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STABILITY OF DELAYED GENERAL HIV-1 WITH CELL-TO-CELL AND INFLAMMATORY CYTOKINES INFECTION MODEL WITH IMMUNE SYSTEM RESPONSE

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Abstract. In this paper, we propose a dynamical model for the immune system's reaction to HIV-1 that includes three temporal delays distributed in the presence of inflammatory cytokines. The model contains six components: uninfected $CD4^{+}T$ cells, infected $CD4^{+}T$ cells, inflammatory cytokines, HIV-1 virus, CTLs, and antibodies. The model contains the general functions for the incidence rates of the healthy $CD4^{+}T$ cells with viruses, infected cells, and inflammatory cytokines. Moreover, the production/proliferation and removal/death rates of the virus and cells are represented by general functions, too. The model was developed to better understand HIV-1 infection by describing two immune responses (antibody and cytotoxic T lymphocyte (CTLs)), three forms of distributed time delays, and inflammatory cytokines in the context of an infection through cell-to-cell (CTC) transmission. First, the system's basic properties were investigated; then the equilibria points of the system were determined. The reproduction numbers for the model have been calculated and investigated, too. The model has five reproduction numbers, basic one ($\Re_0$), CTLs immune response ($\Re_1$), antibody immune response ($\Re_2$), CTLs immune competitive ($\Re_3$), and antibody immune competitive ($\Re_4$). Using the Lyapunov approach and LaSalle's invariance principle, we have demonstrated the global asymptotic stability of all equilibria under a set of restrictions on the general functions and the threshold parameters. The numerical simulations has been performed to demonstrate the theoretical results. By adding a time delay for the model can reduce the quantities of free HIV-1 particles

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and infected cells while also dramatically raising the concentration of healthy CD4$^+$T cells. However, the addition of CTC transmission raises the concentrations of infected cells and free HIV-1 particles while decreasing the concentration of healthy CD4$^+$T cells.

Keywords: HIV-1; inflammatory cytokines; cell-to-cell spread; distributed delays; immune response; Lyapunov method; global stability.

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1. Introduction

Many viruses are affect on the human life and causes many of diseases. the virus HIV-1 is one of the viruses that affect the human body, as it affects the CD4$^+$T cells lymphocytes, which are the backbone of the immune system. In the paper published by the World Health Organisation(WHO) in 2022, there are approximately one $1,180,000$ people infected with human immunodeficiency virus-1 (HIV-1), including $640,000$ men and $540,000$ women [1]. AIDS's most severe stage is known as acquired immune deficiency syndrome. Patients with untreated HIV-1 typically wait many years before developing AIDS. Over this time frame, the CD4$^+$T cell count gradually drops to less than 200 cells/mm^3[2]. Scientists have been interested in recent years to study the HIV-1 analysis using mathematical modelling. The first mathematical model to describe the interaction between infected and uninfected cells and the free viruses of HIV-1 were proposed by Nowak and Bangham [3]. A number of biological characteristics, such as time delay [4]-[9], CTLs immune response [3], pharmacological treatments [10]- [11], antibody immune response [12]-[13], reaction-diffusion [14], and stochastic effects [15], have been taken into account by additional mathematical models to developed and expand the fundamental HIV-1 mathematical model. A model with humoral immunity was examined by Murase et al. [16] on the assumption that the incidence rate of infection is bilinear. In actuality, the dynamics of the virus over the entire period of infection cannot be accurately described by a bilinear incidence rate. Recently, Wang et al. [17] have proposed the following model:

$$\frac{dP}{dt} = \alpha - \omega P - \Psi_1\left(P(t), V(t)\right)V,$$
$$\frac{dT}{dt} = \Psi_1\left(P, V\right)V - \xi_T T(t),$$
$$\frac{dV}{dt} = \beta T - \xi_V V(t) - \psi V(t) Z^V(t), \tag{1}$$

$$\frac{dZ^V}{dt} = \varpi V(t) Z^V(t) - \xi_{Z^V} Z^V(t),$$

where $P = P(t)$, $T = T(t)$, $V = V(t)$ and $Z^V = Z^V(t)$ are the concentrations of uninfected $CD4^+T$ cells, infected $CD4^+T$ cells, free HIV-1 particles and antibodies at time t, respectively. α denotes the production rate of healthy $CD4^+T$ cells. Parameter β refers to production rate of the HIV-1 particles. ϖ represents the stimulation rate of antibodies while ψ donates the neutralization rate of HIV-1. All the compartments have natural death rates that are, respectively, ω, ξ_T, ξ_V and ξ_{Z^V}. Nonlinear generic expressions $\Psi_1(P,V)$, indicate the impact of viral infection virus-to-cell (VTC). The model has been extended by Wang and Liu [18] and Wang et al. [19] to include, respectively, two forms of distributed delays and a saturated incidence rate. This model was expanded by Elaiw and AlShamrani to account for two different kinds of distributed time delays [20]:

$$\begin{aligned}
\frac{dP}{dt} &= ¥(P(t)) - \Psi_1(P(t), T(t), V(t))V, \\
\frac{dT}{dt} &= \int_0^{\varkappa_1} B_1(\varphi) e^{-\kappa_1 \varphi} \Psi_1\left(P_\varphi, T_\varphi, V_\varphi\right) V_\varphi d\varphi - \xi_T T(t), \\
\frac{dV}{dt} &= \beta \int_0^{\varkappa_2} B_2(\varphi) e^{-\kappa_2 \varphi} T_\varphi d\varphi - \xi_V V(t) - \psi V(t) Z^V(t), \\
\frac{dZ^V}{dt} &= \varpi V(t) Z^V(t) - \xi_{Z^V} Z^V(t).
\end{aligned} \tag{2}$$

For the purpose of simplification, if we assume $¥(P(t)) = \alpha - \omega P$ and we use the notation $P_\varphi = P(t-\varphi)$, $T_\varphi = T(t-\varphi)$ and $V_\varphi = V(t-\varphi)$. Here κ_i and $\varkappa_i$ for $i = 1,2$ are positive constants. The delay parameter φ is random taken from a probability distribution function $B_i(\varphi)$ over the time interval $[0, \varkappa_i]$, $i = 1,2$, where $\varkappa_i$ is the limit superior of this delay period. The factor $B_1(\varphi)e^{-\kappa_1\varphi}$ represents the probability that the $CD4^+T$ cell contacted by viruses at time $t-\varphi$ will survive φ time units and become active HIV-1-infected at time t. $B_2(\varphi)e^{-\kappa_2\varphi}$ demonstrates the probability that new immature HIV-1 particles at time $t-\varphi$ will survive for φ time units and become mature at time t. In addition to virus infection from cell to cell, researchers have discovered in recent years that there is another way for viruses to infiltrate the human body: cell-to-cell transmission (CTC) (see e.g. [21]-[22]). As a result, even during antiviral therapy, CTC transmission contributes significantly to the HIV-1 infection process

[23]. Two discrete-time delays and the CTC transmission were added by Guo et al. [24] to the same Yan and Wang [25] model. Adaptive immune response A model of HIV-1 dynamics that includes two distributed-time delays and both VTC and CTC transmission was examined in [26].

Although highly active anti-retroviral medication is very effective at limiting the spread of HIV-1 and preventing its reproduction, it is unable to eradicate the virus entirely from the body. The major barrier to HIV-1 clearance is the silent (latent) $CD4^{+}T$ and Inflammatory cytokines cells. Increased $CD4^{+}T$ cell death and increased $CD4^{+}T$ cell recruitment to inflammatory areas are caused by inflammatory cytokines (cytokine-enhanced HIV-1 infection) [27]. Cytokine-enhanced viral infection models were recently created and examined while taking age structure [28] and reaction–diffusion [29]-[32] into account. A mathematical model is created by Aeshah A. et al. [33] to investigate the dynamics of HIV-1 infection with inflammatory cytokines. The model includes two immune responses antibody and cytotoxic T lymphocyte (CTLs), two infection modalities (viral and cellular), and two kinds of distributed-time delays.

In this paper, the research conducted on the three distributed time delays and both VTC and CTC transmissions mentioned above are expanded upon. The incidence rates of cytokine-enhanced viral infection, active HIV-1-infected cells, and healthy $CD4^{+}T$ cells with free HIV-1 particles are provided by general functions. Additionally, general functions represent the production/proliferation and removal/death rates of every compartment. This is the structure of the remainder of the paper: We develop an HIV-1 dynamics model in Section 2. The nonnegativity and roundedness of the suggested model's solutions are demonstrated in Section 3. Next, we examine whether all of the model's potential equilibria which rely on five threshold parameters exist in Section 4. In Section 5, Lyapunov functions and LaSalle's invariance principle are used to demonstrate the global stability of all equilibria. We demonstrate the dependability of our theoretical findings with a few numerical simulations in Section 6.

