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A NEW GAMMA REGRESSION ESTIMATE OF FEMALE BODY FAT PERCENTAGE: ADVANCED STATISTICAL MODELING

AMERA M. EL-MASRY[1], ASMAA A. ABU-HAMED[2], MOHAMED R. ABONAZEL[3], TAREK M. OMARA[4,*], IBRAHIM G. KHATTAB[5]

[1]Department of Mathematics and Statistics, Faculty of Management Technology and Information Systems, Port Said University, Port Said, Egypt

[2]Department of Statistics, Faculty of Politics and Economics, Beni-Suef University, Beni-Suef, Egypt

[3]Department of Applied Statistics and Econometrics, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt

[4]Department of Mathematics, Faculty of Sciences, Islamic University of Madinah, Madinah, Saudi Arabia

[5]Department of Statistics, Mathematics, and Insurance, Faculty of Business, Alexandria University, Alexandria, Egypt

Abstract: In this paper, a new gamma two-step shrinkage (GTSS) estimator is proposed for the gamma regression model to address the issue of multicollinearity. We derived the mean squared error (MSE) for the proposed estimator and defined the necessary and sufficient conditions for its superiority over five existing estimators. A comprehensive comparison is conducted between the proposed GTSS estimator and traditional maximum likelihood (ML) estimator, Liu estimator, and other conventional estimators, using a matrix mean squared error criterion. The results from the Monte Carlo simulation demonstrate the advantages of the proposed GTSS estimator under various conditions, particularly in the presence of severe multicollinearity. Additionally, we analyzed a real-world dataset on body fat to illustrate the practical relevance and effectiveness of our new estimator, showing significant reductions in the

*Corresponding author
E-mail address: Tarek_em@iu.edu.sa
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estimated mean squared error. As a result, the precision of the estimated parameters improves substantially, fulfilling one of the primary objectives for practitioners in this field.

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1. INTRODUCTION

Many fields of study, like medical science, social, and economic studies, automobile insurance claims, used gamma regression for modeling authoritative data questions, investigations by [1], [2], [3]. When there is a positive skewness for the response variable and it follows gamma distribution, the GRM is suitable, see [4], [5], [6].

The issue of multicollinearity arises when there is a significant correlation between the independent variables in a regression model. This phenomenon can pose challenges to accurate parameter estimation. In their study, Dunder et al. [3] highlighted that the use of the MLE for estimating coefficients in GRM exacerbates variance when there is intercorrelation among the predictors. This underscores the importance of addressing multicollinearity to ensure reliable estimation results. Literature has offered a plethora of methods to address the problem of multicollinearity. When it came to handling multicollinearity, Hoerl and Kennard [7] presented the ridge regression estimator in 1970 as a competitive substitute for MLE. Since then, statistical modelling has commonly used this technique to stabilize parameter estimates when correlated predictors are present.

The ridge estimator's applicability to GLM was expanded by [8]. Schaefer et al. [9] used this ridge estimation on the logistic regression model. This modification made it possible to handle multicollinearity in logistic regression analyses in an efficient manner, which enhanced model stability and parameter estimates. The ridge estimator was further developed in the negative binomial regression model by [10] and the Poisson regression model by [11]. Kurtoglu and Ozkale [12] developed ridge and Liu (presented by Liu [13]) estimators in the GRM. Moreover, Algamal and Asar [14] developed the Liu-type estimator in GRM. In addition, Omara [15] presented the penalized estimators for modified Log-Bilal regression. These variations illustrate the versatility and effectiveness of these statistical methods in different regression settings. Based on Batah et al. [16] and Algamal [6] presented the jackknife method to address the bias of the generalized ridge estimator GRE. Furthermore, Lukeman et al. [17] proposed a modified ridge-type estimator designed for linear regression model, which further extends the methods available for regression analysis. This development highlights the ongoing efforts to improve and consume statistical methods to overcome regression modelling challenges.

Recently, Qasim et al. [18] proposed an efficient estimator to address the multicollinearity problem in the linear regression model. Since this estimator is an efficient estimator among two-parameter estimators in the linear model, and since there is no similar analysis for two-parameter estimators in the GRM, it makes sense to update this estimator in the GRM. In other words, this paper focuses on extending and developing the estimator proposed by [18] for the GRM.

The remainder of this paper is organized as follows. In Section 2, the GRM and the existing estimators. The proposed estimator is presented in Section 3. The theoretical comparisons among the proposed estimator and the other estimators are made in Section 4. In section 5, the optimal (k and d) estimators of the biasing parameter for the proposed estimator are presented. In Section 6, the results of the simulation study are discussed. In Section 7, we attempt a real-world application. Section 8 concludes the paper.

2. THE GAMMA MODEL AND EXISTING ESTIMATORS

Certainly, the gamma distribution is commonly represented as so $Y_i \sim \text{Gamma}(\vartheta,\tau)$; where Y_i is the response variable. Its probability density function (PDF) is typically defined as

$$f(y_i) = \frac{\tau}{\Gamma\vartheta}(\tau y_i)^{\vartheta-1}e^{-\tau y_i}; \quad y_i \geq 0 \tag{1}$$

where the expected value of y is denoted as $E(y) = \vartheta/\tau = \theta$ Similarly, the variance of y is represented as $Var(y) = \vartheta/\tau^2 = \theta^2/\vartheta$, the shape parameter ϑ is constrained to be non-negative, and likewise, the scale parameter τ is also non-negative. Given that $\tau = \vartheta/\theta$. Equation (1) can be reformulated by parameterizing the average (θ) and the shape parameter (ϑ), using an exponential function:

$$f(y_i) = \exp\left\{\frac{y_i(-1/\theta)-\log(-1/\theta)}{1/\vartheta} + c(y_i,\vartheta)\right\} \tag{2}$$

In this context, the canonical link function equals $-1/\theta$, the dispersion parameter equals $\varphi = 1/\vartheta$, and $c(y_i, \vartheta) = \nu\log(\vartheta) + \vartheta\log(y_i) - \log(y_i) - \log(\Gamma(\vartheta))$.

The GRM is often represented using the canonical link function, typically the reciprocal function, $\theta_i = -1/x_i^T\beta$ where $x_i = (x_{i1},\ldots,x_{ip})^T$ denotes a linear combination of variables. Alternatively, to ensure $\theta_i > 0$, the log link function $\theta_i = \exp(x_i^T\beta)$ can be utilized. The maximum likelihood (ML) method, as outlined in Equation (2), is widely employed for estimating coefficients in this model. It assumes independence among elements and, $\theta_i = -1/x_i^T\beta$. The log-likelihood function is then formulated to reflect this assumption as follows:

$$l(\beta) = \sum_{i=1}^{n}\left\{\frac{y_i x_i^T\beta - \log(x_i^T\beta)}{1/\vartheta} + c(y_i, \vartheta)\right\} \tag{3}$$

Then the MLE is acquired by derivation of the Equation (3) Then putting 0 for the first derivative, as

$$\frac{\partial l(\beta)}{\partial \beta} = \frac{1}{\vartheta}\sum_{i=1}^{n}\left[y_i - \frac{1}{x_i^T\beta}\right]x_i = 0 \tag{4}$$

regrettably, due to the nonlinearity of Equation (4) in phrases of (beta), the primary derivative cannot be solved analytically. Hence, we lodge to iterative methods including the iteratively weighted least squares algorithm or the Fisher-scoring method to reap the MLEs for the parameters of the GRM. These iterations may be defined as follows:

$$\beta^{(k+1)} = \beta^{(k)} + I^{-1}\left(\beta^{(k)}\right)S\left(\beta^{(k)}\right)$$

where $S(\beta)$ equals $\partial\ell(\beta)/\partial\beta$ and $I^{-1}\left(\beta^{(k)}\right)$ equals $(-E(\partial^2 l(\beta)/\partial\beta\,\partial\beta^T)^{-1}$, the coefficients estimation last stage is described as:

$$\hat{\beta}_{ML} = (X^T\hat{Z}X)^{-1}X^T\hat{Z}\hat{u} \tag{5}$$

where $\hat{Z} = \text{diag}\left(\hat{\theta}_i^2\right)$ and $\hat{z}$ represents a vector, whose *i*th element is equal to $\hat{u}_i = \hat{\theta}_i + \left((y_i - \hat{\theta}_i)/\hat{\theta}_i^2\right)$.

With a covariance matrix that is compatible with the inverse of the Hessian matrix, the MLE has an almost regularly distributed distribution.

$$\text{COV}\left(\hat{\beta}_{ML}\right) = \vartheta^{-1}\left(X^T\hat{Z}X\right)^{-1} \tag{6}$$

The MSE of the Equation (5) obtained as

$$\text{MSE}\left(\hat{\beta}_{ML}\right) = E\left(\hat{\beta}_{ML} - \hat{\beta}\right)^T\left(\hat{\beta}_{ML} - \hat{\beta}\right) = \varphi\,\text{tr}\left[\left(X^T\hat{Z}X\right)^{-1}\right] = \varphi\sum_{j=1}^{p}\frac{1}{\lambda_j} \tag{7}$$

where g_j is the eigenvalue of the $\Lambda = X^T\hat{Z}X$ matrix and $\varphi = 1/\vartheta$. Moreover, the eigenvalue section of the matrix G is regarded as following: $G = Q\Omega Q$ and Q is an orthogonal matrix consisting of the eigenvectors of G and $\Omega = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. It's obvious that if one or more eigenvalues of the Hessian matrix are close to zero, it indicates that the MLE may exhibit high variance. This can result in inflated MSE and adversely affect the accuracy of the regression coefficient estimates.

For modelling the relation through the response variable and one or more independent factors, researchers commonly employ LRM when the dependent variable is normally distributed. Conversely, when the dependent variable exhibits skewness, the GRM is often preferred. Parameter estimation in both models relies on the MLE. However, the presence of multicollinearity leads to instability in MLE estimation for both models. Consequently, researchers have proposed various alternative estimators and advocated parameter biasing to enhance the accuracy of regression parameter estimation, particularly in the context of GRM and multicollinearity.

