Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2025, 2025:99
https://doi.org/10.28919/cmbn/9423
ISSN: 2052-2541

LOCAL BIFURCATION ANALYSIS OF AN ECOSYSTEM UNDER THE EFFECT OF TOXICITY AND SELF-DEFENSE

ZINA KH. ALABACY[1], QAMAR Q. MOHAMMED[1], ZAINAB JALIL TAHER[1], NADIA M. G. AL-SAIDI[2,*]

[1]Department of Control and Systems Engineering, University of Technology, Iraq

[2]Department of Applied Sciences, University of Technology, Baghdad, Iraq

Abstract. The local bifurcation phenomena in a nonlinear ecosystem model incorporating toxicity effects and species' self-defense strategies are explored in this work. The influence of the ecological parameters on the stability of the equilibrium point is investigated**.** Bifurcation analysis has significant importance because it reveals the changes that occur at the equilibrium points, as well as how specific parameters affect the system. This paper achieved sufficient conditions that ensure the appearance of local bifurcation (LB), pitchfork bifurcation (PFB), transcritical bifurcation (TB), saddle-node bifurcation (SNB), and Hopf bifurcation (HB) of the system, which consists of prey, middle predator, and top predator with toxin effects and self-defense in the ecosystems. Sotomayor's theorem and Hopf bifurcation conditions help characterize the system's sensitivity to parameter variations**.** We observed that near the first equilibrium point, PFB occurs. At the free top predator equilibrium points, there is TB, and for the positive equilibrium points, there is SNB. HB close to the positive equilibrium point has also been studied. Numerical analysis is employed to validate the primary theoretical findings and to illustrate how alterations in parameters have a substantial impact on maintaining biodiversity and ecological balance.

Keywords: local bifurcation; pitchfork bifurcation; transcritical bifurcation; saddle-node bifurcation; Hopf bifurcation.

2020 AMS Subject Classification: 37B35, 34C23.

[*]Corresponding author
E-mail address: nadia.m.ghanim@uotechnology.edu.iq
Received June 14, 2025

INTRODUCTION

Nonlinear differential equations are analysed by dynamical systems to find their fixed points and stability. Although maintaining system stability is essential, the stability of these fixed points can be changed by specific characteristics. We call this phenomenon bifurcation. Bifurcation analysis is essential to know the type of change in solutions. For example, a bifurcation occurs when the behaviour of a stable fixed point changes to become another stable point, or fixed points are created or destroyed. There are many types of bifurcation. The SNB occurs when two fixed points appear or are killed, a TB takes place when two fixed points swap their stabilities, and the PFB occurs in many models where a fixed point changes into three fixed points due to the effect of certain parameters. The HB occurs when an equilibrium point has two pairs of purely imaginary eigenvalues. It is especially useful in disciplines such as biology, engineering, physics, and economics, as changes in parameter values often reveal unexpected behavior in complex systems. Therefore, bifurcation can be applied in many fields [1-4].

For the importance of bifurcation, many researchers in the biology field, such as Zhang and Luo [5] studied a Leslie-type predator-prey model that incorporates prey harvesting and group defense. They delved into the system's dynamics by analyzing SNB and HB around the equilibrium points. Aldosary and Ahmed [6] studied a discrete-time prey-predator system and recognized that it exhibits chaotic behavior using a good constant technique with maximum Lyapunov graphs and bifurcation diagrams. Naik et al. [7] investigated the specific significance of Neimark-Sacker and period-doubling bifurcations in discrete-time prey-predator models. Li et al. [8] investigated the Allee effect and maturation delay in the prey population with a delay in the predator population. They found that at the tipping point, an SNB occurs, and an HB occurs when delays cross the curves from a stable to an unstable region. Almatrafi and Berkal [9] studied the Neimark-Sacker bifurcation and a period-doubling bifurcation on a predator-prey model with the Allee effect. Gros [10] introduced important concepts, including bifurcations and global bifurcations, among others.

Many researchers have studied the bifurcation in the physical field. Tang et al. [11] studied the bifurcation for a model that arises from the connection of nonlinear fiber optics and was also improved by integrating the model along the periodic orbits. That improves the singularly periodic solutions [12-13].

Ecosystems are complex networks of living organisms that interact with their environment and with one another to maintain a delicate balance essential for survival. Toxicology plays a crucial role in these systems, serving as a defense mechanism and a regulatory element that influences the relationships between species. Many species produce toxins to fend off predators, outcompete rivals, or guard against infections, but some have evolved resistance or counterstrategies to lessen the effects of toxins. The dynamic interplay between dangerous substances and self-defense systems, which alters food chains, population dynamics, and ecological stability, highlights the intricate adaptations that preserve biodiversity [12-15]. Majeed and Kadhim [16] investigated the conditions for the appearance of HB at the positive equilibrium point in a food chain model comprising three species, considering the harvest effect on the middle predator and the toxic effects of all species.

Ultimately, this paper's main significant contributions are summarized as follows: investigating the equilibrium states analysis and their stability conditions within the ecological system. The local bifurcation methods have been studied to determine extinction, prey-only survival, prey-middle predator coexistence, and full coexistence scenarios. Important ecological parameters that have a direct impact on ecosystem dynamics and stability have been identified, including predator harvesting rates, mortality from environmental toxicity, and food conversion rates between species. Finally, numerical validation via simulations is employed to validate the theoretical hypotheses and demonstrate the real-world effects of parameter changes on biodiversity, species interactions, and ecological stability, with a focus on the toxicity effect and self-defense strategies.

The remaining parts of this work are organized as follows: Section 2 presents the dynamical model that describes the interactions between prey, the middle predator, and the top predator, as presented in [17]. Section 3 discusses the bifurcation, including its local and global analysis, as well as the existence of equilibrium points. Section 4 discusses numerical simulations used to validate the theoretical results, focusing on local and Hopf bifurcation conditions. Finally, Section 5 concludes the work by discussing the obtained results.

2. DYNAMICAL MODEL [17]

The model that has been suggested, which contains one prey $Z_1(T)$, middle predators $Z_2(T)$ and top predator $Z_3(T)$, of population density at the time T respectively. The parameters are: $a_1 > 0$ is the growth rate of the population prey, $a_2 > 0$ is what the environment provides as living requirements for the prey, $0 < a_3, a_6 < 1$ are the maximum predation rates of a middle predator on the prey and top predator on the middle predator, respectively. $a_4, a_{10}, a_{13} \in (0,1)$ are the rate of susceptibility of the prey to environmental toxicity, and the rate of toxicity of the middle and the top predator, respectively. $a_5, a_{11} \in (0,1)$ are food translation rates when the middle predator devours the prey and when the top predator preys on the middle predator, respectively. $a_7 \in (0,1)$ is a half-saturation constant of the middle predator, and finally, a_8 is the defense efficiency of the middle predator.

By differential equations, which are first-order and nonlinear, the dynamical system has been suggested with the above assumptions as follows:

$$\left.\begin{aligned} \frac{dZ_1}{dT} &= a_1 Z_1\left(1-\frac{Z_1}{a_2}\right) - a_3 Z_1 Z_2 - a_4 Z_1^2 = f_1(Z_1, Z_2, Z_3) \\ \frac{dZ_2}{dT} &= a_5 Z_1 Z_2 - \frac{a_6 Z_2 Z_3}{a_7 + a_8 Z_2^2} - a_9 Z_2 - a_{10} Z_2^2 Z_3 = f_2(Z_1, Z_2, Z_3) \\ \frac{dZ_3}{dT} &= \frac{a_{11} Z_2 Z_3}{a_7 + a_8 Z_2^2} - a_{12} Z_3 - a_{13} Z_2 Z_3^2 = f_3(Z_1, Z_2, Z_3) \end{aligned}\right\} \tag{1}$$

where $Z_1(0) \geq 0$, $Z_2(0) \geq 0$ and $Z_3(0) \geq 0$. Table 1 shows a summary of equilibrium points and their stability.

Table 1: Summary of Equilibrium Points

Equilibrium Point	Biological Interpretation	Stability Condition
M_0 (Extinction)	No species survive	Always unstable
M_1 (Prey-only)	Only prey survives	Stable if $a_9 + a_{12} + 2Z_1\left(\frac{a_1}{a_2} + a_4\right) > a_1 + a_5 Z_1$ $a_1 > 2Z_1(\frac{a_1}{a_2} + a_4)$ $a_5 Z_1 > a_9$
M_2 (Prey –Middle Predator)	Prey and middle predators coexist	Stable if $\frac{a_9}{Z_1} \le a_5 < 2\left(\frac{a_1}{a_2} + a_4\right) < \frac{a_1}{Z_1}$, $\frac{a_{11}}{a_7 + a_8 Z_2^2} < min\left\{a_3, \frac{a_{12}}{Z_2}\right\}$, $Z_1\left(2\left(\frac{a_1}{a_2} + a_4\right) - a_5\right) + Z_2\left(a_3 - \frac{a_{11}}{a_7 + a_8 Z_2^2}\right) + a_9 + a_{12} > a_1$
M_3(Full coexistence)	All species coexist	Stability if $a_5 <$ $min\left\{\frac{a_1}{a_2} + a_4, \frac{1}{Z_1}\left[a_9 + 2a_{10}Z_2Z_3 + \frac{a_6 Z_3 (a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2}\right]\right\}$, $a_7 > a_8 Z_2^2$, $\frac{a_{11}(a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2} < a_3 Z_3$,

3. BIFURCATION

In this section, Sotomayer's theorem [18] to verify the following bifurcation conditions, is utilized and as follows:

For SNB $\Psi^T f_\mu(x^*, \mu_0) \neq 0$, $\Psi^T[D^2 f(x^*, \mu_0)(V, V)] \neq 0$.

For TB $\Psi^T f_\mu(x^*, \mu_0) = 0$, $\Psi^T\left[Df_\mu(x^*, \mu_0)V\right] \neq 0, \Psi^T[D^2 f(x^*, \mu_0)(V, V)] \neq 0.$

For PFB $\Psi^T f_\mu(x^*, \mu_0) = 0$, $\Psi^T\left[Df_\mu(x^*, \mu_0)V\right] \neq 0$, $\Psi^T[D^2 f(x^*, \mu_0)(V, V)] = 0$,

$\Psi^T[D^3 f(x^*, \mu_0)(V, V, V)] \neq 0.$

Where x^* is an equilibrium point, $\mu = \mu_0$ is the bifurcation parameter, V is an eigenvector for the Jacobian matrix J, Ψ is an eigenvector for the matrix J^T.

3.1 The analysis of local bifurcation

The appearance of LB close to the three equilibrium points of system (1) has been studied using Sotomayor's theory.

