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THE BIFURCATION CONDITIONS OF SOKOL-HOWELL PREY-PREDATOR MODEL INVOLVING FEAR AND TOXIN

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Abstract: In this paper, the conditions of the occurrence of local bifurcation (LB) have been founded for a food- chain eco-toxicant model involving three species: prey, first and second predators, incorporating factors such as fear, as well as linear and nonlinear harvesting strategies, including two various functional responses (Lotka-Volterra and Sokol-Howell). This model features six equilibrium points (EPS), all of which are stable under appropriate conditions, the saddle-node bifurcation (SNB) appears near the positive point E_5, transcritical bifurcation (TB) and pitchfork bifurcation (PB) occur near the points E_2 and E_3, while only a transcritical bifurcation takes place near the point E_0, E_1, and E_4. Additionally, the Hopf bifurcation (HB) conditions near the positive equilibrium point (PEP) have been discussed. Finally, the numerical simulation for the hypothetical parameter set confirm our analytical findings about (LB) of this model.

Keywords: food-chain; fear; Sokol-Howell; local bifurcation; Hopf bifurcation.
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1. INTRODUCTION

The prey-predator models have an essential role in understanding the interactions between living populations, attracting substantial attention in mathematical sciences. Furthermore, the

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prey-predator model is a significant subject that covers several fields, including ecology, biology, genetics, and physics. A multitude of researchers have developed and analyzed predator-prey interaction models across numerous fields,[1][2][3][4][5].

Ecology examines the interactions between predators and prey to sustain ecosystem health, illustrating the flexibility of resources and energy, which subsequently affects ecosystem structure,[6][7][8][9][10].

Additionally, several factors, such as fear, toxic substances, and mortality, can influence species interactions within ecosystems. Kadhim and Majeed [11] developed a predator-prey model that has a stage structure in both species incorporating harvesting in the prey population and toxicity affecting all species. Aziz and Majeed [12] study the effect of fear on an ecological model, while many other researchers and academics have also contributed to this field of study, [13][14][15][16][17].

Furthermore, bifurcation theories are a scientific analysis of alterations in the structure of a dynamic system over time. Bifurcation occurs when a minor, gradual modification in a system's parameters results in a sudden alternation in its specific or topological properties. Recently, bifurcation has been the subject of significant research, [18][19][20].

Also, bifurcations are classified into two primary categories: local and global. Local bifurcations including (SNB), (TB), (PB), (HB), and others, occur when parameters surpass critical thresholds and can be examined only by assessing alterations in their local stability characteristics. But when stability and extended invariant sets, such as periodic orbits, intersect, global bifurcation happens, see [21][22][23].

Majeed and Alabacy [24] identified the Hopf bifurcation at the positive equilibria and the criteria for local bifurcation at all equilibrium points in a prey-predator food chain model. Kumar et al. [25] investigated the bifurcation analysis of a predator-prey model involving the Allee effect, the fear effect in prey, and cooperative hunting in predators. and many other studies, see [26][27][28].

Finally, in this paper the conditions needed for the incidence of LB and HB in the proposed model have been discussed.

2. Mathematical Model

Consider the following model [29]

$$\left.\begin{aligned}
\frac{dP}{dT} &= \frac{a_1 P}{1+m_1 P_1} - b_1 P^2 - c_1 P P_1 - h_1 P - \delta_1 P^3, \\
\frac{dP_1}{dT} &= \frac{a_2 P_1}{1+m_2 P_2} - b_2 P_1^2 - \frac{c_2 P_1 P_2}{\alpha+\gamma P_1^2} + l_1 P P_1 - \delta_2 P_1^2 - h_2 P_1^2 - d_1 P_1, \\
\frac{dP_2}{dT} &= \frac{l_2 P_1 P_2}{\alpha+\gamma P_1^2} - \delta_3 P_2 - h_3 P_2^2 - d_2 P_2.
\end{aligned}\right\} \qquad (1)$$

Since the initial conditions are $P(0) \geq 0, P_1(0) \geq 0$, and $P_2(0) \geq 0$. Table 1 explains the biological meaning of system (1) parameters [29]:

Table 1 System (1)'s biologically relevant parameters

Parameters	**Biological description**
$a_i > 0, \quad i = 1,2$	Intrinsic growth rates of the prey and first predator populations, respectively
$b_i > 0, \quad i = 1,2$	Internal competition rates of the prey and first predator, respectively
$m_i > 0, \quad i = 1,2$	Fear rates of the prey and the first predator, respectively
$c_i > 0, \quad i = 1,2$	Attack rates of the prey and first predator, respectively
$l_i > 0, \quad i = 1,2$	Food transfer rates for first and second predators, respectively
$\alpha > 0.$	Half-saturation constant
$\gamma > 0.$	Inverse measure of inhibitory effect
$\delta_i > 0, \quad i = 1,2,3$	Rates of toxicity for the prey, first predator and second predator, respectively
$h_i > 0, \quad i = 1,2,3$	Harvesting rates of prey, first predator and second predator, respectively
$d_i > 0, \quad i = 1,2$	Natural death rates of first predator and second predator, respectively

3. The Local Bifurcation Analysis

In this section, we examined the system's (1) (LB), with particular attention given to the changes that occur around each (EP) as a result of changing the values of the parameters influencing the dynamics. Therefore, the utilization of Sotomayor's theorem [30] is significant in the coming theorems.

Now, with reference to the Jacobian Matrix (JM) $J(P, P_1, P_2)$ of system (1) presented in [29] as follows

$$J = [\mho_{ij}]_{3\times 3}, \qquad (2)$$

$\mho_{11} = \frac{a_1}{1+m_1 P_1} - 2b_1 P - c_1 P_1 - 3\delta_1 P^2 - h_1,$

$\mho_{12} = \frac{-a_1 m_1 P}{(1+ m_1 P_1)^2} - c_1 P$, $\mho_{13} = 0$, $\mho_{21} = l_1 P_1$,

$\mho_{22} = \frac{a_2}{1+ m_2 P_2} - 2b_2 P_1 + l_1 P - \frac{c_2 P_2 (\alpha - \gamma P_1^2)}{(\alpha + \gamma P_1^2)^2} - 2\delta_2 P_1 - d_1 - 2h_2 P_1$,

$\mho_{23} = \frac{-a_2 m_2 P_1}{(1+ m_2 P_2)^2} - \frac{c_2 P_1}{\alpha + \gamma P_1^2}$, $\mho_{31} = 0$, $\mho_{32} = \frac{l_2 P_2 (\alpha - \gamma P_1^2)}{(\alpha + \gamma P_1^2)^2}$,

$\mho_{33} = \frac{l_2 P_1}{\alpha + \gamma P_1^2} - \delta_3 - 2h_3 P_2 - d_2$.

It is essential to understand that for every non - zero vector $N = (n_1, n_2, n_3)^{\mathrm{T}}$, we have

$$D^2 f_\mu(X, \mu)(N, N) = [K_{i1}]_{3\times1}, \tag{3}$$

$$K_{11} = -2\left[(b_1 + 3\delta_1 P) n_1^2 + \left(\frac{a_1 m_1}{(1+m_1 P_1)^2} + c_1\right) n_1 n_2 - \left(\frac{a_1 m_1^2 P}{(1+m_1 P_1)^3}\right) n_2^2\right],$$

$$K_{21} = 2\left[(l_1) n_1 n_2 + \left(\frac{\gamma c_2 P_1 P_2 \left((\alpha + \gamma P_1^2) + 2(\alpha - \gamma P_1^2)\right)}{(\alpha + \gamma P_1^2)^3} - (b_2 + \delta_2 + h_2)\right) n_2^2 - \left(\frac{a_2 m_2}{(1+m_2 P_2)^2} + \frac{c_2 (\alpha - \gamma P_1^2)}{(\alpha + \gamma P_1^2)^2}\right) n_2 n_3 + \left(\frac{a_2 m_2^2 P_1}{(1+m_2 P_2)^3}\right) n_3^2\right],$$

$$K_{31} = -2\left[\left(\frac{\gamma l_2 P_1 P_2 \left((\alpha + \gamma P_1^2) + 2(\alpha - \gamma P_1^2)\right)}{(\alpha + \gamma P_1^2)^3}\right) n_2^2 - \left(\frac{l_2 (\alpha - \gamma P_1^2)}{(\alpha + \gamma P_1^2)^2}\right) n_2 n_3 + (h_3) n_3^2\right].$$

$$D^3 f_\mu(X, \mu)(N, N, N) = [L_{i1}]_{3\times1}, \tag{4}$$

$$L_{11} = -6\left[(\delta_1) n_1^3 - \frac{a_1 m_1^2}{(1+m_1 P_1)^3} \cdot n_1 n_2^2 + \frac{a_1 m_1^3 P}{(1+m_1 P_1)^4} n_2^3\right],$$

$$L_{21} = 2\left[\left(\frac{-2\gamma^2 c_2 P_1 P_2 \left[(\alpha + \gamma P_1^2) + 3P_1 \left((\alpha + \gamma P_1^2) + 2(\alpha - \gamma P_1^2)\right)\right]}{(\alpha + \gamma P_1^2)^4}\right) n_2^3 + \left(\frac{3\gamma c_2 P_1 \left[(\alpha + \gamma P_1^2) + 2(\alpha - \gamma P_1^2)\right]}{(\alpha + \gamma P_1^2)^3}\right) n_2^2 n_3 + \left(\frac{3a_2 m_2^2}{(1+m_2 P_2)^3}\right) n_2 n_3^2 - \left(\frac{3a_2 m_2^3 P_1}{(1+m_2 P_2)^4}\right) n_3^3\right],$$

$$L_{31} = 2\left[\left(\frac{2\gamma^2 l_2 P_1 P_2 \left[(\alpha + \gamma P_1^2) + 3P_1 \left((\alpha + \gamma P_1^2) + 2(\alpha - \gamma P_1^2)\right)\right]}{(\alpha + \gamma P_1^2)^4}\right) n_2^3 - \left(\frac{3\gamma l_2 P_1 \left[(\alpha + \gamma P_1^2) + 2(\alpha - \gamma P_1^2)\right]}{(\alpha + \gamma P_1^2)^3}\right) n_2^2 n_3\right].$$

Where $X = (P, P_1, P_2)^T$ and μ is any parameter.

Theorem 1: For the parameter $a_1^0 = a_1 = h_1$, system (1) exhibits a (TB) at the (EP) E_0 . However, neither a (SNB) nor (PB) may occur at E_0.

Proof: by putting $E_0 = (0, 0, 0)$ into the (JM) in equation (8) in [29] with the parameter value

$(a_1^0 = a_1 = h_1)$.

Then, the characteristic equation of J_0^0 possesses an eigenvalue of zero, which is $(\lambda_{0P} = 0)$ at $a_1^0 = a_1$.

Let $N^{[0]} = (n_1^{[0]}, n_2^{[0]}, n_3^{[0]})^T$ be the eigenvector relating with the eigenvalue $(\lambda_{0P} = 0)$. Thus, $(J_0^0 - \lambda_{0P} I)N^{[0]} = 0$, where $J_0^0 = J(E_0, a_1^0)$.

So, $n_2^{[0]} = 0, n_3^{[0]} = 0$, and $n_1^{[0]} \neq 0$ real number.

Now, let $\Phi^{[0]} = \left(\emptyset_1^{[0]}, \emptyset_2^{[0]}, \emptyset_3^{[0]}\right)^T$ be the eigenvector of J_0^{0T} relating with the eigenvalue $(\lambda_{0P} = 0)$. Thus, $(J_0^{0T} - \lambda_{0P} I)\Phi^{[0]} = 0$. That gives,

$\Phi^{[0]} = (\emptyset_1^{[0]}, 0, 0)^T$, where $\emptyset_1^{[0]} \neq 0$ real number.

Since, $\frac{\partial f}{\partial a_1} = f_{a_1}(X, a_1) = \left(\frac{\partial f_1}{\partial a_1}, \frac{\partial f_2}{\partial a_1}, \frac{\partial f_3}{\partial a_1}\right)^T = \left(\frac{P}{1+m_1 P_1}, 0, 0\right)^T$.

Then, $f_{a_1}(E_0, a_1^0) = (0,0,0)^T$ and hence $\left(\Phi^{[0]}\right)^T f_{a_1}(E_0, a_1^0) = 0$.