2.1 Gamma Ridge Estimator

The well – known ridge regression estimator was introduced by Segerstedt [8] for the GLMs. The gamma ridge estimator (GRE) is defined as follows:

$$\hat{\beta}_{GR} = (\Lambda + kI)^{-1} X^T \hat{Z} \hat{u} = \Lambda_k^{-1} \Lambda \hat{\beta}_{ML} \ = F_K \hat{\beta}_{ML}; \quad k > 0 \tag{8}$$

where $\Lambda = X^T \hat{Z} X$ and $\Lambda_k = (\Lambda + kI)\ and\ F_K = (\Lambda + kI)^{-1}(X^T \hat{Z} X)$. Then the covariance matrix and vector of bias for GRE can be written as follows:

$$cov(\hat{\beta}_{GR}) = \varphi \Lambda_k^{-1} \Lambda \Lambda_k^{-1} \tag{9}$$

$$b_{GRE} = bias(\hat{\beta}_{GR}) = -k \Lambda_k \beta \tag{10}$$

Then, the MSE matrix (MSEM) and MSE for GRE are obtained as follows:

$$MSEM\ (\hat{\beta}_{GR}) = \mathrm{COV}(\hat{\beta}_{GR}) + b_{GRE} b_{GRE}^T = \varphi \Lambda_k^{-1} \Lambda \Lambda_k^{-1} + k^2 \Lambda_k^{-1} \beta \beta^T \Lambda_k^{-1} \tag{11}$$

$$MSE(\hat{\beta}_{GR}) = tr[MSEM(\hat{\beta}_{GR})] = \varphi \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j + k)^2} + k^2 \sum_{j=1}^{p} \frac{\gamma_j^2}{(\lambda_j + k)^2} \tag{12}$$

where $\gamma = (\gamma_1, \dots, \gamma_p\ )^T = Q^T \beta$.

2.2 Gamma Liu estimator

Kurtoglu and Ozkale [12] introduced the Liu estimator within the framework of generalized linear models (GLMs) to address issues related to estimation efficiency and robustness, particularly in the presence of gamma-dependent variables. The gamma Liu estimator (GLE) is written as:

$$\hat{\beta}_{GL} = (\Lambda + I_p)^{-1} (\Lambda + d I_p) \hat{\beta}_{ML} = F_d \hat{\beta}_{ML} \tag{13}$$

where $F_d = (\Lambda + I_p)^{-1} (\Lambda + d I_p)$, $0 < d < 1$. Then the covariance matrix and vector of bias for GLE can be written as follows:

$$\mathrm{COV}(\hat{\beta}_{GL}) = \varphi F_d \Lambda^{-1} {F_d}^T \tag{14}$$

$$b_{GLE} = bias(\hat{\beta}_{GL}) = -(1-d)(\Lambda + I_p)^{-1}\beta \tag{15}$$

Then, the MSEM and MSE for the GLE are obtained as follows:

$$MSEM\ (\hat{\beta}_{GL}) = \mathrm{COV}(\hat{\beta}_{GL}) + b_{GL}b_{GL}^T$$

$$= \varphi F_d \Lambda^{-1} F_d^T + (1-d)^2\ (\Lambda + I_p)^{-1}\beta\beta^T(\Lambda + I_p)^{-1} \tag{16}$$

$$MSE(\hat{\beta}_{GL}) = tr[MSEM(\hat{\beta}_{GL})] = \varphi\sum_{j=1}^{p}\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2} + (1-d)^2\sum_{j=1}^{p}\frac{\gamma_j^2}{(\lambda_j+1)^2} \tag{17}$$

2.3 Gamma KL estimator

Lukman et al. [17] obtained the gamma Kibria-Lukman estimator (GKLE) as follows:

$$\hat{\beta}_{GKL} = (\Lambda + kI_p)^{-1}(\Lambda - kI_p)\hat{\beta}_{ML} = \Lambda_K^{-1}\mathrm{R}_k\widehat{\beta}_{\mathrm{ML}};\ \ k > 0,$$

where $\Lambda_{\mathrm{K}} = (\mathrm{I} + \mathrm{K}\Lambda^{-1}),\ \ \mathrm{R}_{\mathrm{k}} = (\mathrm{I} - \mathrm{k}\Lambda^{-1})$.

The covariance matrix and vector of bias for GKL can be written as follows:

$$\mathrm{COV}(\widehat{\beta}_{\mathrm{GKL}}) = \varphi\Lambda_K^{-1}\mathrm{R}_k\Lambda^{-1}\mathrm{R}_{\mathrm{k}}^{\mathrm{T}}(\Lambda_K^{-1})^T$$

$$b_{GKL} = \mathrm{bias}(\widehat{\beta}_{\mathrm{GKL}}) = (\Lambda_K^{-1}\mathrm{R}_{\mathrm{k}} - \mathrm{I}_{\mathrm{p}})\beta.$$

Then, the MSEM and MSE for the GKL are obtained as follows:

$$\mathrm{MSEM}(\widehat{\beta}_{\mathrm{GKL}}) = \mathrm{COV}(\widehat{\beta}_{\mathrm{GKL}}) + b_{GKL}b_{GKL}^T$$

$$= \varphi\Lambda_K^{-1}\mathrm{R}_k\Lambda^{-1}\mathrm{R}_{\mathrm{k}}^{\mathrm{T}}(\Lambda_K^{-1})^T + (\Lambda_K^{-1}\mathrm{R}_{\mathrm{k}} - \mathrm{I}_{\mathrm{p}})\beta\beta^{\mathrm{T}}(\Lambda_K^{-1}\mathrm{R}_{\mathrm{k}} - \mathrm{I}_{\mathrm{p}})^{\mathrm{T}}$$

$$\mathrm{MSE}(\widehat{\beta}_{\mathrm{GKL}}) = \mathrm{tr}\left(\mathrm{MSEM}(\widehat{\beta}_{\mathrm{GkL}})\right) = \varphi\sum_{\mathrm{j}=1}^{\mathrm{p}}\frac{(\lambda_{\mathrm{j}}-\mathrm{k})^2}{\lambda_{\mathrm{j}}(\lambda_{\mathrm{j}}+\mathrm{k})^2} + 4\mathrm{k}^2\sum_{\mathrm{j}=1}^{\mathrm{p}}\frac{\gamma_{\mathrm{j}}^2}{(\lambda_{\mathrm{j}}+\mathrm{k})^2}$$

2.4 Gamma two– parameter estimator

Asar and Algamal [19] proposed to adopt the estimator defined by Ozkale and Kaciranlar [20] in the gamma regression model, The gamma two – parameter estimator (GTPE) is written as:

$$\hat{\beta}_{GTP} = (\Lambda + \mathrm{kI_P})^{-1}(\Lambda + \mathrm{kdI_P})\hat{\beta}_{ML} = \Lambda_k^{-1}\Lambda_{kd}\hat{\beta}_{ML} = F_{kd}\hat{\beta}_{ML} \tag{18}$$

where $k > 0\ and\ 0 < d < 1$, $\Lambda_{kd} = (\Lambda + kdI)$ and thus $F_{kd} = \Lambda_k^{-1}\Lambda_{kd}$. Then the covariance matrix and vector of bias for GTPE can be written respectively as follows:

$$\mathrm{COV}(\hat{\beta}_{GTP}) = \varphi F_{kd}\Lambda^{-1}\ F_{kd}^T \tag{19}$$

$$b_{GTPE} = bias(\hat{\beta}_{GTP}) = k(d-1)(\Lambda + kI_P)^{-1}\beta \tag{20}$$

Then, the MSEM and MSE for GTPE are obtained as follows:

$$MSEM\left(\hat{\beta}_{GTP}\right) = \mathrm{COV}\left(\hat{\beta}_{GTP}\right) + b_{GTPE}b_{GTPE}^{T}$$
$$= \varphi F_{kd}\Lambda^{-1}F_{kd}^{T} + (d+k)^{2}\Lambda_{k}^{-1}\beta\beta^{T}\Lambda_{k}^{-1} \quad (21)$$

$$MSE\left(\hat{\beta}_{GTP}\right) = tr\left[MSEM\left(\hat{\beta}_{GTP}\right)\right] = \varphi\sum_{j=1}^{p}\frac{(\lambda_j+kd)^2}{\lambda_j(\lambda_j+k)^2} + k^2(\mathrm{d}-1)^2\sum_{j=1}^{p}\frac{\gamma_j^2}{(\lambda_j+k)^2} \quad (22)$$

3 PROPOSED ESTIMATOR

Qasim et al [18] proposed a class of two-step shrinkage (TSS) estimator by augmenting equation of classical linear regression model with $\left(\frac{-\mathrm{kd}}{\mathrm{K}^{1/2}}\right)\hat{\beta}_{\mathrm{OLS}} = \mathrm{K}^{1/2}\beta + \varepsilon, \mathrm{k} > 0,\ 0 < \mathrm{d} < 1$ and then applied the ordinary least squares (OLS) estimator ($\hat{\beta}_{\mathrm{OLS}} = (\mathrm{X^TX})^{-1}X^TY$), where ε is p-vector of random errors. They proposed a new estimator namely $\hat{\beta}_{\mathrm{TSS}}$ that written as follows:

$$\hat{\beta}_{\mathrm{TSS}} = \left\{\mathrm{I_p} - \mathrm{k}(1+\mathrm{d})\ (\mathrm{X^TX} + \mathrm{kI_p})^{-1}\right\}\hat{\beta}_{\mathrm{OLS}};\quad \mathrm{k} > 0,\ 0 < \mathrm{d} < 1 \quad (23)$$

In this paper, we suggest adopting the TSS estimator in the gamma regression model and propose the following gamma class of two-step shrinkage estimator (GTSSE):

$$\hat{\beta}_{\mathrm{GTSS}} = \left\{\mathrm{I_p} - \mathrm{k}(1+\mathrm{d})\ (\mathrm{X^T}\hat{Z}\mathrm{X} + \mathrm{kI_p})^{-1}\right\}\hat{\beta}_{\mathrm{ML}} = R\ \hat{\beta}_{\mathrm{ML}} \quad (24)$$

where $k > 0, 0 < \mathrm{d} < 1$, and $R = \left\{\mathrm{I_p} - \mathrm{k}(1+\mathrm{d})\ (\mathrm{X}^T\hat{Z}\mathrm{X} + \mathrm{kI_p})^{-1}\right\}$. Then the covariance matrix and the vector of bias for the proposed GTSSE can be written, respectively, as follows:

$$\mathrm{COV}\left(\hat{\beta}_{\mathrm{GTSS}}\right) = \varphi R\Lambda^{-1}R^{T} \quad (25)$$

$$b_{GTSSE} = bias\left(\hat{\beta}_{\mathrm{GTSS}}\right) = -K(d+1)\left(\Lambda + \mathrm{k}I_p\right)^{-1}\beta \quad (26)$$

