma+\frac{\mu_1^2}{\mu_2+\gamma}\Big), \mu_2+\gamma-\frac{2\nu}{\varpi B}, \varpi\frac{B}{2} - \frac{\nu^2}{\mu_1}\Big\} \text{ and} \end{aligned}$$

$M^* = \frac{1}{2}(i^* c^2\rho_1 + s^* c^2 \rho_2)$ if $G(x) < c$ for all $x \in [0,\infty)$, or $M^* = \frac{1}{2}(i^*\rho_1\rho_2 + s^*\rho_2^2)$ otherwise, where c is a positive constant.

In order to investigate the proof of Theorem 6, we need the following lemma.

Lemma 3. *Let $X(t,\varphi)$ the solution of* (4) *for any $\varphi \in \mathscr{H}^3(\Delta)$. Define in D the followings functionals by*

$$\begin{aligned} V_{0,\varphi}^{E^*}(t,x) &= s^* g\Big(\frac{x_1}{s^*}\Big) + i^* g\Big(\frac{x_2}{i^*}\Big) + \int_0^h f(\tau)\int_{t-\tau}^t g\Big(\frac{G(\varphi_2(u))}{G(i^*)}\Big) du d\tau \\ V_{1,\varphi}^{E^*}(t,x) &= \frac{\nu}{2\gamma}(x_3 - r^*)^2 \\ V_{2,\varphi}^{E^*}(t,x) &= \frac{1}{2}C\Big[(x_1-s^*)+(x_2-i^*)+(x_3-r^*)+\frac{\mu_2-\mu_1}{2}(x_3-r^*)\Big]^2, \\ V_{3,\varphi}^{E^*}(t,x) &= \frac{\nu}{4\mu_1 s^*}\Big[(x_1-s^*)+(x_2-i^*)+(x_3-r^*)\Big]^2 \end{aligned}$$

and

$$V_\varphi^{E^*}(t,x) = \begin{cases} V_{0,\varphi}^{E^*}(t,x) + V_{1,\varphi}^{E^*}(t,x) + V_{2,\varphi}^{E^*}(t,x), & \textit{if either } \mu_1 < min\{\mu_2,\mu_3\} \\ V_{0,\varphi}^{E^*}(t,x) + V_{1,\varphi}^{E^*}(t,x) + V_{3,\varphi}^{E^*}(t,x), & \textit{if } \mu_1 = \mu_2 = \mu_3, \end{cases} \tag{28}$$

where $x_i = \varphi_i(0)$, $i = 1,2,3$; $g(x) = x - 1 - \ln(x) \geq 0$ *and* $C = \frac{\nu\gamma}{[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)]s^*}$.
Let $R_0 > 1$ *and* $\mu_1 s^* - \nu r^* \geq 0$, *then we have*

$$\mathscr{L}U_{\varphi}^{E^*}(t,X(t)) \leq -A(S(t)-s^*)^2 - B(R(t)-r^*)^2 + \sigma^2 M^*, \tag{29}$$

where

$$\begin{aligned} B &= \frac{\nu(\mu_2-\mu_1)[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_3+\nu)](r^*)^2}{\gamma[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)]s^*} + \frac{\eta(\mu_3+\gamma+\nu)(r^*)^2}{\gamma s^*}, \\ A &= \frac{\mu_1 s^* - \nu r^*}{s_0} \text{ if either } \mu_1 < min\{\mu_1,\mu_2\} \text{ and} \\ A &= \frac{\mu_1 s^* - \nu r^*}{s_0}, B = \frac{\nu(r^*)^2}{\gamma s^*}(\mu_1+\nu) \text{ if } \mu_1 = \mu_2 = \mu_3; \end{aligned}$$

Proof of Lemma 3. Let us put for $n^* = s^* + i^* + r^*$

$$s_t = \frac{S(t)}{s^*}, \quad i_t = \frac{I(t)}{i^*}, \quad r_t = \frac{R(t)}{r^*}, \quad \tilde{y}_{t,\tau} = \frac{G(I(t-\tau)}{G(i^*)}, \quad n_t = \frac{N(t)}{n^*}.$$

By virtue of (7), we get that for any $\varphi \in \mathscr{H}^3(\Delta)$ and $t \in [0,\infty)$

$$\mathscr{L}V_{0,\varphi}^{E^*}(t,X(t)) = \frac{\partial V_{0,\varphi}^{E^*}(t,X(t))}{\partial t} + \sum_{i=1}^{3} \frac{\partial V_{0,\varphi}^{E^*}(t,X(t))}{\partial x_i} a_i(t,\varphi) + \frac{1}{2} trace[b^T(t,\varphi)\nabla^2 V_{0,\varphi}^{E^*}(t,X(t))b(t,\varphi)]$$

where $b(t,\varphi)$ and $a(t,\varphi)$ are defined as in the Remark 1.