In agreement with Sotomayor's theorem, achieving the (SNB) at E_0 is not possible. This indicates that the initial requirement for (TB) is met. Moreover,

$$Df_{a_1}(X, a_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Here, $Df_{a_1}(X, a_1)$ denotes the derivative of $f_{a_1}(X, a_1)$ with $X = (P, P_1, P_2)^T$. Furthermore, it is noted that:

$$Df_{a_1}(E_0, a_1^0)N^{[0]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1^{[0]} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} n_1^{[0]} \\ 0 \\ 0 \end{bmatrix},$$

So, we obtain that:

$$\left(\Phi^{[0]}\right)^T \left[Df_{a_1}(E_0, a_1^0)N^{[0]}\right] = \left(\emptyset_1^{[0]}, 0, 0\right)^T \left(n_1^{[0]}, 0, 0\right) = \emptyset_1^{[0]} n_1^{[0]} \neq 0.$$

By using $N^{[0]}$ in equation (3), we get

$$D^2 f_{a_1}(E_0, a_1^0)\left(N^{[0]}, N^{[0]}\right) = [K_{i1}]_{1 \le i \le 3} = \begin{bmatrix} -2b_1 \left(n_1^{[0]}\right)^2 \\ 0 \\ 0 \end{bmatrix}.$$

So, $\left(\Phi^{[0]}\right)^T \left[D^2 f_{a_1}(E_0, a_1^0)\left(N^{[0]}, N^{[0]}\right)\right] = -2b_1 \left(n_1^{[0]}\right)^2 \emptyset_1^{[0]} \neq 0$.

Then in accordance to the Sotomayor theorem, system (1) reveals a (TB) at E_0 with $a_1 = a_1^0$ and (PB) cannot occur.

Theorem 2: Assume that reversing condition (12) as in [29] and the next condition hold.

$$l_1\breve{A} \neq (b_2 + \delta_2 + h_2) \qquad (5)$$

where,

$$\breve{A} = \frac{\breve{P}\,(a_1 m_1 + c_1)}{a_1 - [\breve{P}(2b_1 + 3\delta_1\breve{P}) + h_1]},$$

then system (1) near E_1 exhibits a (TB) at the parameter value $\left(\breve{d}_1 = d_1 = a_2 + l_1\breve{P}\right)$, but neither (SNB) nor (PB) present near E_1.

Proof: by putting $E_1 = \left(\breve{P}, 0, 0\right)$ in the (JM) in equation (10) in [29] with the parameter value $\breve{d}_1 = d_1 = a_2 + l_1\breve{P}$.

Then, the characteristic equation of $\breve{J}_1$ where $\breve{J}_1 = J\left(E_1, \breve{d}_1\right)$ possesses an eigenvalue zero, which is $(\lambda_{1P_1} = 0)$ at $\breve{d}_1 = d_1$.

Let $\breve{N}^{[1]} = (\breve{n}_1^{[1]}, \breve{n}_2^{[1]}, \breve{n}_3^{[1]})^T$ be the eigenvector relating with the eigenvalue $(\lambda_{1P_1} = 0)$. So, $\left(\breve{J}_1 - \lambda_{1P_1} I\right)N^{[1]} = 0$, where $\breve{J}_1 = J\left(E_1, \breve{d}_1\right)$.

Then, $\breve{n}_1^{[1]} = \breve{A}\breve{n}_2^{[1]}, \breve{n}_3^{[1]} = 0, where\ \breve{A}$ is in the state of the theorem and $\breve{n}_2^{[1]} \neq 0$ real number.

Now, $\breve{\Phi}^{[1]} = (\breve{\emptyset}_1^{[1]}, \breve{\emptyset}_2^{[1]}, \breve{\emptyset}_3^{[1]})^T$ be the eigenvector of $\breve{J}_1^T$ relating with eigenvalue $\lambda_{1P_1} = 0$. Thus, $\left(\breve{J}_1^T - \lambda_{1P_1} I\right)\breve{\emptyset}^{[1]} = 0$. Which gives,

$\breve{\Phi}^{[1]} = (0, \breve{\emptyset}_2^{[1]}, 0)^T$, where $\breve{\emptyset}_2^{[1]} \neq 0$ real number.

Now, since $\frac{\partial f}{\partial d_1} = f_{d_1}(X, d_1) = \left(\frac{\partial f_1}{\partial d_1}, \frac{\partial f_2}{\partial d_1}, \frac{\partial f_3}{\partial d_1}\right)^T = (0, -P_1, 0)^T$.

Then, $f_{d_1}\left(E_1, \breve{d}_1\right) = (0,0,0)^T$ and hence $\left(\breve{\Phi}^{[1]}\right)^T f_{d_1}\left(E_1, \breve{d}_1\right) = 0$.

In agreement with Sotomayor's theorem, achieving the (SNB) at E_1 is not possible. The first requirement for (TB) is therefore achieved. Moreover,

$$Df_{d_1}(X, d_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Here, $Df_{d_1}(X, d_1)$ denotes the derivative of $f_{d_1}(X, d_1)$ with $X = (P, P_1, P_2)^T$. Furthermore, it is noted that:

$$Df_{d_1}(E_1, \check{d}_1)\breve{N}^{[1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \breve{A}\breve{n}_2^{[1]} \\ -\breve{n}_2^{[1]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\breve{n}_2^{[1]} \\ 0 \end{bmatrix},$$

So, we obtain that:

$$(\breve{\Phi}^{[1]})^T [Df_{d_1}(E_1, \check{d}_1)\breve{N}^{[1]}] = \left(0, \breve{\emptyset}_2^{[1]}, 0\right)^T \left(0, -\breve{n}_2^{[1]}, 0\right) = -\breve{n}_2^{[1]}\breve{\emptyset}_2^{[1]} \neq 0.$$

By using $\breve{N}^{[1]}$ in equation (3), we get

$$D^2 f_{d_1}(E_1, \check{d}_1)(N^{[1]}, N^{[1]}) = [\breve{K}_{i1}]_{3\times1}$$

$$= \begin{bmatrix} -2\left(\breve{n}_2^{[1]}\right)^2 \left[\breve{A}^2(b_1 + 3\delta_1\breve{P}) + \breve{A}(a_1 m_1 + c_1) - (a_1 m_1^2 \breve{P})\right] \\ 2\left(\breve{n}_2^{[1]}\right)^2 \left[l_1\breve{A} - (b_2 + \delta_2 + h_2)\right] \\ 0 \end{bmatrix}.$$

So,

$$(\breve{\emptyset}^{[1]})^T [D^2 f_{d_1}(E_1, \check{d}_1)(\breve{N}^{[1]}, \breve{N}^{[1]})] = 2\breve{\emptyset}_2^{[1]}\left(\breve{n}_2^{[1]}\right)^2 \left[l_1\breve{A} - (b_2 + \delta_2 + h_2)\right] \neq 0.$$

Then in accordance to the Sotomayor theorem, system (1) reveals a (TB) at E_1 with $d_1 = \check{d}_1$. If condition (5) not satisfised then by using $\breve{N}^{[1]}$ in equation (4), we get

$$D^3 f_{d_1}(E_1, \check{d}_1)(N^{[1]}, N^{[1]}, N^{[1]}) = [\breve{L}_{i1}]_{3\times1}$$

$$= \begin{bmatrix} -6\left(\breve{n}_2^{[1]}\right)^3 \left[\breve{A}^3\,\delta_1 - \breve{A}a_1 m_1^2 + a_1 m_1^3 \breve{P}\right] \\ 0 \\ 0 \end{bmatrix}$$

$$(\breve{\Phi}^{[1]})^T [D^3 f_{d_1}(E_1, \check{d}_1)(\breve{N}^{[1]}, \breve{N}^{[1]}, \breve{N}^{[1]})] = 0.$$

Therefore, by Sotomayor theorem the (PB) cannot happen at E_1

Theorem 3: Assume that the conditions (16) and (17) as in [29] and the next conditions hold:

$$\delta_3 < \frac{l_2\bar{P}_1}{\alpha + \gamma\bar{P}_1^2}, \quad (6)$$

$$\alpha > \gamma\bar{P}_1^2, \quad (7)$$

$$h_3 \neq H\left(\frac{l_2(\alpha - \gamma\bar{P}_1^2)}{(\alpha + \gamma\bar{P}_1^2)^2}\right), \quad (8)$$

where,

$H = \frac{-\bar{u}_{23}}{\bar{u}_{22}} < 0,$

Then system (1) near E_2 exhibits a (TB) and (PB) at the parameter value ($\bar{d}_2 = d_2 = \frac{l_2 \bar{P}_1}{\alpha + \gamma \bar{P}_1^2} - \delta_3$), while (SNB) not present near E_2 at $\bar{d}_2$.

Proof: by putting $E_2 = (0, \bar{P}_1, 0)$ in the J.M. in equation (14) in [29], $\bar{J}_2 = J_2(E_2, \bar{d}_2) = [\bar{u}_{ij}]_{3\times3}$, where $\bar{u}_{ij} = u_{ij}$, except $\bar{u}_{33} = 0$.

Then, the characteristic equation of $\bar{J}_2$ where $\bar{J}_2 = J(E_2, \bar{d}_2)$ has an eigenvalue zero, which is ($\lambda_{2P_2} = 0$) at $\bar{d}_2 = d_2 = \frac{l_2 \bar{P}_1}{\alpha + \gamma \bar{P}_1^2} - \delta_3$.

Let $\bar{N}^{[2]} = (\bar{n}_1^{[2]}, \bar{n}_2^{[2]}, \bar{n}_3^{[2]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{2P_2} = 0$. Thus, $(\bar{J}_2 - \lambda_{2P_2} I) N^{[2]} = 0$, where $\bar{J}_2 = J(E_2, \bar{d}_2)$.

Then, $\bar{n}_1^{[2]} = 0$, $\bar{n}_2^{[2]} = H \bar{n}_3^{[2]}$, where H is in the state of the theorem and $\bar{n}_3^{[2]} \neq 0$ real number.

Now, let $\bar{\Phi}^{[2]} = (\bar{\emptyset}_1^{[2]}, \bar{\emptyset}_2^{[2]}, \bar{\emptyset}_3^{[2]})^T$ be the eigenvector of $\bar{J}_2^T$ relating with the eigenvalue $\lambda_{2P_2} = 0$. Thus, $(\bar{J}_2^T - \lambda_{2P_2} I) \bar{\Phi}^{[2]} = 0$. Which gives,

$\bar{\Phi}^{[2]} = (0, 0, \bar{\emptyset}_3^{[2]})^T$, where $\bar{\emptyset}_3^{[2]} \neq 0$ real number.

Now, since $\frac{\partial f}{\partial d_2} = f_{d_2}(X, d_2) = \left(\frac{\partial f_1}{\partial d_2}, \frac{\partial f_2}{\partial d_2}, \frac{\partial f_3}{\partial d_2}\right)^T = (0, 0, -P_2)^T$.

Then, $f_{d_2}(E_2, \bar{d}_2) = (0,0,0)^T$ and hence $\left(\bar{\Phi}^{[2]}\right)^T f_{d_2}(E_2, \bar{d}_2) = 0$.

In agreement with Sotomayor's theorem, achieving the (SNB) at E_2 is not possible. The first requirement for (TB) is therefore achieved. Moreover,

$$Df_{d_1}(X, d_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

Here, $Df_{d_2}(X, d_2)$ denotes the derivative of $f_{d_2}(X, d_2)$ with $X = (P, P_1, P_2)^T$. Furthermore, it is noted that:

$$Df_{d_2}(E_2,\bar{d}_2)\bar{N}^{[2]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ H\bar{n}_3^{[2]} \\ \bar{n}_3^{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\bar{n}_3^{[2]} \end{bmatrix},$$

So, we obtain that:

$$(\bar{\Phi}^{[2]})^T [Df_{d_2}(E_2,\bar{d}_2)\bar{N}^{[2]}] = \left(0,0,\bar{\emptyset}_3^{[2]}\right)^T \left(0,0,-\bar{n}_3^{[2]}\right) = -\bar{n}_3^{[2]}\bar{\emptyset}_3^{[2]} \neq 0.$$

By using $\bar{N}^{[2]}$ in equation (3), we get

$$D^2 f_{d_2}(E_2,\bar{d}_2)(N^{[2]},N^{[2]}) = [\bar{K}_{i1}]_{3\times 1}$$

$$= \begin{bmatrix} 0 \\ 2\left(\bar{n}_3^{[2]}\right)^2 \left[a_2 m_2^2 \bar{P}_1 - H\left(H(b_2+\delta_2+h_2) + a_2 m_2 + \frac{c_2(\alpha-\gamma\bar{P}_1^{\,2})}{(\alpha+\gamma\bar{P}_1^{\,2})^2}\right)\right] \\ -2\left(\bar{n}_3^{[2]}\right)^2 \left[h_3 - H\left(\frac{l_2(\alpha-\gamma\bar{P}_1^2)}{(\alpha+\gamma\bar{P}_1^{\,2})^2}\right)\right] \end{bmatrix},$$