2. MODEL DEVELOPMENT

We formulate a six-dimensional system of delayed differential equations (DDEs) as follows:

$$(3)\qquad \frac{dP}{dt} = ¥\left(P\left(t\right)\right) - \Psi_1\left(P\left(t\right), V\left(t\right)\right) - \Psi_2\left(P\left(t\right), T\left(t\right)\right) - \Psi_3\left(P\left(t\right), E\left(t\right)\right),$$

$$\frac{dT}{dt} = \int_0^{\varkappa_1} B_1(\varphi)e^{-\kappa_1\varphi}\left[\Psi_1\left(P_\varphi, V_\varphi\right) + \Psi_2\left(P_\varphi, T_\varphi\right) + \Psi_3\left(P_\varphi, E_\varphi\right)\right]d\varphi$$

$$- \left(\mu_1 + \xi_T\right)\aleph_1\left(T\left(t\right)\right) - \lambda\aleph_1\left(T\left(t\right)\right)\aleph_4\left(Z^T\left(t\right)\right), \tag{4}$$

$$\frac{dE}{dt} = \mu_2\int_0^{\varkappa_2} B_2(\varphi)e^{-\kappa_2\varphi}\aleph_1\left(T_\varphi\right)d\varphi - \xi_E\aleph_2\left(E\left(t\right)\right), \tag{5}$$

$$\frac{dV}{dt} = \beta\int_0^{\varkappa_3} B_3(\varphi)e^{-\kappa_3\varphi}\aleph_1\left(T_\varphi\right)d\varphi - \xi_V\aleph_3\left(V\left(t\right)\right) - \psi\aleph_3\left(V\left(t\right)\right)\aleph_5\left(Z^V\left(t\right)\right), \tag{6}$$

$$\frac{dZ^T}{dt} = \sigma\aleph_1\left(T\left(t\right)\right)\aleph_4\left(Z^T\left(t\right)\right) - \xi_{Z^T}\aleph_4\left(Z^T\left(t\right)\right), \tag{7}$$

$$\frac{dZ^V}{dt} = \varpi\aleph_3\left(V\left(t\right)\right)\aleph_5\left(Z^V\left(t\right)\right) - \xi_{Z^V}\aleph_5\left(Z^V\left(t\right)\right), \tag{8}$$

where $P = P(t)$, $T = T(t)$, $E = E(t)$, $V = V(t)$, $Z^T = Z^T(t)$, and $Z^V = Z^V(t)$ are the concentrations of uninfected CD4$^+$T cells, infected CD4$^+$T cells, inflammatory cytokines, free HIV-1 particles, (CTLs) and antibodies at time t, respectively. Function $¥\left(P\left(t\right)\right)$ is the intrinsic growth rate of healthy CD4$^+$T cells that takes into consideration both natural mortality and production. Nonlinear generic expressions $\Psi_2\left(P, T\right)$, and $\Psi_3\left(P, E\right)$ indicate the impact of CTC and cytokine-enhanced viral infection, respectively. $\mu_1\aleph_1\left(T\right)$ is the rate at which pyroptosis kills infected CD4$^+$T cells. The natural death rates in each compartment are, respectively, $\xi_T\aleph_1\left(T\right)$, $\xi_E\aleph_2\left(E\right)$, $\xi_V\aleph_3\left(V\right)$, $\xi_{Z^T}\aleph_4\left(Z^T\right)$, and $\xi_{Z^V}\aleph_5\left(Z^V\right)$. The rate at which CTLs immune cells eliminate the infected cells is described by the phrase $\lambda\aleph_1\left(T\right)\aleph_4\left(Z^T\right)$. Where as $\psi\aleph_3\left(V\right)\aleph_5\left(Z^V\right)$ refers to the rate at which the antibodies neutralise HIV-1 particles. For the purpose of simplification, we use the notation $E_\varphi = E(t-\varphi)$. Here κ_i and $\varkappa_i$ for $i = 1,2,3$ are positive constants. The delay parameter φ is random taken from a probability distribution function $B_i(\varphi)$ over the time interval $[0, \varkappa_i]$, $i = 1,2,3$ where $\varkappa_i$ is the limit superior of this delay period. The factor $B_2(\varphi)e^{-\kappa_2\varphi}$ indicates that HIV-1-infected cells survive for $t-\varphi$ time units to release new inflammatory cytokines at time t. The term $B_3(\varphi)e^{-\kappa_3\varphi}$ demonstrates the probability that new immature HIV-1 particles at time $t-\varphi$ will survive for φ time units and become mature at time t. The function $B_i(\varphi)$, $i = 1,2,3$ satisfies $B_i(\varphi) > 0$,

$$\int_0^{\varkappa_i} B_i(\varphi)d\varphi = 1 \text{ and } \int_0^{\varkappa_i} B_i(\varphi)e^{-z\varphi}d\varphi < \infty, \quad i = 1,2,3,$$

where $z > 0$. Let us denote

$$\tilde{\mathscr{B}}_i(\varphi) = B_i(\varphi)e^{-\kappa_i\varphi}, \quad \mathscr{B}_i = \int_0^{\varkappa_i} \tilde{\mathscr{B}}_i(\varphi)\,d\varphi, \quad i = 1,2,3.$$

Then, $0 < \mathscr{B}_i \leq 1$, $i = 1,2,3$. The initial conditions of system (3)-(8) are given by:

$$P(\theta) = \ell_1(\theta),\ T(\theta) = \ell_2(\theta),\ E(\theta) = \ell_3(\theta),\ V(\theta) = \ell_4(\theta),$$

$$Z^T(\theta) = \ell_5(\theta),\ Z^V(\theta) = \ell_6(\theta),\ \ell_j(\theta) \geq 0,\ \theta \in [-\varkappa^*, 0],\ j = 1,2,...,6, \tag{9}$$

where $\varkappa^* = \max\{\varkappa_1, \varkappa_2, \varkappa_3\}$, $\ell_j(\theta) \in \mathscr{C}([-\varkappa^*, 0], \mathbb{R}_{\geq 0})$, $j = 1,2,...,6$ and $\mathscr{C}$ is the Banach space of continuous functions mapping the interval $[-\varkappa^*, 0]$ into $\mathbb{R}_{\geq 0}$ with norm $\left\|\ell_j\right\| \sup_{-\varkappa^* \leq \varepsilon \leq 0}$
$= \left|\ell_j(\varepsilon)\right|$ for $\ell_j \in \mathscr{C}$. Therefore, based on the fundamental theory of functional differential equations, we can say that system (3)-(8) with initial conditions (9) has a singular solution. The functions ¥, Ψ_i ,$i = 1,2,3$ and $\aleph_{\Bbbk}$, $\Bbbk = 1,2,3,4,5$, are continuously differentiable and satisfy the following conditions in(see [34]-[36]):

Condition (H1).

(i) there exists P_0 such that $¥(P_0) = 0$ and $¥(P) > 0$ for $P \in [0, P_0)$,

(ii) $\grave{¥}(P) < 0$ for all $P > 0$,

(iii) there are $\alpha > 0$ and $\omega_0 > 0$ such that $¥(P) \leq \alpha - \omega_0 P$ for $P \geq 0$.

Condition (H2).

(i) $\Psi_i(P,U) > 0$ and $\Psi_i(0,U) = \Psi_i(P,0) = 0$ for all $P > 0, U > 0$, $i = 1,2,3$,

(ii) $\frac{\partial \Psi_i(P,U)}{\partial P} > 0$, $\frac{\partial \Psi_i(P,U)}{\partial U} > 0$, and $\frac{\partial \Psi_i(P,U)}{\partial U}|_{U=0} > 0$ for all $P > 0$, $U > 0$, $i = 1,2,3$,

(iii) $\frac{d}{dP}(\frac{\partial \Psi_i(P,U)}{\partial U}|_{U=0}) > 0$ for all $P > 0$, $i = 1,2,3$.

Condition (H3).

(i) $\aleph_{\Bbbk}(\rho) > 0$ for all $\rho > 0$, $\aleph_{\Bbbk}(0) = 0$, $\Bbbk = 1,2,3,4,5$,

(ii) $\aleph'_{\Bbbk}(\rho) > 0$ for all $\rho > 0$, $\Bbbk = 1,2,3,4,5$. Further, $\aleph'_{\Bbbk}(0) > 0$, $\Bbbk = 1,2,3,4,5$,

(iii) there are $\omega_{\Bbbk} > 0$ such that $\aleph_{\Bbbk}(\rho) \geq \omega_{\Bbbk}\rho$ for all $\rho \geq 0$, $\Bbbk = 1,2,3,4,5$.

Condition (H4).

$\frac{\partial}{\partial V}\left(\frac{\aleph_1(P,V)}{\aleph_3(V)}\right) \leq 0$, $\frac{\partial}{\partial E}\left(\frac{\aleph_3(P,E)}{\aleph_1(E)}\right) \leq 0$, and $\frac{\partial}{\partial T}\left(\frac{\aleph_2(P,T)}{\aleph_2(T)}\right) \leq 0$ for all $P, T, E, V > 0$.

3. Non-Negativity and Boundedness of Solutions

Proposition 1 Suppose that Conditions **H1** and **H3** are satisfied. Then the compact set Θ is positively invariant for system (3)-(8).