In view of the (26) we obtain that

$$\begin{aligned} \frac{1}{2} trace[b^T(t,\varphi)\nabla^2 V_{0,\varphi}^{E^*}(t,X(t))b(t,\varphi)] &= \frac{1}{2}\sigma^2\Big(s^* I^2(t) + i^* S^2(t)\Big)\Big(\int_0^h f(\tau)G(I(t-\tau))d\tau\Big)^2 \\ &\leq \frac{1}{2}\sigma^2(i^*\rho_1\rho_2 + s^*\rho_2^2). \end{aligned}$$

Moreover, if $G(x) < c$ for all $x \in [0,\infty)$, we have

$$\frac{1}{2} trace[b^T(t,\varphi)\nabla^2 V_{0,\varphi}^{E^*}(t,X(t))b(t,\varphi)] \leq \frac{1}{2}\sigma^2(i^* c^2\rho_1 + c^2 s^*\rho_2).$$

It follows that

$$\begin{aligned} \mathscr{L}V_{0,\varphi}^{E^*}(t,X(t)) &= \Big(1 - \frac{s^*}{S(t)}\Big)\Big(b - \mu_1 S(t) + \beta S(t)\int_0^h f(\tau)G(I(s-\tau))d\tau + \nu R(s)\Big) \\ &\quad + \Big(1 - \frac{i^*}{I(t)}\Big)\Big(\beta S(t)\int_0^h f(\tau)G(I(s-\tau))d\tau - (\mu_2+\gamma)I(t)\Big) \\ &\quad + \int_0^h f(\tau)(g(\tilde{y}_t) - g(\tilde{y}_{t,\tau}))d\tau + \sigma^2 M^* \end{aligned}$$

$$
\begin{aligned}
&= \Big(1-\frac{s^*}{S(t)}\Big)\Big(-\mu_1 s^*\Big(\frac{S(t)}{s^*}-1\Big)+\nu r^*\Big(\frac{R(t)}{r^*}-1\Big)\Big)\\
&\quad +\beta s^* G(i^*)\int_0^h f(\tau)\Big(1-\frac{s^*}{S(t)}\Big)\Big(1-\frac{S(t)}{s^*}\frac{G(I(s-\tau))}{G(i^*)}\Big)d\tau\\
&\quad +\beta s^* G(i^*)\int_0^h f(\tau)\Big(1-\frac{i^*}{I(t)}\Big)\Big(\frac{S(t)}{s^*}\frac{G(I(s-\tau))}{G(i^*)}-\frac{I(t)}{i^*}\Big)d\tau\\
&\quad +\int_0^h f(\tau)(g(\tilde{y}_t)-g(\tilde{y}_{t,\tau}))d\tau+\sigma^2 M^*\\
&= -\mu_1 s^*\Big(1-\frac{1}{s_t}\Big)(s_t-1)+\nu r^*\Big(1-\frac{1}{s_t}\Big)(r_t-1)+\int_0^h f(\tau)(g(\tilde{y}_t)-g(\tilde{y}_{t,\tau}))d\tau+\sigma^2 M^*\\
&\quad +\beta s^* G(i^*)\int_0^h f(\tau)\Big[\Big(1-\frac{1}{s_t}\Big)(1-s_t\tilde{y}_{t,\tau})d\tau+\Big(1-\frac{1}{i_t}\Big)\Big(s_t\tilde{y}_{t,\tau}-i_t\Big)\Big]d\tau,
\end{aligned}
$$

On the other hand, we observe that,

$$
\Big(1-\frac{1}{s_t}\Big)(1-s_t\tilde{y}_{t,\tau})d\tau+\Big(1-\frac{1}{i_t}\Big)\Big(s_t\tilde{y}_{t,\tau}-i_t\Big)=-g\Big(\frac{1}{s_t}\Big)-g\Big(\frac{s_t\tilde{y}_{t,\tau}}{i_t}\Big)-(g(i_t)-g(\tilde{y}_{t,\tau}))
$$