So, by condition (7) and (8)

$$(\bar{\Phi}^{[2]})^T [D^2 f_{d_2}(E_2,\bar{d}_2)(\bar{N}^{[2]},\bar{N}^{[2]})] = -2\bar{\emptyset}_3^{[2]}\left(\bar{n}_3^{[2]}\right)^2 \left[h_3 - H\left(\frac{l_2(\alpha-\gamma\bar{P}_1^2)}{(\alpha+\gamma\bar{P}_1^{\,2})^2}\right)\right] \neq 0.$$

Then accordance to the Sotomayor theorem, system (1) reveals a (TB) at E_2 with $d_2 = \bar{d}_2$. If condition (8) not met, then

$$D^3 f_{d_2}(E_2,\bar{d}_2)(N^{[2]},N^{[2]},N^{[2]}) = [\bar{L}_{i1}]_{3\times 1}$$

$$= \begin{bmatrix} 0 \\ 2\left(\bar{n}_3^{[2]}\right)^3 \left[H^2\left(\frac{3\gamma c_2\bar{P}_1[(\alpha+\gamma\bar{P}_1^2)+2(\alpha-\gamma\bar{P}_1^2)]}{(\alpha+\gamma\bar{P}_1^2)^3}\right) + 3Ha_2m_2^2 - 3a_2m_2^3\bar{P}_1\right] \\ 2\left(\bar{n}_3^{[2]}\right)^3 \left[-H^2\left(\frac{3\gamma l_2\bar{P}_1[(\alpha+\gamma\bar{P}_1^2)+2(\alpha-\gamma\bar{P}_1^2)]}{(\alpha+\gamma\bar{P}_1^2)^3}\right)\right] \end{bmatrix},$$

$$(\bar{\Phi}^{[2]})^T [D^3 f_{d_2}(E_2,\bar{d}_2)(\bar{N}^{[2]},\bar{N}^{[2]},\bar{N}^{[2]})] = -2\bar{\emptyset}_3^{[2]}\left(\bar{n}_3^{[2]}\right)^3 H^2\left(\frac{3\gamma l_2\bar{P}_1[(\alpha+\gamma\bar{P}_1^2)+2(\alpha-\gamma\bar{P}_1^2)]}{(\alpha+\gamma\bar{P}_1^2)^3}\right).$$

Therefore, by Sotomayor theorem the (PB) occurs at E_2.

Theorem 4: Assume that the conditions $(22)\ and\ (23)$ as in [29] and the next conditions hold.

$$\alpha > \gamma\ddot{P}_1^2, \tag{9}$$

$$d_2 < \frac{l_2\ddot{P}_1}{\alpha+\gamma\ddot{P}_1^2}, \tag{10}$$

$$h_3 \neq B_2\left(\frac{l_2(\alpha-\gamma\ddot{P}_1^2)}{(\alpha+\gamma\ddot{P}_1^{\,2})^2}\right), \tag{11}$$

where,

$B_2 = \dfrac{\vartheta_{12}\vartheta_{23}}{\vartheta_{12}\vartheta_{21} - \vartheta_{11}\vartheta_{22}} > 0,$

then system (1) near E_3 exhibits a (TB) and (PB) at the parameter value ($\ddot{\delta}_3 = \delta_3 = \frac{l_2\ddot{P}_1}{\alpha+\gamma\ddot{P}_1^2} - d_2$) , while (SNB) not present near E_3 at $\ddot{\delta}_3$.

Proof: by putting $E_3 = (\ddot{P}, \ddot{P}_1, 0)$ in the J.M. in equation (19) in [29], $\ddot{J}_3 = J_3(E_3, \ddot{\delta}_3) = [\ddot{\vartheta}_{ij}]_{3\times 3}$, where $\ddot{\vartheta}_{ij} = \vartheta_{ij}$, except $\ddot{\vartheta}_{33} = 0$.

Then, the characteristic equation of $\ddot{J}_3$ where $\ddot{J}_3 = J(E_3, \ddot{\delta}_3)$ has an eigenvalue zero, which is $\lambda_{3P_2} = 0$ at $\ddot{\delta}_3 = \delta_3 = \frac{l_2\ddot{P}_1}{\alpha+\gamma\ddot{P}_1^2} - d_2$.

Let $\ddot{N}^{[3]} = (\ddot{n}_1^{[3]}, \ddot{n}_2^{[3]}, \ddot{n}_3^{[3]})^T$ be the eigenvector relating with the eigenvalue $\lambda_{3P_2} = 0$. Thus, $(\ddot{J}_3 - \lambda_{3P_2} I)N^{[3]} = 0$, gives

$\ddot{n}_1^{[3]} = B_1\ddot{n}_3^{[3]}$, $\ddot{n}_2^{[3]} = B_2\ddot{n}_3^{[3]}$, *where* $B_1 = \frac{-\vartheta_{12}\vartheta_{23}}{\vartheta_{12}\vartheta_{21} - \vartheta_{11}\vartheta_{22}} < 0$, where B_2 is in the state of the theorem and $\ddot{n}_3^{[3]} \neq 0$ real number.

Now, let $\ddot{\Phi}^{[3]} = (\ddot{\emptyset}_1^{[3]}, \ddot{\emptyset}_2^{[3]}, \ddot{\emptyset}_3^{[2]})^T$ be the eigenvector of $\ddot{J}_3^T$ relating with the eigenvalue $\lambda_{3P_2} = 0$. Thus, $(\ddot{J}_3^T - \lambda_{3P_2} I)\ddot{\Phi}^{[3]} = 0$, gives

$\ddot{\Phi}^{[3]} = (0, 0, \ddot{\emptyset}_3^{[3]})^T$, where $\ddot{\emptyset}_3^{[3]} \neq 0$ real number.

Now, since $\frac{\partial f}{\partial \delta_3} = f_{\delta_3}(X, \delta_3) = \left(\frac{\partial f_1}{\partial \delta_3}, \frac{\partial f_2}{\partial \delta_3}, \frac{\partial f_3}{\partial \delta_3}\right)^T = (0, 0, -P_2)^T$.

Then, $f_{\delta_3}(E_3, \ddot{\delta}_3) = (0,0,0)^T$ and hence $(\ddot{\Phi}^{[3]})^T f_{\delta_3}(E_3, \ddot{\delta}_3) = 0$.

According to Sotomayor's theorem, achieving the (SNB) at E_3 is not possible. The initial condition for (TB) is therefore met. Moreover,

$$Df_{\delta_3}(X, \delta_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

Here, $Df_{\delta_3}(X, \delta_3)$ denotes the derivative of $f_{\delta_3}(X, \delta_3)$ with $X = (P, P_1, P_2)^T$. Furthermore, it is noted that:

$$Df_{\delta_3}(E_3,\ddot{\delta}_3)\ddot{N}^{[3]}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}B_1\ddot{n}_3^{[3]}\\B_2\ddot{n}_3^{[3]}\\\ddot{n}_3^{[3]}\end{bmatrix}=\begin{bmatrix}0\\0\\-\ddot{n}_3^{[3]}\end{bmatrix},$$

So, we obtain that:

$$\left(\ddot{\Phi}^{[3]}\right)^T\left[Df_{\delta_3}(E_3,\ddot{\delta}_3)\ddot{N}^{[3]}\right]=\left(0,0,\ddot{\phi}_3^{[3]}\right)^T\left(0,0,-\ddot{n}_3^{[3]}\right)=-\ddot{n}_3^{[3]}\ddot{\phi}_3^{[3]}\neq 0.$$

By using $\ddot{N}^{[2]}$ in equation (3), we get

$$D^2f_{\delta_3}(E_3,\ddot{\delta}_3)\left(N^{[3]},N^{[3]}\right)=\left[\ddot{K}_{i1}\right]_{3\times1},$$

$$\ddot{K}_{11}=-2\left(\ddot{n}_3^{[3]}\right)^2\left[B_1^2(b_1+3\delta_1\ddot{P})+B_1B_2\left(\frac{a_1m_1}{(1+m_1\ddot{P}_1)^2}+c_1\right)-B_2^2\left(\frac{a_1m_1^2\ddot{P}}{(1+m_1\ddot{P}_1)^3}\right)\right],$$

$$\ddot{K}_{21}=2\left(\ddot{n}_3^{[3]}\right)^2\left[B_1B_2(l_1)+a_2m_2^2\ddot{P}_1-B_2\left(a_2m_2+\frac{c_2(\alpha-\gamma\ddot{P}_1^2)}{(\alpha+\gamma\ddot{P}_1^2)^2}\right)-B_2^2(b_2+\delta_2+h_2)\right],$$

$$\ddot{K}_{31}=-2\left(\ddot{n}_3^{[3]}\right)^2\left[h_3-B_2\left(\frac{l_2(\alpha-\gamma\ddot{P}_1^2)}{(\alpha+\gamma\ddot{P}_1^2)^2}\right)\right].$$

So, by conditions (10) and (11),

$$\left(\ddot{\Phi}^{[3]}\right)^T\left[D^2f_{\delta_3}(E_3,\ddot{\delta}_3)\left(\ddot{N}^{[3]},\ddot{N}^{[3]}\right)\right]=-2\ddot{\phi}_3^{[3]}\left(\ddot{n}_3^{[3]}\right)^2\left[h_3-B_2\left(\frac{l_2(\alpha-\gamma\ddot{P}_1^2)}{(\alpha+\gamma\ddot{P}_1{}^2)^2}\right)\right]\neq 0.$$

Then based on the Sotomayor theorem, system (1) indicates a (TB) at E_3 with $\delta_3=\ddot{\delta}_3$. If condition (11) not satisfised then

$$D^3f_{\delta_3}(E_3,\ddot{\delta}_3)\left(N^{[3]},N^{[3]},N^{[3]}\right)=\left[\ddot{L}_{i1}\right]_{3\times1},$$

$$\ddot{L}_{11}=-6\left(\ddot{n}_3^{[3]}\right)^3\left[B_1^3\delta_1-\frac{B_1B_2^2(a_1m_1^2)}{\left(1+m_1\ddot{P}_1\right)^3}+\frac{B_2^3(a_1m_1^3\ddot{P})}{\left(1+m_1\ddot{P}_1\right)^4}\right],$$

$$\ddot{L}_{21}=2\left(\ddot{n}_3^{[3]}\right)^3\left[\frac{3B_2^2\gamma c_2\ddot{P}_1\left[(\alpha+\gamma\ddot{P}_1^2)+2(\alpha-\gamma\ddot{P}_1^2)\right]}{\left(\alpha+\gamma\ddot{P}_1^2\right)^3}+3B_2a_2m_2^2-3a_2m_2^3\ddot{P}_1\right],$$

$$\ddot{L}_{31}=2\left(\ddot{n}_3^{[3]}\right)^3\left[-\frac{3B_2^2\gamma l_2\ddot{P}_1\left[(\alpha+\gamma\ddot{P}_1^2)+2(\alpha-\gamma\ddot{P}_1^2)\right]}{\left(\alpha+\gamma\ddot{P}_1^2\right)^3}\right].$$

$$\left(\ddot{\Phi}^{[3]}\right)^T\left[D^3f_{\delta_3}(E_3,\ddot{\delta}_3)\left(\ddot{N}^{[3]},\ddot{N}^{[3]},\ddot{N}^{[3]}\right)\right]=-2\ddot{\phi}_3^{[3]}\left(\ddot{n}_3^{[3]}\right)^3\left(\frac{3B_2^2\gamma l_2\ddot{P}_1\left[(\alpha+\gamma\ddot{P}_1^2)+2(\alpha-\gamma\ddot{P}_1^2)\right]}{\left(\alpha+\gamma\ddot{P}_1^2\right)^3}\right).$$

Therefore, by Sotomayor theorem the (PB) occurs at E_3.

Theorem 5: Assume that the conditions (27),(28) and (29) as in [29], and the next condition hold:

$$c_1\tilde{P}_1<\frac{a_1}{1+m_1\tilde{P}_1}\,, \qquad (12)$$

Then system (1) at the equilibrium point E_4 has a (TB) only, while neither (SNB) nor (PB) can be occurred with the parameter value $\tilde{h}_1 = h_1 = \frac{a_1}{1+m_1\tilde{P}_1} - c_1\tilde{P}_1$.

Proof: by putting $E_4 = (0, \tilde{P}_1, \tilde{P}_2)$ in the J.M. in equation (24) in [29], $\tilde{J}_4 = J_4(E_4, \tilde{h}_1) = [\tilde{\varkappa}_{ij}]_{3\times3}$, where $\tilde{\varkappa}_{ij} = \varkappa_{ij}$, except $\tilde{\varkappa}_{11} = 0$.