Proof: First, we show the nonnegativity of solutions. The proof is similar to the one given in [37]. System (3)-(8) can be written as: $\dot{X}(t) = \mathbb{Z}(X(t))$, where $X(t) = (P(t), T(t), E(t), V(t), Z^T(t), Z^V(t))^T$,

$\mathbb{Z} = (\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6)^T$, and

$$\begin{pmatrix} \mathbb{Z}_1 X(t) \\ \mathbb{Z}_2 X(t) \\ \mathbb{Z}_3 X(t) \\ \mathbb{Z}_4 X(t) \\ \mathbb{Z}_5 X(t) \\ \mathbb{Z}_6 X(t) \end{pmatrix} = \begin{pmatrix} ¥(P(t)) - \Psi_1(P(t), V(t)) - \Psi_2(P(t), T(t)) - \Psi_3(P(t), E(t)) \\ \int_0^{\varkappa_1} B_1(\varphi) e^{-\kappa_1 \varphi} \left[\Psi_1\left(P_\varphi, V_\varphi\right) + \Psi_2\left(P_\varphi, T_\varphi\right) + \Psi_3\left(P_\varphi, E_\varphi\right)\right] d\varphi \\ -(\mu_1 + \xi_T) \aleph_1(T(t)) - \lambda \aleph_1(T(t)) \aleph_4\left(Z^T(t)\right) \\ \mu_2 \int_0^{\varkappa_2} B_2(\varphi) e^{-\kappa_2 \varphi} \aleph_1\left(T_\varphi(t)\right) d\varphi - \xi_E \aleph_2(E(t)) \\ \beta \int_0^{\varkappa_3} B_3(\varphi) e^{-\kappa_3 \varphi} \aleph_1\left(T_\varphi\right) d\varphi - \xi_V \aleph_3(V(t)) - \psi \aleph_3(V(t)) \aleph_5\left(Z^V(t)\right) \\ \sigma \aleph_1(T(t)) \aleph_4\left(Z^T(t)\right) - \xi_{Z^T} \aleph_4\left(Z^T(t)\right) \\ \varpi \aleph_3(V(t)) \aleph_5\left(Z^V(t)\right) - \xi_{Z^V} \aleph_5\left(Z^V(t)\right) \end{pmatrix}$$

It is easy to see that the function $\mathbb{Z}$ satisfies the following condition

$$\mathbb{Z}(X(t)) \mid_{X_i(t)=0,\, X(t) \in \mathscr{C}^6_{\geq 0}} \geq 0, i = 1, 2, ..., 6.$$

Due to Lemma 2 in [38], any solution of system (3)-(8) with initial conditions (9) satisfies $X(t) \in \mathbb{R}_{\geq 0}$ for all $t \geq 0$. It means that model (3)-(8) is considered biologically acceptable as long as no population declines. In addition, the orthant $\mathbb{R}_{\geq 0}$ is positively invariant for system (3)-(8). The boundedness of the model's solutions is then established. The nonnegativity of the model's solution together with condition **H1** implies that $\limsup_{t \to \infty} P(t) \leq \frac{\alpha}{\omega_0}$. Further we let

$$\Omega_1 = \int_0^{\varkappa_1} \tilde{\mathscr{B}}_1(\varphi) P_\varphi d\varphi + T + \frac{\lambda}{\sigma} Z^T.$$

Determine the equation above, we get

$$\begin{aligned} \dot{\Omega}_1 &= \int_0^{\varkappa_1} \tilde{\mathscr{B}}_1(\varphi) ¥\left(P_\varphi\right) d\varphi - (\mu_1 + \xi_T) \aleph_1(T(t)) - \frac{\lambda}{\sigma} \xi_{Z^T} \aleph_4\left(Z^T(t)\right) \\ &\leq \alpha \mathscr{B}_1 - \omega_0 \int_0^{\varkappa_1} \tilde{\mathscr{B}}_1(\varphi) P_\varphi d\varphi - \omega_1 (\mu_1 + \xi_T) T(t) - \omega_2 \frac{\lambda \xi_{Z^T}}{\sigma} Z^T(t) \end{aligned}$$

$$\leq \alpha - \phi_1 \left(\int_0^{\varkappa_1} \tilde{\mathscr{B}}_1(\varphi) P_\varphi d\varphi + T + \frac{\lambda}{\sigma} Z^T \right) = \alpha - \phi_1 \Omega_1,$$

where $\varepsilon_1 = \min\{\omega_0, \omega_1(\mu_1 + \xi_T), \omega_2 \xi_{Z^T}\}$. Hence, $\lim_{t\to\infty}\sup \Omega_1(t) \leq \mathscr{A}_1$, where $\mathscr{A}_1 = \frac{\alpha}{\varepsilon_1}$. Therefore, we can obtain that $\lim_{t\to\infty}\sup T(t) \leq \mathscr{A}_1$ and $\lim_{t\to\infty}\sup Z^T(t) \leq \frac{\sigma}{\lambda}\mathscr{A}_1 = \mathscr{A}_2$. Then Eq. (5) implies that

$$\dot{E} = \mu_2 \int_0^{\varkappa_2} B_2(\varphi) e^{-\kappa_2 \varphi} \aleph_1 \left(T_\varphi(t)\right) d\varphi - \xi_E \aleph_2 (E(t)) \leq \mu_2 \mathscr{B}_1 \aleph_1 (\Omega_1) - \omega_3 \xi_E E(t),$$

which confirms that $\lim_{t\to\infty}\sup C(t) \leq \mathscr{A}_3$, where $\mathscr{A}_3 = \frac{\mu_2 \mathscr{A}_1}{\xi_E}$. Further, we assume that $\Omega_2 = V + \frac{\psi}{\varpi} Z^V$. Then, from Eqs. (6) and (8), we have

$$\begin{aligned}\dot{\Omega}_2 &= \beta \int_0^{\varkappa_3} B_3(\varphi) e^{-\kappa_3 \varphi} \aleph_1 \left(T_\varphi\right) d\varphi - \xi_V \aleph_3 (V(t)) - \frac{\psi}{\varpi} \xi_{Z^V} \aleph_5 \left(Z^V(t)\right) \\ &\leq \beta \mathscr{B}_3 \aleph_1 (\Omega_1) - \omega_4 \xi_V V(t) - \omega_5 \xi_{Z^V} \frac{\psi}{\varpi} Z^V(t) \\ &\leq \beta \aleph_1 (\Omega_1) - \phi_2 \left(V + \frac{\psi}{\varpi} Z^V\right) = \beta \aleph_1 (\Omega_1) - \phi_2 \Omega_2,\end{aligned}$$

where $\varepsilon_2 = \min\{\omega_4 \xi_V, \omega_5 \xi_{Z^V}\}$. Hence, $\lim_{t\to\infty}\sup \Omega_2(t) \leq \mathscr{A}_4$, where $\mathscr{A}_4 = \frac{\beta \mathscr{A}_1}{\varepsilon_2}$. Therefore, we can obtain that $\lim_{t\to\infty}\sup V(t) \leq \mathscr{A}_4$, and $\lim_{t\to\infty}\sup Z^V(t) \leq \frac{\varpi}{\psi}\mathscr{A}_3 = \mathscr{A}_5$.

Based on Proposition 1, one can establish the compact set

$$\Xi = \left\{(P,T,E,V,Z^T,Z^V) \in \mathscr{C}^6_{\geq 0} : \|P\| \leq \mathscr{A}_1, \|T\| \leq \mathscr{A}_1, \|E\| \leq \mathscr{A}_2, \|V\| \leq \mathscr{A}_4, \left\|Z^T\right\| \leq \mathscr{A}_3, \left\|Z^V\right\| \leq \mathscr{A}_5\right\},$$

which can be easily proved that it is positively invariant with respect to system (3)-(8).

4. Existence of Equilibrium Points

In this section, we study the equilibria of the model (3)-(8) and derive the conditions for their existence. Let (P,T,E,V,Z^T,Z^V) be any equilibrium satisfying the following system of algebraic equations:

(10) $$0 = \yen(P) - \Psi_1(P,V) - \Psi_2(P,T) - \Psi_3(P,E),$$

(11) $$0 = \mathscr{B}_1 [\Psi_1(P,V) + \Psi_2(P,T) + \Psi_3(P,E)] - (\mu_1 + \xi_T) \aleph_1(T) - \lambda \aleph_1(T) \aleph_4\left(Z^T\right),$$

(12) $$0 = \mu_2 \mathscr{B}_2 \aleph_1(T) - \xi_E \aleph_2(E),$$

(13) $$0 = \beta \mathscr{B}_3 \aleph_1(T) - \xi_V \aleph_3(V) - \psi \aleph_3(V) \aleph_5\left(Z^V\right),$$

(14) $$0 = \sigma \aleph_1(T) \aleph_4\left(Z^T\right) - \xi_{Z^T} \aleph_4\left(Z^T\right),$$

$$0=\varpi\aleph_3(V)\,\aleph_5\left(Z^V\right)-\xi_{Z^V}\,\aleph_5\left(Z^V\right). \tag{15}$$

Obviously, eqs. (14) and (15) admits two solutions $\aleph_5\left(Z^V\right)=0$ and $\aleph_4\left(Z^T\right)=0$. Let us consider the case when $\aleph_5\left(Z^V\right)=\aleph_4\left(Z^T\right)=0$, we get two equilibria for system (3)-(8) as follows:

(I) Infection-free equilibrium point, $Đ_0=(P_0,0,0,0,0,0)$, where $¥(P_0)=0$. This case describes the situation of healthy state where the infection is absent.