So, we get

$$
\begin{aligned}
\mathscr{L}V_{0,\varphi}^{E^*}(t,X(t)) &= -\mu_1 s^*\frac{(s_t-1)^2}{s_t}+\nu r^*\Big(1-\frac{1}{s_t}\Big)(r_t-1)\\
&\quad -\beta s^* G(i^*)\int_0^h f(\tau)\Big[g\Big(\frac{1}{s_t}\Big)+g\Big(\frac{s_t\tilde{y}_{t,\tau}}{i_t}\Big)+(g(i_t)-g(\tilde{y}_{t,\tau}))\Big]d\tau. \qquad (30)
\end{aligned}
$$

Similarly, the calculation of $\mathscr{L}V_{1,\varphi}^{E^*}(t,X(t))$, is given by

$$
\begin{aligned}
\mathscr{L}V_{1,\varphi}^{E^*}(t,X(t)) &= \frac{\nu}{\gamma s^*}(R(t)-r^*)[\gamma I(t)-(\mu_2+\nu)R(t)]\\
&= \frac{\nu}{\gamma s^*}(R(t)-r^*)[\gamma(N(t)-S(t)-R(t))-(\mu_2+\nu)R(t)]\\
&= \frac{\nu}{\gamma s^*}(R(t)-r^*)[\gamma(N(t)-n^*)-\gamma(S(t)-s^*)-(\mu_2+\gamma+\nu)(R(t)-r^*)]\\
&= \frac{\nu r^* n^*}{\gamma s^*}(r_t-1)(n_t-1)-\nu r^*(r_t-1)(s_t-1)-\frac{\nu(r^*)^2(\mu_2+\gamma+\nu)}{\gamma s^*}(r_t-1)^2. \qquad (31)
\end{aligned}
$$

The calculation of $\mathscr{L}V_{2,\varphi}^{E^*}(t,X(t))$ is obtained under the assumption $\mu_1\le\min\{\mu_2,\mu_3\}$ by

$$
\begin{aligned}
\mathscr{L}V_{2,\varphi}^{E^*}(t,X(t)) &= C\Big[(N(t)-n^*)+\frac{\mu_2-\mu_1}{\gamma}(R(t)-r^*)\Big]\\
&\quad \times\Big[b-\mu_1 S(t)-\mu_2 I(t)-\mu_3 R(t)+\frac{\mu_2-\mu_1}{\gamma}\Big(\gamma I(t)-(\mu_3+\nu)R(t)\Big)\Big]\\
&= C\Big[(N(t)-n^*)+\frac{\mu_2-\mu_1}{\gamma}(R(t)-r^*)\Big]\\
&\quad \times\Big[b-\mu_1(N(t)-I(t)-R(t))-\mu_2 I(t)-\mu_3 R(t)+\frac{\mu_2-\mu_1}{\gamma}\Big(\gamma I(t)-(\mu_3+\nu)R(t)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= C\Big[(N(t)-n^*)+\frac{\mu_2-\mu_1}{\gamma}(R(t)-r^*)\Big] \\
&\quad\times\Big[b-\mu_1 N(t)-(\mu_2-\mu_1)I(t)-(\mu_3-\mu_1)R(t)+\frac{\mu_2-\mu_1}{\gamma}\Big(\gamma I(t)-(\mu_3+\nu)R(t)\Big)\Big] \\
&= C\Big[(N(t)-n^*)+\frac{\mu_2-\mu_1}{\gamma}(R(t)-r^*)\Big] \\
&\quad\times\Big[b-\mu_1 N(t)-\Big((\mu_3-\mu_1)+\frac{(\mu_2-\mu_1)(\mu_3+\nu)}{\gamma}\Big)R(t)\Big].
\end{aligned}
$$