The Jacobian matrix [17] of the system (1) is given as:

$$J_i = [\hat{S}_{ij}]_{3\times 3}, \tag{2}$$

where $i, j = 1,2,3$ and $\hat{S}_{11} = a_1 - 2Z_1\left(a_4 - \frac{a_1}{a_2}\right) - a_3 Z_2$, $\hat{S}_{12} = -a_3 Z_1$, $\hat{S}_{13} = 0$,

$$\hat{S}_{21} = a_5 Z_2 \,, \hat{S}_{22} = a_5 Z_1 - a_9 - 2a_{10} Z_2 Z_3 - \frac{a_6 Z_3 (a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2} \,,$$

$$\hat{S}_{23} = -Z_2\left(\frac{a_6}{a_7 + a_8 Z_2^2} + a_{10} Z_2\right) , \hat{S}_{31} = 0 \,, \hat{S}_{32} = Z_3\left[\frac{a_{11}(a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2} - a_{13} Z_3\right] ,$$

$$\hat{S}_{33} = \frac{a_{11} Z_2}{a_7 + a_8 Z_2^2} - a_{12} - 2a_{13} Z_2 Z_3.$$

Clearly, for any nonzero vector $\breve{K} = \left(\breve{K}_1, \breve{K}_2, \breve{K}_3\right)^T$

$$DF(\tilde{X}, \mu)\breve{K} = [\breve{N}_{i1}]_{3\times 1}, \tag{3}$$

Where $\tilde{X} = (Z_1, Z_2, Z_3)$, μ be any parameter that causes bifurcation,

$$\breve{N}_{11} = \left[a_1 - 2Z_1\left(a_4 - \frac{a_1}{a_2}\right) - a_3 Z_2\right]\breve{K}_1 - a_3 Z_1 \breve{K}_2,$$

$$\breve{N}_{21} = a_5 Z_2 \breve{K}_1 + [a_5 Z_1 - a_9 - 2a_{10} Z_2 Z_3 - \frac{a_6 Z_3 (a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2}]\breve{K}_2$$

$$- Z_2 \breve{K}_3 \left(\frac{a_6}{a_7 + a_8 Z_2^2} + a_{10} Z_2\right),$$

$$\breve{N}_{31} = Z_2 \breve{K}_2 \left[\frac{a_{11}(a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2} - a_{13} Z_3\right] + [\frac{a_{11} Z_2}{a_7 + a_8 Z_2^2} - a_{12} - 2a_{13} Z_2 Z_3]\breve{K}_3,$$

and $$D^2F(\tilde{X}, \mu)(\breve{K}, \breve{K}) = [\widehat{M}_{i1}]_{3\times 1}, \tag{4}$$

where:

$$\breve{M}_{11} = -2\left(a_4 - \frac{a_1}{a_2}\right)\breve{K}_1^2 - 2a_3 \breve{K}_1 \breve{K}_2,$$

$$\breve{M}_{21} = -[2a_{10} Z_3 + \frac{2a_6 a_8 Z_2 Z_3 [2(a_7 + a_8 Z_2^2) - 1]}{(a_7 + a_8 Z_2^2)^3}\breve{K}_2^2 - [2a_{10} Z_2$$

$$+ \frac{a_7(1 - a_6) + a_8 Z_2^2 (1 - 3a_6)}{(a_7 + a_8 Z_2^2)^2}]\breve{K}_2 \breve{K}_3,$$

$$\breve{M}_{31} = \frac{-a_8 a_{11} Z_2 Z_3 [(a_7 + a_8 Z_2^2)^2 + (a_7 - a_8 Z_2^2)^2]}{(a_7 + a_8 Z_2^2)^4} \breve{K}_2^2 - 2a_{13} Z_2 \breve{K}_3^2$$

$$+ \left[\frac{2a_{11}(a_7 - a_8 Z_2^2)^2}{(a_7 + a_8 Z_2^2)^2} - 4a_{13} Z_3\right] \breve{K}_2 \breve{K}_3.$$

and $D^3 G(\tilde{X}, \mu)(\breve{K}, \breve{K}, \breve{K}) = [\hat{P}_{i1}]_{3\times 1}$,

where $\breve{P}_{11} = 0, \breve{P}_{21} = \delta_1 \breve{K}_2^3 - (\delta_2 + \delta_3)\breve{K}_2^2 \breve{K}_3, \breve{P}_{31} = \delta_4 \breve{K}_2^3 - 6a_{13} \breve{K}_2 \breve{K}_3^2 - (\delta_5 - \delta_6)\breve{K}_2^2 \breve{K}_3$.

Here $\delta_1 = \frac{2a_6 a_8 Z_3 (2a_7 - 6a_8 Z_2^2 - 1)[a_7 - 5a_8 Z_2^2]}{(a_7 + a_8 Z_2^2)^4}$, $\delta_2 = 2a_{10} + \frac{2a_8 Z_2 [Z_2(1 - 3a_6(1 + 2a_8 Z_2)) - 2a_6 a_7]}{(a_7 + a_8 Z_2^2)^2}$,

$$\delta_3 = -[2a_{10} Z_3 + \frac{2a_6 a_8 Z_2 [2(a_7 - a_8 Z_2^2) - 1]}{(a_7 + a_8 Z_2^2)^3}] ,$$

$$\delta_4 = \frac{-a_8 a_{11} Z_3 [(a_7 + a_8 Z_2^2)^2 - (a_7 + 3a_8 Z_2^2) - 1]}{(a_7 + a_8 Z_2^2)^4}$$

$$+ \frac{2a_8^2 a_{11} Z_2^2 Z_3 [(a_7 + a_8 Z_2^2)^2 + (a_7 - a_8 Z_2^2)]}{(a_7 + a_8 Z_2^2)^5} ,$$

$$\delta_5 = \frac{4a_8 a_{11} Z_2 [1 + a_7 - a_8 Z_2^2]}{(a_7 + a_8 Z_2^2)^3} , \delta_6 = -\frac{2a_8 a_{11} Z_2 \left[(a_7 + a_8 Z_2^2)^2 + (a_7 - a_8 Z_2^2)\right]}{(a_7 + a_8 Z_2^2)^4} .$$

In the following theorems, the LB conditions close to the three equilibrium points have been determined. Table 2 helps identify the exact parameters where bifurcations occur and their impact on stability.

Table 2: The parameters' effect on the system behavior

parameters	Equilibrium point	Meaning	Bifurcation Type	Effect on System Behavior
a_9	M_1	The middle predator's harvesting rate	PFB	Alter stability hierarchy
a_{12}	M_2	top predator's mortality rate	TB	Triggers oscillations
a_5	M_3	conversion rate of food from the prey to the middle predator	SNB	Creates equilibria
a_{11}	M_3	conversion rate of food from the middle predator to the top predator	HB	Introduces a new equilibrium

In the following theorem, the effect of the harvest rate a_9 on the stability of the point M_1 was studied under some conditions.

Theorem 3.1: Let the conditions of local stability [17] shown in Table 1 hold with

$$a_1 \neq 2Z_1\left(\frac{a_1}{a_2} + a_4\right), \tag{5}$$

Then the system (1) at $M_1 = (Z_1, 0,0)$ with $\breve{a}_9 = a_5 Z_1$ has PFB, which occurs at M_1.

Proof: Let Eq. (2) of the system (1) at M_1 has $\lambda = 0$ (say $\lambda_{2z_1} = 0$) at $\breve{a}_9 = a_9 > 0$. System's (1) Jacobian matrix with $\breve{a}_9 = a_9$ becomes:

$$\hat{J}_1 = J(M_1, \breve{a}_9) = \begin{bmatrix} a_1 - 2Z_1(\frac{a_1}{a_2} + a_4) & -a_3 Z_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a_{12} \end{bmatrix},$$

Now, let $\widehat{K}^{[1]} = \left(\widehat{K}_1^{[1]}, \widehat{K}_2^{[1]}, \widehat{K}_3^{[1]}\right)^T$ be the eigenvector of $\hat{J}_1$ for $\lambda_{2Z_1} = 0$.

Thus $\left(\hat{J}_1 - \lambda_{2Z_1} I\right)\widehat{K}^{[1]} = 0$, which gives: $\widehat{K}^{[1]} = \left(\widehat{K}_1^{[1]}, \mu_1 \widehat{K}_1^{[1]}, 0\right)^{\mathrm{T}}$ where $\widehat{K}_1^{[1]} \neq 0$ any real number and $\mu_1 = \frac{1}{a_3}\ [a_1 - 2Z_1(\frac{a_1}{a_2} + a_4)]$.

Let $\hat{\psi}^{[1]} = \left(\hat{\psi}_1^{[1]}, \hat{\psi}_2^{[1]}, \hat{\psi}_3^{[1]}\right)^T$ be the eigenvector of $\left(\hat{J}_1\right)^T$ for $\lambda_{2Z_1} = 0$.

Let $\left(\left(\hat{J}_1\right)^T - \lambda_{2Z_1} I\right)\hat{\psi}^{[1]} = 0$, then $\hat{\psi}^{[1]} = \left(0, \hat{\psi}_2^{[1]}, 0\right)^{\mathrm{T}}, \hat{\psi}_2^{[1]} \neq 0, \hat{\psi}_2^{[1]} \in R$.

Now, consider:

$$\frac{\partial f}{\partial a_9} = f_{a_9}\left(\tilde{X}, a_9\right) = \left(\frac{\partial \hat{f}_1}{\partial a_9}, \frac{\partial \hat{f}_2}{\partial a_9}, \frac{\partial \hat{f}_3}{\partial a_9}\right)^{\mathrm{T}} = (0, -Z_2, 0)^{\mathrm{T}}.$$

So, $f_{a_9}(M_1, \breve{a}_9) = (0,0,0)^{\mathrm{T}}$ and hence $\left(\hat{\psi}^{[1]}\right)^T f_{a_9}(M_1, \breve{a}_9) = 0$.

Therefore, at M_1, the SNB condition cannot occur based on Sotomayor’s theorem. Now the TB's first condition can be applied as follows:

$Df_{a_9}\left(\tilde{X}, a_9\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where it is the derivative of $f_{a_9}\left(\tilde{X}, a_9\right)$ for $\tilde{X} = (Z_1, Z_2, Z_3)^T$. So,

$Df_{a_9}(M_1,\breve{\alpha}_9)\widehat{K}^{[1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \mu_1 \widehat{K}_1^{[1]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\mu_1 \widehat{K}_1^{[1]} \\ 0 \end{bmatrix}$, hence

$\left(\hat{\psi}^{[1]}\right)^{\mathrm{T}}\left[Df_{a_9}(M_1,\breve{\alpha}_9)\widehat{K}^{[1]}\right] = -\mu_1 \widehat{K}_1^{[1]} \hat{\psi}_2^{[1]} \neq 0$ if condition (5) holds.

Now by substituting $\widehat{K}^{[1]}$ in Eq. (3) get:

$D^2 f_{a_9}(M_1,\breve{\alpha}_9)\left(\widehat{K}^{[1]},\widehat{K}^{[1]}\right) = \begin{bmatrix} \mu_2 \left(\widehat{K}_1^{[1]}\right)^2 \\ 0 \\ 0 \end{bmatrix}$, where $\mu_2 = -2[\left(a_4 - \frac{a_1}{a_2}\right) + a_3 \mu_1]$.

Thus $\left(\hat{\psi}^{[1]}\right)^{\mathrm{T}}\left[D^2 f_{a_9}(M_1,\breve{\alpha}_9)\left(\widehat{K}^{[1]},\widehat{K}^{[1]}\right)\right] = 0$ at M_1 with $\breve{\alpha}_9 = a_9$ has no TB according to Sotomayor's theorem. Now by, substituting $\widehat{K}^{[1]}$ in Eq. (4), we get:

$D^3 g(M_1,\breve{\alpha}_9)\left(\widehat{K}^{[1]},\widehat{K}^{[1]},\widehat{K}^{[1]}\right) = \begin{bmatrix} 0 \\ \mu_1^5 \left(\widehat{K}_1^{[1]}\right)^5 \\ 0 \end{bmatrix}$,

Hence, $\left(\hat{\psi}^{[1]}\right)^{\mathrm{T}}\left[D^3 g(M_1,\breve{\alpha}_9)\left(\widehat{K}^{[1]},\widehat{K}^{[1]},\widehat{K}^{[1]}\right)\right] = \mu_1^5 \left(\widehat{K}_1^{[1]}\right)^5 \hat{\psi}_2^{[1]} \neq 0$, if condition (5) holds. Therefore, at M_1 with $\breve{\alpha}_9 = a_9$ system (1) has PFB, which means the harvest rate $\breve{\alpha}_9 = a_9$ affected the stability of the point M_1. Therefore, the PEF has appeared.