(II) Point of balance for chronic infections with dormant immunological responses, $Đ_1=(P_1,T_1,E_1,V_1,0,0)$, from eqs (10)-(13) we get

$$\begin{aligned}¥(P)=\Psi_1(P,V)+\Psi_2(P,T)+\Psi_3(P,E)&=\frac{\mu_1+\xi_T}{\mathscr{B}_1}\aleph_1(T)\\&=\frac{\xi_E(\mu_1+\xi_T)}{\mu_2\mathscr{B}_1\mathscr{B}_2}\aleph_2(E)=\frac{\xi_V(\mu_1+\xi_T)}{\beta\mathscr{B}_1\mathscr{B}_3}\aleph_3(V).\end{aligned} \tag{16}$$

Condition **H3** implies that $\aleph_{\Bbbk}^{-1}$ exists, is continuous and strictly increasing. From Eq. (16), we obtain

$$\begin{aligned}T&=\aleph_1^{-1}\left(\frac{\mathscr{B}_1¥(P)}{\mu_1+\xi_T}\right)=f_1(P),\quad E=\aleph_2^{-1}\left(\frac{\mu_2\mathscr{B}_1\mathscr{B}_2¥(P)}{\xi_E(\mu_1+\xi_T)}\right)=f_2(P),\\V&=\aleph_3^{-1}\left(\frac{\beta\mathscr{B}_1\mathscr{B}_3¥(P)}{\xi_V(\mu_1+\xi_T)}\right)=f_3(P).\end{aligned} \tag{17}$$

From condition **H1**, $f_i(P)=0$ for all $P\in[0,P_0)$ and $f_i(P_0)=0$, $i=1,2,3$. Let us define

$$\digamma_1(P)=\Psi_1(P,f_3(P))+\Psi_2(P,f_1(P))+\Psi_3(P,f_2(P))-\frac{\xi_V(\mu_1+\xi_T)}{\beta\mathscr{B}_1\mathscr{B}_3}\aleph_3(f_3(P))=0.$$

Then from conditions **H1-H3**, we have

$$\digamma_1(0)=-\frac{\xi_V(\mu_1+\xi_T)}{\beta\mathscr{B}_1\mathscr{B}_3}\aleph_3(f_3(0))<0,\qquad \digamma_1(P_0)=0.$$

Moreover,

$$\digamma_1'(P)=\frac{\partial\Psi_1}{\partial P}+f_3'(P)\frac{\partial\Psi_1}{\partial V}+\frac{\partial\Psi_2}{\partial P}+f_1'(P)\frac{\partial\Psi_2}{\partial T}+\frac{\partial\Psi_3}{\partial P}+f_2'(P)\frac{\partial\Psi_3}{\partial E}-\frac{\xi_V(\mu_1+\xi_T)}{\beta\mathscr{B}_1\mathscr{B}_3}\aleph_3'(f_3(P))\,f_3'(P),$$

then

$$\begin{aligned}\digamma_1'(P_0)=&\frac{\partial\Psi_1(P_0,0)}{\partial P}+f_3'(P_0)\frac{\partial\Psi_1(P_0,0)}{\partial V}+\frac{\partial\Psi_2(P_0,0)}{\partial P}+f_1'(P_0)\frac{\partial\Psi_2(P_0,0)}{\partial T}\\&+\frac{\partial\Psi_3(P_0,0)}{\partial P}+f_2'(P_0)\frac{\partial\Psi_3(P_0,0)}{\partial E}-\frac{\xi_V(\mu_1+\xi_T)}{\beta\mathscr{B}_1\mathscr{B}_3}\aleph_3'(0)\,f_3'(P_0).\end{aligned}$$

Condition (**H2**) implies that $\frac{\partial \Psi_i(P_0,0)}{\partial U} = 0, i = 1,2,3$. Also, from Condition (**H3**), we have $\aleph_3'(0) > 0$, then

$$F_1'(P_0) = \frac{\xi_V(\mu_1+\xi_T)}{\beta \mathscr{B}_1\mathscr{B}_3} \aleph_3'(0) f_3'(P_0) \left(\frac{\beta \mathscr{B}_1\mathscr{B}_3}{\xi_V(\mu_1+\xi_T)\aleph_3'(0)} \frac{\partial \Psi_1(P_0,0)}{\partial V} + \frac{\beta \mathscr{B}_1\mathscr{B}_3 f_1'(P_0)}{\xi_V(\mu_1+\xi_T)\aleph_3'(0) f_3'(P_0)} \right.$$
$$\left. \times \frac{\partial \Psi_2(P_0,0)}{\partial T} + \frac{\beta \mathscr{B}_1\mathscr{B}_3 f_2'(P_0)}{\xi_V(\mu_1+\xi_T)\aleph_3'(0) f_3'(P_0)} \frac{\partial \Psi_3(P_0,0)}{\partial E} - 1 \right).$$

From Eqs. (16) and (17), then

$$F_1'(P_0) = ¥'(P_0) \left(\frac{\beta \mathscr{B}_1\mathscr{B}_3}{\xi_V(\mu_1+\xi_T)\aleph_3'(0)} \frac{\partial \Psi_1(P_0,0)}{\partial V} + \frac{\mathscr{B}_1}{(\mu_1+\xi_T)\aleph_1'(0)} \frac{\partial \Psi_2(P_0,0)}{\partial E} \right.$$
$$\left. + \frac{\mu_2 \mathscr{B}_1\mathscr{B}_2}{\xi_E(\mu_1+\xi_T)\aleph_2'(0)} \frac{\partial \Psi_3(P_0,0)}{\partial T} - 1 \right).$$

From Condition (**H1**), we have $¥'(P_0) < 0$. Therefore, if

$$\frac{\beta \mathscr{B}_1\mathscr{B}_3}{\xi_V(\mu_1+\xi_T)\aleph_3'(0)} \frac{\partial \Psi_1(P_0,0)}{\partial V} + \frac{\mathscr{B}_1}{(\mu_1+\xi_T)\aleph_1'(0)} \frac{\partial \Psi_2(P_0,0)}{\partial E} + \frac{\mu_2 \mathscr{B}_1\mathscr{B}_2}{\xi_E(\mu_1+\xi_T)\aleph_2'(0)} \frac{\partial \Psi_3(P_0,0)}{\partial T} > 0,$$

then $F_1'(P_0) < 0$ and there exists $P_1 \in (0,P_0)$. such that $F_1(P_1) = 0$. From Eq. (17) and Condition (**H3**), we have

$$T_1 = \aleph_1^{-1}\left(\frac{\mathscr{B}_1 ¥(P)}{\mu_1+\xi_T}\right), \quad E_1 = \aleph_2^{-1}\left(\frac{\mu_2 \mathscr{B}_1\mathscr{B}_2 ¥(P)}{\xi_E(\mu_1+\xi_T)}\right), \quad V_1 = \aleph_3^{-1}\left(\frac{\beta \mathscr{B}_1\mathscr{B}_3 ¥(P)}{\xi_V(\mu_1+\xi_T)}\right).$$

It follows that $Đ_1 = (P_1, T_1, E_1, V_1, 0, 0)$, exists when

$$\frac{\beta \mathscr{B}_1\mathscr{B}_3}{\xi_V(\mu_1+\xi_T)\aleph_3'(0)} \frac{\partial \Psi_1(P_0,0)}{\partial V} + \frac{\mathscr{B}_1}{(\mu_1+\xi_T)\aleph_1'(0)} \frac{\partial \Psi_2(P_0,0)}{\partial E} + \frac{\mu_2 \mathscr{B}_1\mathscr{B}_2}{\xi_E(\mu_1+\xi_T)\aleph_2'(0)} \frac{\partial \Psi_3(P_0,0)}{\partial T} > 0.$$

We refer to $Đ_1$ as an equilibrium of chronic HIV-1 infection with dormant immunological responses. Determining the model's fundamental HIV-1 reproduction number $\Re_0$ is essential. The existence of the chronic HIV-1 infection equilibrium with dormant immune responses is determined by the following method:

$$\Re_0 = \Re_{01} + \Re_{02} + \Re_{03},$$

where

$$\Re_{01} = \frac{\beta \mathscr{B}_1\mathscr{B}_3}{\xi_V(\mu_1+\xi_T)\aleph_3'(0)} \frac{\partial \Psi_1(P_0,0)}{\partial V},$$
$$\Re_{02} = \frac{\mathscr{B}_1}{(\mu_1+\xi_T)\aleph_1'(0)} \frac{\partial \Psi_2(P_0,0)}{\partial E},$$
$$\Re_{03} = \frac{\mu_2 \mathscr{B}_1\mathscr{B}_2}{\xi_E(\mu_1+\xi_T)\aleph_2'(0)} \frac{\partial \Psi_3(P_0,0)}{\partial T}.$$

It is notable that the equilibrium $Đ_1$ exists when $\Re_0 > 1$, It ultimately establishes whether a chronic infection can be proven. In fact, $\Re_{01}$, $\Re_{02}$, and $\Re_{03}$ refer to the role of viral infection, inflammatory cytokines, and cellular infection, respectively.