By the definition of the endemic equilibrium E^* of (1), we remark that

$$
b=\frac{(\mu_2-\mu_1)(\mu_3+\nu)}{\gamma}r^*+(\mu_3-\mu_1)r^*+\mu_1 n^*
$$

Therefore, we obtain

$$
\begin{aligned}
\mathscr{L}V_{2,\varphi}^{E^*}(t,X(t)) &= C\Big[(N(t)-n^*)+\frac{\mu_2-\mu_1}{\gamma}(R(t)-r^*)\Big] \\
&\quad\times\Big[-\mu_1(N(t)-n^*)-\Big((\mu_3-\mu_1)+\frac{(\mu_2-\mu_1)(\mu_3+\nu)}{\gamma}\Big)(R(t)-r^*)\Big] \\
&= -\mu_1 C(n^*)^2(n_t-1)^2-n^*r^*C\Big((\mu_3-\mu_1)+\frac{(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)}{\gamma}\Big)(n_t-1)(r_t-1) \\
&\quad-\frac{(r^*)^2C(\mu_2-\mu_1)}{\gamma}\Big((\mu_3-\mu_1)+\frac{(\mu_2-\mu_1)(\mu_3+\nu)}{\gamma}\Big)(r_t-1)^2 \\
&= -\frac{\mu_1\gamma\nu(n^*)^2}{[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)]s^*}(n_t-1)^2-\frac{\nu n^*r^*}{s^*}(n_t-1)(r_t-1) \\
&\quad-\frac{\nu(\mu_2-\mu_1)[(\mu_3-\mu_1)\gamma+(\mu_2-\mu_1)(\mu_3+\nu)](r^*)^2}{\gamma[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)]s^*}(r_t-1)^2 \qquad (32)
\end{aligned}
$$

For $\mu_1=\mu_2=\mu_3$, $\mathscr{L}V_{3,\varphi}^{E^*}(t,X(t))$ is given by

$$
\mathscr{L}V_{3,\varphi}^{E^*}(t,X(t)) = \frac{\nu}{4\mu_1 s^*}(N(t)-n^*)(b-\mu_1N(t))=\frac{\nu}{4\mu_1}(N(t)-n^*)^2=\frac{\nu(n^*)^2}{4\mu_1}(n_t-1)^2 \qquad (33)
$$

Therefore, by combining (30), (31) and (32) for $\mu_1\le\min\{\mu_2,\mu_3\}$, we get

$$
\begin{aligned}
\mathscr{L}V_{\varphi}^{E^*}(t,X(t)) &= -\mu_1 s^*\frac{(s_t-1)^2}{s_t}+\nu r^*\Big(1-\frac{1}{s_t}\Big)(r_t-1) \\
&\quad\frac{\nu r^*n^*}{\gamma s^*}(r_t-1)(n_t-1)-\nu r^*(r_t-1)(s_t-1)-\frac{\nu(r^*)^2(\mu_2+\gamma+\nu)}{\gamma s^*}(r_t-1)^2 \\
&\quad-\frac{\mu_1\gamma\nu(n^*)^2}{[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)]s^*}(n_t-1)^2-\frac{\nu n^*r^*}{s^*}(n_t-1)(r_t-1) \\
&\quad-\frac{\nu(\mu_2-\mu_1)[(\mu_3-\mu_1)\gamma+(\mu_2-\mu_1)(\mu_3+\nu)](r^*)^2}{\gamma[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)]s^*}(r_t-1)^2
\end{aligned}
$$

$$-\int_0^h f(\tau)\Big[g\Big(\frac{1}{s_t}\Big)+g\Big(\frac{s_t\tilde{y}_{t,\tau}}{i_t}\Big)+g(i_t)-g(\tilde{y}_t)\Big]d\tau+\sigma^2M^*$$

On other hand, using the convexity of the function $g(x)$ and a simple calculation, we have

$$g(i_t)-g(\tilde{y}_t)=\frac{G(I(t))-G(i^*)}{i^*}\Big(\frac{I(t)}{G(I(t))}-\frac{i^*}{G(i^*)}\Big)\geq 0$$

$$\Big(1-\frac{1}{s_t}\Big)(r_t-1)-(r_t-1)(s_t-1)=\frac{(s_t-1)^2}{s_t}(r_t-1)$$

Therefore we get

$$\begin{aligned}
&\mathscr{L}V_\varphi^{E^*}(t,X(t))\\
= \;& -(\mu_1 s^*+\nu r^*(r_t-1))\frac{(s_t-1)^2}{s_t}-\frac{\mu_1\delta\gamma(N^*)^2}{\gamma[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)]s^*}(n_t-1)^2\\
&-\Big[\frac{\delta(\mu_2-\mu_1)\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_3+\delta)(r^*)^2}{\gamma\{\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)\}s^*}+\frac{\eta(\mu_3+\gamma+\nu)(r^*)^2}{\gamma s^*}\Big](r_t-1)^2\\
&-\int_0^h f(\tau)\Big[g\Big(\frac{1}{s_t}\Big)+g\Big(\frac{s_t\tilde{y}_{t,\tau}}{i_t}\Big)+g(i_t)-g(\tilde{y}_t)\Big]d\tau+\sigma^2M^*\\
\leq \;& -\Big[\frac{\nu(\mu_2-\mu_1)\{\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_3+\nu)\}(r^*)^2}{\gamma[\gamma(\mu_3-\mu_1)+(\mu_2-\mu_1)(\mu_1+\mu_3+\nu)]s^*}+\frac{\eta(\mu_3+\gamma+\nu)(r^*)^2}{\gamma s^*}\Big](R(t)-r^*)^2\\
&-\frac{\mu_1 s^*-\nu r^*}{\rho_1}(S(t)-s^*)^2+\sigma^2M^*
\end{aligned}$$