In the following theorem, the effect of the top predator mortality rate a_{12} (due to poisoning) on the point M_2 was studied.

Theorem 3.2*:* Assume that the following conditions, with the conditions mentioned in Table 1, are held.

$$a_1 > 2Z_1\left(\frac{a_1}{a_2} + a_4\right) + a_3 Z_2\,, \tag{6}$$

$$a_9 < Z_1, \tag{7}$$

$$a_7 < a_8 Z_2^2, \tag{8}$$

$$b_5 \neq 0\,, \tag{9}$$

Then the system (1) at $M_2 = \left(Z_1, \bar{\bar{\bar{Z}}}_2, 0\right)$ with $\breve{\alpha}_{12} = a_{12} = \frac{a_{11}Z_2}{a_7 + a_8 Z_2^2}$has only TB.

Proof: Let Eq. (2) of the system (1) at M_2 has $\lambda = 0$ (say $\lambda_{2Z_2} = 0$) at $\breve{\alpha}_{12} = a_{12}$.

Jacobian matrix $J_2(M_2)$ with $\breve{a}_{12} = a_{12}$ becomes $J(M_2, \breve{a}_{12})$, such that,

$$\breve{J}_2 = J(M_2, \breve{a}_{12}) = \begin{bmatrix} a_1 - 2Z_1(\frac{a_1}{a_2} + a_4) - a_3Z_2 & -a_1Z_1 & 0 \\ a_5Z_2 & a_5Z_1 - a_9 & -Z_2\left(\frac{a_6}{a_7 + a_8Z_2^2} + a_{10}Z_2\right) \\ 0 & 0 & 0 \end{bmatrix},$$

Now, let $\breve{K}^{[2]} = \left(\breve{K}_1^{[2]}, \breve{K}_2^{[2]}, \breve{K}_3^{[2]}\right)^T$ be the eigenvector of $\breve{J}_2$ for $\lambda_{2Z_2} = 0$.

Thus $\left(\breve{J}_2 - \lambda_{2Z_2}I\right)\breve{K}^{[2]} = 0$, which gives: $\breve{K}^{[2]} = \left(\breve{K}_1^{[2]}, b_1\breve{K}_1^{[2]}, b_2\breve{K}_1^{[2]}\right)^T$ where $\breve{K}_1^{[2]} \neq 0$ any real number and $b_1 = \frac{a_1 - 2Z_1(\frac{a_1}{a_2} + a_4) - a_3Z_2}{a_1Z_1}$, $b_2 = \frac{(a_7 + a_8Z_2^2)}{Z_2[a_6 + a_{10}Z_2(a_7 + a_8Z_2^2)]}[a_5Z_2 + b_1(Z_1 - a_9)]$.

For $\left(\breve{J}_2\right)^T$ let the eigenvector be $\breve{\psi}^{[2]} = \left(\breve{\psi}_1^{[2]}, \breve{\psi}_2^{[2]}, \breve{\psi}_3^{[2]}\right)^T$ and $\lambda_{2Z_2} = 0$, we get

$\left(\left(\breve{J}_2\right)^T - \lambda_{2Z_2}I\right)\breve{\psi}^{[2]} = 0$. So, $\breve{\psi}^{[2]} = \left(0,0,\breve{\psi}_3^{[2]}\right)^T$, $\breve{\psi}_3^{[2]} \neq 0$, $\breve{\psi}_3^{[2]} \in R$.

Let $\frac{\partial f}{\partial a_{12}} = f_{a_{12}}\left(\tilde{X}, a_{12}\right) = \left(\frac{\partial \hat{f}_1}{\partial a_{12}}, \frac{\partial \hat{f}_2}{\partial a_{12}}, \frac{\partial \hat{f}_3}{\partial a_{12}}\right)^{\mathrm{T}} = (0,0,-Z_3)^{\mathrm{T}}$.

Therefore, $f_{a_{12}}(M_2, \breve{a}_{12}) = (0,0,0)^{\mathrm{T}}$ and $\left(\breve{\psi}^{[2]}\right)^T f_{a_{12}}(M_2, \breve{a}_{12}) = 0$.

Thus, at M_2 by the condition of the Sotomayor theorem, SNB cannot occur. But the first condition of the Sotomayor theorem for TB is proved, as follows:

For $\tilde{X} = (Z_1, Z_2, Z_3)^T, Df_{a_{12}}\left(\tilde{X}, \breve{a}_{12}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, where $Df_{a_{12}}\left(\tilde{X}, \breve{a}_{12}\right)$ is the derivative of $f_{a_{12}}\left(\tilde{X}, \breve{a}_{12}\right)$.

Additionally, it observes that

$$Df_{a_{12}}(M_2, \breve{a}_{12})\breve{K}^{[2]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \breve{K}_1^{[2]} \\ b_1\breve{K}_1^{[2]} \\ b_2\breve{K}_1^{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -b_2\breve{K}_2^{[2]} \end{bmatrix}, \text{ hence}$$

$\left(\breve{\psi}^{[2]}\right)^{\mathrm{T}}\left[Df_{a_{12}}(\mathrm{M}_2, \breve{a}_{12})\breve{K}^{[2]}\right] = -b_2\breve{K}_2^{[2]}\breve{\psi}_2^{[2]} \neq 0$, if condition (6) holds.

Furthermore, by substituting $\breve{K}^{[2]}$ in Eq. (3), we get:

$$D^2 f_{a_{12}}(M_2,\breve{\alpha}_{12})\left(\breve{K}^{[2]},\breve{K}^{[2]}\right)=\begin{bmatrix} b_3\left(\breve{K}_1^{[2]}\right)^2 \\ b_4\left(\breve{K}_1^{[2]}\right)^2 \\ b_5\left(\breve{K}_1^{[2]}\right)^2 \end{bmatrix},$$

where, $b_3=-2\left[a_4-\frac{a_1}{a_2}+a_3b_1\right], b_4=\left[\frac{a_7(1-a_6)+a_8Z_2^2(1-3a_6)}{\left(a_7+a_8Z_2^2\right)^2}-2a_{10}Z_2\right]b_1b_2$, and

$b_5=\frac{2a_{11}(a_7-a_8Z_2^2)b_1b_2}{\left(a_7+a_8Z_2^2\right)^2}-2a_{13}b_2^2$.

Thus $\left(\breve{\psi}^{[2]}\right)^{\mathrm{T}}\left[D^2 f_{a_{12}}(\mathrm{M}_2,\breve{\alpha}_{12})\left(\breve{K}^{[2]},\breve{K}^{[2]}\right)\right]=b_5\left(\breve{K}_1^{[2]}\right)^2\breve{\psi}_3^{[2]}\neq 0$ If the conditions (6-9) are met. Therefore, at M_2 with $\breve{\alpha}_{12}=a_{12}$ system (1) has TB, but PFB cannot occur according to Sotomayor's theorem. So, the top predator death rate $\breve{\alpha}_{12}=a_{12}$ (due to poisoning) was affecting the point M_2 causing it to fluctuate in its stability.

In the following theorem, the effect of the conversion rate of food from the prey to the middle predator on the stability of the point M_3 was studied.

Theorem 3.3: Assume that the local stability conditions given in Table 1, with the condition $Z_2\neq c_1Z_3,$ is held, where $c_1=\frac{-Z_1}{a_3Z_2}\left(\frac{a_1}{a_2}+a_4\right)$. Then system (1) has only SNB occur at $M_3=(Z_1,Z_2,Z_3)$ with the parameter $\bar{\bar{a}}_5=a_5=\frac{1}{Z_1}\left[a_9+2a_{10}Z_2Z_3-\frac{a_6Z_3(a_7-a_8Z_2^2)}{\left(a_7+a_8Z_2^2\right)^2}\right]$.

Proof: Eq. (2) of the system (1) at M_3 has $\lambda=0$ (say $\lambda_{3Z_3}=0$) at $\bar{\bar{a}}_5=a_{5,}$ Therefore,

$$\bar{\bar{J}}_3=J(M_3,\bar{\bar{a}}_5)=$$

$$\begin{bmatrix} -Z_1\left(\frac{a_1}{a_2}+a_4\right) & -a_3Z_2 & 0 \\ \frac{Z_2}{Z_1}\left[a_9+2a_{10}Z_2Z_3+\frac{a_6Z_3^*(a_7-a_8Z_2^2)}{\left(a_7+a_8Z_2^2\right)^2}\right] & 0 & -Z_2\left(\frac{a_6}{a_7+a_8Z_2^2}+a_{10}Z_2\right) \\ 0 & Z_3\left(\frac{a_{11}(a_7-a_8Z_2^2)}{\left(a_7+a_8Z_2^2\right)^2}-a_{13}Z_3\right) & -a_{13}Z_2Z_3 \end{bmatrix},$$

Now, for $\bar{\bar{J}}_3$, let the eigenvector be $\bar{\bar{K}}^{[3]}=\left(\bar{\bar{K}}_1^{[3]},\bar{\bar{K}}_2^{[3]},\bar{\bar{K}}_3^{[3]}\right)^T$ for $\lambda_{3Z_3}=0$.

Thus $\left(\bar{\bar{J}}_3-\lambda_{3Z_3}I\right)\bar{\bar{K}}^{[3]}=0$, which gives: $\bar{\bar{K}}^{[3]}=\left(\bar{\bar{K}}_1^{[3]},c_1\bar{\bar{K}}_1^{[3]},c_2\bar{\bar{K}}_1^{[3]}\right)^T$ where $\bar{\bar{K}}_1^{[3]}\neq 0$ be any real number and $c_1=-\frac{Z_1}{a_3Z_2}\left(\frac{a_1}{a_2}+a_4\right), c_2=\frac{c_1}{a_{13}Z_2}\left[\frac{a_{11}(a_7-a_8Z_2^2)}{\left(a_7+a_8Z_2^2\right)^2}-a_{13}Z_3\right]$.

Let $\bar{\bar{\psi}}^{[3]} = \left(\bar{\bar{\psi}}_1^{[3]}, \bar{\bar{\psi}}_2^{[3]}, \bar{\bar{\psi}}_3^{[3]}\right)^T$ be the eigenvector of $\left(\bar{\bar{J}}_3\right)^T$ for $\lambda_{3Z_3} = 0,$ we get

$\left(\left(\bar{\bar{J}}_3\right)^T - \lambda_{3Z_3} I\right)\bar{\bar{\psi}}^{[3]} = 0$. Let $\bar{\bar{\psi}}^{[3]} = \left(c_3\bar{\bar{\psi}}_1^{[3]}, c_4\bar{\bar{\psi}}_3^{[3]}, \bar{\bar{\psi}}_3^{[3]}\right)^T$, where $\bar{\bar{\psi}}_3^{[3]} \neq 0$ be any real number and $c_3 = \frac{Z_3}{a_{13}Z_2}\left[\frac{a_{11}(a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2} - a_{13}Z_3\right]$, $c_4 = \frac{-a_{13}Z_2Z_3}{Z_1\left(\frac{a_6}{a_7 + a_8 Z_2^2} + a_{10}Z_2\right)}$.