(III) Point of equilibrium for chronic infection with just CTLs response, $Đ_2 = (P_2, T_2, E_2, V_2, Z_2^T, 0)$, where $Z_2^T \neq 0$ and $\aleph_1(T) = \frac{\xi_{Z^T}}{\sigma}$, then $T_2 = \aleph_1^{-1}\left(\frac{\xi_{Z^T}}{\sigma}\right)$ From Eqs. (10) -(11) we get

$$\yen(P) = \Psi_1(P,V) + \Psi_2(P,T) + \Psi_3(P,E) = \left(\frac{\mu_1 + \xi_T + \lambda\,\aleph_4(Z^T)}{\mathscr{B}_1}\right)\aleph_1(T), \tag{18}$$

according to condition **H3** and from Eqs. (12)-(13) we have

$$\mu_2\mathscr{B}_2\aleph_1(T_2) = \xi_E\,\aleph_2(E_2),$$

$$\beta\mathscr{B}_3\aleph_1(T_2) = \xi_V\,\aleph_3(V_2).$$

Then

$$E_2 = \aleph_2^{-1}\left(\frac{\mu_2\mathscr{B}_2\aleph_1(T_2)}{\xi_E}\right) = \aleph_2^{-1}\left(\frac{\mu_2\mathscr{B}_2\xi_{Z^T}}{\sigma\xi_E}\right) > 0,$$

$$V_2 = \aleph_3^{-1}\left(\frac{\beta\mathscr{B}_3\aleph_1(T_2)}{\xi_V}\right) = \aleph_3^{-1}\left(\frac{\beta\mathscr{B}_3\xi_{Z^T}}{\sigma\xi_V}\right) > 0.$$

From we using Eq. (18), we define

$$\digamma_2(P) = \yen(P) - \Psi_1(P,V_2) - \Psi_2(P,T_2) - \Psi_3(P,E_2) = 0.$$

H1 and **H2** imply that $\digamma_2(0) = \yen(0) > 0$ and

$$\digamma_2(P_0) = -\left(\Psi_1(P_0,V_2) + \Psi_2(P_0,T_2) + \Psi_3(P_0,E_2)\right) < 0,$$

which also means that there exists $P_2 \in (0, P_0)$ such that $\digamma_2(P_2) = 0$. According to Eq. (18), we can get

$$Z_2^T = \aleph_4^{-1}\left(\frac{(\mu_1+\xi_T)}{\lambda\mathscr{B}_1}\left[\frac{\mathscr{B}_1\Psi_1(P_2,V_2)}{(\mu_1+\xi_T)\,\aleph_1(T_2)} + \frac{\mathscr{B}_1\Psi_2(P_2,T_2)}{(\mu_1+\xi_T)\,\aleph_1(T_2)} + \frac{\mathscr{B}_1\Psi_3(P_2,E_2)}{(\mu_1+\xi_T)\,\aleph_1(T_2)} - 1\right]\right).$$

where

$$\Re_1 = \frac{\sigma\mathscr{B}_1\Psi_1(P_2,V_2)}{\xi_{Z^T}(\mu_1+\xi_T)} + \frac{\sigma\mathscr{B}_1\Psi_2(P_2,T_2)}{\xi_{Z^T}(\mu_1+\xi_T)} + \frac{\sigma\mathscr{B}_1\Psi_3(P_2,E_2)}{\xi_{Z^T}(\mu_1+\xi_T)}.$$

Here, $\Re_1$ is the CTLs response activation number. Obviously, $Đ_2$ exists if $\Re_1 > 1$. Depending on the value of the parameter $\Re_1$, the CTLs response is either activated or not.

(IV) Point of equilibrium for a chronic infection with only an antibody reaction, $Đ_3 = (P_3, T_3, E_3, V_3, 0, Z_3^V)$, here $Z_3^V \neq 0$ and $\aleph_3(V_3) = \frac{\xi_{Z^V}}{\varpi}$, then $V_3 = \aleph_3^{-1}\left(\frac{\xi_{Z^V}}{\varpi}\right)$ From Eqs. (10) -(13) we get

$$\yen(P)=\Psi_1(P,V)+\Psi_2(P,T)+\Psi_3(P,E)=\frac{\mu_1+\xi_T}{\mathscr{B}_1}\aleph_1(T)=\left(\frac{\xi_E(\mu_1+\xi_T)}{\mu_2\mathscr{B}_1\mathscr{B}_2}\right)\aleph_2(E)$$

$$=\frac{\mu_1+\xi_T}{\beta\mathscr{B}_1\mathscr{B}_3}\left[\xi_V+\psi\aleph_5\left(Z^V\right)\right]\aleph_3(V). \tag{19}$$

From Eq. (19), we get

$$T=\aleph_1^{-1}\left(\frac{\mathscr{B}_1\yen(P)}{\mu_1+\xi_T}\right)=f_4(P),\quad E=\aleph_2^{-1}\left(\frac{\mu_2\mathscr{B}_1\mathscr{B}_2\yen(P)}{\xi_E(\mu_1+\xi_T)}\right)=f_5(P). \tag{20}$$

From **H1**, we know $f_i(P_0)=0$ and $f_i(P)>0$ for all $P\in[0,P_0)$, $i=4,5$, define

$$\digamma_3(P)=\yen(P)-\Psi_1(P,V_2)-\Psi_2(P,f_4(P))-\Psi_3(P,f_5(P)).$$

Then, we get

$$\digamma_3(0)=\yen(0)>0,\quad \digamma_3(P_0)<0.$$

Since $\digamma_3(P)$ is continuous on $[0,P_0)$, then there exists $P_3\in[0,P_0)$, such that $\digamma_3(P_3)=0$, from **H3** and Eq. (19), we can get

$$Z_3^V=\aleph_5^{-1}\left(\frac{\xi_V}{\psi}\left[\frac{\beta\mathscr{B}_1\mathscr{B}_3(\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3))}{\xi_V(\mu_1+\xi_T)\aleph_3(V_3)}-1\right]\right).$$

Clearly, $Z_3^V>0$ if and only if $\frac{\beta\mathscr{B}_1\mathscr{B}_3\left(\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3)\right)}{\xi_V(\mu_1+\xi_T)\aleph_3(V_3)}>1$. The antibody immunological response reproductive number is defined as:

$$\Re_2=\frac{\varpi\beta\mathscr{B}_1\mathscr{B}_3(\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3))}{\xi_{Z^V}\xi_V(\mu_1+\xi_T)}.$$

Thus, $Z_3^V=\aleph_5^{-1}\left(\frac{\xi_V}{\psi}[\Re_2-1]\right)$. Depending on the value $\Re_2$, the antibody response is either initiated or not. Therefore, the existence of the equilibrium $Ð_3$ is ensured by the condition $\Re_2>1$.

(V) Point of equilibrium for chronic infections with both CTLs and antibody responses $Ð_4=(P_4,T_4,E_4,V_4,Z_4^T,Z_4^V)$, where

$$T_4=\aleph_1^{-1}\left(\frac{\xi_{Z^T}}{\sigma}\right),\ E_4=\aleph_2^{-1}\left(\frac{\mu_2\mathscr{B}_2\xi_{Z^T}}{\sigma\xi_E}\right),\ V_4=\aleph_3^{-1}\left(\frac{\xi_{Z^V}}{\varpi}\right),$$

from eqs. (10) -(11) we get

$$\yen(P)=\Psi_1(P,V)+\Psi_2(P,T)+\Psi_3(P,E)=\left(\frac{\mu_1+\xi_T+\lambda\aleph_4\left(Z^T\right)}{\mathscr{B}_1}\right)\aleph_1(T),$$

$$\beta\mathscr{B}_3\aleph_1(T)=\left(\xi_V+\psi\aleph_5\left(Z^V\right)\right)\aleph_3(V). \tag{21}$$

Let $T_4 = T, E_4 = E, V_4 = V$ and define

$$\digamma_4(P) = \yen(P) - \Psi_1(P, V_4) - \Psi_2(P, T_4) - \Psi_3(P, E_4).$$

Then, we have $\digamma_4(0) = \yen(0) > 0$ and $\digamma_4(P_0) = -(\Psi_1(P_0, V_4) + \Psi_2(P_0, T_4) + \Psi_3(P_0, E_4)) < 0$. Thus, there exists $P_4 \in (0, P_0)$ such that $\digamma_4(P_4) = 0$. Moreover, from Eq. (21), we have

$$Z_4^T = \aleph_4^{-1}\left(\frac{(\mu_1+\xi_T)\mathscr{B}_1}{\lambda}\left[\frac{\sigma(\Psi_1(P_4,V_4)+\Psi_2(P_4,T_4)+\Psi_3(P_4,E_4))}{\xi_{Z^T}(\mu_1+\xi_T)} - 1\right]\right),$$

then, we get

$$Z_4^V = \aleph_5^{-1}\left[\frac{\xi_V}{\psi}\left(\frac{\varpi\xi_{Z^T}\beta\mathscr{B}_3}{\sigma\xi_V\xi_{Z^V}} - 1\right)\right].$$

where $\mathfrak{R}_3$ and $\mathfrak{R}_4$ represent the competitive reproductive numbers of CTLs and antibodies, respectively, and they are provided as

$$\mathfrak{R}_3 = \frac{\sigma(\Psi_1(P_4,V_4)+\Psi_2(P_4,T_4)+\Psi_3(P_4,E_4))}{\xi_{Z^T}(\mu_1+\xi_T)}, \quad \mathfrak{R}_4 = \frac{\varpi\xi_{Z^T}\beta\mathscr{B}_3}{\sigma\xi_V\xi_{Z^V}}.$$