Combining (30), (31) and (33) for $\mu_1=\mu_2=\mu_3$, we get

$$\begin{aligned}
\mathscr{L}V_\varphi^{E^*}(t,X(t)) &= -(\mu_1 s^*+\nu r^*(r_t-1))\frac{(s_t-1)^2}{s_t}-\frac{\nu}{s^*}\Big[r^*(r_t-1)-\frac{N^*}{2}(n_t-1)\Big]^2\\
&\quad -\int_0^h f(\tau)\Big[g\Big(\frac{1}{s_t}\Big)+g\Big(\frac{s_t\tilde{y}_{t,\tau}}{i_t}\Big)\Big]d\tau,\\
&\quad -\frac{\nu(r^*)^2}{\gamma s^*}(\mu+\nu)(r_t-1)^2+\sigma^2M^*\\
&\leq -\frac{\mu_1 s^*-\nu r^*}{\rho_1}(S(t)-s^*)^2-\frac{\nu(r^*)^2}{\gamma s^*}(\mu_1+\nu)(R(t)-r^*)^2+\sigma^2M^*
\end{aligned}$$

and the proof is now complete. □

Proof of Theorem 6. For any $(x_1,x_2,x_3)\in\mathbb{R}^3$, let us put $H(x_1,x_2,x_3)=(x_1-s^*,x_2-i^*,x_3-r^*)$ and $z(t)=H(X(t))$, where $X(t)=(S(t),I(t),R(t))$.

By using Itô's formula, we have

$$dz(t)=a(z_t+E^*)dt+b(z_t+E^*)dB(t),$$

where a, b and the three-dimensional brownian motion $B(t)$ are defined by the Remark 1.
Let put $I=\{(x_1,x_2,x_2)\in\mathbb{R}^3/-s^*<x_1<s_0-s^*,-i^*<x_2<s_0-i^*,-r^*<x_3<s_0-r^*\}$ and define for all $(x_1,x_2,x_2)\in I$, the function

$$W_\varphi(x)=(x_1+x_2)^2+\varpi U_\varphi^{E^*}(t,x+E^*),\ \text{where },\varpi\in(0,\infty).$$

It is clear that $W_\varphi(x)\geq 0$. By (7) we have

$$\begin{aligned}
&\mathscr{L}W_\varphi(t,z_t)\\
=\ &2(z_1(t)+z_2(t))[-\mu_1(z_1(t)+s^*)-(\mu_2+\gamma)(z_2(t)+i^*)+\nu(z_3(t)+r^*)]+\varpi\mathscr{L}U_\varphi^{E^*}(t,z(t)+E^*)\\
=\ &2(z_1(t)+z_2(t))[-\mu_1 z_1-(\mu_2+\gamma)z_2(t)+\nu z_3(t)]\\
&+2(z_1(t)+z_2(t))[(-\mu_1 s^*-\beta s^*G(i^*)+\nu r^*)+(\beta s^*G(i^*)-(\mu_2+\gamma)i^*)]+\varpi\mathscr{L}U_\varphi^{E^*}(t,z(t)+E^*).
\end{aligned}$$

Now, in view of the Lemma 3 and the fact that the endemic equilibrium $E^*=(s^*,i^*,r^*)$ of the deterministic model (1) is such that $-\mu_1 s^*-\beta s^*G(i^*)+\nu r^*=0,\quad \beta s^*G(i^*)-(\mu_2+\gamma)i^*=0,$ and $\gamma i^*-(\mu_3+\nu)r^*=0$, we get