Then, the characteristic equation of $\tilde{J}_4$ where $\tilde{J}_3 = J(E_4, \tilde{h}_1)$ has an eigenvalue zero, which is $(\lambda_{4P} = 0)$ at $\left(\tilde{h}_1 = h_1 = \frac{a_1}{1+m_1\tilde{P}_1} - c_1\tilde{P}_1\right)$.

Let $\tilde{N}^{[4]} = (\tilde{n}_1^{[4]}, \tilde{n}_2^{[4]}, \tilde{n}_3^{[4]})^T$ be the eigenvector relating with the eigenvalue $\lambda_{4P} = 0$. Thus, $(\tilde{J}_4 - \lambda_{4P} I)N^{[4]} = 0$, which gives

$$\tilde{n}_2^{[4]} = \Theta_1 \tilde{n}_1^{[4]}, \tilde{n}_3^{[4]} = \Theta_2 \tilde{n}_1^{[4]}, where \ \ \Theta_1 = \frac{-\varkappa_{21}\varkappa_{33}}{\varkappa_{22}\varkappa_{33} - \varkappa_{23}\varkappa_{32}} < 0,\ \Theta_2 = \frac{\varkappa_{21}\varkappa_{32}}{\varkappa_{22}\varkappa_{33} - \varkappa_{23}\varkappa_{32}} > 0$$

and $\tilde{n}_1^{[4]} \neq 0$ real number.

Now, let $\tilde{\Phi}^{[4]} = (\tilde{\emptyset}_1^{[4]}, \tilde{\emptyset}_2^{[4]}, \tilde{\emptyset}_3^{[4]})^T$ be the eigenvector of $\tilde{J}_4^T$ relating with the eigenvalue $(\lambda_{4P} = 0)$. Thus, $(\tilde{J}_4^T - \lambda_{4P} I)\tilde{\emptyset}^{[4]} = 0$. Which gives,

$\tilde{\Phi}^{[4]} = (\tilde{\emptyset}_1^{[4]}, 0, 0)^T$, where $\tilde{\emptyset}_1^{[4]} \neq 0$ real number.

Now, since $\frac{\partial f}{\partial h_1} = f_{h_1}(X, h_1) = \left(\frac{\partial f_1}{\partial h_1}, \frac{\partial f_2}{\partial h_1}, \frac{\partial f_3}{\partial h_1}\right)^T = (-P, 0, 0)^T$.

Then, $f_{h_1}(E_4, \tilde{h}_1) = (0,0,0)^T$ and hence $\left(\tilde{\Phi}^{[4]}\right)^T f_{h_1}(E_4, \tilde{h}_1) = 0$.

In agreement with Sotomayor's theorem, achieving the (SNB) at E_4 is not possible. The first requirement for (TB) is therefore achieved. Moreover,

$$Df_{h_1}(X, \tilde{h}_1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Here, $Df_{h_1}(X, \tilde{h}_1)$ denotes the derivative of $f_{h_1}(X, \tilde{h}_1)$ with $X = (P, P_1, P_2)^T$. Furthermore, it is noted that:

$$Df_{h_1}(E_4, \tilde{h}_1)\tilde{N}^{[4]} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{n}_1^{[4]} \\ \Theta_1 \tilde{n}_1^{[4]} \\ \Theta_2 \tilde{n}_1^{[4]} \end{bmatrix} = \begin{bmatrix} -\tilde{n}_1^{[4]} \\ 0 \\ 0 \end{bmatrix},$$

So, we obtain that:

$$\left(\widetilde{\Phi}^{[4]}\right)^T\left[Df_{h_1}\left(E_4,\tilde{h}_1\right)\widetilde{N}^{[4]}\right]=\left(\widetilde{\emptyset}_1^{[4]},0,0\right)^T\left(-\tilde{n}_1^{[4]},0,0\right)=-\tilde{n}_1^{[4]}\widetilde{\emptyset}_1^{[4]}\neq 0.$$

By using $\widetilde{N}^{[4]}$ in equation (3), we get

$$D^2f_{h_1}\left(E_4,\tilde{h}_1\right)\left(N^{[4]},N^{[4]}\right)=\left[\widetilde{K}_{i1}\right]_{3\times 1}$$

$$\widetilde{K}_{11}=-2\left(\tilde{n}_1^{[4]}\right)^2\left[b_1+\left(\frac{a_1m_1}{(1+m_1\tilde{P}_1)^2}+c_1\right)\theta_1\right],$$

$$\widetilde{K}_{21}=2\left(\tilde{n}_1^{[4]}\right)^2\Bigg[(l_1)\theta_1+\frac{\theta_1^2[\gamma c_2\tilde{P}_1\tilde{P}_2(\alpha+\gamma\tilde{P}_1^2)+2(\alpha-\gamma\tilde{P}_1^2)]}{(\alpha+\gamma\tilde{P}_1^2)^3}-\theta_1^2(b_2+\delta_2+h_2)-\frac{a_2m_2\theta_1\theta_2}{(1+m_2\tilde{P}_2)^2}-\frac{c_2(\alpha-\gamma\tilde{P}_1^2)\theta_1\theta_2}{(\alpha+\gamma\tilde{P}_1^2)^2}+\frac{a_2m_2^2\tilde{P}_1\theta_2^2}{(1+m_2\tilde{P}_2)^3}\Bigg],$$

$$\widetilde{K}_{31}=-2\left(\tilde{n}_1^{[4]}\right)^2\left[\frac{\theta_1^2[\gamma l_2\tilde{P}_1\tilde{P}_2(\alpha+\gamma\tilde{P}_1^2)+2(\alpha-\gamma\tilde{P}_1^2)]}{(\alpha+\gamma\tilde{P}_1^2)^3}-\frac{\theta_1\,\theta_2\left(l_2(\alpha-\gamma\tilde{P}_1^2)\right)}{(\alpha+\gamma\tilde{P}_1^2)^2}+\theta_2h_3\right].$$

So,

$$\left(\widetilde{\Phi}^{[4]}\right)^T\left[D^2f_{h_1}\left(E_4,\tilde{h}_1\right)\left(\widetilde{N}^{[4]},\widetilde{N}^{[4]}\right)\right]=-2\widetilde{\emptyset}_1^{[4]}\left(\tilde{n}_1^{[4]}\right)^2\left[b_1+\left(\frac{a_1m_1}{(1+m_1\tilde{P}_1)^2}+c_1\right)\theta_1\right]\neq 0.$$

Then based on the Sotomayor theorem, system (1) indicates a (TB) at E_4 with $h_1=\tilde{h}_1$.

Theorem 6: Assume that the locally conditions $(32),(33)$ and (35) in [29] and the following conditions hold:

$$\frac{\S_{12}\S_{21}\S_{33}}{\S_{23}\S_{32}-\S_{22}\S_{33}}+\frac{a_1}{1+m_1P_1^*}>P^*(2b_1+3\delta_1P^*)+c_1P_1^*, \tag{13}$$

$$Y_5\neq Y_6\,, \tag{14}$$

where,

$$Y_5=\Bigg((b_1+3\delta_1P^*)Y_1^2Y_3+(b_2+\delta_2+h_2)Y_2^2Y_4+h_3-(l_1)Y_1Y_2Y_4+\left(\frac{\gamma l_2P_1^*P_2^*\left((\alpha+\gamma P_1^{*2})+2(\alpha-\gamma P_1^{*2})\right)}{\left(\alpha+\gamma P_1^{*2}\right)^3}\right)Y_2^2-\left(\frac{l_2\left(\alpha-\gamma P_1^{*2}\right)}{\left(\alpha+\gamma P_1^{*2}\right)^2}\right)Y_2\Bigg),$$

$$Y_6=\Bigg(\left(\frac{a_1m_1^2P^*}{(1+m_1P_1^*)^3}\right)Y_2^2Y_3-\left(\frac{a_1m_1}{(1+m_1P_1^*)^2}+c_1\right)Y_1Y_2Y_3+\left(\frac{a_2m_2^2P_1^*}{(1+m_2P_2^*)^3}\right)Y_4-\left(\frac{a_2m_2}{(1+m_2P_2^*)^2}+\frac{c_2\left(\alpha-\gamma P_1^{*2}\right)}{\left(\alpha+\gamma P_1^{*2}\right)^2}\right)Y_2Y_4+\left(\frac{\gamma c_2P_1^*P_2^*\left((\alpha+\gamma P_1^{*2})+2(\alpha-\gamma P_1^{*2})\right)}{(\alpha+\gamma P_1^{*2})^3}\right)Y_2^2Y_4+\Bigg),$$

And,

$Y_1 = \frac{\S_{12}\S_{23}}{\S_{11}\S_{22}-\S_{12}\S_{21}} > 0, Y_2 = \frac{-\S_{11}\S_{23}}{\S_{11}\S_{22}-\S_{12}\S_{21}} < 0, Y_3 = \frac{\S_{21}\S_{32}}{\S_{22}\S_{11}-\S_{12}\S_{21}} > 0, Y_4 = \frac{-\S_{11}\S_{32}}{\S_{22}\S_{22}-\S_{12}\S_{21}} > 0.$

Then system (1) at E_5 exhibits just a (SNB) and neither (TB) nor (PB) occurs at $h_1 = h_1^* = \frac{\S_{12}\S_{21}\S_{33}}{\S_{23}\S_{32}-\S_{22}\S_{33}} + \frac{a_1}{1+m_1 P_1^*} - [P^*(2b_1 + 3\delta_1 P^*) + c_1 P_1^*].$

Proof: From the characteristic equation $\lambda^3 + \beta_1\lambda^2 + \beta_2\lambda + \beta_3 = 0$ of the Jacobian matrix J_5, as presented in Eq. (31) in [29], system (1) at the (EP) E_5 possesses an eigenvalue equal to zero, denoted as $(\lambda_{5P} = 0)$.Then E_5 will be non-hyperbolic equilibrium point if and only if $\beta_3 = 0$. Then J_5 with $h_1 = h_1^*$ becomes $J_5^* = J_5(E_5, h_1^*) = [\S_{ij}^*]_{3\times 3}$, where $\S_{ij}^* = \S_{ij}$; $i,j = 1,2,3$ which are given in Eq. (30) in [29] except $\S_{11}^* = \frac{-\S_{12}^*\S_{21}^*\S_{33}^*}{\S_{23}^*\S_{32}^*-\S_{22}^*\S_{33}^*}$.

Let $N^{[5]} = (n_1^{*[5]}, n_2^{*[5]}, n_3^{*[5]})^T$ be the eigenvector relating with the eigenvalue $(\lambda_{5P} = 0)$. So, $(J_5^* - \lambda_{5P} I)N^{[5]} = 0$, which gives

$n_1^{*[5]} = Y_1 n_3^{*[5]}, n_2^{*[5]} = Y_2 n_3^{*[5]}, where$ Y_1 and Y_2 are given in the state of the theorem and $n_3^{*[5]} \neq 0$ real number.

Now, let $\Phi^{*[5]} = (\emptyset_1^{*[5]}, \emptyset_2^{*[5]}, \emptyset_3^{*[5]})^T$ be the eigenvector of J_5^{*T} relating with the eigenvalue $(\lambda_{5P} = 0)$. Thus, $(J_5^{*T} - \lambda_{5P} I)\Phi^{*[5]} = 0$. Which gives,

$\emptyset_1^{*[5]} = Y_3\emptyset_3^{*[5]}$, $\emptyset_2^{*[5]} = Y_4\emptyset_3^{*[5]}$, where Y_3 and Y_4 are given in the state of the theorem and $\emptyset_3^{*[5]} \neq 0$ real number.