Parameters determine the efficacy of the CTLs and antibody immune responses $\mathfrak{R}_3$ and $\mathfrak{R}_4$. Therefore, the existence of the equilibrium Đ$_4$ is ensured by the condition $\mathfrak{R}_3$ and $\mathfrak{R}_4 > 1$ and we can write $Z_4^T = \aleph_4^{-1}\left(\frac{(\mu_1+\xi_T)\mathscr{B}_1}{\lambda}[\mathfrak{R}_3 - 1]\right), Z_4^V = \aleph_5^{-1}\left(\frac{\xi_V}{\psi}[\mathfrak{R}_4 - 1]\right)$. The threshold parameters are given as follows:

$$\mathfrak{R}_0 = \frac{\beta\mathscr{B}_1\mathscr{B}_3}{\xi_V(\mu_1+\xi_T)\aleph_3'(0)}\frac{\partial\Psi_1(P_0,0)}{\partial V} + \frac{\mathscr{B}_1}{(\mu_1+\xi_T)\aleph_1'(0)}\frac{\partial\Psi_2(P_0,0)}{\partial E} + \frac{\mu_2\mathscr{B}_1\mathscr{B}_2}{\xi_E(\mu_1+\xi_T)\aleph_2'(0)}\frac{\partial\Psi_3(P_0,0)}{\partial T},$$

$$\mathfrak{R}_1 = \frac{\sigma\mathscr{B}_1\Psi_1(P_2,V_2)}{\xi_{Z^T}(\mu_1+\xi_T)} + \frac{\sigma\mathscr{B}_1\Psi_2(P_2,T_2)}{\xi_{Z^T}(\mu_1+\xi_T)} + \frac{\sigma\mathscr{B}_1\Psi_3(P_2,E_2)}{\xi_{Z^T}(\mu_1+\xi_T)},$$

(22)

$$\mathfrak{R}_2 = \frac{\varpi\beta\mathscr{B}_1\mathscr{B}_3(\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3))}{\xi_V\xi_{Z^V}(\mu_1+\xi_T)},$$

$$\mathfrak{R}_3 = \frac{\sigma(\Psi_1(P_4,V_4)+\Psi_2(P_4,T_4)+\Psi_3(P_4,E_4))}{\xi_{Z^T}(\mu_1+\xi_T)}, \ \mathfrak{R}_4 = \frac{\varpi\xi_{Z^T}\beta\mathscr{B}_3}{\sigma\xi_V\xi_{Z^V}}.$$

5. Global Stability

In this part, we illustrate the global asymptotic stability of all equilibrium using Lyapunov functional theory. Define $\chi(\Lambda) = \Lambda - 1 - \ln(\Lambda)$. Furthermore, a Lyapunov functional candidate is defined $\Phi_i(P,T,E,V,Z^T,Z^V)$ and let Γ_i' be the largest invariant subset of

$$\Gamma_i = \left\{(P,T,E,V,Z^T,Z^V) : \frac{d\Phi_i}{dt} = 0\right\}, \qquad i = 0,1,2,3,4,$$

and define

$$\mathscr{F}_1(P) = \lim_{V\to 0^+}\frac{\Psi_1(P,V)}{\aleph_3(V)}, \quad \mathscr{F}_2(P) = \lim_{T\to 0^+}\frac{\Psi_2(P,T)}{\aleph_1(T)}, \quad \mathscr{F}_3(P) = \lim_{E\to 0^+}\frac{\Psi_3(P,E)}{\aleph_2(E)}. \tag{23}$$

Based on Conditions (**H2**) and (**H3**), we obtain

$$\mathscr{F}_1(P) = \frac{1}{\aleph_3'(0)} \frac{\partial \Psi_1(P,0)}{\partial V} > 0,$$
$$\mathscr{F}_2(P) = \frac{1}{\aleph_1'(0)} \frac{\partial \Psi_2(P,0)}{\partial T} > 0,$$
$$\mathscr{F}_3(P) = \frac{1}{\aleph_2'(0)} \frac{\partial \Psi_3(P,0)}{\partial E} \text{ for any } P > 0,$$

moreover,

$$\mathscr{F}_i'(P) > 0, \quad i = 1,2,3. \tag{24}$$

Thus, the basic reproduction number $\Re_0$ can be rewritten as

$$\Re_0 = \frac{\beta \mathscr{B}_1 \mathscr{B}_3}{\xi_V(\mu_1+\xi_T)} \mathscr{F}_1(P_0) + \frac{\mathscr{B}_1}{(\mu_1+\xi_T)} \mathscr{F}_2(P_0) + \frac{\mu_2 \mathscr{B}_1 \mathscr{B}_2}{\xi_E(\mu_1+\xi_T)} \mathscr{F}_3(P_0),$$

to investigate the next Theorem 1, we give the following condition [39]:

Condition (H5)

(**i**) The supremum of $\frac{\mathscr{F}_2(P)}{\mathscr{F}_1(P)}$ is achieved at $P = P_0$ for all $P \in (0, P_0]$,

(**ii**) The supremum of $\frac{\mathscr{F}_3(P)}{\mathscr{F}_1(P)}$ is achieved at $P = P_0$ for all $P \in (0, P_0]$.

Theorem 1. If $\Re_0 \leq 1$ and conditions **(H1)-(H5)** $Ð_0$ be satisfied, then is globally asymptotically stable (G.A.S).

Proof. Construct a Lyapunov functional as:

$$\begin{aligned}
\Phi_0 = {} & P - P_0 - \int_{P_0}^{P} \frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(\theta)} d\theta + \frac{1}{\mathscr{B}_1} T + \frac{\mathscr{F}_3(P_0)}{\xi_E} E + \frac{\mathscr{F}_1(P_0)}{\xi_V} V + \frac{\lambda}{\sigma \mathscr{B}_1} Z^T + \frac{\psi \mathscr{F}_1(P_0)}{\varpi \xi_V} Z^V \\
& + \frac{1}{\mathscr{B}_1} \int_0^{\varkappa_1} \tilde{\mathscr{B}}_1(\varphi) \int_{t-\varphi}^{t} \left[\Psi_1(P(\phi), V(\phi)) + \Psi_2(P(\phi), T(\phi)) + \Psi_3(P(\phi), E(\phi))\right] d\phi d\varphi \\
& + \frac{\mu_2 \mathscr{F}_3(P_0)}{\xi_E} \int_0^{\varkappa_2} \tilde{\mathscr{B}}_2(\varphi) \int_{t-\varphi}^{t} \aleph_1(T(\phi)) d\phi d\varphi + \frac{\beta \mathscr{F}_1(P_0)}{\xi_V} \int_0^{\varkappa_3} \tilde{\mathscr{B}}_3(\varphi) \int_{t-\varphi}^{t} \aleph_1(T(\phi)) d\phi d\varphi.
\end{aligned}$$

We note that, $\Phi_0(P,T,E,V,Z^T,Z^V) > 0$ for all $(P,T,E,V,Z^T,Z^V) > 0$, and $\Phi_0(P_0,0,0,0,0,0) = 0$. We calculate $\frac{d\Phi_0}{dt}$ along the solutions of model (3)-(8) as:

$$\begin{aligned}
\frac{d\Phi_0}{dt} = {} & \left(1 - \frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}\right) \left(¥(P) - \Psi_1(P,V) - \Psi_2(P,T) - \Psi_3(P,E)\right) + \frac{1}{\mathscr{B}_1} \\
& \times \left(\int_0^{\varkappa_1} \tilde{\mathscr{B}}_1(\varphi) \left[\Psi_1(P_\varphi, V_\varphi) + \Psi_2(P_\varphi, T_\varphi) + \Psi_3(P_\varphi, E_\varphi)\right] d\varphi - (\mu_1 + \xi_T) \aleph_1(T) - \lambda \aleph_1(T) \aleph_4(Z^T) \right) \\
& + \frac{\mathscr{F}_3(P_0)}{\xi_E} \left(\mu_2 \int_0^{\varkappa_2} \tilde{\mathscr{B}}_2(\varphi) \aleph_1(T_\varphi) d\varphi - \xi_E \aleph_2(E) \right) \\
& + \frac{\mathscr{F}_1(P_0)}{\xi_V} \left(\beta \int_0^{\varkappa_3} \tilde{\mathscr{B}}_3(\varphi) \aleph_1(T_\varphi) d\varphi - \xi_V \aleph_3(V) - \psi \aleph_3(V) \aleph_5(Z^V) \right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda}{\sigma\mathscr{B}_1}\left(\sigma\aleph_1(T)\aleph_4\left(Z^T\right)-\xi_{Z^T}\aleph_4\left(Z^T\right)\right)+\frac{\psi\mathscr{F}_1(P_0)}{\varpi\xi_V}\left(\varpi\aleph_3(V)\aleph_5\left(Z^V\right)-\xi_{Z^V}\aleph_5\left(Z^V\right)\right)\\
&+\frac{1}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\Psi_1(P,V)+\Psi_2(P,T)+\Psi_3(P,E)\right)d\varphi\\
&-\frac{1}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\left(\Psi_1\left(P_\phi,V_\phi\right)+\Psi_2\left(P_\phi,T_\phi\right)+\Psi_3\left(P_\phi,E_\phi\right)\right)d\varphi\\
&+\frac{\mu_2\mathscr{F}_3(P_0)}{\xi_E}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1(T)\,d\varphi-\frac{\mu_2\mathscr{F}_3(P_0)}{\xi_E}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1\left(T_\phi\right)d\varphi\\
&+\frac{\beta\mathscr{F}_1(P_0)}{\xi_V}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1(T)\,d\varphi-\frac{\beta\mathscr{F}_1(P_0)}{\xi_V}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1\left(T_\phi\right)d\varphi. \qquad (25)
\end{aligned}
$$