Now, consider:

$\frac{\partial f}{\partial a_5} = f_{a_5}\left(\tilde{X}, a_5\right) = \left(\frac{\partial \hat{f}_1}{\partial a_5}, \frac{\partial \hat{f}_2}{\partial a_5}, \frac{\partial \hat{f}_3}{\partial a_5}\right)^{\mathrm{T}} = (0, Z_1Z_2, 0)^{\mathrm{T}}$.

So, $f_{a_5}(M_3, \bar{\bar{a}}_5) = (0, Z_1Z_2, 0)^{\mathrm{T}}$ and hence $\left(\bar{\bar{\psi}}^{[3]}\right)^T f_{a_5}(M_3, \bar{\bar{a}}_5) = c_4 Z_1 Z_2 \bar{\bar{\psi}}_3^{[3]} \neq 0$.

Now, $Df_{a_5}\left(\tilde{X}, a_5\right) = \begin{bmatrix} 0 & 0 & 0 \\ Z_2 & Z_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where it is the derivative of $f_{a_5}\left(\tilde{X}, a_5\right)$ for $\tilde{X} = (Z_1, Z_2, Z_3)^T$.

Further, it is observed that

$$Df_{a_5}(M_3, \bar{\bar{a}}_5)\bar{\bar{K}}^{[3]} = \begin{bmatrix} 0 & 0 & 0 \\ Z_2 & Z_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{\bar{K}}_1^{[3]} \\ c_1\bar{\bar{K}}_1^{[3]} \\ c_2\bar{\bar{K}}_1^{[3]} \end{bmatrix} = \begin{bmatrix} 0 \\ (Z_2 + c_1 Z_1)\bar{\bar{K}}_1^{[3]} \\ 0 \end{bmatrix}.$$

Therefore, $\left(\bar{\bar{\psi}}^{[3]}\right)^{\mathrm{T}}\left[Df_{a_5}(M_3, \bar{\bar{a}}_5)\bar{\bar{K}}^{[3]}\right] = c_4(Z_2 + c_1 Z_1)\bar{\bar{K}}_1^{[3]}\bar{\bar{\psi}}_3^{[3]} \neq 0$.

Thus, for M_3 according to the Sotomayor theorem, SNB can appear if condition (10) holds. It was found that the conversion rate of food from the prey to the middle predator $\bar{\bar{a}}_5 = a_5$ creates and destroys the point M_3.

3.2 The analysis of Hopf Bifurcation

The HB appearance has been studied according to the Haque and Venturino methods [19] for the positive equilibrium point of system (1) for $n = 3$ as below:

$P_4(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$

Let the HB's first condition satisfies if and only if $A_i > 0$; $i = 1, 2$. $\Delta = A_1A_2 - A_3 = 0$.

Then, the characteristic equation becomes:

$$P_3(\lambda) = (\lambda + A_1)(\lambda^2 + A_2) = 0 . \tag{11}$$

In Eq. (11), three roots have appeared; $\lambda = -A_1$ and $\lambda_{1,2} = \pm i\sqrt{A_2}$.

To demonstrate the transversal condition of HB, by replacing $\lambda(\mu) = \vartheta_1(\mu) \mp i\vartheta_2(\mu)$ into Eq. (11), then differentiate for μ, $P_3'\big(\lambda(\mu)\big) = 0$, to find the real and imaginary parts, have:

$$\left.\begin{aligned} &\varphi^*(\mu)\vartheta_1'(\mu) - \Phi^*(\mu)\vartheta_2' + \theta^*(\mu) = 0, \\ &\Phi^*(\mu)\vartheta_1'(\mu) + \varphi^*(\mu)\vartheta_2'(\mu) + \Gamma^*(\mu) = 0. \end{aligned}\right\} \tag{12}$$

where:

$$\left.\begin{aligned} &\varphi^*(\mu) = 3(\vartheta_1(\mu))^2 + 2A_1(\mu)\sigma_1(\mu) + Z_2(\mu) - 3(\vartheta_2(\mu))^2 \\ &\Phi^*(\mu) = 6\vartheta_1(\mu)\vartheta_2(\mu) + 2A_1(\mu)\vartheta_2(\mu) \\ &\theta^*(\mu) = (\vartheta_1(\mu))^2 + A_1'(\mu) + A_2'(\mu)\vartheta_1(\mu) + A_3'(\mu) - A_1'(\mu)(\vartheta_2(\mu))^2 \\ &\Gamma^*(\mu) = 2\vartheta_1(\mu)\vartheta_2(\mu)A_1'(\mu) + A_2'(\mu)\vartheta_2(\mu) \end{aligned}\right\} \tag{13}$$

By using Cramer's rule [19] system (12) for $\sigma_1'(\mu)$ and $\sigma_2'(\mu)$ can be solved as

$$\sigma_1'(\mu) = -\frac{\theta^*(\mu)\,\varphi^*(\mu) + \Gamma^*(\mu)\,\Phi^*(\mu)}{\big(\varphi^*(\mu)\big)^2 + \big(\Phi^*(\mu)\big)^2}, \qquad \text{and } \sigma_2'(\mu) = \frac{-\Gamma^*(\mu)\,\varphi^*(\mu) + \theta^*(\mu)\,\Phi^*(\mu)}{(\varphi^*(\mu))^2 + (\Phi^*(\mu))^2}.$$

Therefore, the transversal condition of the HB is $\frac{d}{d\mu}Re(\lambda)\,\Big|_{\mu=\bar{\mu}} = \sigma_1'(\mu)\,\Big|_{\mu=\bar{\mu}} \neq 0$ if and only if: $\theta^*(\mu)\,\varphi^*(\mu) + \Gamma^*(\mu)\,\Phi^*(\mu) \neq 0.$ (14)

In the following theorem, the effect of the rate of food conversion from the middle predator to the top predator (and thus the transfer of toxicity) on the point M_3 is studied for HB.

Theorem: Suppose that the following conditions hold

$$\frac{a_1}{a_2} + a_4 > a_5\,, \tag{15}$$

$$a_7 > a_8 Z_2^2\,, \tag{16}$$

$$a_5 Z_1 < a_9 + 2a_{10}Z_2Z_3 + \frac{a_6Z_3(a_7 - a_8Z_2^2)}{(a_7 + a_8Z_2^2)^2} \tag{17}$$

$$a_3Z_3 > \frac{a_{11}(a_7 - a_8Z_2^2)}{(a_7 + a_8Z_2^2)^2}, \tag{18}$$

$$d_3 = d_1 d_2, \tag{19}$$

Then at M_3 for $a_{11}^* = a_{11}$ system (1) has an HB.

Proof: The characteristic equation of system (1) at M_3 stated in [17].

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{20}$$

To verify HB's necessary and sufficient conditions; $d_i(a_{11}^*) > 0,\ i = 1,2$ and $\Delta(a_{11}^*) = d_1 d_2 - d_3 = 0$.

$\mathrm{d_i}(a_{11}^*) > 0,\ \mathrm{i} = 1,2\ , \Delta(a_{11}^*) = 0$ provided conditions (15-19) hold. Moreover, Eq.(20) has one positive root based on Descartes' rule [19].

Now, at $(a_{11}^* = a_{11})$, Eq. (20) can be rewritten as:

$$P(\lambda) = (\lambda + d_1)(\lambda^2 + d_2) \tag{21}$$

has the roots: $\lambda_1 = -d_1$ *and* $\lambda_{2,3} = \mp i\sqrt{d_2}$.

Now, for a_{11} close to d_2, the three roots are generally given as:

$\lambda_{2,3}\,(a_{11}) = \varepsilon_1(a_{11}) \pm i\varepsilon_2(a_{11})$.

Clearly, $Re\big(\lambda_{2,3}\,(a_{11})\big)\,\big|_{a_{11}^* = a_{11}} = \varepsilon_1(a_{11}^*) = 0$. Now, to prove the transversal condition:

$\theta^*(a_{11}^*)\,\varphi^*(a_{11}^*) + \Gamma^*(a_{11}^*)\,\Phi^*(a_{11}^*) \neq 0$, where $\theta^*, \varphi^*, \Gamma^*$ and Φ^* for $(a_{11}^* = a_{11})$ we get:

$\varepsilon_1 = 0$ and $\varepsilon_2 = \sqrt{c_2}$, substituting ε_2 in Eq. (13) gives:

$$\left.\begin{array}{c} \theta^*(a_{11}^*) = \dfrac{Z_2 Z_3 (a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2}\left(\dfrac{a_6}{a_7 + a_8 Z_2^2} + a_{10} Z_2\right), \\ \varphi^*(a_{11}^*) = 2c_1 c_2, \qquad \Gamma^*(a_{11}^*) = 0 \\ \Phi^*(a_{11}^*) = -2c_2 \end{array}\right\}, \tag{22}$$

Now, substituting Eq. (22) in Eq. (14) gives

$$\theta^*(a_{11}^*)\,\varphi^*(a_{11}^*) + \Gamma^*(a_{11}^*)\,\Phi^*(a_{11}^*)$$
$$= \frac{Z_2 Z_3 (a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2}\left(\frac{a_6}{a_7 + a_8 Z_2^2} + a_{10} Z_2\right)(2c_1 c_2) + 0(-2c_2)$$

$$\theta^*(a_{11}^*)\,\varphi^*(a_{11}^*) + \Gamma^*(a_{11}^*)\,\Phi^*(a_{11}^*) = 2c_1 c_2 \frac{Z_2 Z_3 (a_7 - a_8 Z_2^2)}{(a_7 + a_8 Z_2^2)^2}\left(\frac{a_6}{a_7 + a_8 Z_2^2} + a_{10} Z_2\right) \neq 0$$

Therefore, system (1) at M_3 with the parameter a_{11}^* has an HB if condition (16) holds. It found that the conversion rate of food from the middle predator to the top predator (and thus the transfer of toxicity) $a_{11}^* = a_{11}$ made triggers oscillations and a limit cycle around M_3.