Then, the MSEM and MSE for GTSSE are obtained as follows:

$$MSEM\left(\hat{\beta}_{GTSS}\right) = COV\left(\hat{\beta}_{GTSS}\right) + b_{GTSSE}b_{GTSSE}^{T}$$
$$= \varphi R\Lambda^{-1}R^{T} + K^2(d+1)^2(\Lambda + KI_p)^{-1}\beta\beta^{T}(\Lambda + KI_p)^{-1} \quad (27)$$

$$MSE\left(\hat{\beta}_{GTSS}\right) = tr\left[MSEM\left(\hat{\beta}_{GTSS}\right)\right] = \varphi\sum_{\mathrm{j}=1}^{\mathrm{p}}\frac{(\lambda_{\mathrm{j}}-dk)^2}{(\lambda_{\mathrm{j}}+\mathrm{k})^2\lambda_{\mathrm{j}}} + k^2(d+1)^2\sum_{\mathrm{j}=1}^{\mathrm{p}}\frac{\gamma_j^2}{(\lambda_{\mathrm{j}}+\mathrm{k})^2} \quad (28)$$

4 SUPERIORITY OF THE PROPOSED ESTIMATOR ($\hat{\beta}_{GTSS}$)

In this section, we compare the performance of the proposed GTSSE with the existing estimators in the sense of mean square error (MSE) criteria. The MSE of an estimator $\hat{\alpha}$ of α can be defined as

$$\mathrm{MSE}\ (\hat{\alpha}) = \mathrm{tr}[\mathrm{MSE}M(\hat{\alpha})] = \mathrm{tr}[\mathrm{COV}(\hat{\alpha})] + \left[\left(\mathrm{Bias}(\hat{\alpha})\right)^T\left(\mathrm{Bias}(\hat{\alpha})\right)\right] \qquad (29)$$

where $\mathrm{COV}(\hat{\alpha})$ is represent the covariance matrix of $\hat{\alpha}$ and Bias $(\hat{\alpha}) = \mathrm{E}(\hat{\alpha}) - \alpha$ is the bias vector.

Let $\hat{\alpha}_1$ and $\hat{\alpha}_2$ be the two estimators of α, the estimator $\hat{\alpha}_2$ is said to be superior to the estimator $\hat{\alpha}_1$ if and only if

$$\Delta = \mathrm{MSE}\ (\hat{\alpha}_2) - \mathrm{MSE}\ (\hat{\alpha}_1) \leq 0. \qquad (30)$$

4.1 MSE comparison of $\hat{\beta}_{GTSS}$ and $\hat{\beta}_{ML}$

Theorem1: if $\left(\lambda_j - dk\right)^2 + k^2(1+d)^2{\gamma_j}^2\varphi^{-1}\lambda_j < \left(\lambda_j + k\right)^2\ for\ all\ j$, then $MSE\left(\hat{\beta}_{GTSS}\right) < MSE\left(\hat{\beta}_{ML}\right)$.

Proof: The MSE difference between the GTSSE and the MLE is written as

$$\Delta_1 = MSE\left(\hat{\beta}_{GTSS}\right) - MSE\left(\hat{\beta}_{ML}\right)$$

$$\Delta_1 = \sum_{j=1}^{p} \frac{\varphi(\lambda_j - dk)^2 + k^2(d+1)^2{\gamma_j}^2\lambda_j}{(\lambda_j + k)^2\lambda_j} - \varphi\sum_{j=1}^{p}\frac{1}{\lambda_j}$$

$$\Delta_1 = \varphi\sum_{j=1}^{P}\left[\frac{\left(\lambda_j - dk\right)^2 + k^2(1+d)^2{\gamma_j}^2\varphi^{-1}\lambda_j - \left(\lambda_j + k\right)^2}{\left(\lambda_j + k\right)^2\lambda_j}\right] \qquad (31)$$

In the case of $\left(\lambda_j - dk\right)^2 + k^2(1+d)^2{\gamma_j}^2\varphi^{-1}\lambda_j - \left(\lambda_j + k\right)^2 < 0$ in the Equation (31), it implies that $\left(\lambda_j - dk\right)^2 + k^2(1+d)^2{\gamma_j}^2\varphi^{-1}\lambda_j < \left(\lambda_j + k\right)^2\ for\ all\ j,$ then $MSE\left(\hat{\beta}_{GTSS}\right) < MSE\left(\hat{\beta}_{ML}\right)$. That means that the GTSSE is better than the MLE if $\left(\lambda_j - dk\right)^2 + k^2(1+d)^2{\gamma_j}^2\varphi^{-1}\lambda_j < \left(\lambda_j + k\right)^2 for\ all\ j$.

4.2 MSE comparison of $\hat{\beta}_{GTSS}$ and $\hat{\beta}_{GR}$

Theorem 2: if $\left(\lambda_j - dk\right)^2 + k^2{\gamma_j}^2\varphi^{-1}\lambda_j(d^2 + 2d) < {\lambda_j}^2\ for\ all\ j,$ then $MSE\left(\hat{\beta}_{GTSS}\right) < MSE\left(\hat{\beta}_{GR}\right)$.

Proof: The MSE difference between the GTSSE and the GRE is written as

$$\Delta_2 = MSE\left(\hat{\beta}_{GTSS}\right) - MSE\left(\hat{\beta}_{GR}\right)$$

$$\Delta_2 = \sum_{j=1}^{p} \frac{\varphi(\lambda_j - dk)^2 + k^2(d+1)^2{\gamma_j}^2\lambda_j}{(\lambda_j + k)^2\lambda_j} - \sum_{j=1}^{p}\frac{\varphi\lambda_j + k^2{\gamma_j}^2}{(\lambda_j + k)^2}$$

$$= \varphi \sum_{j=1}^{p} \left[\frac{\left(\lambda_j - dk\right)^2 + k^2(d+1)^2 {\gamma_j}^2 \varphi^{-1} \lambda_j - {\lambda_j}^2 - k^2 {\gamma_j}^2 \hat{\varphi}^{-1} \lambda_j}{\left(\lambda_j + k\right)^2 \lambda_j} \right]$$

$$= \varphi \sum_{j=1}^{p} \left[\frac{\left(\lambda_j - dk\right)^2 + k^2 {\gamma_j}^2 \varphi^{-1} \lambda_j (d^2 + 2d) - {\lambda_j}^2}{\left(\lambda_j + k\right)^2 \lambda_j} \right] \quad (32)$$

In the case of $\left(\lambda_j - dk\right)^2 + k^2 {\gamma_j}^2 \varphi^{-1} \lambda_j (d^2 + 2d) - {\lambda_j}^2 < 0$ in the Equation (32), it implies that$\left(\lambda_j - dk\right)^2 + k^2 {\gamma_j}^2 \varphi^{-1} \lambda_j (d^2 + 2d) < {\lambda_j}^2 \; for\ all\ j.$, then $MSE\left(\hat{\beta}_{\mathrm{GTSS}}\right) < MSE\left(\hat{\beta}_{GR}\right)$. That means that the GTSSE is better than the GRE if $\left(\lambda_j - dk\right)^2 + k^2 {\gamma_j}^2 \varphi^{-1} \lambda_j (d^2 + 2d) < {\lambda_j}^2 \; for\ all\ j.$

4.3 MSE comparison of $\hat{\boldsymbol{\beta}}_{\mathbf{GTSS}}$ and $\hat{\boldsymbol{\beta}}_{\mathbf{GL}}$

Theorem 3: if $\frac{\varphi\left(\lambda_j - dk\right)^2}{\lambda_j \gamma_j^2 \left(\lambda_j + d\right)^2} + \frac{k^2(d+1)^2}{\left(\lambda_j + k\right)^2} < \frac{\varphi\left(\lambda_j + d\right)^2}{\lambda_j \gamma_j^2 \left(\lambda_j + 1\right)^2} + \frac{(d-1)^2}{\left(\lambda_j + 1\right)^2} \; for\ all\ j,$ then $MSE\left(\hat{\beta}_{\mathrm{GTSS}}\right) < MSE\left(\hat{\beta}_{\mathrm{GL}}\right)$.

Proof: The MSE difference between the GTSSE and the GLE is written as

$$\Delta_3 = MSE\left(\hat{\beta}_{\mathrm{GTSS}}\right) - MSE\left(\hat{\beta}_{\mathrm{GL}}\right)$$

$$\Delta_3 = \sum_{j=1}^{p} \frac{\varphi(\lambda_j - dk)^2 + k^2(d+1)^2 {\gamma_j}^2 \lambda_j}{(\lambda_j + k)^2 \lambda_j} - \sum_{j=1}^{p} \frac{\varphi(\lambda_j + d)^2 + (d-1)^2 \lambda_j {\gamma_j}^2}{\lambda_j (\lambda_j + 1)^2}$$

$$= \varphi \sum_{j=1}^{p} \left[\frac{\left(\lambda_j - dk\right)^2 \left(\lambda_j + 1\right)^2 + k^2 \left(\lambda_j + 1\right)^2 (d+1)^2 {\gamma_j}^2 \varphi^{-1} \lambda_j - \left(\lambda_j + d\right)^2 \left(\lambda_j + k\right)^2 - \left(\lambda_j + k\right)^2 (d-1)^2 \lambda_j \varphi^{-1} {\gamma_j}^2}{\left(\lambda_j + k\right)^2 \left(\lambda_j + 1\right)^2 \lambda_j} \right] \quad (33)$$

In the case of $\left(\lambda_j - dk\right)^2 \left(\lambda_j + 1\right)^2 + k^2 \left(\lambda_j + 1\right)^2 (d+1)^2 {\gamma_j}^2 \varphi^{-1} \lambda_j - \left(\lambda_j + d\right)^2 \left(\lambda_j + k\right)^2 - \left(\lambda_j + k\right)^2 (d-1)^2 \lambda_j \varphi^{-1} {\gamma_j}^2 < 0$ in the Equation (33), it implies that $\frac{\varphi\left(\lambda_j - dk\right)^2}{\lambda_j \gamma_j^2 \left(\lambda_j + d\right)^2} + \frac{k^2(d+1)^2}{\left(\lambda_j + k\right)^2} < \frac{\varphi\left(\lambda_j + d\right)^2}{\lambda_j \gamma_j^2 \left(\lambda_j + 1\right)^2} + \frac{(d-1)^2}{\left(\lambda_j + 1\right)^2}$, then $MSE\left(\hat{\beta}_{\mathrm{GTSS}}\right) < MSE\left(\hat{\beta}_{GL}\right)$. That means that the GTSSE is better than GLE if $\frac{\varphi\left(\lambda_j - dk\right)^2}{\lambda_j \gamma_j^2 \left(\lambda_j + d\right)^2} + \frac{k^2(d+1)^2}{\left(\lambda_j + k\right)^2} < \frac{\varphi\left(\lambda_j + d\right)^2}{\lambda_j \gamma_j^2 \left(\lambda_j + 1\right)^2} + \frac{(d-1)^2}{\left(\lambda_j + 1\right)^2}$ for all j.