Since, $\frac{\partial f}{\partial h_1} = f_{h_1}(X, h_1) = \left(\frac{\partial f_1}{\partial h_1}, \frac{\partial f_2}{\partial h_1}, \frac{\partial f_3}{\partial h_1}\right)^T = (-P, 0, 0)^T.$

Then, $f_{h_1}(E_5, h_1^*) = (-P^*, 0,0)^T$ and hence $(\Phi^{*[5]})^T f_{h_1}(E_5, h_1^*) = -Y_3\emptyset_3^{*[5]}P^* \neq 0.$

Moreover, by using $\widetilde{N}^{[4]}$ in equation (3), we get

$D^2 f_{h_1}(E_5, h_1^*)(N^{[5]}, N^{[5]}) = [K_{i1}^*]_{3\times 1}.$

$K_{11}^* = -2\left(n_3^{*[5]}\right)^2 \left[(b_1 + 3\delta_1 P^*)Y_1^2 + \left(\frac{a_1 m_1}{(1+m_1 P_1^*)^2} + c_1\right)Y_1 Y_2 - \left(\frac{a_1 m_1^2 P^*}{(1+m_1 P_1^*)^3}\right)Y_2^2\right],$

$K_{21}^* = 2\left(n_3^{*[5]}\right)^2 \left[(l_1)Y_1 Y_2 + \left(\frac{\gamma c_2 P_1^* P_2^*\left((\alpha+\gamma P_1^{*2}) + 2(\alpha - \gamma P_1^{*2})\right)}{(\alpha+\gamma P_1^{*2})^3} - (b_2 + \delta_2 + h_2)\right)Y_2^2 - \left(\frac{a_2 m_2}{(1+m_2 P_2^*)^2} + \right.\right.$

$$\left.\frac{c_2(\alpha-\gamma P_1^{*2})}{(\alpha+\gamma P_1^{*2})^2}\right)Y_2+\frac{a_2 m_2^2 P_1^*}{(1+m_2P_2^*)^3}\Bigg],$$

$$K_{31}^*=-2\left(\mathrm{n}_3^{*[5]}\right)^2\left[\left(\frac{\gamma l_2 P_1^* P_2^*\left((\alpha+\gamma P_1^{*2})+2(\alpha-\gamma P_1^{*2})\right)}{(\alpha+\gamma P_1^{*2})^3}\right)Y_2^2-\left(\frac{l_2(\alpha-\gamma P_1^{*2})}{(\alpha+\gamma P_1^{*2})^2}\right)Y_2+h_3\right].$$

So, it implies that

$$\left(\Phi^{*[5]}\right)^T\left[D^2 f_{h_1}(E_5,h_1^*)\left(N^{*[5]},N^{*[5]}\right)\right]=2\Bigg[-\Bigg((b_1+3\delta_1P^*)Y_1^2Y_3+(b_2+\delta_2+h_2)Y_2^2Y_4+$$

$$h_3-(l_1)Y_1Y_2Y_4+\left(\frac{\gamma l_2P_1^*P_2^*\left((\alpha+\gamma P_1^{*2})+2(\alpha-\gamma P_1^{*2})\right)}{(\alpha+\gamma P_1^{*2})^3}\right)Y_2^2-\left(\frac{l_2(\alpha-\gamma P_1^{*2})}{(\alpha+\gamma P_1^{*2})^2}\right)Y_2\Bigg)+\Bigg(\left(\frac{a_1m_1^2P^*}{(1+m_1P_1^*)^3}\right)Y_2^2Y_3-$$

$$\left(\frac{a_1m_1}{(1+m_1P_1^*)^2}+c_1\right)Y_1Y_2Y_3+\left(\frac{a_2m_2^2P_1^*}{(1+m_2P_2^*)^3}\right)Y_4-\left(\frac{a_2m_2}{(1+m_2P_2^*)^2}+\frac{c_2(\alpha-\gamma P_1^{*2})}{(\alpha+\gamma P_1^{*2})^2}\right)Y_2Y_4+$$

$$\left(\frac{\gamma c_2P_1^*P_2^*\left((\alpha+\gamma P_1^{*2})+2(\alpha-\gamma P_1^{*2})\right)}{(\alpha+\gamma P_1^{*2})^3}\right)Y_2^2Y_4+\Bigg)\Bigg]\emptyset_3^{*[5]}\left(n_3^{*[5]}\right)^2,$$

So, by reversing condition (34) in [29] and conditions (13) and (14), we get

$$\left(\Phi^{*[5]}\right)^T\left[D^2 f_{h_1}(E_5,h_1^*)\left(N^{*[5]},N^{*[5]}\right)\right]=2[-Y_5+Y_6]\emptyset_3^{*[5]}\left(n_3^{*[5]}\right)^2\neq 0,$$

Then by Sotomayor theorem system (1) has only (SNB) at the positive point E_5 at the parameter $h_1=h_1^*$.

4. The Hopf-bifurcation Analysis

In this section, the next theorem illustrates that when a (HB) occurs at the positive equilibrium point (PEP) E_5 of a system (1), the Hopf bifurcation [31] is suitable for local bifurcation analysis.

Theorem 7: Suppose that the locally conditions (32), (33), and (34) as in [29] and the following conditions hold:

$$\frac{a_1^2}{(1+m_1P_1^*)^2}+4b_1P^*(b_1P^*+c_1P_1^*+h_1)+c_1P_1^*(c_1+2h_1)+h_1^2+(2b_1P^*+c_1P_1^*+h_1)F_1+$$

$$F_2<\frac{a_1[2(2b_1P^*+c_1P_1^*+h_1)+1]}{1+m_1P_1^*}, \tag{15}$$

$$(\S_{22}+\S_{33})[\S_{11}-\beta_1(\delta_1^*)]\neq \S_{12}\S_{21}. \tag{16}$$

Then for the parameter value $\delta_1=\delta_1^*$, system (1) has a (HB) at E_5.

Proof: The characteristic equation of system (1) at E_5 which mentioned in equation (31) in [29] as follows:

$\lambda^3 + \beta_1\lambda^2 + \beta_2\lambda + \beta_3 = 0,$ (17)

Where,

$\beta_1 = -(\S_{11} + \S_{22} + \S_{33}),$

$\beta_2 = \S_{11}(\S_{22} + \S_{33}) + \S_{22}\S_{33} - \S_{12}\S_{21} - \S_{23}\S_{32},$

$\beta_3 = \S_{11}(\S_{23}\S_{32} - \S_{22}\S_{33}) + \S_{12}\S_{21}\S_{33},$

The parameter (δ_1^*) is essential to verify the necessary and sufficient conditions for (HB) to appear at the (PEP) that satisfy: $\beta_i(\delta_1^*) > 0, (i = 1{,}3)$, $\Delta = \Delta_1\beta_3 = 0$ where $\Delta_1(\delta_1^*) = \beta_1\beta_2 - \beta_3$. Provided conditions of locally (32-34) in [29] , $\beta_i(\delta_1^*) > 0, (i = 1{,}3)$, and $(\delta_1^* > 0)$.

It observed that $\Delta_1 = 0$, gives:

$\mu_1\delta_1^{*2} + \mu_2\delta_1^* + \mu_3 = 0,$ (18)

where,

$\mu_1 = 9P^{*4} > 0,$

$$\mu_2 = \frac{-6a_1P^{*2}}{1 + m_1P_1^*} + [6p^2(6b_1P^* + c_1P_1^* + h_1) + 3P^{*2}F_1],$$

$$\mu_3 = \frac{a_1^2}{(1+m_1P_1^*)^2} + 4b_1P^*(b_1P^* + c_1P_1^* + h_1) + c_1P_1^*(c_1 + 2h_1) + h_1^2 + (2b_1P^* + c_1P_1^* + h_1)F_1 + F_2 - \left(\frac{a_1[2(2b_1P^*+c_1P_1^*+h_1)+1]}{1+m_1P_1^*}\right).$$

With

$F_1 = \frac{\S_{12}\S_{21} - (\S_{22}+\S_{33})^2}{(\S_{22}+\S_{33})}$ and $F_2 = \frac{\S_{22}(\S_{22}\S_{33} - \S_{12}\S_{21} - \S_{23}\S_{32}) + \S_{33}(\S_{22}\S_{33} - \S_{23}\S_{32})}{(\S_{22}+\S_{33})}$.

Utilizing Descartes rule of sign, equation (18) possesses a unique positive root (δ_1^*), provided condition (15) hold:

Now, at $(\delta_1 = \delta_1^*)$ the characteristic equation (17) can be expressed as:

$P^*(\lambda) = (\lambda + \beta_1)(\lambda^2 + \beta_2) = 0,$ (19)

Which have two roots:

$\lambda_1 = -\beta_1 < 0$, and $\lambda_{2,3} = \pm i\sqrt{\beta_2}$,

It was noted that at $(\delta_1 = \delta_1^*)$, there exists one real and negative eigenvalue (λ_1) and two pure imaginary eigenvalues $(\lambda_{2,3})$.

Now, for all values of δ_1 in the neighbourhood of δ_1^*, the general expression for the roots is given as follows:

$\lambda_{2,3} = \varsigma_1(\delta_1) \pm \varsigma_2(\delta_1).$

In order to verify the transversality criteria, we have to show that:

$$\Psi^*(\delta_1^*)\Theta^*(\delta_1^*) + \Gamma^*(\delta_1^*)\Phi^*(\delta_1^*) \neq 0,$$

Note that for $\delta_1 = \delta_1^*$ we have $\varsigma_1 = 0$ and $\varsigma_2 = \sqrt{\beta_2}$, substitute the value of ς_2 yields the next simplifications:

$$\Psi^*(\delta_1^*) = 3(\varsigma_1(\delta_1^*))^2 + 2\beta_1(\delta_1^*)\,\varsigma_1(\delta_1^*) + \beta_2(\delta_1^*) - 3(\varsigma_2(\delta_1^*))^2 = -2\beta_2(\delta_1^*),$$

$$\Phi^*(\delta_1^*) = 6\varsigma_1(\delta_1^*)\varsigma_2(\delta_1^*) + 2\beta_1(\delta_1^*)\,\varsigma_2(\delta_1^*) = 2\beta_1(\delta_1^*)\sqrt{\beta_2(\delta_1^*)},$$

$$\Theta^*(\delta_1^*) = (\varsigma_1(\delta_1^*))^2\beta_1'(\delta_1^*) + \beta_2'(\delta_1^*)\varsigma_1(\delta_1^*) + \beta_3'(\delta_1^*) - \beta_1'(\delta_1^*)(\varsigma_2(\delta_1^*))^2$$
$$= -3P^{*2}[(\S_{23}\S_{32} - \S_{22}\S_{33}) + \beta_2(\delta_1^*)],$$

$$\Gamma^*(\delta_1^*) = 2\varsigma_1(\delta_1^*)\varsigma_2(\delta_1^*)\beta_1'(\delta_1^*) + \beta_2'(\delta_1^*)\varsigma_2(\delta_1^*) = -3P^{*2}(\S_{22} + \S_{33})\sqrt{\beta_2(\delta_1^*)}.$$

So,

$$\Psi^*(\delta_1^*)\Theta^*(\delta_1^*) + \Gamma^*(\delta_1^*)\Phi^*(\delta_1^*) = 6P^{*2}\big((\S_{22} + \S_{33})[\S_{11} - \beta_1(\delta_1^*)] - \S_{12}\S_{21}\big)\beta_2(\delta_1^*) \neq 0$$

Thus, under the specified conditions (32, 33, 34) outlined in [29] along with conditions (16), system (1) at E_5 with the parameter δ_1^* has a (HB).

5. Numerical Simulation

For further confirmation of our analytical results and to examine the impact of altering each parameter's values on the system's dynamical behaviour, we numerically analyse the dynamic behaviour of system (1) using mathematical tools. Fig. (1) (a-e) demonstrates that system (1) has a globally asymptotically stable (PEP) under the specified hypothetical parameters that satisfies the stability criteria for this equilibrium point.

$$\left.\begin{array}{l} a_1 = 0.5\,, m_1 = 0.1, b_1 = 0.01, c_1 = 0.7, \delta_1 = 0.03, h_1 = 0.02, a_2 = 0.2, m_2 = 0.5, \\ b_2 = 0.001, l_1 = 0.2, c_2 = 0.5, \alpha = 0.4, \gamma = 0.8, \delta_2 = 0.05, h_2 = 0.3, \\ d_1 = 0.01, l_2 = 0.3, \delta_3 = 0.02, h_3 = 0.06, d_2 = 0.1\ . \end{array}\right\} \quad (20)$$