4. NUMERICAL ANALYSIS

In this section, the results obtained previously in [17] are numerically verified for another set of parameters, as given in (23), along with the local and Hopf bifurcation conditions. It is noticed that system (1) has global stability and converges to $M_3 = (0.563, 0.106, 0.128)$ starting from three different initial points $(7,5,3)$, $(2,4,6)$ and $(1,3,5)$ as shown in Figure 1(a-d).

$$\left.\begin{aligned} a_1 = 0.5, a_2 = 0.9, a_3 = 0.7, a_4 = a_{10} = 0.2, a_5 = a_8 = 0.4, \\ a_6 = a_7 = a_9 = a_{13} = 0.1, a_{11} = a_{12} = 0.09\,. \end{aligned}\right\} \tag{23}$$

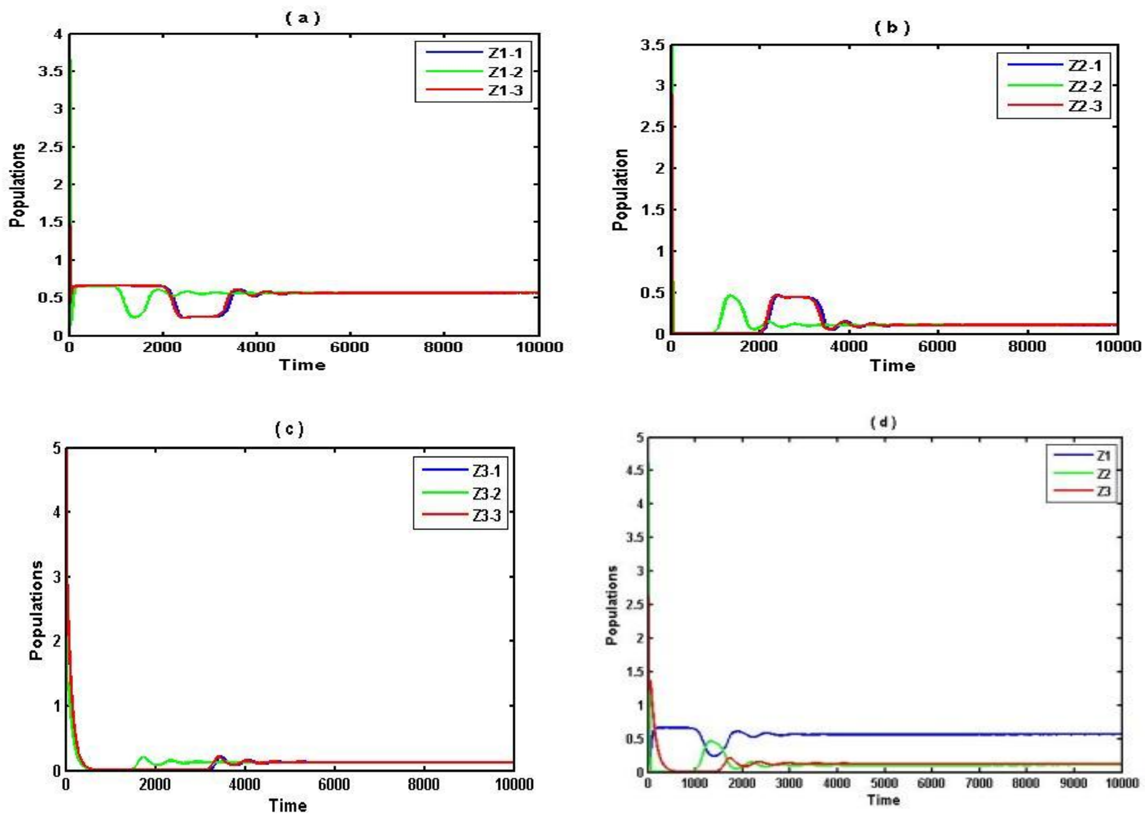

Figure1. Time series of the system's (1) (TS) solution beginning from $(7,5,3)$, $(2,4,6)$ and $(1,3,5)$ for equation (23), the solution converges to $M_3 = (0.563, 0.106, 0.128)$, (a) the prey's paths Z_1, (b) The middle predator's paths Z_2, (c) the top predator's paths of Z_3. (d) The paths from all initial points.

Now, the effectiveness of parameters will be discussed by varying one parameter each time for the set (23), and Table 3 summarizes the results of the set (23).

Table 3: The numerical results of system (1)

Range	Converge	Bifurcation
$0.1 \le a_1 < 0.19$ $0.19 \le a_1 \le 1$	M_2 M_3	 $a_1 = 0.19$
$0.2 < a_2 \le 0.31$ $0.31 < a_2 < 0.4$ $0.4 \le a_2 \le 0.99$	M_1 M_2 M_3	$a_2 = 0.31$ $a_2 = 0.4$
$0.01 \le a_3 < 0.12$ $0.12 \le a_3 < 0.99$	M_2 M_3	 $a_3 = 0.12$
$0.1 \le a_4 < 0.9$ $0.9 \le a_4 \le 0.99$	M_3 M_2	 $a_4 = 0.9$
$0.01 < a_5 \le 0.17$ $0.17 < a_5 \le 0.2$ $0.2 < a_5 \le 1$	M_1 M_2 M_3	$a_5 = 0.17$ $a_5 = 0.2$
$0.01 \le a_6 < 0.4$	M_3	
$0.05 \le a_7 < 0.4$ $0.4 \le a_7 \le 1$	M_3 M_2	 $a_7 = 0.4$
$0.1 \le a_8 < 1$	M_3	
$0.01 \le a_9 \le 0.21$ $0.21 < a_9 \le 0.25$ $0.25 < a_9 \le 0.99$	M_3 M_2 M_1	$a_9 = 0.21$ $a_9 = 0.25$
$0.01 \le a_{10} < 1$	M_3	
$0.01 \le a_{11} \le 0.035$ $0.035 < a_{11} < 0.28$	M_2 M_3	$a_{11} = 0.035$
$0.01 \le a_{12} < 0.13$ $0.13 \le a_{12} < 0.23$ $0.23 \le a_{12} \le 0.99$	M_1 M_3 M_2	 $a_{12} = 0.13$ $a_{12} = 0.23$
$0.01 \le a_{13} < 1$	M_3	

Now, we can explain the results we obtained as shown in Table 3 for equation (23) in the following figures for only these have bifurcation, where Figure 2 shows how a_1 (the prey's population growth rate) has bifurcation at $a_1 = 0.19$ and how it effects in transforming from M_2 to M_3 and keeps all species.

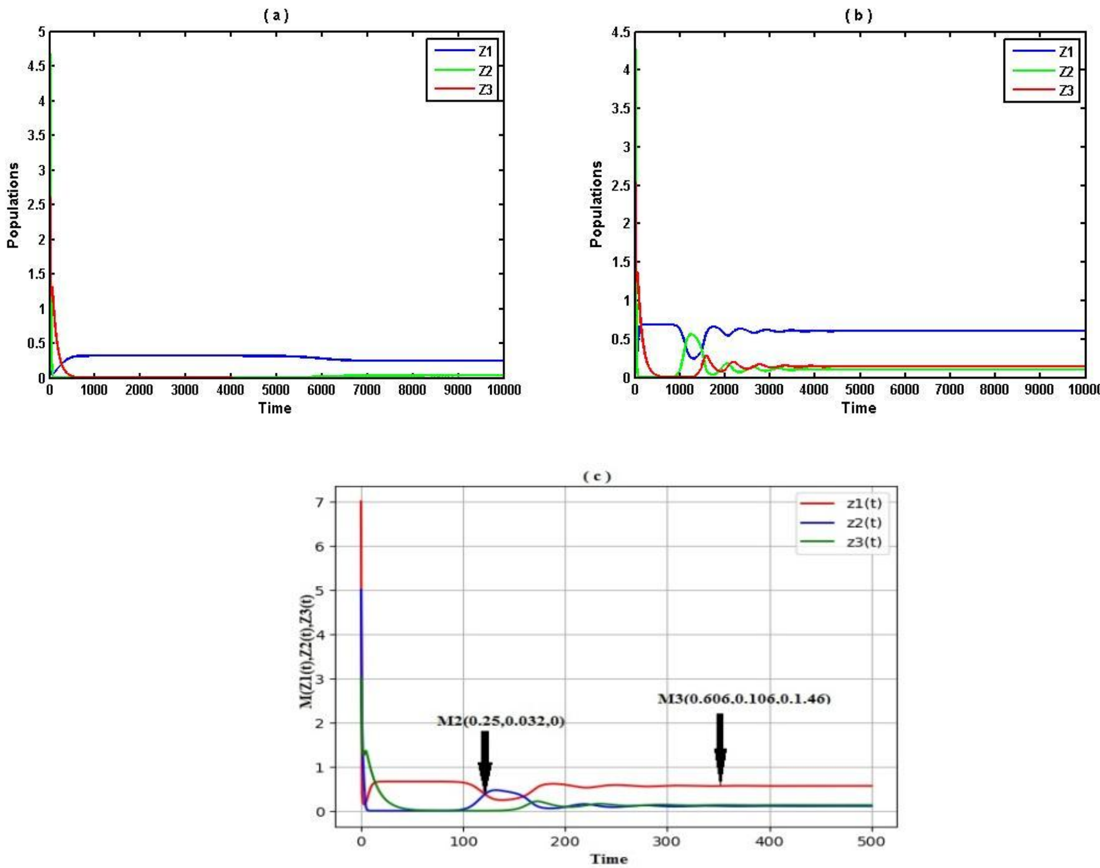

Figure 2: TS of the solution converges to (a) $M_2 = (0.25,0.032,0)$, for $a_1 = 0.1$. (b) $M_3 = (0.606,0.106,0.146)$, for $a_1 = 0.6$, (c) the bifurcation of a_1 .

Figure 3 shows how a_2 (the prey's carrying capacity) has a bifurcation at $a_2 = 0.31$ and $a_2 = 0.4$ and how it effects the transformation from M_1 to M_2 then to M_3 and keeps all species.

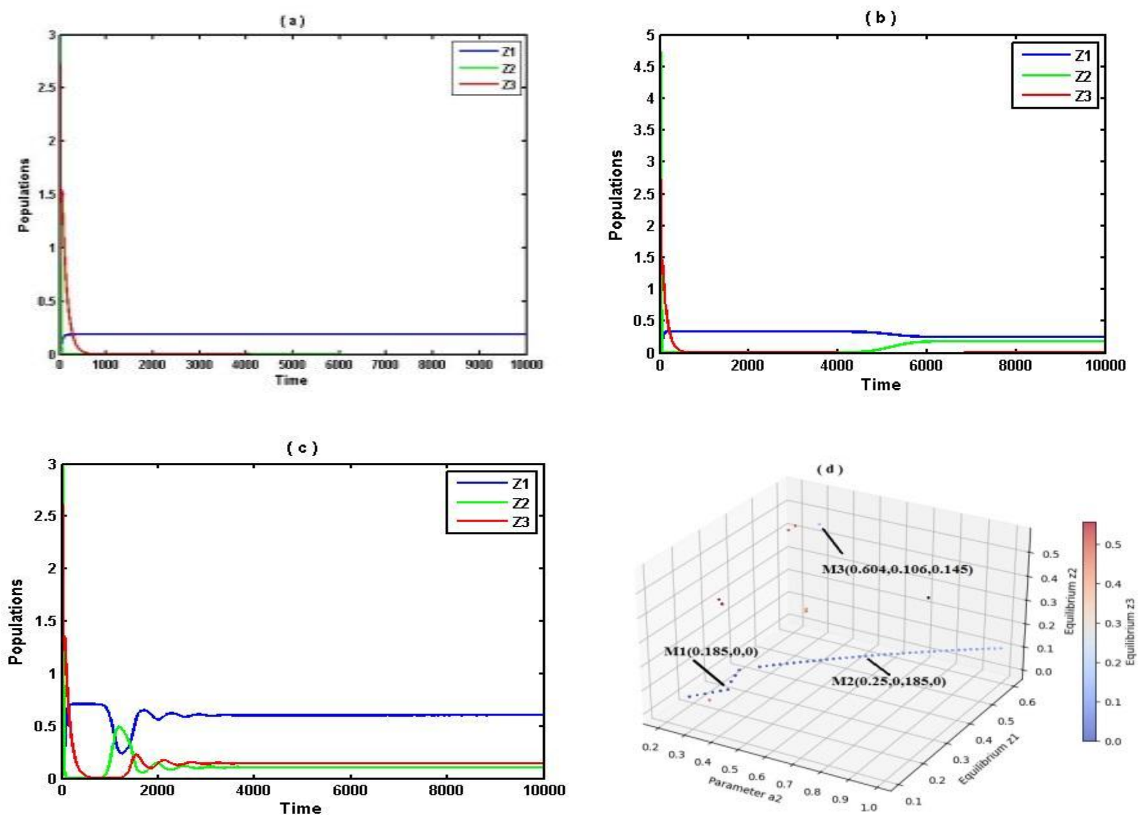

Figure 3: TS of the solution converges to (a) $M_1 = (0.185,0,0)$, for $a_2 = 0.2$ (b) $M_2 = (0.25,0.185,0)$, for $a_2 = 0.3$ (c) $M_3 = (0.604,0.106,0.145)$, for $a_2 = 0.99$, (d) the bifurcation of a_2 .