4.4 MSE comparison of $\hat{\beta}_{GTSS}$ and $\hat{\beta}_{GKL}$

Theorem 4: if $(\lambda_j - dk)^2 + k^2(d+1)^2\gamma_j^2\varphi^{-1}\lambda_j - (\lambda_j - k)^2 < 4k^2\gamma_j^2\varphi^{-1}\lambda_j$ for all j, then the MSE of $\hat{\beta}_{GTSS}$ is less than the MSE of $\hat{\beta}_{GKL}$.

Proof: The difference in MSE across the GTSSE and GKLE is written as

$$\Delta_4 = MSE(\hat{\beta}_{GTSS}) - MSE(\hat{\beta}_{GKL})$$

$$\Delta_4 = \sum_{j=1}^{p} \frac{\varphi(\lambda_j - dk)^2 + k^2(d+1)^2{\gamma_j}^2\lambda_j}{(\lambda_j + k)^2\lambda_j} - \sum_{j=1}^{p} \frac{\varphi(\lambda_j - k)^2 + 4k^2\lambda_j\gamma_j^2}{(\lambda_j + k)^2\lambda_j}$$

$$= \varphi \sum_{j=1}^{p} \left[\frac{(\lambda_j - dk)^2 + k^2(d+1)^2\gamma_j^2\varphi^{-1}\lambda_j - (\lambda_j - k)^2 - 4k^2\gamma_j^2\varphi^{-1}\lambda_j}{(\lambda_j + k)^2\lambda_j}\right] \quad (34)$$

In the scenario of $(\lambda_j - dk)^2 + k^2(d+1)^2\gamma_j^2\varphi^{-1}\lambda_j - (\lambda_j - k)^2 - 4k^2\gamma_j^2\varphi^{-1}\lambda_j < 0$ in the Equation (34), it indicates that $(\lambda_j - dk)^2 + k^2(d+1)^2\gamma_j^2\varphi^{-1}\lambda_j - (\lambda_j - k)^2 < 4k^2\gamma_j^2\varphi^{-1}\lambda_j$ for all j, then $MSE(\hat{\beta}_{GTSS}) < MSE(\hat{\beta}_{GKL})$. That means that the GTSSE is superior than the GKLE if $(\lambda_j - dk)^2 + k^2(d+1)^2\gamma_j^2\varphi^{-1}\lambda_j - (\lambda_j - k)^2 < 4k^2\gamma_j^2\varphi^{-1}\lambda_j$ for all j.

4.5 MSE comparison of $\hat{\beta}_{GTSS}$ and $\hat{\beta}_{GTP}$

Theorem 5: if $(\lambda_j - dk)^2 + k^2\gamma_j^2\varphi^{-1}\lambda_j(d^2-1)^2 < (\lambda_j + dk)^2$ *for all j*, then $MSE(\hat{\beta}_{GTSS}) < MSE(\hat{\beta}_{GTP})$.

Proof: The MSE difference between the GTSSE and the GTPE is written as

$$\Delta_5 = MSE(\hat{\beta}_{GTSS}) - MSE(\hat{\gamma}_{GTP})$$

$$\Delta_5 = \sum_{j=1}^{p} \frac{\varphi(\lambda_j - dk)^2 + k^2(d+1)^2{\gamma_j}^2\lambda_j}{(\lambda_j + k)^2\lambda_j} - \sum_{j=1}^{p} \frac{\varphi(\lambda_j + kd)^2 + \lambda_j k^2(d-1)^2\gamma_j^2}{(\lambda_j + k)^2\lambda_j}$$

$$= \varphi \sum_{j=1}^{p} \left[\frac{(\lambda_j - dk)^2 + k^2(d+1)^2\gamma_j^2\varphi^{-1}\lambda_j - (\lambda_j + kd)^2 - k^2(d-1)^2\gamma_j^2\varphi^{-1}\lambda_j}{(\lambda_j + k)^2\lambda_j}\right]$$

$$= \varphi \sum_{j=1}^{p} \left[\frac{\left(\lambda_j - dk\right)^2 + k^2\gamma_j^2\varphi^{-1}\lambda_j(d^2-1)^2 - \left(\lambda_j + dk\right)^2}{\left(\lambda_j + k\right)^2 \lambda_j} \right] \quad (35)$$

In the case of $\left(\lambda_j - dk\right)^2 + k^2\gamma_j^2\varphi^{-1}\lambda_j(d^2-1)^2 - \left(\lambda_j + dk\right)^2 < 0$ in the Equation (35), it implies that $\left(\lambda_j - dk\right)^2 + k^2\gamma_j^2\varphi^{-1}\lambda_j(d^2-1)^2 < \left(\lambda_j + dk\right)^2 \, for\ all\ j$, then $MSE\left(\hat{\beta}_{GTSS}\right) < MSE\left(\hat{\beta}_{GTP}\right)$. That means that the GTSSE is better than the GTPE if $\left(\lambda_j - dk\right)^2 + k^2\gamma_j^2\varphi^{-1}\lambda_j(d^2-1)^2 < \left(\lambda_j + dk\right)^2 \, for\ all\ j$.

5 Selection of Shrinkage Parameters k and d

The performance of the proposed estimator depends on the suitable value of the shrinkage parameters k and d. Therefore, we derive optimal k and d, and suggest an iterative method for the determination of k and d. The optimal value of d is obtained by taking the derivatives of MSE ($\hat{\beta}_{GTSS}$) with respect to d for fixed k as follows:

$$\frac{\partial MSE(\hat{\beta}_{GTSS})}{\partial d} = \sum_{j=1}^{p} \frac{2\gamma_j^2 k^2 \lambda_j (1+d) - 2k\varphi(\lambda_j - kd)}{\lambda_j\left(\lambda_j + k\right)^2}.$$

For $\frac{\partial MSE(\hat{\beta}_{GTSS})}{\partial d} = 0$, simplifying the numerator of the above expression and solving for d as:

$$\hat{d} = \frac{\sum_{j=1}^{p}\left(\varphi - \gamma_j^2 k\right)}{\sum_{j=1}^{p}\left(\frac{k\varphi}{\lambda_j} + \gamma_j^2 k\right)}, \quad (36)$$

Where γ_j^2 and φ are unknown parameter and we replace these unknown parameters with their unbiased estimators, see [18].

The optimal value of k is determined by differentiating MSE ($\hat{\beta}_{GTSS}$) for k and equating it to be zero: $\frac{\partial MSE(\hat{\beta}_{GTSS})}{\partial k} = \sum_{j=1}^{p} \frac{2(1+d)\left\{d\varphi + \left(\left(\gamma_j^2 d + \gamma_j^2\right)\lambda_j\right)\right\}k - \lambda_j\varphi}{\left(\lambda_j + k\right)^3}$, then

$$k_j = \frac{\lambda_j\varphi}{d\varphi + \gamma_j^2(1+d)\lambda_j}. \quad (37)$$

When d = 0, the expression of k_j in Equation (37) reduces to $k_j = \varphi/\gamma_j^2$, which is suggested by [7] to estimate the ridge parameter k. It can be noted that the value of k_j is always positive. The expression in Equation (37) depends on the unknown parameters γ_j^2, and φ we replaced them with their corresponding unbiased estimators, see [18].

6 MONTE CARLO SIMULATION

This Monte Carlo simulation is carried out to compare the finite sample properties of the proposed estimators with the traditional estimators in different empirically relevant situations. The means square error (MSE) of the estimator is determined based on 5000 replications and the entire process executed 5000 times to compute the simulated MSE as follows:

$$\text{MSE}\left(\hat{\beta}_a\right) = \frac{\sum_{r=1}^{5000}\left(\hat{\beta}_{a(r)} - \beta\right)^T\left(\hat{\beta}_{a(r)} - \beta\right)}{5000}$$

Where $\hat{\beta}_a$ denotes to the used estimator (i.e., a = ML, GR, GL, GKL, GTP, or GTSS) and $\hat{\beta}_{a(r)}$ is the estimated vector $\left(\hat{\beta}_a\right)$ in the r^{th} replication.

6.1 Design of the Simulation

According to [22], [23], [24], [25], [26] and [27], the independent variables are obtained from a multivariate normal distribution MVN (1, Σ_x), where the matrix of variance and covariance Σ_x is defined as follows {diag (Σ_x) $= 1$ and off-diag (Σ_x) $= \rho$}. The response variable of the gamma model is obtained by using the gamma distribution. The parameters β_j are set so that $\beta^T\beta = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_p$, which are typical restrictions in many simulation studies, as mentioned by [11], [28], [29], [30], and [31]. Furthermore, these constraints aim to simplify the analysis and assure variable balance, enabling result interpretation and minimizing the complexity of the simulated models.

The performance of the GTSS estimator and other existing estimators (GRE, GLE, GKLE, and GTPE) is examined by simulation study. The parameters k and d for the GRE, GLE, GKLE, and GTPE are chosen as follows:

- $\hat{k}_{GR}$ of the GRE is, [21], $\hat{k}_{GR} = \frac{p\hat{\varphi}}{\sum_{j=1}^{p}\gamma_j^2}$.
- $\hat{d}_{GL}$ of the GLE is, [12], $\hat{d}_{GL} = \frac{1}{2}min\left(\frac{\gamma_j^2}{\frac{\hat{\varphi}}{\lambda_j}+\gamma_j^2}\right)_{j=1}^{p}$.
- $\hat{k}_{GKL}$ of the GKLE is, [13], $\hat{k}_{GKL} = \frac{p\,\hat{\varphi}}{\sum_{j=1}^{p}\left(\frac{\hat{\varphi}}{\lambda_j}+2\gamma_j^2\right)}$.