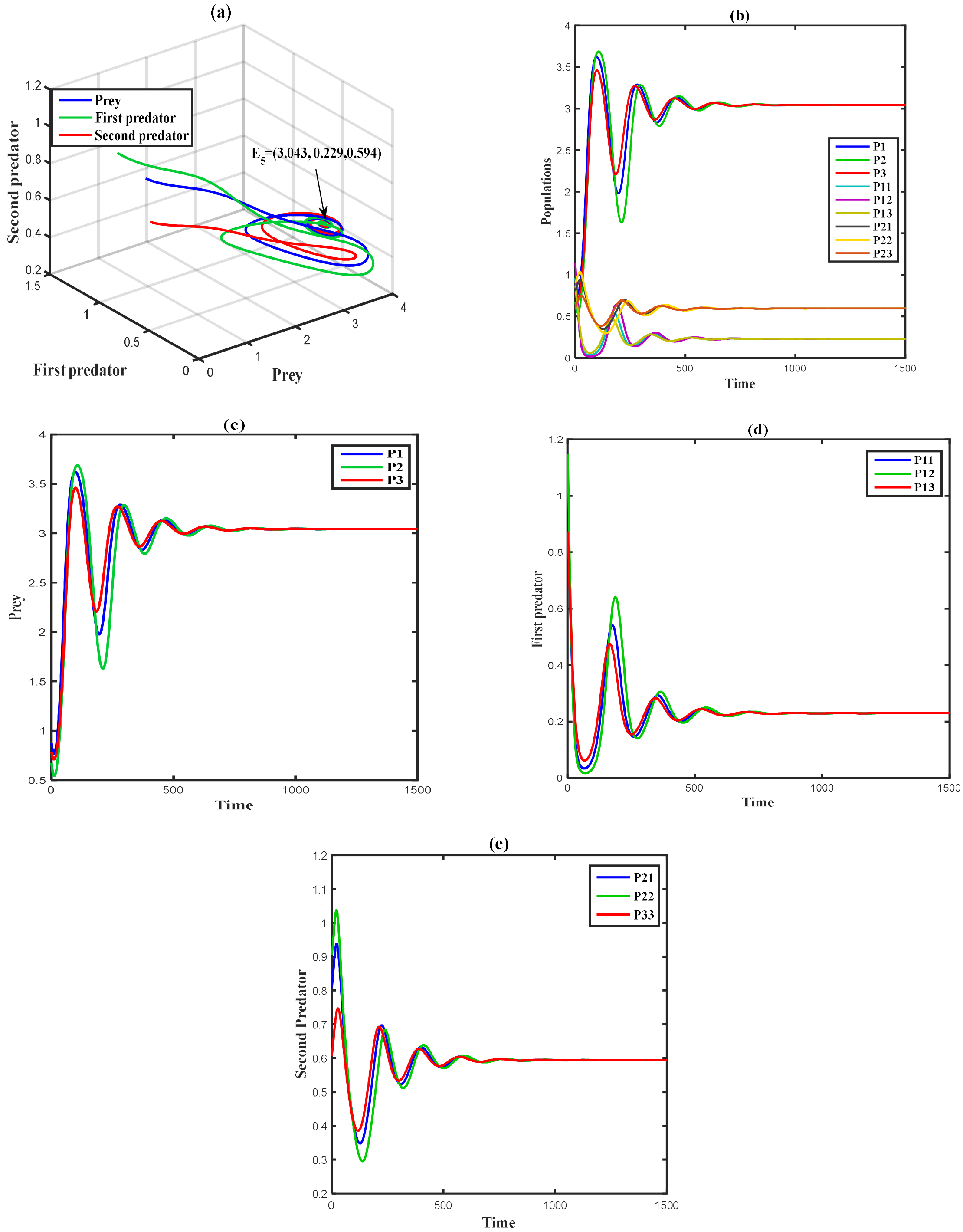


Fig. 1. Time series of the solution (T.S.) to system (1) initiated by three different beginning points (0.9, 1, 0.8), (0.7, 1.2, 0.9) and (0.8, 0.9, 0.6) for the data provided by (20). (a) The solution approach to the (EP) $E_5 = (3.043, 0.229, 0.594)$, (b) Trajectories of P, P_1, P_2 from different initial points, (c) Trajectory of P over time, (d) Trajectory of P_1 over time, (e) Trajectory of P_2 over time.

To investigate the influence of parameter values on the system's dynamical behavior, we will alter each parameter independently while preserving the others constant (20). The findings are presented in Table 2.

Table 2: The dynamical behaviour of system (1) parameters

Parameter range	Stable point	Bifurcation Point
$0.001 \leq a_1 < 0.14$ $0.14 \leq a_1 < 1.2$ $1.2 \leq a_1 < 1.71$ $1.71 \leq a_1 < 2$	E_4 E_5 Periodic E_5	$a_1 = 0.14$
$0.0001 \leq m_1 < 1$	E_5	
$0.001 \leq b_1 < 2$	E_5	
$0.1 \leq c_1 < 0.4$ $0.4 \leq c_1 < 2$	Periodic E_5	
$0.0001 \leq \delta_1 < 0.0076$ $0.0076 \leq \delta_1 < 2$	Periodic E_5	
$0.001 \leq h_1 < 0.3999$ $0.3999 \leq h_1 < 1$	E_5 E_4	$h_1 = 0.3999$
$0.01 \leq a_2 < 0.66$ $0.66 \leq a_2 < 1.075$ $1.075 \leq a_2 < 1.104$	E_5 Periodic E_4	$a_2 = 1.075$
$0.001 \leq m_2 < 2$	E_5	
$0.001 \leq b_2 < 2$	E_5	
$0.001 \leq l_1 < 0.7$	E_5	
$0.3 \leq c_2 < 2$	E_5	
$0.21 \leq \alpha < 1.5$ $1.5 \leq \alpha < 2$	E_5 E_3	$\alpha = 1.5$
$0.01 \leq \gamma < 1$	E_5	
$0.01 \leq \delta_2 < 2$	E_5	
$0.001 \leq h_2 < 1$	E_5	
$0.001 \leq d_1 < 0.781$ $0.781 \leq d_1 < 0.95$ $0.95 \leq d_1 < 1$	E_5 E_3 E_1	$d_1 = 0.781$ $d_1 = 0.95$
$0.0001 \leq l_2 < 0.129$ $0.129 \leq l_2 < 0.5$	E_3 E_5	$l_2 = 0.129$
$0.001 \leq \delta_3 < 0.179$ $0.179 \leq \delta_3 < 1$	E_5 E_3	$\delta_3 = 0.179$
$0.0001 \leq h_3 < 1$	E_5	
$0.0001 \leq d_2 < 0.260$ $0.260 \leq d_2 < 1$	E_5 E_3	$d_2 = 0.260$

The influence of altering the intrinsic rate of growth (a_1) of the prey P in the range $(0.001 \leq a_1 < 0.14)$ the solution goes to E_4, as illustrated in Fig. (2) (a-a1) with typical value $a_1 = 0.001$ while, in the range of $(0.14 \leq a_1 < 1.2)$ the solution goes to the (PEP) E_5 as illustrated in Fig. (2)(b-b1) with the typical value $a_1 = 0.15$. Also, keep change the parameter a_1 as in $(1.2 \leq a_1 < 1.71)$ system (1) approaches to periodic dynamic in $Int. R_+^3$, as it illustrated in Fig. (3) (a-b), further for $(1.71 \leq a_1)$ the solution still has a stable (PEP)

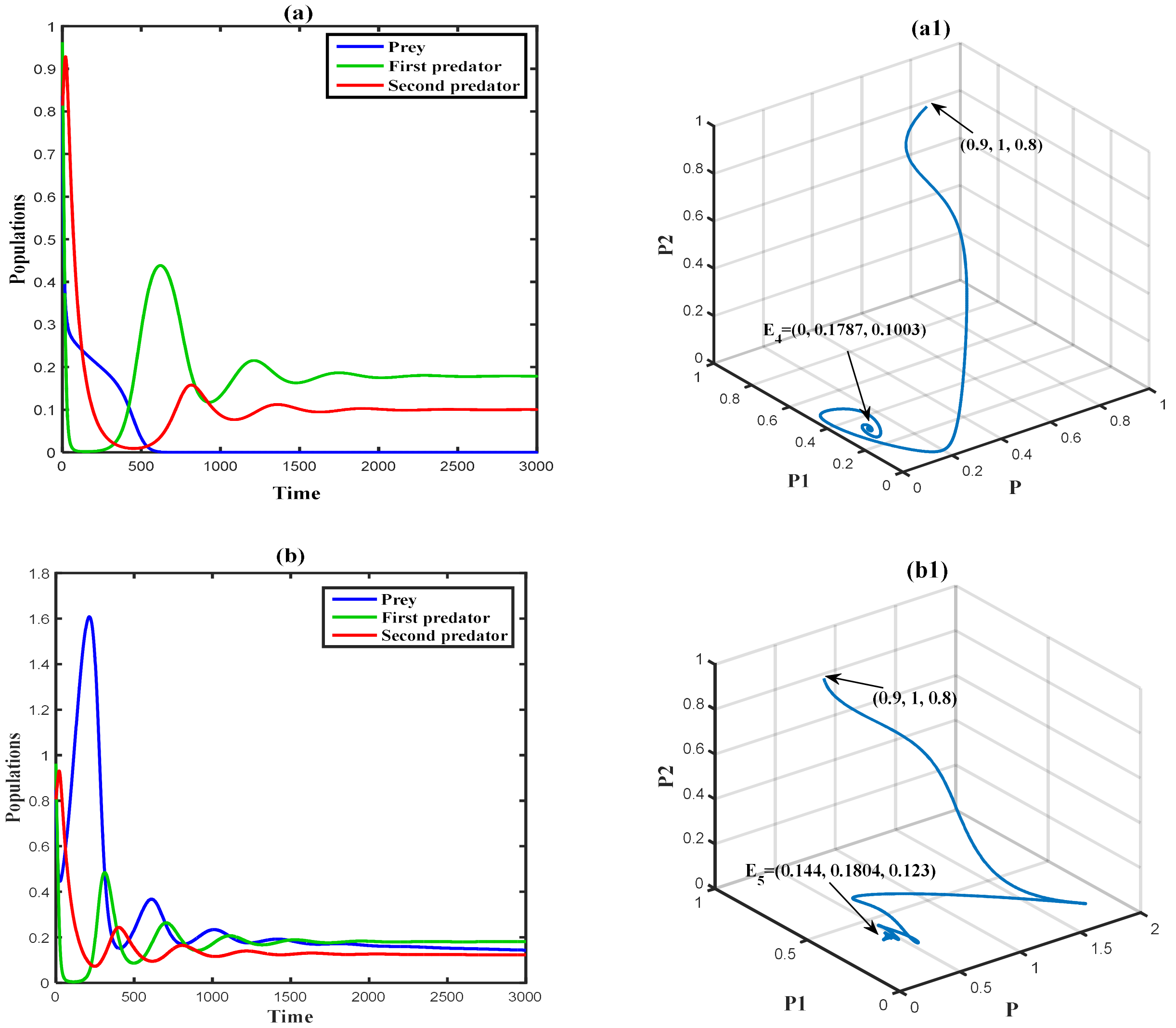


Fig. 2. (a) (T.S) of solution which goes to $E_4 = (0, 0.1787, 0.1003)$ when $a_1 = 0.001$, $(a1)$ 3D phase portrait of (a), (b) (T.S) of solution which goes to $E_5 = (0.144, 0.1804, 0.123)$ with $a_1 = 0.15$, $(b1)$ 3D phase portrait of (b).

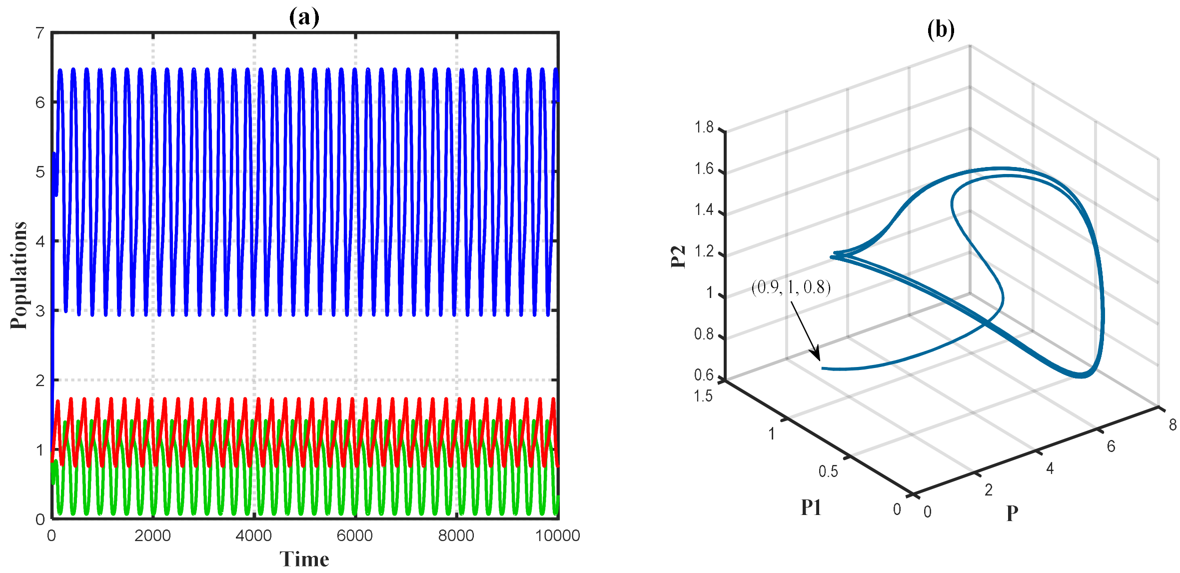


Fig. 3. (a) (T.S) of solution for the data given in (20) with $a_1 = 1.399$, (b) Periodic dynamics of solution.