Figure 4 shows how a_3 (the middle predators' maximum predation rate over the prey) has a bifurcation at $a_3 = 0.12$ and how it affects the transformation from M_2 to M_3 and keeps all species.

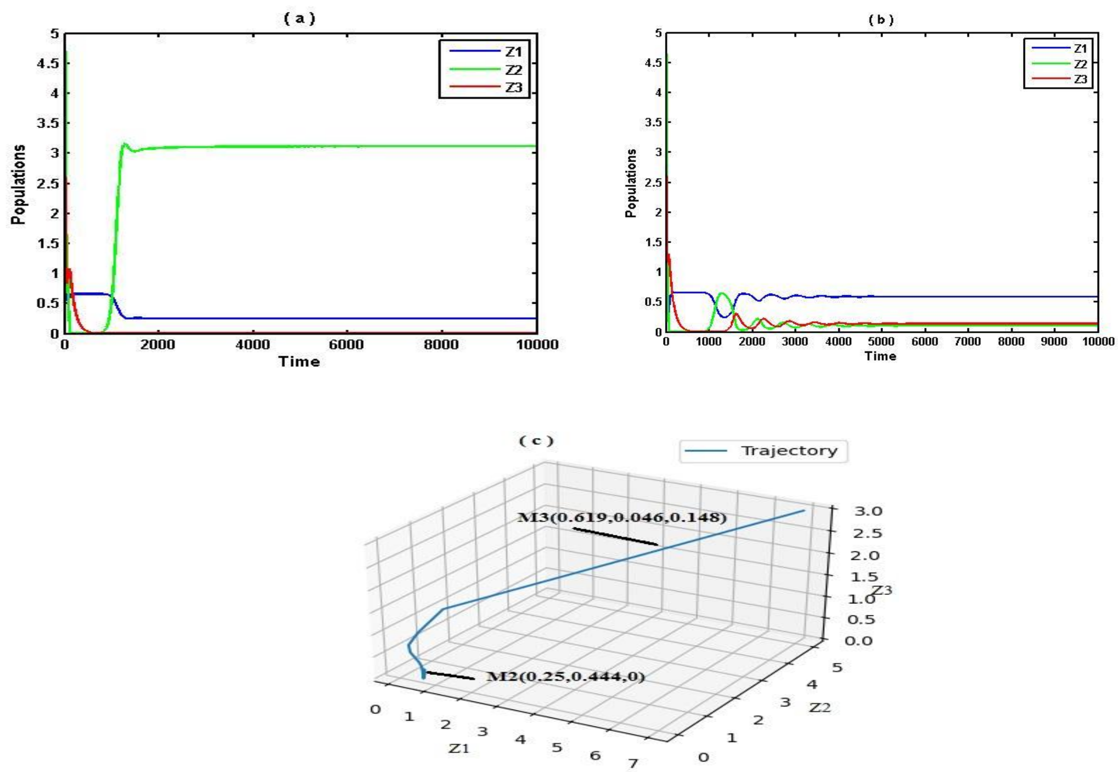

Figure 4: TS of the solution converges to (a) $M_2 = (0.25,3.111,0)$, for $a_3 = 0.1$ (b) $M_3 = (0.501,0.106,0.140)$, for $a_3 = 0.5$, (c) the bifurcation of a_3 .

Figure 5 shows how a_4 (The toxicant environment rate on the prey) has a bifurcation at $a_4 = 0.9$ and how it effects the transformation from M_3 to M_2 and keeps only the prey and middle predator alive.

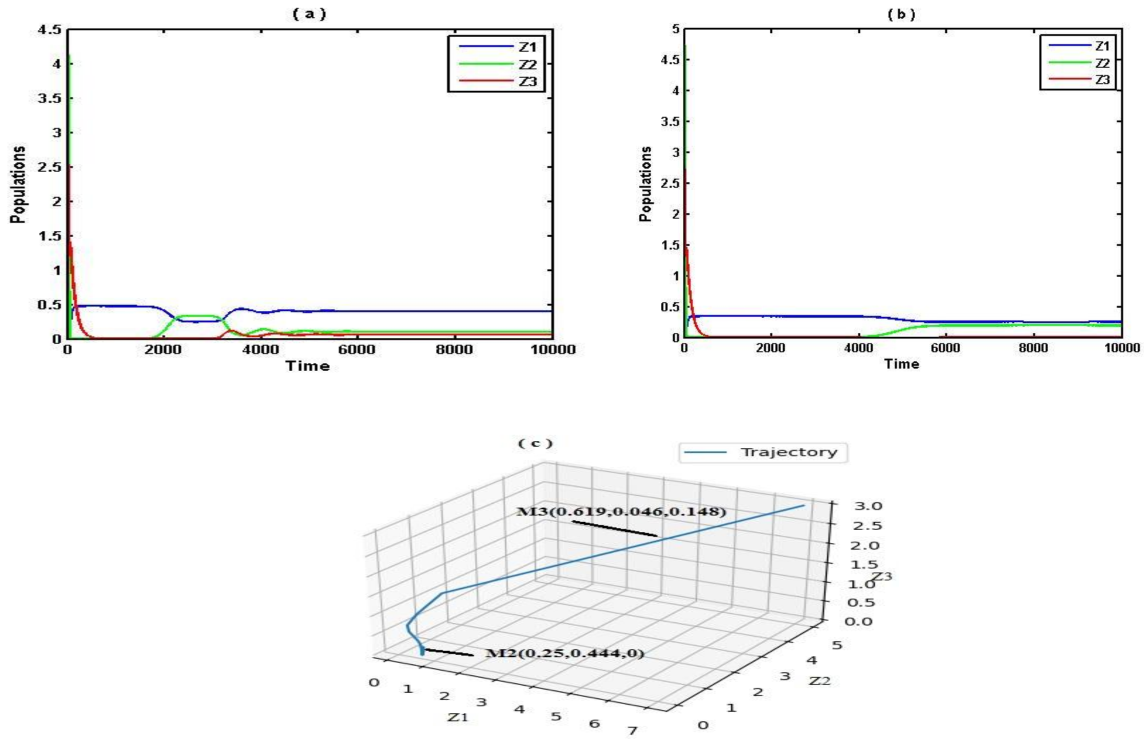

Figure 5: TS of the solution converges to (a) $M_3 = (0.404,0.105,0.663)$, for $a_4 = 0.5$. (b) $M_2 = (0.25,0.194,0)$, for $a_4 = 0.9$, (c) the bifurcation of a_4 .

Figure 6 shows how a_5 (The rate of food conversion from the prey to the middle predator) has a bifurcation at $a_5 = 0.17$ and $a_5 = 0.2$ and how it effects the transformation from M_1 to M_2 then to M_3 and keeps all species.

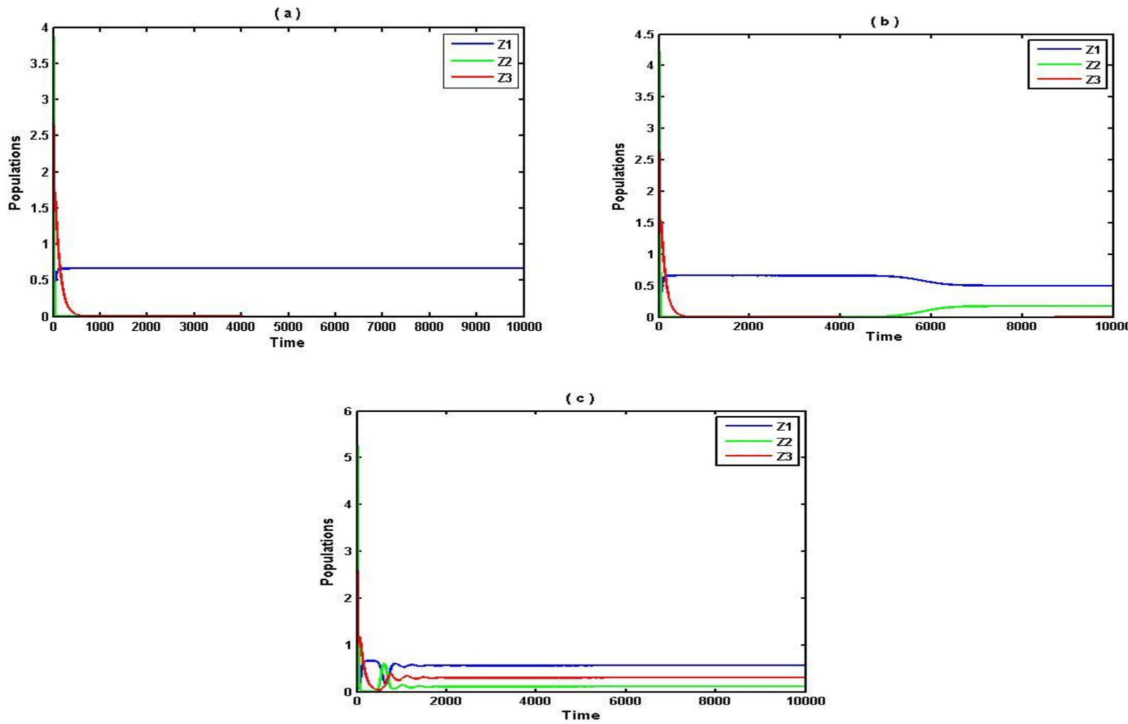

Figure 6: TS of the solution converges to (a) $M_1 = (0.662,0,0)$, for $a_5 = 0.1$ (b) $M_2 = (0.5,0.175,0)$, for $a_5 = 0.2$. (c) $M_3 = (0.561,0.108,0.3)$, for $a_5 = 0.7$.

Figure 7 shows how a_7 (the middle predator's half-saturation constant) has a bifurcation at $a_7 = 0.4$, and how it effects transforming from M_3 to M_2 and keeps only the prey and middle predator alive.