- $\hat{d}_{GTP}$ and $\hat{k}_{GTP}$ of the GTPE are, [20], $\hat{d}_{GTP} = \hat{d}_{GL} = \frac{1}{2} min\left(\frac{\gamma_j^2}{\frac{\hat{\varphi}}{\lambda_j}+\gamma_j^2}\right)_{j=1}^{p}$; $\hat{k}_{GTP} = \frac{p\,\hat{\varphi}}{\sum_{j=1}^{p}\left[\gamma_j^2-\hat{d}_{GTP}\left(\frac{\hat{\varphi}}{\lambda_j}+\gamma_j^2\right)\right]}$.

We can suggest the following estimators of k_{GTSS} and d_{GTSS} of the proposed GTSSE:

$$\hat{d}_{\text{GTSS}} = \hat{d}_{GL} = \hat{d}_{GTP} = \frac{1}{2} min\left(\frac{\gamma_j^2}{\frac{\hat{\varphi}}{\lambda_j}+\gamma_j^2}\right)_{j=1}^{p}$$

and three estimators for k_{GTSS} will be proposed as follows:

$$\hat{k}_{\text{GTSS.1}} = \left(\hat{k}_{GR}\right)^{1/p} = \left(\frac{p\hat{\varphi}}{\sum_{j=1}^{p}\gamma_j^2}\right)^{1/p}$$

$$\hat{k}_{\text{GTSS.2}} = \frac{1}{p}\sum_{j=1}^{p}\left(\frac{\lambda_j\hat{\varphi}}{\hat{d}_{\text{GTSS}}\,\hat{\varphi}+\gamma_j^2(1+\hat{d}_{\text{GTSS}})\lambda_j}\right)$$

$$\hat{k}_{\text{GTSS.3}} = \max_{j}\left(\frac{\lambda_j\hat{\varphi}}{\hat{d}_{\text{GTSS}}\,\hat{\varphi}+\gamma_j^2(1+\hat{d}_{\text{GTSS}})\lambda_j}\right)$$

6.2 Results of the simulation

Tables 1 through 6 present simulation study results for different estimators, organized in such a way as to understand how different factors affect estimation accuracy. The tables are divided into three groups, with each group focusing on a specific number of explanatory variables (p), and including various sample sizes, such as $(\text{n} = 50, 75, 100, 150, 200, \text{and } 400)$. This setup allows for an observation of how sample size impacts the performance of different estimators. Additionally, correlation coefficient values (ρ) the correlation coefficient values are simulated in the following values $(\rho = 0.80, 0.85, 0.90, 0.95, and\ 0.99)$, allowing for a detailed analysis of how the strength of correlation between variables impacts estimator performance .The simulation also includes various levels of dispersion parameter ($\varphi = 0.75$, $1\ and\ 1.25$) to assess the stability of estimators under different data dispersion conditions. This organization provides a broad perspective on how each estimator responds to these changing factors. Specifically, Tables 1 to 3 present the AMES values when the number of variables is set to p = 3, corresponding to different sample sizes, correlation coefficients, and the dispersion parameter. While Tables 4 to 6 detail AMES values for p = 9, again for the various sample sizes, correlation coefficients, and dispersion parameter levels. This structured approach provides a comprehensive overview of how each estimator responds to these varying conditions.

Upon examining Tables 1 to 3, we observe a clear impact of sample size and correlation coefficient values on the average mean squared error (MSE) of the estimators. It appears that as sample size increases, estimation accuracy improves, as reflected by a general decrease in MSE values. For instance, with a small sample size like $(n = 50)$, MSE values are higher compared to when the sample size increases to $(n = 100)\ or (n = 400)$. Additionally, the tables reveal that higher correlation coefficients (ρ) lead to increased MSE values, suggesting that strong correlations between variables negatively impact the performance of traditional estimators, potentially inflating estimation error. These results underscore the challenges that multicollinearity poses to estimator accuracy, especially as (ρ) approaches 0.99, where the correlation effect intensifies, making it harder to obtain precise estimates. Despite the increase in the dispersion parameter values from 0.75 to 1 and then to 1.25 in Tables 1 to 3, while keeping the number of variables ($p = 3$), sample sizes, and correlation coefficients constant, there remains a consistent decline in MSE values with increasing sample size and decreasing correlation coefficients. This trend is also applicable to other tables, where similar patterns in MSE changes are observed across different sample sizes and correlation levels, suggesting that these results align with the outcomes in other groups. This trend also applies to the other tables, where similar patterns in MSE changes are observed across various sample sizes and correlation levels.

Tables 1 to 3 provides a detailed comparison of several estimators used in regression models, including GMLE, GRE, GLE, GKLE, GTPE, and the new GTSSE class, which includes versions GTSSE.1, GTSSE.2, and GTSSE.3. The table illustrates how MSE for each estimator is affected by changes in sample size and correlation.

The GMLE demonstrates relatively good performance with smaller sample sizes, but as (ρ) increases, its MSE values rise significantly. For example, when $(\rho = 0.99)\ and\ (n = 50)$, the MSE for GMLE reaches approximately 4.45693, indicating its high sensitivity to strong multicollinearity. GRE shows slightly better performance than GMLE in cases of moderate correlation, reaching an MSE of 2.63983 at ($\rho = 0.99$) and ($n = 50$), which, although lower than GMLE, remains high under strong correlation. GLE and GKLE perform similarly and are reasonably effective with medium sample sizes. For instance, when ($n = 100$) and ($\rho = 0.85$), their MSE values are around 0.16471 and 0.16770, respectively. However, their MSE values increase significantly when ($\rho = 0.99$), reaching approximately 1.15430 and 1.57001 for GLE and GKLE, respectively, indicating weaker performance under high multicollinearity. GTPE strikes a decent balance with moderate correlation levels, but its MSE values surge with increasing (ρ). At (ρ = 0.99) and (n = 50), its MSE is 1.57001, highlighting the limitations of this estimator in highly correlated conditions. The new GTSS estimator includes three versions—GTSSE.1, GTSSE.2, and GTSSE.3—designed to improve estimation accuracy and mitigate the effects of multicollinearity. GTSSE.1 and GTSSE.2 show substantial performance improvement over traditional estimators, with MSE values decreasing as sample size increases. For instance, at (ρ = 0.99) and (n = 50), GTSSE.2 achieves an MSE of 0.58006, significantly lower than traditional estimators, reflecting its relative stability even with high correlations. GTSSE.3, however, demonstrates clear superiority in nearly all scenarios, recording the lowest MSE values in cases of high correlation (ρ = 0.99) and larger sample sizes (n = 400). For example, when (ρ = 0.99) and (n = 50), the MSE for GTSSE.3 is just 0.36232, the lowest among all estimators in the table. At (ρ = 0.80) and (n = 50), the MSE for GTSSE.3 further decreases to 0.14787, establishing it as the most efficient and stable option.

Based on the values presented in Table 1, the new GTSS estimator, particularly the GTSSE.3 version, offers the best performance among all estimators. GTSSE.3 consistently shows the lowest MSE values across various conditions, demonstrating its high effectiveness in delivering accurate and stable estimates, especially in cases of high correlation and larger sample sizes. These results suggest that GTSSE.3 is the optimal choice among the estimators, combining estimation accuracy and robust performance in the face of strong multicollinearity.

Table 1: MSE results of the deferent estimators when $p = 3$ and $\varphi = 0.75$

n	P	GMLE	GRE	GLE	GKLE	GTPE	GTSSE.1	GTSSE.2	GTSSE.3
50	**0.80**	0.24246	0.20927	0.20107	0.20814	0.20473	0.18667	0.15517	**0.14787**
	0.85	0.42733	0.33730	0.29543	0.33260	0.31271	0.26040	0.21427	**0.20691**
	0.90	0.43149	0.33877	0.30044	0.33360	0.31640	0.26443	**0.22321**	0.22436
	0.95	1.12203	0.77323	0.53688	0.74920	0.62733	0.37712	0.35957	**0.27192**
	0.99	4.45693	2.63983	1.15430	2.48771	1.57001	**0.22892**	0.58006	0.36232
75	**0.80**	0.16877	0.15139	0.14941	0.15088	0.15039	0.13927	0.12134	**0.11392**
	0.85	0.23469	0.20081	0.19378	0.19967	0.19693	0.17658	0.13367	**0.12423**
	0.90	0.29926	0.24387	0.22758	0.24148	0.23461	0.20249	0.15286	**0.14505**
	0.95	0.47427	0.35597	0.30524	0.34937	0.32601	0.25579	0.20421	**0.18429**
	0.99	2.89975	1.72996	0.88290	1.63232	1.15033	**0.24865**	0.51002	0.25522
100	**0.80**	0.12263	0.11162	0.11195	0.11138	0.11205	0.10419	0.08911	**0.08044**
	0.85	0.19585	0.16853	0.16471	0.16770	0.16663	0.15001	0.11960	**0.10969**
	0.90	0.26347	0.21594	0.20358	0.21412	0.20895	0.18124	0.12028	**0.10215**
	0.95	0.41981	0.32225	0.28270	0.31743	0.29899	0.24264	0.19120	**0.17125**
	0.99	2.03478	1.22108	0.71441	1.15437	0.88905	0.26397	0.44211	**0.21432**
150	**0.80**	0.06951	0.06586	0.06658	0.06581	0.06639	0.06328	0.06012	**0.05572**
	0.85	0.11408	0.10379	0.10465	0.10359	0.10455	0.09671	0.08213	**0.07052**
	0.90	0.15976	0.13800	0.13694	0.13738	0.13775	0.12317	0.09151	**0.07708**
	0.95	0.24181	0.19480	0.18512	0.19293	0.18946	0.16264	0.10770	**0.09662**
	0.99	1.54384	0.94496	0.60732	0.89753	0.72925	0.27447	0.39990	**0.19991**
200	**0.80**	0.05868	0.05584	0.05655	0.05580	0.05636	0.05378	0.05222	**0.04902**
	0.85	0.07547	0.07055	0.07161	0.07048	0.07133	0.06698	0.06262	**0.05478**
	0.90	0.10213	0.09255	0.09361	0.09237	0.09342	0.08612	0.07332	**0.06393**
	0.95	0.18979	0.15786	0.15469	0.15680	0.15645	0.13636	0.08972	**0.07187**
	0.99	1.02459	0.64271	0.45841	0.61377	0.52830	0.24610	0.29334	**0.16731**
400	**0.80**	0.03061	0.02976	0.03005	0.02976	0.02997	0.02915	0.02936	**0.02847**
	0.85	0.03443	0.03334	0.03370	0.03333	0.03360	0.03255	0.03276	**0.03152**
	0.90	0.05123	0.04839	0.04926	0.04836	0.04902	0.04636	0.04590	**0.04149**
	0.95	0.08439	0.07629	0.07760	0.07614	0.07730	0.07078	0.05838	**0.04741**
	0.99	0.43141	0.30629	0.26841	0.29929	0.28417	0.20545	0.13289	**0.09929**

Note: Bold highlights indicate the lowest MSE in the row.