For the parameters values specified in Eq. (20) with c_1 varying within the range $0.1 \leq c_1 < 0.4$ the solution goes to the periodic dynamics in the interior of positive octant of $PP_1P_2 - space$ as illustrated in Fig. (4) (a-b), finally for $c_1 \geq 0.4$ the solution goes to the (PEP) E_5.

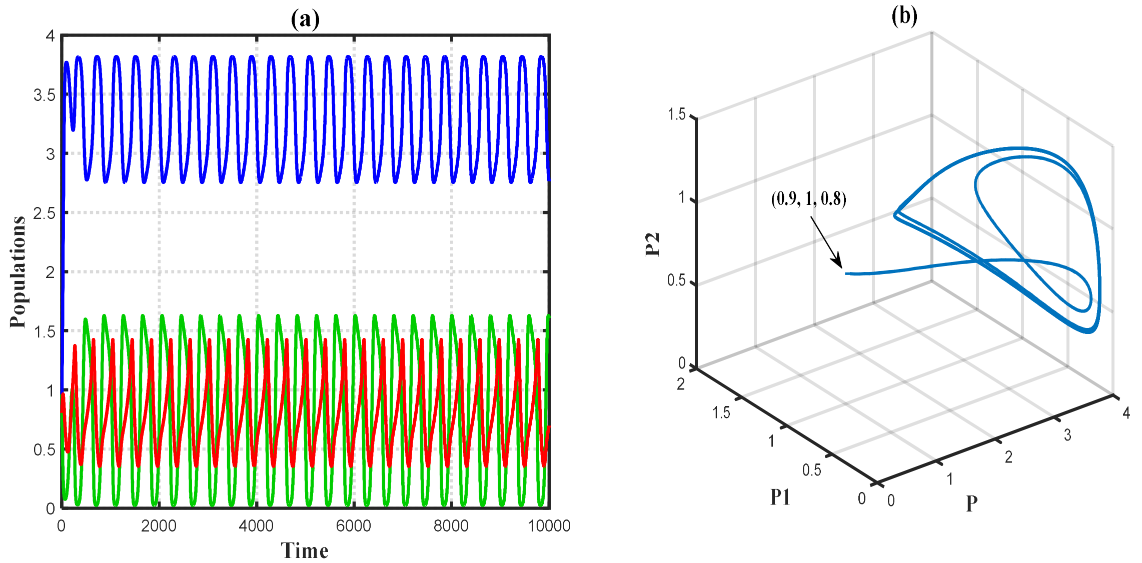


Fig. 4. (a) (T.S) of the solution for the data given by (20) with $c_1 = 0.1$, (b) Periodic dynamics of solution.

By changing the toxin rate of the prey δ_1 while maintaining the values of the other parameters as specified in Eq. (19) within the defined range $0.0001 \leq \delta_1 < 0.0076$ the system exhibits periodic dynamics in the $Int.\ R_+^3$, see Fig. (5) (a-b). While keeping increasing δ_1 for $0.0076 \leq \delta_1 < 2$ the solution goes to the (PEP) E_5.

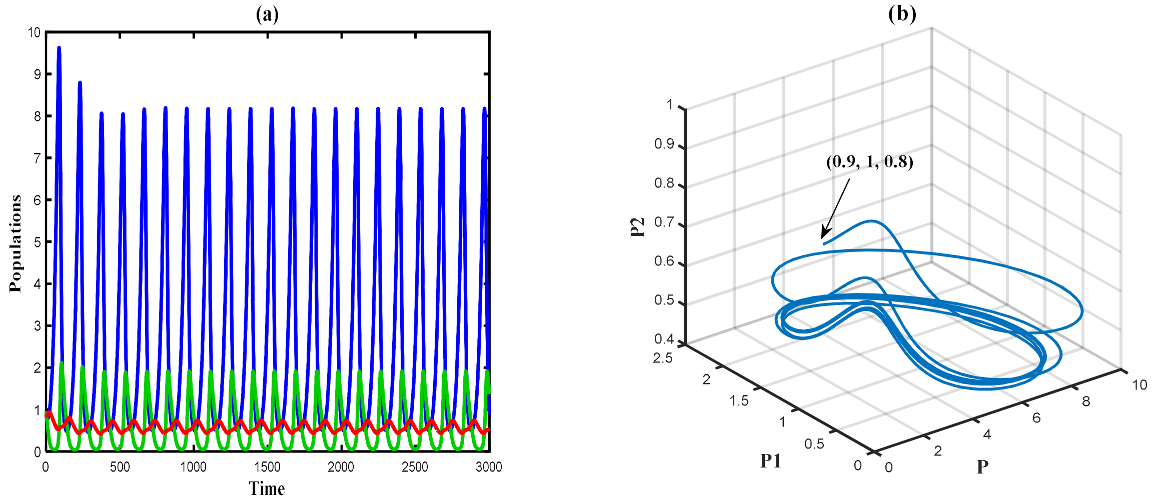
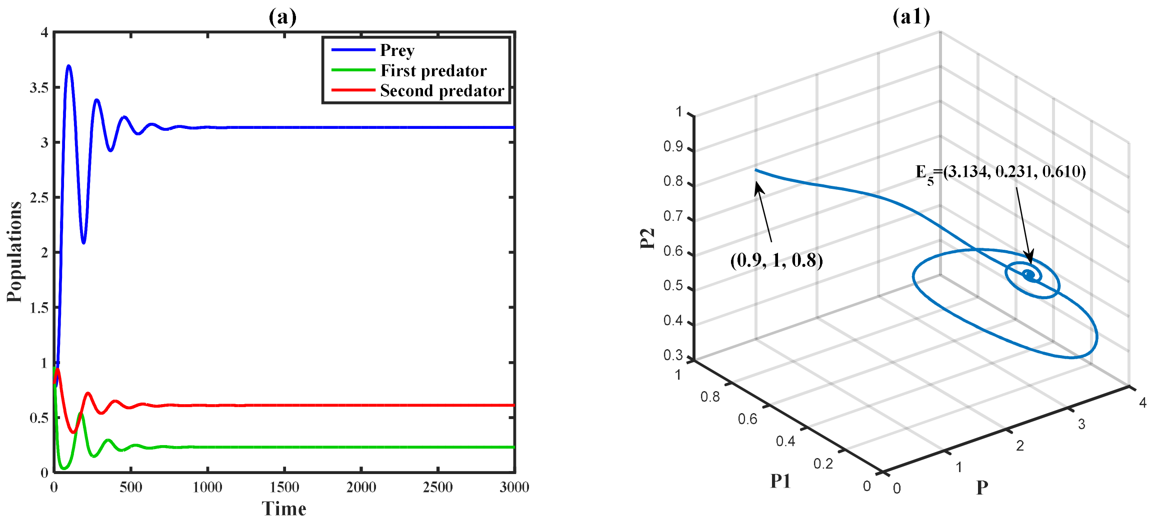


Fig. 5. (a) (T.S) of the solution for system (1) according to the data set given by Eq. (20) with $\delta_1 = 0.0001$, (b) Periodic dynamics of solution.

The influence of varying the linear harvesting rate of the prey h_1 with the data given in Eq. (20) the solution for $0.001 \leq h_1 < 0.3999$ goes to E_5 as illustrated in Fig. (6) (a-a1), with typical value $h_1 = 0.001$, while for the range $0.3999 \leq h_1 < 1$ the solution goes to E_4 as illustrated in Fig. (6) (b-b1), with typical value $h_1 = 0.5$.



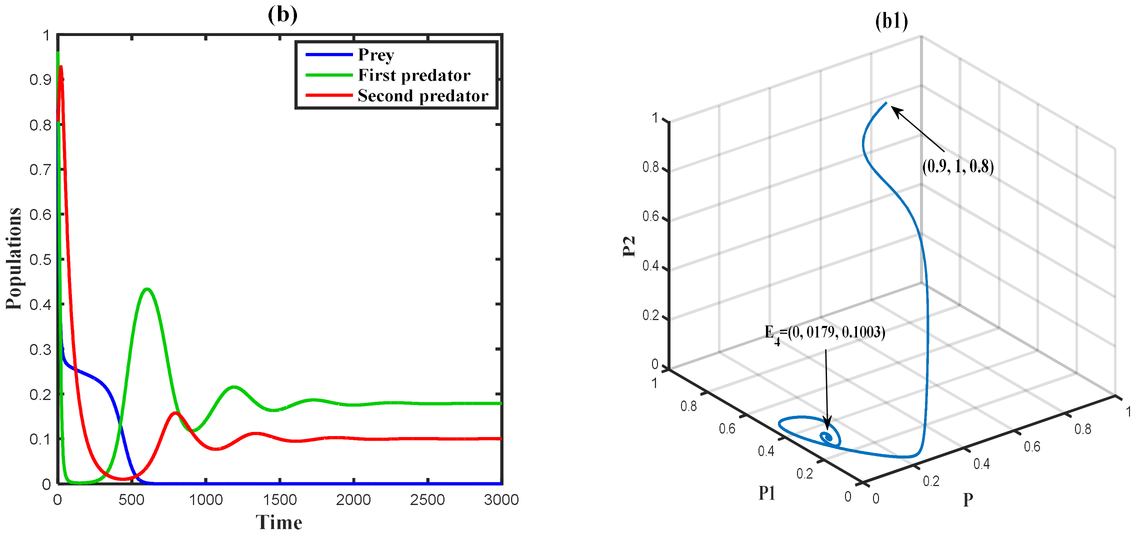


Fig. 6. (a) (T.S) of solution which goes to $E_5 = (3.134, 0.231, 0.610)$ when $h_1 = 0.001$, $(a1)$ 3D phase portrait of (a), (b) (T.S) of solution which goes to $E_4 = (0, 0.179, 0.1003)$ when $h_1 = 0.5$, $(b1)$ 3D phase portrait of (b),

The influence of altering the intrinsic growth rate a_2 of first predator P_1 with the data given in Eq. (20) in range $0.01 \leq a_2 < 0.66$ the solution goes to the (PEP) E_5, with typical value $a_2 = 0.01$. while continues increasing a_2 as in the range $0.66 \leq a_2 < 1.075$ the system has periodic dynamics in the $int. R_+^3$ as it illustrated in Fig. (7) (a-b), finally for $1.075 \leq a_2 < 1.104$ the solution of system (1) goes to E_4, see Table (2).