$$\begin{aligned}
\mathscr{L}W_\varphi(t,z_t)\ &\leq\ 2(z_1(t)+z_2(t))[-\mu_1 z_1(t)-(\mu_2+\gamma)z_2(t)+\nu z_3(t)]+\varpi\mathscr{L}U_\varphi^{E^*}(z(t)+E^*)\\
&\leq\ -2\mu_1 z_1^2(t)-2(\mu_2+\gamma)z_2^2(t)-2(\mu_1+\mu_2+\gamma)z_1(t)z_2(t)\\
&\quad+2\nu z_1(t)z_3(t)+2\nu z_2(t)z_3(t)+\varpi(-Az_1^2-Bz_3^2+\sigma^2M^*)\\
&\leq\ -(\varpi A+\mu_1)z_1^2(t)-(\mu_2+\gamma)z_2^2(t)-\varpi\frac{B}{2}z_3^2(t)-(\mu_2+\gamma)\Big(z_2(t)+\frac{\mu_1+\mu_2+\gamma}{(\mu_2+\gamma)}z_1(t)\Big)^2\\
&\quad-\mu_1\Big(z_1(t)-\frac{\nu}{\mu_1}z_3(t)\Big)^2-\varpi\frac{B}{2}\Big(z_3(t)-\frac{2\nu}{\varpi B}z_2(t)\Big)^2\\
&\quad+\frac{(\mu_1+\mu_2+\gamma)^2}{(\mu_2+\gamma)}z_1^2(t)+\frac{2(\nu)^2}{\varpi B}z_2^2+\frac{\nu^2}{\mu_1}z_3^2(t)+\varpi\sigma^2M^*
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{L}W_\varphi(t,z_t)\ &\leq\ -\Big[\varpi A-\Big(\mu_1-\frac{(\mu_1+\mu_2+\gamma)^2}{(\mu_2+\gamma)}\Big)\Big]z_1^2(t)-\Big(\mu_2+\gamma-\frac{2\nu}{\varpi B}\Big)z_2^2(t)\\
&\quad-\Big(\varpi\frac{B}{2}-\frac{\nu^2}{\mu_1}\Big)z_3^2(t)+\varpi\sigma^2M^*\\
&\leq\ -\Big[\varpi A-\Big(\mu_1+\mu_2+\gamma+\frac{\mu_1^2}{\mu_2+\gamma}\Big)\Big]z_1^2(t)-\Big(\mu_2+\gamma-\frac{2\nu}{\varpi B}\Big)z_2^2(t)\\
&\quad-\Big(\varpi\frac{B}{2}-\frac{\nu^2}{\mu_1}\Big)z_3^2(t)+\varpi\sigma^2M^*.
\end{aligned}$$

By setting $\varpi > \max\left\{\frac{\mu_1+\mu_2+\gamma+\frac{\mu_1^2}{\mu_2+\gamma}}{A}, \frac{2\nu}{(\mu_2+\gamma)B}, \frac{2\nu^2}{\mu_1 B}\right\}$, we obtain that

$$\Theta_1 = \varpi A - \left(\mu_1+\mu_2+\gamma+\frac{\mu_1^2}{\mu_2+\gamma}\right) > 0, \quad \Theta_2 = \mu_2+\gamma-\frac{2\nu}{\varpi B} > 0, \quad \Theta_3 = \varpi\frac{B}{2}-\frac{\nu^2}{\mu_1} > 0.$$

Hence, we have

$$\begin{aligned}\mathscr{L}W_\varphi(t,z_t) &\leq -\Theta_1 z_1^2(t) - \Theta_2 z_2^2 - \Theta_3 z_3^2(t) + \varpi\sigma^2 s_0^4. \\ &\leq -\min\{\Theta_1,\Theta_2,\Theta_3\}|z(t)|^2 + \varpi\sigma^2 M^*.\end{aligned}$$

By virtue of Itô's formula, we derive

$$W_\varphi(t,z_t) - W_\varphi(z(0)) = \int_0^t \mathscr{L}W_\varphi(z(s))ds + M(t).$$

It follows that

$$W_\varphi(t,z_t) - W_\varphi(z(0)) \leq -\zeta\int_0^t |z(t)|^2 ds + \varpi\sigma^2 Dt + M(t) \tag{34}$$

where $\zeta = \min\{\Theta_1,\Theta_2,\Theta_3\}$ and
$M(t) = \varpi\sigma\int_0^t \Big(z_2(s)(z_1(s)+s^*) + z_1(s)(z_2(s)+i^*)\Big)\int_0^h f(\tau)G(z_2(s-\tau)+i^*)d\tau dW(s)$.
In view of Theorem 2, $|z(t)| = |X(t)-E^*|$ is bounded, hence the quadratic variation of the martingale M is locally bounded. So, by the strong law of large number for local martingales (see e.g. [2]) we have $\lim_{t\to\infty}\frac{M(s)}{t} = 0$. By virtue of (34), we have

$$\liminf_{t\to\infty}\frac{1}{t}\left[\varpi\sigma^2 Dt - \zeta\int_0^t |z(t)|^2 ds\right] ds \geq 0.$$