Table 2: MSE results of the deferent estimators when $p = 3$ and $\varphi = 1$

n	ρ	GMLE	GRE	GLE	GKLE	GTPE	GTSSE.1	GTSSE.2	GTSSE.3
50	**0.80**	0.12836	0.12517	0.11853	0.12515	0.12206	0.11569	0.11224	**0.10197**
	0.85	0.22735	0.21740	0.18986	0.21727	0.20436	0.18555	0.16959	**0.14802**
	0.90	0.23055	0.21936	0.19242	0.21918	0.20649	0.18741	0.16563	**0.14550**
	0.95	0.59499	0.54718	0.39130	0.54606	0.46838	0.36775	0.37623	**0.29948**
	0.99	2.35608	1.98990	0.90180	1.97217	1.32947	**0.51632**	1.31181	0.88187
75	**0.80**	0.09250	0.09122	0.08783	0.09122	0.08967	0.08633	0.08765	**0.08420**
	0.85	0.12767	0.12489	0.11727	0.12487	0.12140	0.11459	0.11184	**0.10135**
	0.90	0.16389	0.15900	0.14407	0.15896	0.15205	0.14080	0.13003	**0.11537**
	0.95	0.25869	0.24709	0.20650	0.24694	0.22820	0.20296	0.18193	**0.15419**
	0.99	1.56165	1.38604	0.70797	1.37958	1.01349	**0.53504**	1.01687	0.77876
100	**0.80**	0.06637	0.06571	0.06379	0.06571	0.06485	0.06294	0.06389	**0.06261**
	0.85	0.10680	0.10499	0.09892	0.10498	0.10232	0.09738	0.09548	**0.08946**
	0.90	0.14246	0.13915	0.12649	0.13913	0.13352	0.12464	0.11557	**0.10296**
	0.95	0.23129	0.22342	0.19029	0.22335	0.20853	0.18882	0.17054	**0.14597**
	0.99	1.12194	1.02127	0.57234	1.01838	0.78622	**0.48802**	0.77884	0.63213
150	**0.80**	0.03816	0.03798	0.03738	0.03798	0.03771	**0.03710**	0.03784	0.03753
	0.85	0.06302	0.06249	0.06061	0.06249	0.06167	0.05989	0.06153	**0.05985**
	0.90	0.08859	0.08741	0.08273	0.08741	0.08543	0.08167	0.08256	**0.07859**
	0.95	0.13272	0.12990	0.11739	0.12988	0.12438	0.11600	0.10541	**0.09604**
	0.99	0.84840	0.79226	0.47972	0.79109	0.63900	**0.44781**	0.59616	0.49947
200	**0.80**	0.03215	0.03203	0.03155	0.03203	0.03182	**0.03138**	0.03197	0.03183
	0.85	0.04217	0.04196	0.04107	0.04196	0.04159	**0.04080**	0.04181	0.04146
	0.90	0.05638	0.05597	0.05420	0.05597	0.05521	**0.05365**	0.05522	0.05391
	0.95	0.10623	0.10463	0.09693	0.10462	0.10140	0.09582	0.09386	**0.08752**
	0.99	0.56808	0.53821	0.35206	0.53774	0.45293	0.35045	0.40797	**0.35002**
400	**0.80**	0.01712	0.01709	0.01694	0.01709	0.01703	**0.01692**	0.01709	0.01707
	0.85	0.01913	0.01910	0.01891	0.01910	0.01902	**0.01887**	0.01909	0.01906
	0.90	0.02861	0.02853	0.02802	0.02853	0.02834	**0.02795**	0.02851	0.02843
	0.95	0.04714	0.04688	0.04530	0.04688	0.04624	**0.04504**	0.04636	0.04548
	0.99	0.24290	0.23714	0.19064	0.23711	0.21796	0.19442	0.17658	**0.14696**

Note: Bold highlights indicate the lowest MSE in the row.

Table 3: MSE results of the deferent estimators when $p = 3$ and $\varphi = 1.25$

n	ρ	GMLE	GRE	GLE	GKLE	GTPE	GTSSE.1	GTSSE.2	GTSSE.3
50	**0.80**	0.07959	0.07549	0.07566	0.07544	0.07574	0.07181	0.06878	**0.06350**
	0.85	0.14172	0.12787	0.12509	0.12758	0.12654	0.11550	0.09310	**0.07778**
	0.90	0.14403	0.12937	0.12702	0.12904	0.12833	0.11716	0.09530	**0.08586**
	0.95	0.36877	0.29916	0.26392	0.29654	0.27885	0.22783	0.15677	**0.12269**
	0.99	1.46652	0.94781	0.59353	0.90967	0.71906	0.28660	0.43256	**0.22357**
75	**0.80**	0.05849	0.05644	0.05659	0.05642	0.05658	0.05435	0.05410	**0.05120**
	0.85	0.08036	0.07566	0.07609	0.07561	0.07604	0.07127	0.06855	**0.05927**
	0.90	0.10360	0.09526	0.09499	0.09512	0.09529	0.08764	0.07630	**0.06358**
	0.95	0.16344	0.14269	0.13851	0.14218	0.14051	0.12499	0.09080	**0.06942**
	0.99	0.97767	0.67013	0.47189	0.64999	0.54728	0.28942	0.34804	**0.22350**
100	**0.80**	0.04162	0.04040	0.04058	0.04039	0.04054	0.03921	0.03896	**0.03726**
	0.85	0.06726	0.06368	0.06399	0.06364	0.06394	0.06028	0.05738	**0.05065**
	0.90	0.08914	0.08259	0.08233	0.08250	0.08256	0.07632	0.06793	**0.05776**
	0.95	0.14641	0.13082	0.12723	0.13051	0.12892	0.11619	0.08858	**0.07050**
	0.99	0.71096	0.51476	0.38966	0.50371	0.43877	0.27246	0.26262	**0.18150**
150	**0.80**	0.02412	0.02372	0.02380	0.02372	0.02378	0.02334	0.02359	**0.02325**
	0.85	0.03996	0.03872	0.03896	0.03872	0.03889	0.03754	0.03799	**0.03643**
	0.90	0.05636	0.05360	0.05388	0.05358	0.05383	0.05092	0.04945	**0.04435**
	0.95	0.08387	0.07727	0.07733	0.07718	0.07744	0.07099	0.06411	**0.05238**
	0.99	0.53644	0.40304	0.32420	0.39653	0.35582	0.24799	0.19095	**0.13680**
200	**0.80**	0.02031	0.02001	0.02006	0.02001	0.02005	**0.01971**	0.01994	0.01973
	0.85	0.02698	0.02641	0.02652	0.02640	0.02648	0.02584	0.02620	**0.02566**
	0.90	0.03571	0.03461	0.03479	0.03460	0.03474	0.03353	0.03355	**0.03228**
	0.95	0.06794	0.06364	0.06390	0.06359	0.06388	0.05942	0.05622	**0.04818**
	0.99	0.36093	0.28194	0.24361	0.27861	0.25938	0.19330	0.13064	**0.08921**
400	**0.80**	0.01092	0.01083	0.01085	0.01083	0.01085	**0.01075**	0.01082	0.01078
	0.85	0.01216	0.01205	0.01207	0.01205	0.01206	**0.01193**	0.01203	0.01197
	0.90	0.01828	0.01796	0.01804	0.01796	0.01802	0.01766	0.01790	0.01769
	0.95	0.03008	0.02914	0.02931	0.02913	0.02926	0.02820	0.02861	**0.02743**
	0.99	0.15588	0.13491	0.12948	0.13440	0.13188	0.11468	0.07036	**0.04713**

Note: Bold highlights indicate the lowest MSE in the row.

Table 4: MSE results of the deferent estimators when $p = 9$ and $\varphi = 0.75$