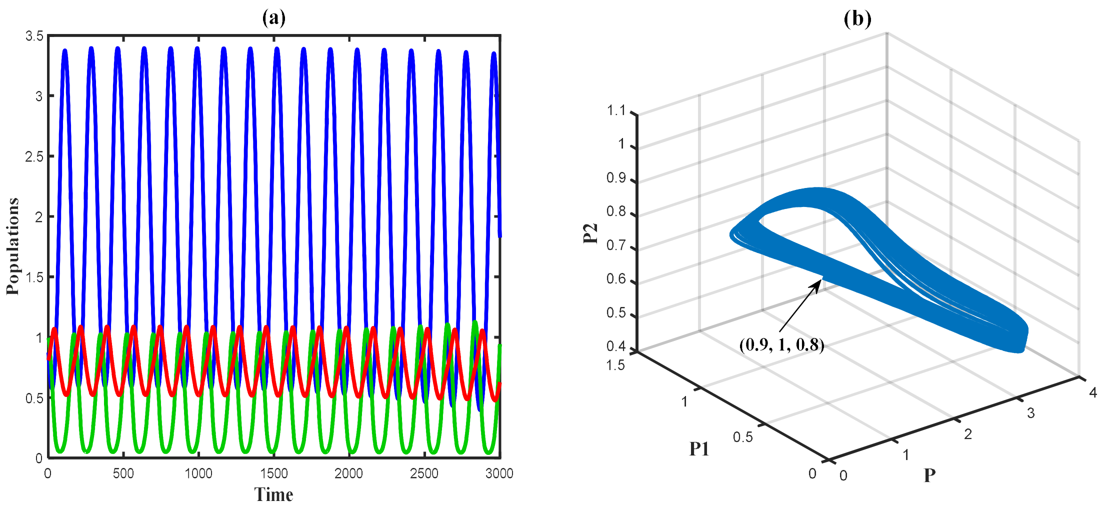
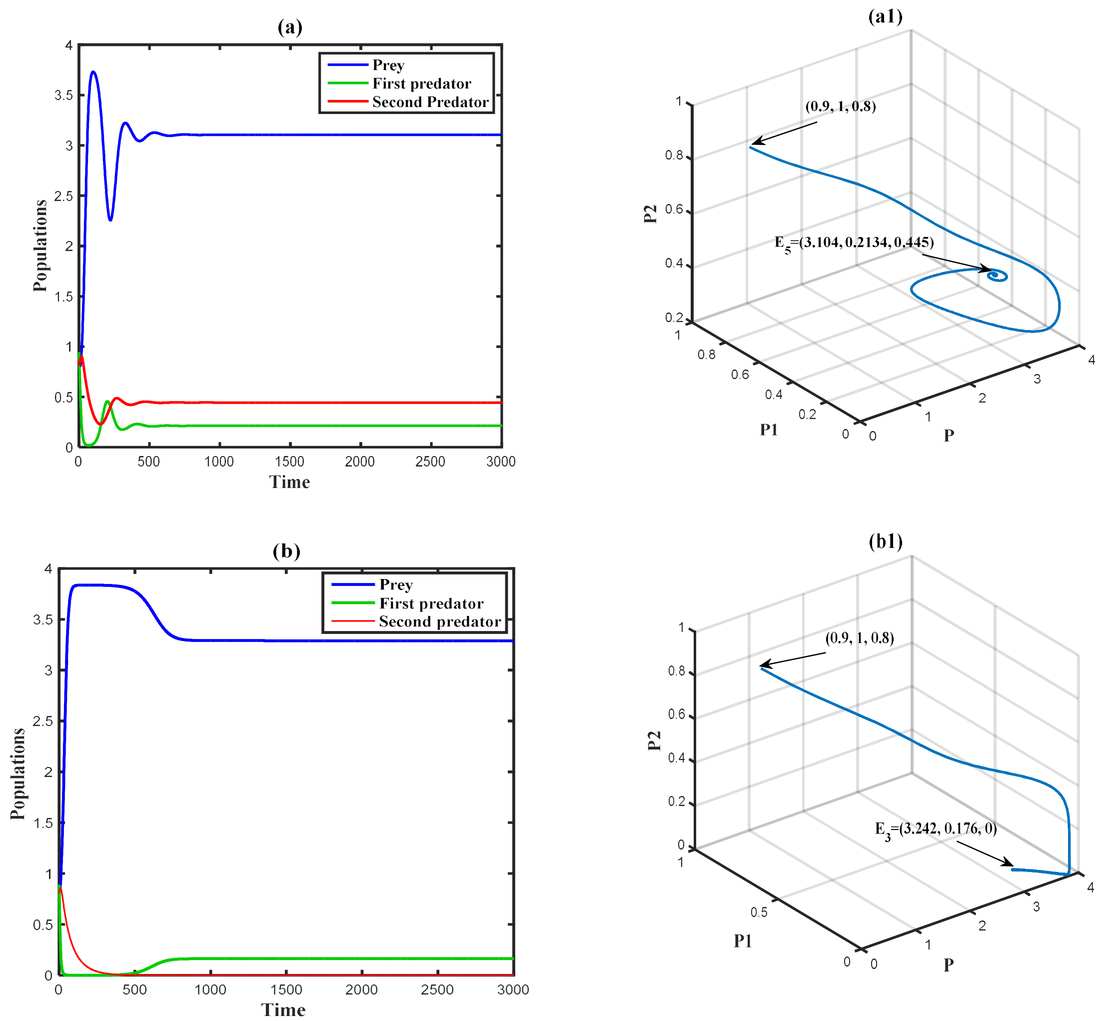


Fig. 7. (a) (T.S) of the solution for the given data by Eq. (20) with $a_2 = 0.66$, (b) Periodic dynamics of solutions.

Additionally, the effect of altering the natural death rate d_1 of the first predator P_1 has been studied while maintaining the other parameters values as specified in (20), It has been noted that for $0.001 \leq d_1 < 0.781$ the solution goes to the (PEP) E_5, as it illustrated in Fig. (8) (a-a1) with typical value $d_1 = 0.2$. while increasing d_1 in the range $0.781 \leq d_1 < 0.95$ the solution goes to E_3, as it illustrated in Fig. (8) (b-b1) with the typical value $d_1 = 0.781$, where the bifurcation occurred. Finally, when $0.95 \leq d_1 < 1$ the solution goes to E_1, as it illustrated in Fig. (8) (c-c1) with typical value $d_1 = 0.999$.



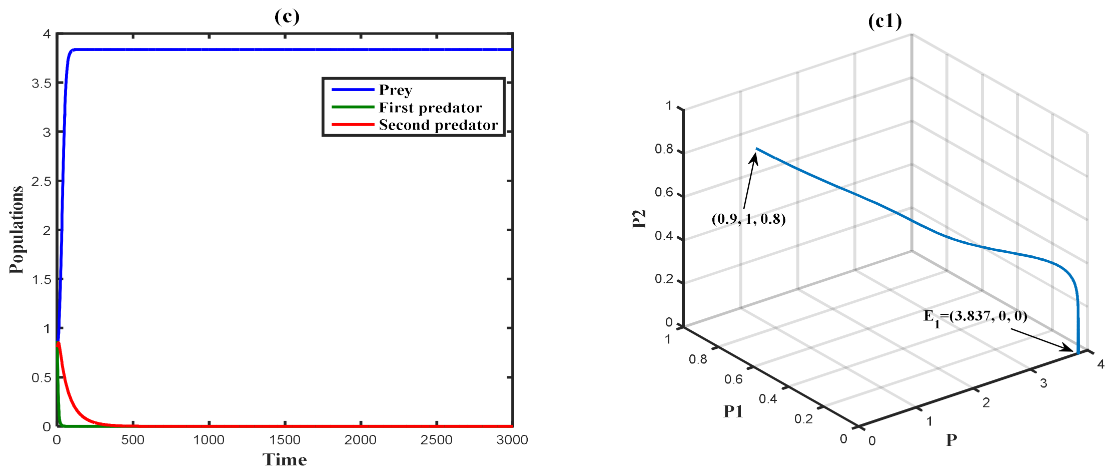


Fig. 8. (a) (T.S) of solution which goes to $E_5 = (3.104, 0.2134, 0.445)$ when $d_1 = 0.2$, $(a1)$ 3D phase portrait of (a), (b) (T.S) of solution of system (1) which goes to $E_3 = (3.242, 0.176, 0)$ when $d_1 = 0.781$, $(b1)$ 3D phase portrait of (b), (c) (T.S) of solution which goes to $E_1 = (3.837, 0, 0)$ when $d_1 = 0.999$, $(c1)$ 3D phase portrait of (c),

Also, the influence of altering the intrinsic growth rate a_1 of the prey P and the food transfer rate l_2 of the second predator P_2 in the same time while maintaining the other parameters unchanged as specified in Eq. (20) in the range $(0.001 \leq a_1 < 0.5)$, $(0.001 \leq l_2 < 0.2)$ the solution goes to E_2 as illustrated in Fig. (9) (a-a1), with typical values $a_1 = 0.1, l_2 = 0.1$.

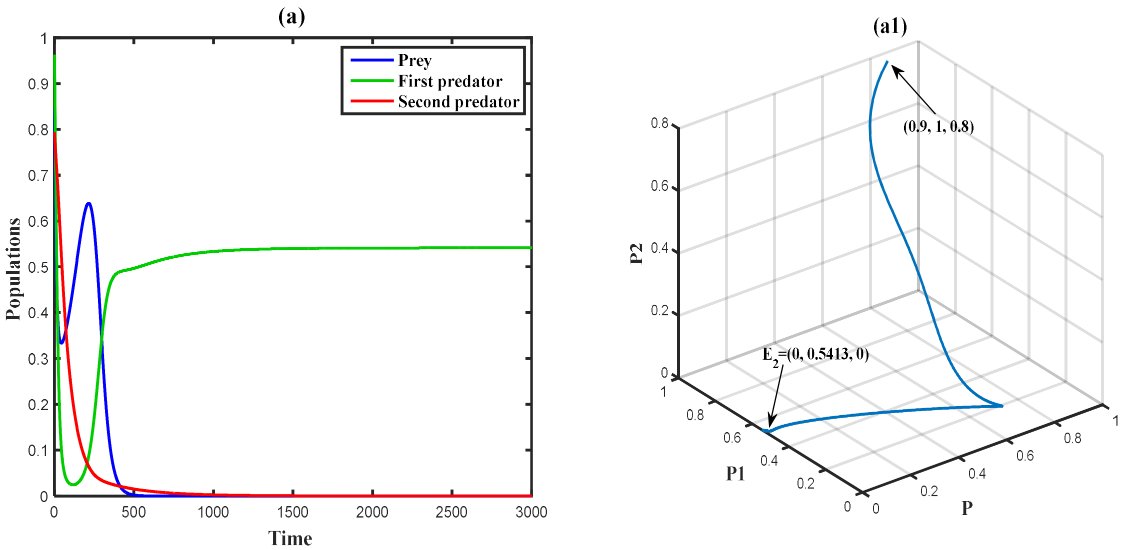


Fig. 9. (a) (T.S) of solution which goes to $E_2 = (0, 0.5413, 0)$ when $a_1 = 0.1$ and $l_2 = 0.1$, $(a1)$ 3D phase portrait of (a).

Finally, the influence of altering the intrinsic growth rate a_1 of the prey P and the natural death rate d_1 of the first predator P_1 in the same time while maintaining the other parameters unchanged as specified in Eq. (20) in the range $(0.0001 \leq a_1 < 0.005)$, $(0.3 \leq d_1 < 1)$ leads to the extinction of all species, with the solution approaching E_0, as illustrated in Fig. (10) (a-a1), with typical values $a_1 = 0.001, d_1 = 0.7$.

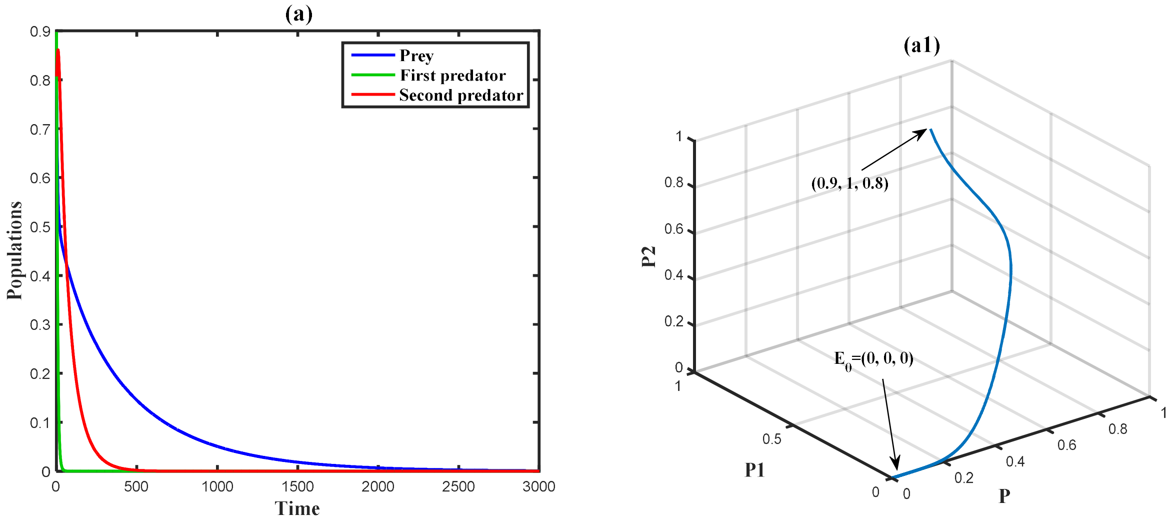


Fig. 10. (a) (T.S) of solution which goes to $E_0 = (0,0,0)$ when $a_1 = 0.001$ and $d_1 = 0.7$, $(a1)$ 3D phase portrait of (a).

6. The Conclusion and Discussions:

This work discussed the occurrence of local bifurcation under specific conditions in a food chain prey-predator model that includes a prey species and two predator species in the presence of toxic substances that impact the populations of all organisms, including predators and prey, and under the influence of essential factors such as fear, harvesting and others. It is note that a saddle-node bifurcation occurs around the positive equilibrium point E_5, while transcritical and pitchfork bifurcations exhibit near points E_2 and E_3 . Furthermore, only a transcritical bifurcation occurs near points E_0 , E_1 and E_4. Also, studies of the Hopf bifurcation at the toxin rate of prey (δ_1^*) near to the positive equilibrium have been done. Ultimately, numerical simulations have been conducted for three distinct initial points and one dataset illustrated in (20). The outcomes demonstrated that:

- The system exhibits periodic dynamics in the $Int. R_+^3$ at the parameters $(a_1, c_1, \delta_1, a_2)$.
- The parameters most effective for achieving system stability are

$(a_1, h_1, a_2, \alpha, d_1, l_2, \delta_3, d_2)$.

- The stability of system (1) is not influenced by the parameters $(m_{1,} b_1, m_2, b_2, l_2, c_2, \gamma, \delta_2, h_2, h_3)$ so that the solutions consistently approach to the positive equilibrium point.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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