Therefore

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t |z(s)|^2 ds \leq \frac{\varpi\sigma^2 M^*}{\zeta}$$

where $|z(s)|^2 = z_1(t)^2 + z_2(t)^2 + z_3(t)^2 = (S(t)-s^*)^2 + (I(t)-i^*)^2 + (R(t)-r^*)^2$. In view of the Lemma 3 we conclude.

□

6. Conclusion and numerical simulation

In this paper, we have considered a stochastic delayed differential equation standing for a stochastic model of a deterministic SIRS epidemic model with delay. Firstly, we have proved the global positivity of the solution (see Theorem 2) and the almost sure exponential stability of the disease-free equilibrium E^0 of the stochastic model (4) (see Theorem 3). Then, we have investigated the almost sure asymptotic stability under a suitable condition (see Theorem 4). Finally, in Theorem 6, under the condition $R_0 > 1$ and the unstability of the disease-free equilibrium E^0, we have studied the asymptotic behaviour of the solution of the stochastic model around the endemic equilibrium state E^* of the deterministic model. We have showed that the solution will oscillate around this endemic equilibrium position E^* and the amplitudes of this fluctuation increase with σ.

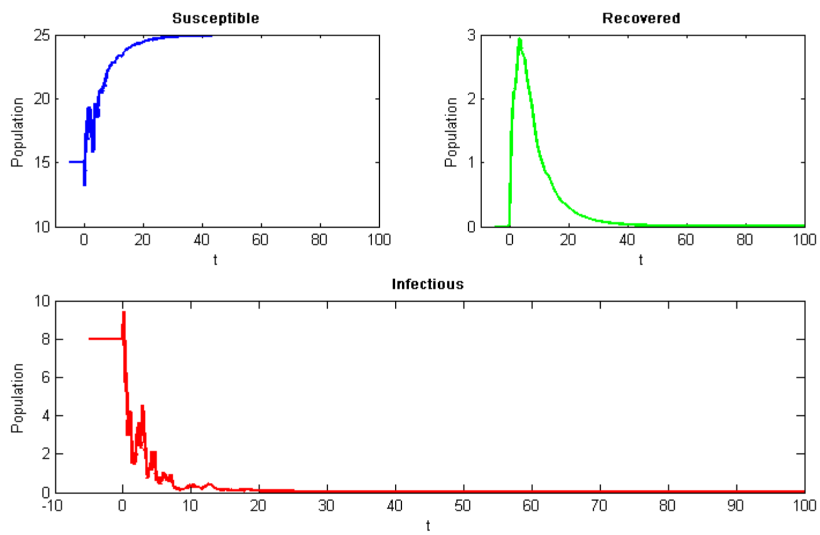

FIGURE 1. Sample paths of the stochastic SIRS epidemic models (4) with $G(x) = x/(1+x)$. The initial values are: $S(\theta) = 15$, $I(\theta) = 8$, $R(\theta) = 0$ for $\theta \in [-5,0]$. The values of the parameters are given by: h=5, $\Lambda = 5$, $\mu_1 = \mu_3 = 0.2$, $\mu_2 = 0.3$, $\beta = 0.016$, $\gamma = 0.4$, $\sigma = 0.03$, $\nu = 0.06$. The conditions of the theorem 3 are checked $2\mu_1 = 0.6 > \max\{\gamma - \nu, s_0\beta + \nu\} = 0.46$ and $s_0\beta = 0.4 < \mu_2 + \gamma - \frac{1}{2}(s_0\sigma)^2 = 0.419$.

We give an illustration of the stability result by numerical simulation. We use Euler-Maruyama method (see e.g. [32]) to simulate the path of the model (4) with G(x)=x/(1+x) (i.e $G(x) \leq 1$) for all $x \in [0,\infty)$ and $f(s) = 1/h$ for all $s \in [0,h]$ and null otherwise. We see that the numerical simulations agree with the analytical results of Theorem 3 (see Figure 1). Nevertheless, we notice that the disease-free equilibrium E^0 seems to be asymptotic stable without condition (i) of Theorem 3, as shown in the numerical simulation Figure 2.