n	ρ	GMLE	GRE	GLE	GKLE	GTPE	GTSSE.1	GTSSE.2	GTSSE.3
50	**0.80**	2.01084	1.52088	1.35053	1.50190	1.37558	1.25271	0.52534	**0.39193**
	0.85	3.14732	2.22485	1.65386	2.17823	1.73179	1.50284	0.66280	**0.43544**
	0.90	4.55284	3.15645	1.99098	3.09250	2.14256	1.79585	0.70804	**0.42778**
	0.95	7.18276	4.92326	2.38228	4.82357	2.68270	2.09521	0.89535	**0.44145**
	0.99	37.68948	25.12950	3.06617	24.55953	4.21233	1.82430	2.46109	**0.93043**
75	**0.80**	1.26879	0.94281	0.94109	0.93116	0.94261	0.88524	0.34719	**0.34063**
	0.85	1.55394	1.14002	1.09280	1.12536	1.10030	1.02721	0.35431	**0.30026**
	0.90	2.30522	1.61279	1.37929	1.58404	1.41048	1.27589	0.44826	**0.32412**
	0.95	4.55263	3.07860	1.94824	3.02140	2.08790	1.78176	0.60176	**0.28388**
	0.99	21.05423	13.27822	2.51435	12.96144	3.25237	1.83063	1.23817	**0.36880**
100	**0.80**	0.86713	0.66316	0.69203	0.65646	0.68944	0.65719	0.28111	**0.27209**
	0.85	8.13789	4.86023	1.86883	4.66574	2.08387	1.38142	0.80044	**0.36777**
	0.90	1.63370	1.16888	1.12195	1.15358	1.12903	1.05509	0.33831	**0.25800**
	0.95	3.30935	2.20864	1.64877	2.16832	1.71890	1.53150	0.45104	**0.24771**
	0.99	17.69790	11.03241	2.43182	10.76822	3.06956	1.88033	1.02198	**0.29429**
150	**0.80**	0.53353	0.43048	0.46738	0.42843	0.46414	0.44837	**0.21821**	0.21877
	0.85	0.70160	0.55015	0.59233	0.54696	0.58850	0.56631	0.24356	**0.23737**
	0.90	1.03631	0.76821	0.80736	0.76141	0.80384	0.76158	0.27801	**0.21712**
	0.95	2.23317	1.50050	1.31552	1.47610	1.33949	1.22974	0.36310	**0.21312**
	0.99	10.51072	6.31500	2.11360	6.16344	2.49519	1.84084	0.80382	**0.20750**
200	**0.80**	0.37515	0.31608	0.34353	0.31524	0.34128	0.33113	0.18835	**0.18559**
	0.85	1.26244	0.90270	0.62694	1.24204	0.65443	0.61571	0.26066	**0.20498**
	0.90	0.74779	0.57364	0.62039	0.56987	0.61609	0.59157	0.22588	**0.18528**
	0.95	12.12451	7.68382	2.94962	7.38074	2.94944	1.72487	2.03055	**0.58251**
	0.99	7.13147	4.32446	2.00230	4.23214	2.24572	1.81334	0.63017	**0.17136**
400	**0.80**	0.16798	0.15240	0.16242	0.15230	0.16174	0.15743	0.11896	**0.08456**
	0.85	0.20883	0.18593	0.19974	0.18574	0.19876	0.19322	0.13084	**0.09644**
	0.90	0.33307	0.28227	0.30763	0.28169	0.30560	0.29679	0.16179	**0.13648**
	0.95	0.68609	0.53039	0.57893	0.52758	0.57452	0.55270	0.20400	**0.14519**
	0.99	3.28651	2.09205	1.61137	2.05956	1.67001	1.49203	0.44914	**0.16500**

Note: Bold highlights indicate the lowest MSE in the row.

Table 5: MSE results of the deferent estimators when $p = 9$ and $\varphi = 1$

n	ρ	GMLE	GRE	GLE	GKLE	GTPE	GTSSE.1	GTSSE.2	GTSSE.3
50	**0.80**	1.00699	0.91340	0.83079	0.91204	0.84626	0.78296	0.43912	**0.30512**
	0.85	1.59030	1.40076	1.14318	1.39619	1.18842	1.03779	0.63749	**0.38913**
	0.90	2.29757	1.99682	1.47721	1.99019	1.56690	1.33743	0.80981	**0.44509**
	0.95	3.58637	3.07477	1.94605	3.06296	2.12599	1.68574	1.12567	**0.48746**
	0.99	19.09173	15.80051	3.44182	15.70799	4.52551	2.10646	3.66355	**1.05605**
75	**0.80**	0.65426	0.61298	0.57057	0.61253	0.57881	0.54156	0.38065	**0.30031**
	0.85	0.80039	0.74960	0.68134	0.74909	0.69471	0.64319	0.44452	**0.27060**
	0.90	1.19824	1.09689	0.93617	1.09539	0.96569	0.86184	0.57372	**0.34443**
	0.95	2.36958	2.12276	1.52153	2.11897	1.62983	1.37310	0.93605	**0.44819**
	0.99	10.95984	9.53986	3.05461	9.51156	3.84429	2.00523	3.49537	**1.18543**
100	**0.80**	0.44200	0.42483	0.40637	0.42474	0.41009	0.38889	0.30986	**0.23131**
	0.85	5.62867	4.64733	1.53648	3.85448	1.74994	1.14453	1.03892	**0.38548**
	0.90	0.86330	0.81406	0.72451	0.81364	0.74213	0.68459	0.46822	**0.29213**
	0.95	1.74367	1.60974	1.23495	1.60820	1.30685	1.13489	0.80455	**0.39290**
	0.99	9.31641	8.29121	2.91937	8.27406	3.64877	2.03846	3.35234	**1.25280**
150	**0.80**	0.28489	0.27793	0.27023	0.27791	0.27177	0.25990	0.22883	**0.17643**
	0.85	0.37830	0.36788	0.35206	0.36784	0.35528	0.33878	0.27841	**0.20517**
	0.90	0.55656	0.53591	0.49757	0.53582	0.50536	0.47452	0.35673	**0.23740**
	0.95	1.19794	1.13003	0.93391	1.12946	0.97391	0.86136	0.66392	**0.38385**
	0.99	5.64920	5.17279	2.37902	5.16695	2.84958	1.85327	2.57754	**1.19980**
200	**0.80**	0.20401	0.20075	0.19724	0.20075	0.19796	0.19115	0.18204	**0.15030**
	0.85	0.63744	0.49738	0.32908	0.48865	0.34576	0.32312	0.23316	**0.16763**
	0.90	0.40555	0.39501	0.37445	0.39497	0.37882	0.35952	0.29237	**0.20658**
	0.95	10.59653	7.09096	2.46294	6.85618	2.82581	1.37237	2.28298	**0.66576**
	0.99	3.88898	3.62898	2.01564	3.62669	2.32429	1.69971	1.86260	**0.94985**
400	**0.80**	0.09240	0.09191	0.09118	0.09191	0.09134	0.08960	0.09051	**0.08601**
	0.85	0.11506	0.11427	0.11312	0.11427	0.11336	0.11074	0.11089	**0.10373**
	0.90	0.18377	0.18193	0.17830	0.18192	0.17908	0.17325	0.16692	**0.13989**
	0.95	0.37822	0.37167	0.35271	0.37166	0.35686	0.33816	0.29511	**0.21019**
	0.99	1.81982	1.75233	1.29892	1.75204	1.40176	1.17183	1.09810	**0.62343**

Note: Bold highlights indicate the lowest MSE in the row.

Table 6: MSE results of the deferent estimators when $p = 9$ and $\varphi = 1.25$

n	ρ	GMLE	GRE	GLE	GKLE	GTPE	GTSSE.1	GTSSE.2	GTSSE.3
50	**0.80**	0.60770	0.48630	0.53346	0.48356	0.52971	0.51290	**0.21790**	0.22427
	0.85	0.96464	0.70692	0.76306	0.69696	0.75859	0.71704	0.28363	**0.26201**
	0.90	1.39444	0.98571	1.01802	0.97178	1.01651	0.95259	0.30122	**0.22631**
	0.95	2.15874	1.45649	1.36948	1.43177	1.38202	1.27160	0.37862	**0.24549**
	0.99	11.60600	7.04112	2.50754	6.85355	2.91401	2.20052	0.81492	**0.20142**
75	**0.80**	0.40107	0.33572	0.36503	0.33449	0.36271	0.35107	0.18588	**0.16959**
	0.85	0.49006	0.40483	0.43746	0.40326	0.43477	0.42106	0.20404	**0.18549**
	0.90	0.73793	0.57182	0.61695	0.56730	0.61310	0.58646	0.23515	**0.20070**
	0.95	1.46088	1.04867	1.04944	1.03727	1.05100	0.99085	0.29159	**0.18131**
	0.99	6.75575	4.25490	2.18189	4.16711	2.40528	1.93004	0.73559	**0.16926**
100	**0.80**	0.27466	0.24275	0.25900	0.24240	0.25778	0.25063	0.15997	**0.12917**
	0.85	3.47702	2.53471	1.06828	2.48280	1.20552	0.78994	0.48019	**0.19425**
	0.90	0.53605	0.43954	0.47288	0.43782	0.47002	0.45372	0.20031	**0.15728**
	0.95	1.07931	0.81447	0.83671	0.80824	0.83529	0.79415	0.25844	**0.18653**
	0.99	5.77598	3.71718	2.07692	3.64943	2.26161	1.87458	0.67748	**0.18841**
150	**0.80**	0.17766	0.16223	0.17127	0.16211	0.17064	0.16591	0.12469	**0.09595**
	0.85	0.23749	0.21304	0.22597	0.21283	0.22501	0.21870	0.14638	**0.10548**
	0.90	0.34831	0.30059	0.32130	0.30004	0.31962	0.30999	0.15843	**0.11920**
	0.95	0.74903	0.58870	0.62024	0.58542	0.61754	0.58847	0.21648	**0.13121**
	0.99	3.53718	2.36257	1.67285	2.32928	1.75731	1.54428	0.52007	**0.17308**
200	**0.80**	0.12850	0.12043	0.12565	0.12039	0.12531	0.12216	0.10635	**0.08530**
	0.85	0.27728	0.22564	0.17339	0.22311	0.17735	0.18149	0.13474	**0.08914**
	0.90	0.25503	0.22704	0.24083	0.22679	0.23976	0.23286	0.13982	**0.09454**
	0.95	10.33637	6.84318	2.61290	6.61787	2.81877	1.14548	2.19662	**0.58967**
	0.99	2.45251	1.69768	1.41743	1.67982	1.45286	1.31805	0.40600	**0.15900**
400	**0.80**	0.05850	0.05670	0.05800	0.05670	0.05792	0.05702	0.05558	**0.05123**
	0.85	0.07295	0.07019	0.07215	0.07018	0.07203	0.07069	0.06787	**0.05943**
	0.90	0.11655	0.10979	0.11414	0.10977	0.11386	0.11110	0.09900	**0.07610**
	0.95	0.23961	0.21514	0.22716	0.21496	0.22622	0.21987	0.13681	**0.08860**
	0.99	1.15699	0.87912	0.87590	0.87434	0.87704	0.82788	0.26359	**0.12283**

Note: Bold highlights indicate the lowest MSE in the row.

7 REAL DATA APPLICATION: BODY FAT DATA

Within this section, we examined a dataset to demonstrate the discoveries outlined in the paper. The dataset referred to as "body fat data" was initially examined in [32] and subsequently utilized in gamma regression modeling by [29, 33, 34]. This dataset consists of 71 observations of healthy female subjects. There are 9 explanatory variables: age in years (x_1), waist circumference (x_2), hip circumference (x_3), breadth of the elbow (x_4), breadth of the knee (x_5), and the sum of logarithms of three anthropometric measurements divided into four groups (x_6, x_7, x_8, and x_9). The response variable of this data is the body fat (DEXfat).