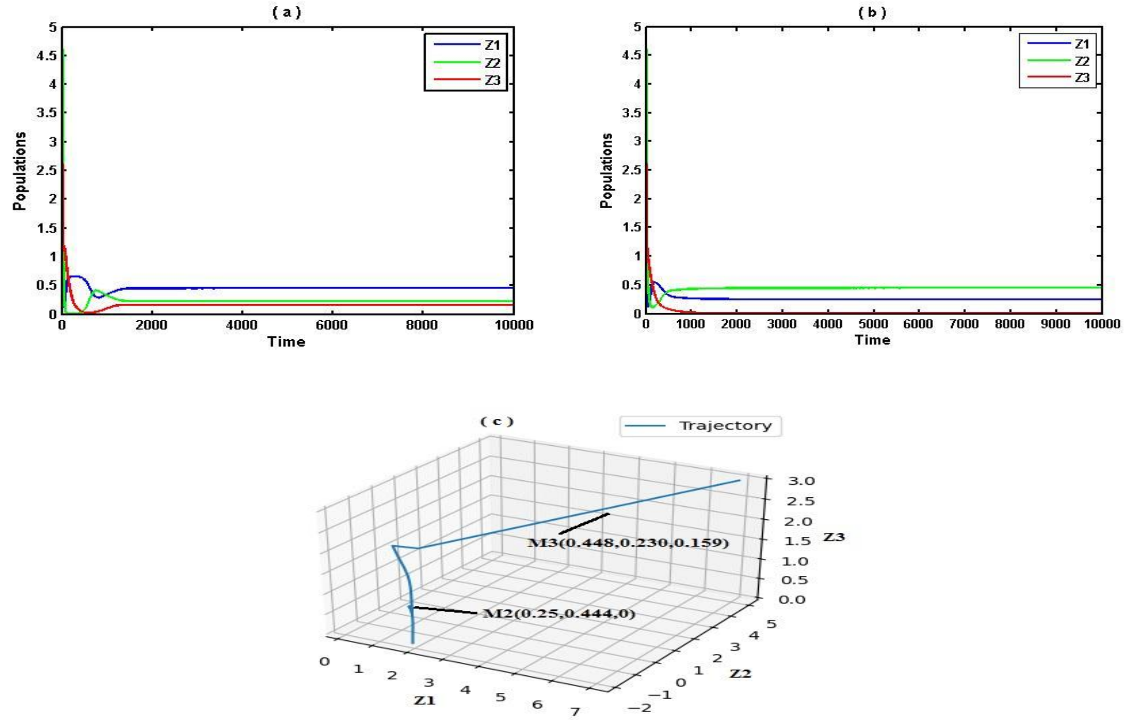

Figure 7: TS of the solution converges to (a) $M_3 = (0.448, 0.230, 0.159)$, for $a_7 = 0.2$. (b) $M_2 = (0.25, 0.444, 0)$, for $a_7 = 0.5$, (c) the bifurcation of a_7 .

Figure 8 shows how a_9 (the middle predator's harvesting rate) has a bifurcation at $a_9 = 0.21$ and $a_9 = 0.25$, and how it effects transforming from M_3 to M_2 then to M_1 and keeps only the prey alive.

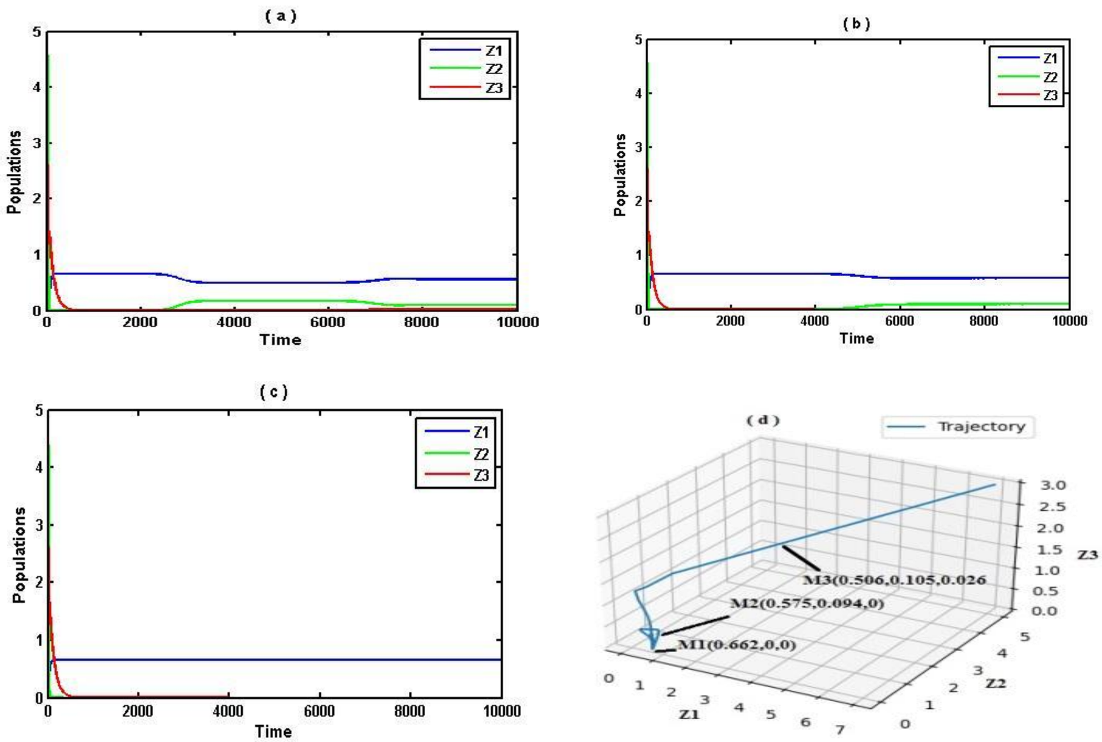

Figure 8: TS of the solution converges to (a) $M_3 = (0.565,0.105,0.026)$, for $a_9 = 0.2$ (b) $M_2 = (0.575,0.094,0)$, for $a_9 = 0.23$ (c) $M_1 = (0.662,0,0)$, for $a_9 = 0.7$, (d) the bifurcation of a_9 .

Figure 9 shows how a_{11} (the rate of food conversion from the middle predator to the top predator) has a bifurcation at $a_{11} = 0.035$ and how it affects transforming from M_2 to M_3 and keeps all species.

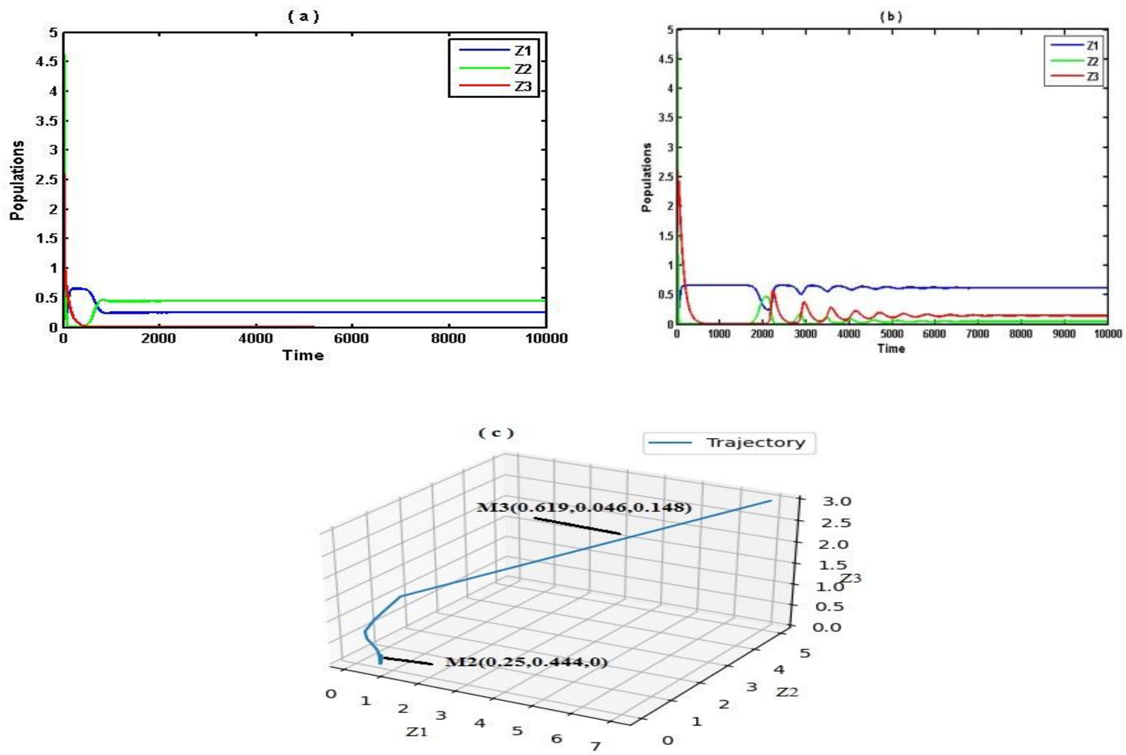

Figure 9: TS of the solution converges to (a) $M_2 = (0.25,0.444,0)$, for $a_{11} = 0.01$,(b) $M_3 = (0.619,0.046,0.148)$, for $a_{11} = 0.2$, (c) the bifurcation of a_{11} .

Figure 10 shows how a_{12} (the top predator's mortality rate) has a bifurcation at $a_{12} = 0.13$ and $a_{12} = 0.23$, and how it effects transforming from $M_1$to M_3then to M_2 and keeps only the prey and middle predator alive.

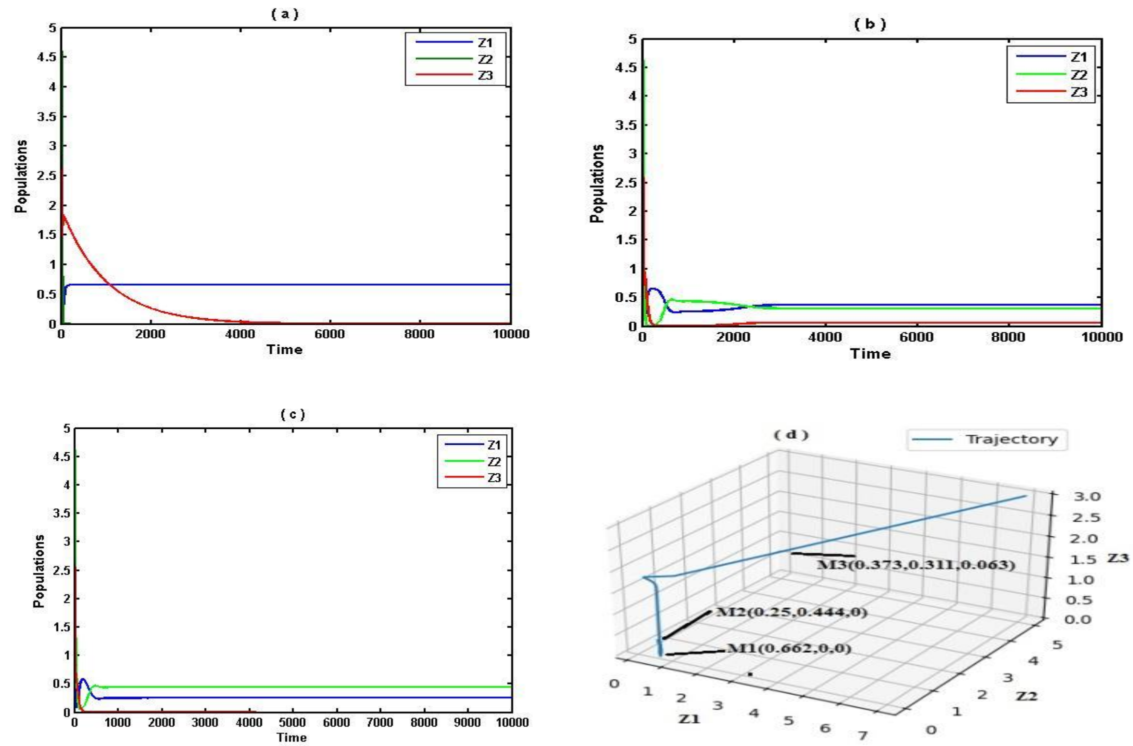

Figure 10: TS of the solution converges to (a) $M_1 = (0.662,0,0)$, for $a_{12} = 0.01$. (b) $M_3 = (0.373,0.311,0.063)$, for $a_{12} = 0.2$. (c) $M_2 = (0.25,0.444,0)$, for $a_{12} = 0.3$, (d) the bifurcation of a_{12} .