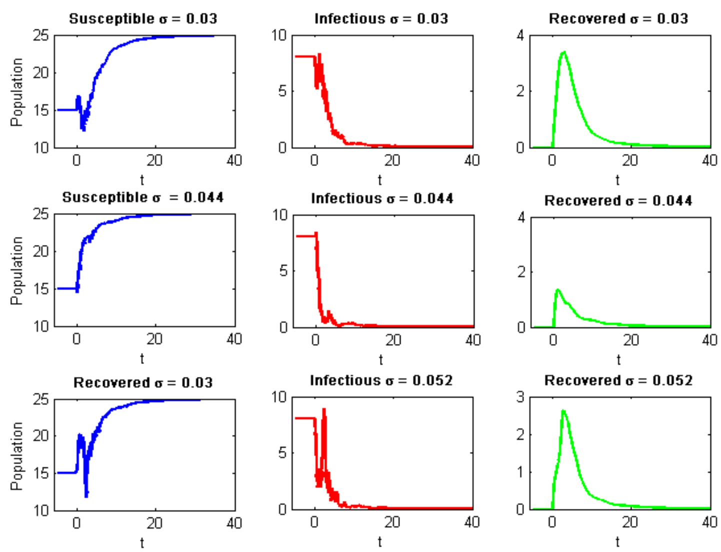

FIGURE 2. Three (3) sample paths of the stochastic SIRS epidemic models (4) with three different values of σ, $\nu = 0.3$ and the remainder parameters are as in the Figure. (1). The condition (i) of the theorem 3 is not checked $2\mu_1 = 0.6 < \max\{\gamma - \nu, s_0\beta + \nu\} = 0.7$, and the condition (ii) $s_0\beta = 0.4 < \mu_2 + \gamma - \frac{1}{2}(s_0\sigma)^2 = 0.419$ is checked only for $\sigma = 0.03$.

Theorem 6 suggests that if $R_0 > 1$ and the disease-free equilibrium E^0 of the model (4) is unstable, the solution of the stochastic model (4) oscillates around the endemic equilibrium E^* of the deterministic model and the amplitude of this oscillations increases with the intensity of the noise level. The following numerical simulation with G(x)=x/(1+x) shown in the Figure 3 clearly support this result. Let us note that when G(x)=x/(1+x), the endemic equilibrium E^* is given by

$$s^* = \frac{(\mu_2+\gamma)(1+i^*)}{\beta}, \quad i^* = \frac{\mu_1(\mu_3+\delta)(\mu_2+\gamma)(R_0-1)}{(\mu_2+\gamma)(\mu_3+\delta)(\beta+\mu_1)-\beta\gamma\delta}, \quad r^* = \frac{\gamma i^*}{\mu_3+\delta}$$

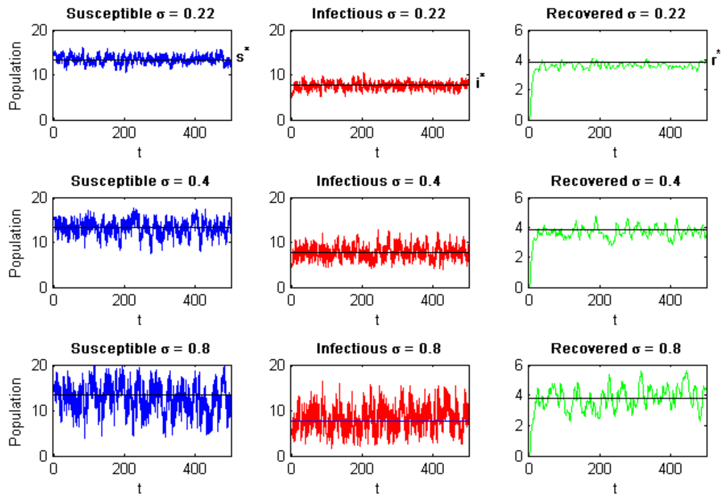

FIGURE 3. Three (3) sample paths of the stochastic SIRS epidemic models (4) with three different values of σ. The initial values are: $S(\theta) = 15, \quad I(\theta) = 5, \quad R(\theta) = 0$ for $\theta \in [-5, 0]$. The values of the parameters are given by: h=5, $\Lambda = 5, \quad \mu_1 = \mu_3 = 0.2, \quad \mu_2 = 0.21, \quad \beta = 0.2, \quad \gamma = 0.1, \quad \nu = 0.01$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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