Figure 1 examines the distribution of DEXfat graphically. Looking at Figure 1, we can observe that the data clearly follows a gamma distribution pattern. This correspondence is evident through the orange curve representing the Gamma distribution, which fits well with the histogram bars of the actual data. This alignment confirms that the Gamma distribution model is appropriate for describing this dataset. The characteristics shown in the data match the typical features of a Gamma distribution, where we can see positive skewness (tail extending to the right), all values are positive, and most of the data is concentrated on the left side of the distribution. Furthermore, we can observe that the area under the orange curve covers most of the histogram values consistently, which reinforces our conclusion that the Gamma distribution is an appropriate statistical model for this data. Moreover, Asar and Korkmaz [34] employed the Anderson-Darling (AD) test to examine the distribution of the DEXfat variable. They discovered that the AD test statistic yielded a value of 0.361, with a corresponding p-value of 0.886. These results indicate that the gamma regression model is appropriate for analyzing this dataset.

The Pearson correlation coefficient, variance inflation factor (VIF), and condition number (CN) are the three measures used to test the multicollinearity between the nine explanatory variables. According to the correlation matrix, as shown in Figure 2, there are several high correlations that indicate serious multicollinearity issues. The most notable are among the anthropometric variables (x_6, x_7, x_8, and x_9), which show extremely strong correlations ranging from 0.88 to 0.98. Additionally, x_2 exhibits strong correlations with several other variables, particularly with x_3 at 0.87 and x_5 at 0.73. These high correlation coefficients, especially those exceeding the 0.8 threshold, strongly suggest the presence of multicollinearity among these explanatory variables.

We fit the model including the intercept term and compute the CN based on eigenvalues of $\left(X^T \hat{Z} X\right)$ matrix: $CN = \sqrt{\lambda_{max}/\lambda_{min}} = 3394.30$, where λ_{max} is the maximum eigenvalue and λ_{min} is the minimum eigenvalue. Since the value of CN (3394.30) is greater than 30, it strongly suggests the presence of multicollinearity. Moreover, VIF values provide crucial insights into the multicollinearity issue. Severe multicollinearity: x_9 (108.03), x_7 (46.87), and x_6 (39.04) have extremely high VIF values, indicating severe multicollinearity. Moderate multicollinearity: x_8 (8.91), x_3 (6.45), and x_2 (5.63) have VIF values above 5, suggesting moderate to high multicollinearity. Low multicollinearity: x_1 (1.21), x_4 (1.45), and x_5 (3.18) have VIF values below 5, indicating low multicollinearity. Given that VIF values above 5 indicate multicollinearity issues, many variables in this dataset exhibit this problem.

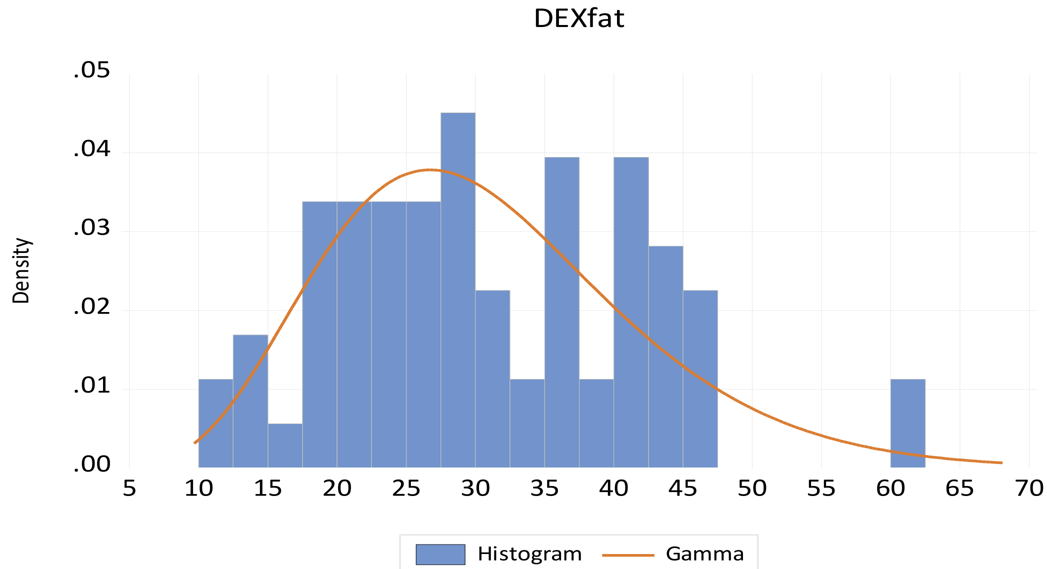

Figure 1: Histogram and theoretical gamma distribution of the response variable (DEXfat)

Table 7: Estimated coefficients, MSE, k, and d values for each estimator in body fat data

Variable	GMLE	GRE	GLE	GKLE	GTPE	GTSSE.1	GTSSE.2	GTSSE.3
Intercept	-0.21947	-0.02451	-0.215	0.040276	-0.04273	-0.21099	-0.09905	-0.06304
X1	0.001606	0.001499	0.001602	0.001408	0.001495	0.001599	0.001512	0.001491
X2	0.004196	0.00491	0.004212	0.00508	0.004815	0.004226	0.004608	0.004726
X3	0.010772	0.010316	0.010755	0.009869	0.010293	0.010739	0.010342	0.010254
X4	0.014118	0.000344	0.013763	-0.00582	0.001297	0.013444	0.00481	0.002264
X5	0.042557	0.035268	0.04244	0.035107	0.036451	0.042336	0.039181	0.03788
X6	-0.12272	0.004097	-0.12037	0.022173	-0.01207	-0.11826	-0.05661	-0.03294
X7	0.140434	0.105326	0.140437	0.118409	0.112437	0.140423	0.131807	0.123956
X8	0.129183	0.130408	0.129433	0.14245	0.133255	0.129658	0.135571	0.136465
X9	0.163677	0.091274	0.16173	0.061827	0.097356	0.159992	0.115933	0.103235
MSE	0.12907	0.067382	0.124057	0.110783	0.060144	0.119718	0.055747	0.055679
k	------	284.1197	------	61.63497	203.8405	1.759341	47.65945	88.56533
d	------	------	0.045775	------	0.045775	0.045775	0.045775	0.045775

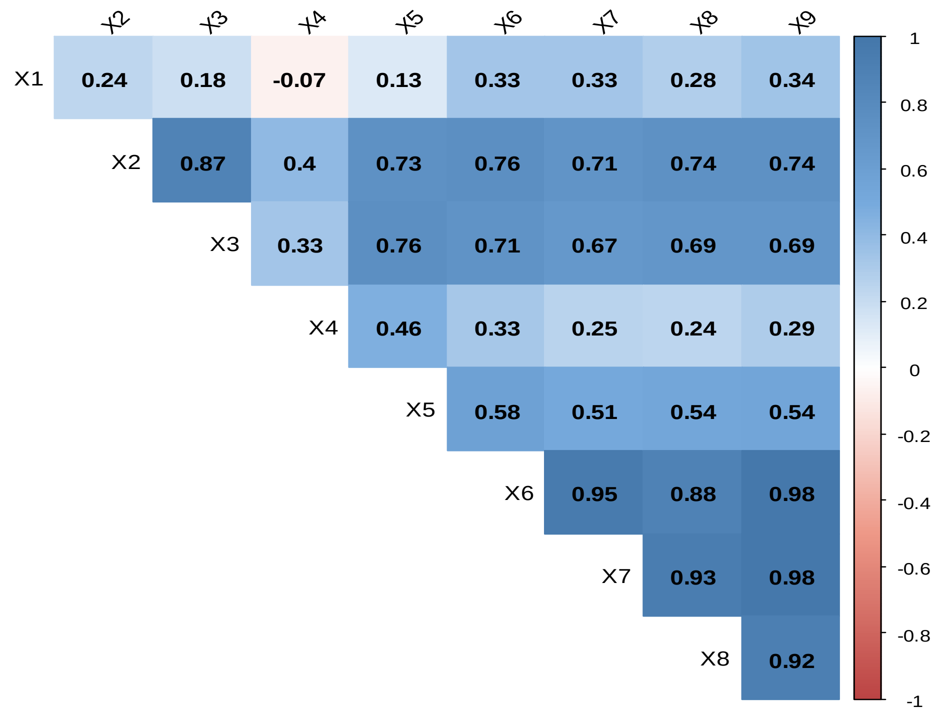

Figure 2: Correlation coefficient matrix heatmap for the explanatory variables

Table 7 presents estimated coefficients for body fat data using various estimation methods, each affected differently by multicollinearity. By examining the MSE values, we see that GTSSE.2 and GTSSE.3 methods yield the lowest MSEs (0.055747 and 0.055679), which points to their superior accuracy in comparison to other methods. In contrast, the GMLE method has the highest MSE (0.12907), suggesting it may be less effective in handling multicollinearity. This aligns with findings from the simulation study, where GTSSE estimators showed a clear advantage in terms of minimizing error under multicollinear conditions. Thus, the GTSSE estimators are likely the best choice for this dataset, providing precise parameter estimates that support the broader conclusions of the simulation study.

8 CONCLUSIONS

In this study, a new class of two-step shrinkage estimator (GTSS) was introduced within the gamma regression model framework to address the issue of multicollinearity. The Monte Carlo simulation results demonstrated that the proposed estimator outperformed the maximum likelihood estimator (ML) as well as other estimators like GLE, GRE, and GTPE in reducing mean squared error (MSE) under various conditions, particularly when multicollinearity was severe.

Additionally, real-world data applications highlighted that the proposed estimator significantly improved the accuracy of parameter estimates by substantially reducing standard errors, enhancing its practical relevance. Thus, it can be concluded that the GTSS estimator is a highly effective and promising option for improving estimation accuracy in regression models facing strong multicollinearity issues. In further research, we may create a robust version of the proposed GTSS estimator to address gamma models that include outliers and multicollinearity. As future work, we can develop a new robust version of the GTSS estimator to address outliers and multicollinearity problems together in the gamma regression model, as suggested by [35, 36, 37, 38, 39, 40, 41] in other regression models.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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