5. CONCLUSION

In this paper, the bifurcation analysis provides rich information and a clear picture of the effect of parameters on the stability of the equilibrium points, as well as the essential conditions for the appearance of this bifurcation. It was noted:

1- For M_1, it is possible for the PFB to appear with the harvest rate $\check{a}_9 = a_9$ affected the stability of the point M_1 So, a new point was created.

2- The second point M_2 TB can appear with the top predator's death rate $\check{a}_{12} = a_{12}$ (due to poisoning) was affecting the point M_2, causing it to fluctuate in its stability

3- The third point M_3 the SNB can appear with the conversion rate of food from the prey to the middle predator $\bar{\bar{a}}_5 = a_5$ it causes the creation and destruction of the point M_3

4- At the positive point M_3, it is possible for the HB to appear with that conversion rate of food from the middle predator to the top predator (and thus the transfer of toxicity) $a_{11}^{*} = a_{11}$ made triggers oscillations and a limit cycle around M_3.

5- Changing the set as shown in Table 4 made some parameters effective, as bifurcation appeared when they were changed, such as a_1, a_2, a_3, a_4 and a_7.

6- Some parameters showed a different effect in set 2 than in set 1, as shown in Table 4, such as a_9 and a_{12}.

7- Some parameters did not show any effect after changing the set, such as a_6, a_8, a_{10} *and* a_{13}.

Table 4 presents the effect of parameters on the appearance of bifurcation, comparing Set (23) with the set provided in [17].

Table 4: The comparison results

Set.1 [15] initial points (5,3,1) , (2,3,5) *and* (1,2,3) $M_3 = (0.059, 2.657, 0.093)$ $\left.\begin{matrix} a_1 = 1, a_2 = a_3 = 0.3, a_4 = a_6 = 0.1, a_5 = 0.25, a_7 = a_8 = 0.5, \\ a_9 = a_{10} = a_{12} = 0.01, a_{11} = 0.09, a_{13} = 0.2 \end{matrix}\right\}$			Set.2 initial points (7,5,3) , (2,4,6) *and* (1,3,5) $M_3 = (0.563, 0.106, 0.128)$ $\left.\begin{matrix} a_1 = 0.5, a_2 = 0.9, a_3 = 0.7, a_4 = a_{10} = 0.2, a_5 = a_8 = 0.4, \\ a_6 = a_7 = a_9 = a_{13} = 0.1, a_{11} = a_{12} = 0.09\,. \end{matrix}\right\}$		
Range	Converge	Bifurcation	Range	Converge	Bifurcation
$0.1 \le a_1 < 1.5$	M_3		$0.1 \le a_1 < 0.19$ $0.19 \le a_1 \le 1$	M_2 M_3	 $a_1 = 0.19$
$0.2 \le a_2 < 1$	M_3		$0.2 < a_2 \le 0.31$ $0.31 < a_2 < 0.4$ $0.4 \le a_2 \le 0.99$	M_1 M_2 M_3	$a_2 = 0.31$ $a_2 = 0.4$
$0.3 \le a_3 < 1$	M_3		$0.01 \le a_3 < 0.12$ $0.12 \le a_3 < 0.99$	M_2 M_3	 $a_3 = 0.12$
$0.1 \le a_4 < 1$	M_3		$0.1 \le a_4 < 0.9$ $0.9 \le a_4 \le 0.99$	M_3 M_2	 $a_4 = 0.9$
$0.01 < a_5 \le 0.05$ $0.05 < a_5 \le 0.08$ $0.08 < a_5 \le 0.3$	M_1 M_2 M_3	$a_5 = 0.05$ $a_5 = 0.08$	$0.01 < a_5 \le 0.17$ $0.17 < a_5 \le 0.2$ $0.2 < a_5 \le 1$	M_1 M_2 M_3	$a_5 = 0.17$ $a_5 = 0.2$
$0.01 \le a_6 < 0.5$	M_3		$0.01 \le a_6 < 0.4$	M_3	

$0.1 \le a_7 \le 1$	M_3		$0.05 \le a_7 < 0.4$ $0.4 \le a_7 \le 1$	M_3 M_2	$a_7 = 0.4$
$0.1 \le a_8 \le 1$	M_3		$0.1 \le a_8 < 1$	M_3	
$0.01 \le a_9 < 0.07$ $0.07 \le a_9 < 1$	M_3 M_1	$a_9 = 0.07$	$0.01 \le a_9 \le 0.21$ $0.21 < a_9 \le 0.25$ $0.25 < a_9 \le 0.99$	M_3 M_2 M_1	$a_9 = 0.21$ $a_9 = 0.25$
$0.01 \le a_{10} < 1$	M_3		$0.01 \le a_{10} < 1$	M_3	
$0.01 \le a_{11} \le 0.04$ $0.04 < a_{11} \le 0.1$	M_2 M_3	$a_{11} = 0.04$	$0.01 \le a_{11} \le 0.035$ $0.035 < a_{11} < 0.28$	M_2 M_3	$a_{11} = 0.035$
$0.01 \le a_{12} < 0.06$ $0.06 \le a_{12} < 1$	M_3 M_2	$a_{12} = 0.06$	$0.01 \le a_{12} < 0.13$ $0.13 \le a_{12} < 0.23$ $0.23 \le a_{12} \le 0.99$	M_1 M_3 M_2	$a_{12} = 0.13$ $a_{12} = 0.23$
$0.1 \le a_{13} < 1$	M_3		$0.01 \le a_{13} < 1$	M_3	

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] S.J. Gill, M.H. Lowenberg, S.A. Neild, B. Krauskopf, G. Puyou, E. Coetzee, Upset Dynamics of an Airliner Model: a Nonlinear Bifurcation Analysis, J. Aircr. 50 (2013), 1832-1842. https://doi.org/10.2514/1.c032221.

[2] S.H. Salih, Z.K. Alabacy, S. Khlil, S.N. Shihab, N.M.G. Al-Saidi, Fear-Driven Disease Control: A Two-Host, One-Parasite Model with SI Dynamics and Behavioral Change, Commun. Math. Biol. Neurosci. 2024 (2024), 130. https://doi.org/10.28919/cmbn/8888.

[3] A. Ali, J. Ahmad, S. Javed, S. Alkarni, N.A. Shah, Investigate the Dynamic Nature of Soliton Solutions and Bifurcation Analysis to a New Generalized Two-Dimensional Nonlinear Wave Equation with Its Stability, Results Phys. 53 (2023), 106922. https://doi.org/10.1016/j.rinp.2023.106922.

[4] S. Khlil, H. Al-Khazraji, Z. Alabacy, Solving Assembly Production Line Balancing Problem Using Greedy Heuristic Method, IOP Conf. Ser.: Mater. Sci. Eng. 745 (2020), 012068. https://doi.org/10.1088/1757-899x/745/1/012068.

[5] Y. Zhang, J. Luo, Bifurcation Analysis of a Leslie-Type Predator–prey System with Prey Harvesting and Group Defense, Front. Phys. 12 (2024), 1392446. https://doi.org/10.3389/fphy.2024.1392446.

[6] S.F. Aldosary, R. Ahmed, Stability and Bifurcation Analysis of a Discrete Leslie Predator-Prey System via

Piecewise Constant Argument Method, AIMS Math. 9 (2024), 4684-4706. https://doi.org/10.3934/math.2024226.

[7] P.A. Naik, Z. Eskandari, H.E. Shahkari, K.M. Owolabi, Bifurcation Analysis of a Discrete-Time Prey-Predator Model, Bull. Biomath. (2023), 111-123. https://doi.org/10.59292/bulletinbiomath.2023006.

[8] S. Li, S. Yuan, Z. Jin, H. Wang, Bifurcation Analysis in a Diffusive Predator-Prey Model with Spatial Memory of Prey, Allee Effect and Maturation Delay of Predator, J. Differ. Equ. 357 (2023), 32-63. https://doi.org/10.1016/j.jde.2023.02.009.

[9] M.B. Almatrafi, M. Berkal, Bifurcation Analysis and Chaos Control for Prey-Predator Model with Allee Effect, Int. J. Anal. Appl. 21 (2023), 131. https://doi.org/10.28924/2291-8639-21-2023-131.

[10] C. Gros, Bifurcations and Chaos in Dynamical Systems, in: Complex and Adaptive Dynamical Systems, Springer International Publishing, Cham, 2015: pp. 43-77. https://doi.org/10.1007/978-3-319-16265-2_2.

[11] A. Arenas, A. Garijo, S. Gómez, J. Villadelprat, Bifurcation Analysis of the Microscopic Markov Chain Approach to Contact-Based Epidemic Spreading in Networks, Chaos Solitons Fractals 166 (2023), 112921. https://doi.org/10.1016/j.chaos.2022.112921.

[12] L. Tang, A. Biswas, Y. Yıldırım, A.A. Alghamdi, Bifurcation Analysis and Optical Solitons for the Concatenation Model, Phys. Lett. 480 (2023), 128943. https://doi.org/10.1016/j.physleta.2023.128943.

[13] K.Q. Al-Jubouri, R.M. Hussien, N.M.G. Al-Saidi, F. Nazarimehr, Global Stability Analysis and Persistence for an Ecological Food Web-Model, Ital. J. Pure Appl. Math. 49 (2023), 90-103.

[14] K. AL-Jubouri, R. Hussien, N. Al-Saidi, Modeling of an Eco-Epidemiological System Involving Various Epidemic Diseases with Optimal Harvesting, Eurasian J. Math. Comput. Appl. 8 (2020), 4-27. https://doi.org/10.32523/2306-6172-2020-8-2-4-27.

[15] A.J. Kadhim, A.A. Majeed, The Bifurcation Analysis of an Epidemiological Model Involving Two Diseases in Predator, Int. J. Nonlinear Anal. Appl. 13 (2022). 2195-2217. https://doi.org/10.22075/ijnaa.2022.5915.

[16] A.A. Majeed, A.J. Kadhim, The Bifurcation Analysis and Persistence of the Food Chain Ecological Model with Toxicant, J. Phys.: Conf. Ser. 1818 (2021), 012191. https://doi.org/10.1088/1742-6596/1818/1/012191.

[17] Z.K. Alabacy, S.H. Salih, S. Khlil, N.M. Al-Saidi, Stabilizing Ecosystem Dynamics: The Toxicity Effect on an Ecosystem in the Presence of Self-Defense, Commun. Math. Biol. Neurosci. 2024 (2024), 62. https://doi.org/10.28919/cmbn/8575.

[18] Z.K. Alabacy, A.A. Majeed, The Local Bifurcation Analysis of Two Preys Stage-Structured Predator Model with Anti-Predator Behavior, J. Phys.: Conf. Ser. 2322 (2022), 012061. https://doi.org/10.1088/1742-6596/2322/1/012061.

[19] A.A. Majeed, Z.K. Alabacy, The Persistence and Bifurcation Analysis of an Ecological Model with Fear Effect Involving Prey Refuge and Harvesting, AIP Conf. Proc. 2394 (2022), 070001. https://doi.org/10.1063/5.0121877.