Collecting terms of Eq. (25), we get

$$
\begin{aligned}
\frac{d\Phi_0}{dt}&=¥(P)\left(1-\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}\right)+\Psi_1(P,V)\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}+\Psi_2(P,T)\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}+\Psi_3(P,E)\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}\\
&-\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T)-\mathscr{F}_3(P_0)\aleph_2(E)-\mathscr{F}_1(P_0)\aleph_3(V)-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right) \qquad (26)\\
&-\frac{\psi\mathscr{F}_1(P_0)}{\varpi\xi_V}\xi_{Z^V}\aleph_5\left(Z^V\right)+\frac{\mu_2\mathscr{F}_3(P_0)}{\xi_E}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1(T)\,d\varphi+\frac{\beta\mathscr{F}_1(P_0)}{\xi_V}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1(T)\,d\varphi.
\end{aligned}
$$

From condition **H4** we get

$$
\begin{aligned}
&\frac{\Psi_1(P,V)}{\aleph_3(V)}\le\lim_{V\to0^+}\frac{\Psi_1(P,V)}{\aleph_3(V)}=\mathscr{F}_1(P),\quad \frac{\Psi_2(P,T)}{\aleph_1(T)}\le\lim_{T\to0^+}\frac{\Psi_2(P,T)}{\aleph_1(T)}=\mathscr{F}_2(P),\\
&\frac{\Psi_3(P,E)}{\aleph_2(E)}\le\lim_{E\to0^+}\frac{\Psi_3(P,E)}{\aleph_2(E)}=\mathscr{F}_3(P).
\end{aligned}
$$

Then,

$$
\Psi_1(P,V)\le\mathscr{F}_1(P)\aleph_3(V),\quad \Psi_2(P,T)\le\aleph_1(T)\mathscr{F}_2(P),\quad \Psi_3(P,E)\le\aleph_2(E)\mathscr{F}_3(P).
$$

Therefore, Eq. (26) will become

$$
\begin{aligned}
\frac{d\Phi_0}{dt}&\le¥(P)\left(1-\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}\right)+\aleph_1(T)\mathscr{F}_1(P_0)\frac{\mathscr{F}_2(P)}{\mathscr{F}_1(P)}+\aleph_2(E)\mathscr{F}_1(P_0)\frac{\mathscr{F}_3(P)}{\mathscr{F}_1(P)}\\
&-\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T)-\mathscr{F}_3(P_0)\aleph_2(E)-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right)-\frac{\psi\mathscr{F}_1(P_0)}{\varpi\xi_V}\xi_{Z^V}\aleph_5\left(Z^V\right)\\
&+\frac{\mu_2\mathscr{F}_3(P_0)}{\xi_E}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1(T)\,d\varphi+\frac{\beta\mathscr{F}_1(P_0)}{\xi_V}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1(T)\,d\varphi.
\end{aligned}
$$

Condition **H5** implies that

$$
\frac{\mathscr{F}_2(P)}{\mathscr{F}_1(P)}\mathscr{F}_1(P_0)\le\frac{\mathscr{F}_2(P_0)}{\mathscr{F}_1(P_0)}\mathscr{F}_1(P_0)=\mathscr{F}_2(P_0),\frac{\mathscr{F}_3(P)}{\mathscr{F}_1(P)}\mathscr{F}_1(P_0)\le\frac{\mathscr{F}_3(P_0)}{\mathscr{F}_1(P_0)}\mathscr{F}_1(P_0)=\mathscr{F}_3(P_0)
$$

then

$$
\frac{d\Phi_0}{dt}\le¥(P)\left(1-\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}\right)+\aleph_1(T)\mathscr{F}_2(P_0)+\aleph_2(E)\mathscr{F}_3(P_0)-\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T)
$$

$$-\mathscr{F}_3(P_0)\aleph_2(E)-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right)-\frac{\psi\mathscr{F}_1(P_0)}{\varpi\xi_V}\xi_{Z^V}\aleph_5\left(Z^V\right)$$
$$+\frac{\mu_2\mathscr{F}_3(P_0)}{\xi_E}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1(T)\,d\varphi+\frac{\beta\mathscr{F}_1(P_0)}{\xi_V}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1(T)\,d\varphi.$$

Substituting inequality and using $¥(P_0)=0$, we get

$$\frac{d\Phi_0}{dt}\le(¥(P)-¥(P_0))\left(1-\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}\right)$$
$$+\frac{(\mu_1+\xi_T)}{\mathscr{B}_1}\left(\frac{\beta\mathscr{B}_1\mathscr{B}_3\mathscr{F}_1(P_0)}{\xi_V(\mu_1+\xi_T)}+\frac{\mathscr{B}_1\mathscr{F}_2(P_0)}{(\mu_1+\xi_T)}+\frac{\mu_2\mathscr{B}_1\mathscr{B}_2\mathscr{F}_3(P_0)}{\xi_E(\mu_1+\xi_T)}-1\right)\aleph_1(T)$$
$$-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right)-\frac{\psi\mathscr{F}_1(P_0)}{\varpi\xi_V}\xi_{Z^V}\aleph_5\left(Z^V\right)$$
$$=(¥(P)-¥(P_0))\left(1-\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}\right)+\frac{(\mu_1+\xi_T)}{\mathscr{B}_1}(\Re_0-1)\aleph_1(T)-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right)$$
$$-\frac{\psi\mathscr{F}_1(P_0)}{\varpi\xi_V}\xi_{Z^V}\aleph_5\left(Z^V\right).$$

Conditions **H1**, **H2** and provide that $¥(P_0)=0$ is a strictly decreasing function of P, while $\mathscr{F}_1(P)$ is a strictly increasing function of P . Then,

$$(¥(P)-¥(P_0))\left(1-\frac{\mathscr{F}_1(P_0)}{\mathscr{F}_1(P)}\right)\le 0.$$

If $\Re_0\le 1$ then $\frac{d\Phi_0}{dt}\le 0$ for all $(P,T,E,V,Z^T,Z^V)>0$. Moreover, $\frac{d\Phi_0}{dt}=0$ when $P=P_0$, $T=Z^T=Z^V=0$. The solutions of system (3)-(8) converge to Γ_0' and $(P_0,0,E,V,0,0)\in\Gamma_0'$. According to LaSalle's invariance principle we have $\lim\limits_{t\to\infty}P(t)=P_0$ and $\lim\limits_{t\to\infty}T(t)=\lim\limits_{t\to\infty}Z^T(t)=\lim\limits_{t\to\infty}Z^V(t)=0$. Then, $\dot{P}(t)=0$ and $\dot{T}(t)=\dot{Z}^T(t)=\dot{Z}^V(t)=0$. From the Eq. (5) and the Eq. (6), we have $\dot{E}=-\xi_E\aleph_2(E)<0$ and $\dot{V}=-\xi_V\aleph_3(V(t))<0$, this leads $\lim\limits_{t\to\infty}E(t)=\lim\limits_{t\to\infty}V(t)=0$. This implies that the largest invariant subset of $\Gamma_0'=\{Đ_0\}$. LaSalle's invariance principle (L.I.P.) reveals that $Đ_0$ is G.A.S [41]. □

Remark 1. From Condition **H2** and Condition **H4**, we can get

$$(27)\qquad (\Psi_1(P,V)-\Psi_1(P,V_i))\left(\frac{\Psi_1(P,V)}{\aleph_3(V)}-\frac{\Psi_1(P,V_i)}{\aleph_3(V_i)}\right)\le 0,\quad P,V,V_i>0,\quad i=1,2,3,4,$$

which leads to

$$(28)\qquad \left(1-\frac{\Psi_1(P,V_i)}{\Psi_1(P,V)}\right)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_i)}-\frac{\aleph_3(V)}{\aleph_3(V_i)}\right)\le 0,\quad P,V,V_i>0,\quad i=1,2,3,4.$$

Define the following functions [39]:

$$(29)\qquad \mathfrak{F}_i^T(P,T)=\frac{\Psi_2(P,T)}{\Psi_1(P,V_i)},\qquad \mathfrak{F}_i^E(P,E)=\frac{\Psi_3(P,E)}{\Psi_1(P,V_i)},\quad i=1,2,3,4.$$

We state the following condition:

Condition (**H6).**

(i) $\left(1-\frac{\mathfrak{F}_i^T(P_i,T_i)}{\mathfrak{F}_i^T(P,T)}\right)\left(\frac{\mathfrak{F}_i^T(P,T)}{\mathfrak{F}_i^T(P_i,T_i)}-\frac{\aleph_1(T)}{\aleph_1(T_i)}\right)\leq 0,\quad P\in(0,P_0),\quad P_i,T,T_i>0,\ i=1,2,3,4.$

(ii) $\left(1-\frac{\mathfrak{F}_i^E(P_i,E_i)}{\mathfrak{F}_i^T(P,E)}\right)\left(\frac{\mathfrak{F}_i^E(P,E)}{\mathfrak{F}_i^E(P_i,E_i)}-\frac{\aleph_2(E)}{\aleph_2(E_i)}\right)\leq 0,\quad P\in(0,P_0),\quad P_i,E,E_i>0,\quad i=1,2,3,4.$

Using the definition of $\mathfrak{F}_i^T(P,T)$ given in condition (**H6**), we obtain

$$\begin{aligned}
&\frac{\Psi_2(P,T)\Psi_1(P_i,V_i)}{\Psi_2(P_i,T_i)\Psi_1(P,V_i)}-\frac{\aleph_1(T)}{\aleph_1(T_i)}-1+\frac{\Psi_1(P,V_i)\Psi_2(P_i,T_i)\aleph_1(T)}{\Psi_1(P_i,V_i)\Psi_2(P,T)\aleph_1(T_i)}\\
&=\left(1-\frac{\mathfrak{F}_i^T(P_i,T_i)}{\mathfrak{F}_i^T(P,T)}\right)\left(\frac{\mathfrak{F}_i^T(P,T)}{\mathfrak{F}_i^T(P_i,T_i)}-\frac{\aleph_1(T)}{\aleph_1(T_i)}\right),\quad i=1,2,3,4,
\end{aligned}\tag{30}$$

and

$$\begin{aligned}
&\frac{\Psi_3(P,E)\Psi_1(P_i,V_i)}{\Psi_3(P_i,E_i)\Psi_1(P,V_i)}-\frac{\aleph_2(E)}{\aleph_2(E_i)}-1+\frac{\Psi_1(P,V_i)\Psi_3(P_i,E_i)\aleph_2(E)}{\Psi_1(P_i,V_i)\Psi_3(P,E)\aleph_2(E_i)}\\
&=\left(1-\frac{\mathfrak{F}_i^E(P_i,E_i)}{\mathfrak{F}_i^E(P,E)}\right)\left(\frac{\mathfrak{F}_i^E(P,E)}{\mathfrak{F}_i^E(P_i,E_i)}-\frac{\aleph_2(E)}{\aleph_2(E_i)}\right),\quad i=1,2,3,4.
\end{aligned}\tag{31}$$

The following equalities are taken into consideration for usage in the following theorems:

$$\begin{aligned}
\ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)}{\Psi_1(P,V)}\right)&=\ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_i)}{\Psi_1(P_i,V_i)\aleph_1(T)}\right)+\ln\left(\frac{\Psi_1(P_i,V_i)}{\Psi_1(P,V_i)}\right)+\ln\left(\frac{\aleph_1(T)\aleph_2(E_i)}{\aleph_1(T_i)\aleph_2(E)}\right)\\
&+\ln\left(\frac{\aleph_2(E)\aleph_3(V_i)}{\aleph_2(E_i)\aleph_3(V)}\right)+\ln\left(\frac{\Psi_1(P,V_i)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_i)}\right),\\
\ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)}{\Psi_2(P,T)}\right)&=\ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_i)}{\Psi_2(P_i,T_i)\aleph_1(T)}\right)+\ln\left(\frac{\Psi_1(P_i,V_i)}{\Psi_1(P,V_i)}\right)\\
&+\ln\left(\frac{\Psi_1(P,V_i)\Psi_2(P_i,T_i)\aleph_1(T)}{\Psi_1(P_i,V_i)\Psi_2(P,T)\aleph_1(T_i)}\right),\\
\ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)}{\Psi_3(P,E)}\right)&=\ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_i)}{\Psi_3(P_i,E_i)\aleph_1(T)}\right)+\ln\left(\frac{\Psi_1(P_i,V_i)}{\Psi_1(P,V_i)}\right)\\
&+\ln\left(\frac{\aleph_1(T)\aleph_2(E_i)}{\aleph_1(T_i)\aleph_2(E)}\right)+\ln\left(\frac{\Psi_1(P,V_i)\Psi_3(P_i,E_i)\aleph_2(E)}{\Psi_1(P_i,V_i)\Psi_3(P,E)\aleph_2(E_i)}\right),\\
\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)&=\ln\frac{\aleph_1(T_\varphi)\aleph_2(E_i)}{\aleph_1(T_i)\aleph_2(E)}+\ln\left(\frac{\aleph_1(T_i)\aleph_2(E)}{\aleph_1(T)\aleph_2(E_i)}\right),\\
\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)&=\ln\frac{\aleph_1(T_\varphi)\aleph_3(V_i)}{\aleph_1(T_i)\aleph_3(V)}+\ln\left(\frac{\aleph_1(T_i)\aleph_3(V)}{\aleph_1(T)\aleph_3(V_i)}\right),\qquad i=1,2,3,4.
\end{aligned}\tag{32}$$

Theorem 2. If $\Re_0>1$, $\Re_1\leq 0$ and $\Re_2\leq 0$ conditions **(H1)-(H4),(H6)** Đ$_1$ be satisfied, then is (G.A.S).

Proof. Construct a Lyapunov functional as:

$$\begin{aligned}
\Phi_1&=P-P_1-\int_{P_1}^{P}\frac{\Psi_1(P_1,V_1)}{\Psi_1(\theta,V_1)}d\theta+\frac{1}{\mathscr{B}_1}\left(T-T_1-\int_{T_1}^{T}\frac{\aleph_1(T_1)}{\aleph_1(\theta)}d\theta\right)\\
&+\frac{\Psi_3(P_1,E_1)}{\xi_E\aleph_2(E_1)}\left(E-E_1-\int_{E_1}^{E}\frac{\aleph_2(E_1)}{\aleph_2(\theta)}d\theta\right)+\frac{\Psi_1(P_1,V_1)}{\xi_V\aleph_3(V_1)}\left(V-V_1-\int_{V_1}^{V}\frac{\aleph_3(V_1)}{\aleph_3(\theta)}d\theta\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda}{\sigma\mathscr{B}_1}Z^T+\frac{\psi\Psi_1(P_1,V_1)}{\varpi\xi_V\aleph_3(V_1)}Z^V+\frac{\Psi_1(P_1,V_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_1(P(\phi),V(\phi))}{\Psi_1(P_1,V_1)}\right]d\phi d\varphi\\
&+\frac{\Psi_2(P_1,T_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_2(P(\phi),T(\phi))}{\Psi_2(P_1,T_1)}\right]d\phi d\varphi\\
&+\frac{\Psi_{.3}(P_1,E_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_3(P(\phi),E(\phi))}{\Psi_3(P_1,E_1)}\right]d\phi d\varphi\\
&+\frac{\mu_2\Psi_3(P_1,E_1)\aleph_1(T_1)}{\xi_E\aleph_2(E_1)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\aleph_1(T(\phi))}{\aleph_1(T_1)}\right]d\phi d\varphi\\
&+\frac{\beta\Psi_1(P_1,V_1)\aleph_1(T_1)}{\xi_V\aleph_3(V_1)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\aleph_1(T(\phi))}{\aleph_1(T_1)}\right]d\phi d\varphi.
\end{aligned}
$$

We calculate $\frac{d\Phi_1}{dt}$ and summing terms, we derive

$$
\begin{aligned}
\frac{d\Phi_1}{dt}&=\yen(P)\left(1-\frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}\right)+\Psi_1(P,V)\frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}+\Psi_2(P,T)\frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}\\
&+\Psi_3(P,E)\frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}-\frac{1}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_1)}{\aleph_1(T)}d\varphi\\
&-\frac{1}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_1)}{\aleph_1(T)}d\varphi-\frac{1}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_1)}{\aleph_1(T)}d\varphi\\
&-\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T)+\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T_1)+\frac{1}{\mathscr{B}_1}\lambda\aleph_1(T_1)\aleph_4(Z^T)\\
&-\frac{\Psi_3(P_1,E_1)}{\xi_E\aleph_2(E_1)}\mu_2\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\frac{\aleph_1(T_\varphi)\aleph_2(E_1)}{\aleph_2(E)}d\varphi-\Psi_3(P_1,E_1)\frac{\aleph_2(E)}{\aleph_2(E_1)}+\Psi_3(P_1,E_1)\\
&-\frac{\Psi_1(P_1,V_1)}{\xi_V\aleph_3(V_1)}\beta\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\frac{\aleph_1(T_\varphi)\aleph_3(V_1)}{\aleph_3(V)}d\varphi-\Psi_1(P_1,V_1)\frac{\aleph_3(V)}{\aleph_3(V_1)}+\Psi_1(P_1,V_1)\\
&+\frac{\Psi_1(P_1,V_1)}{\xi_V}\psi\aleph_5(Z^V)-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4(Z^T)-\frac{\psi\Psi_1(P_1,V_1)}{\varpi\xi_V\aleph_3(V_1)}\xi_{Z^V}\aleph_5(Z^V)\\
&+\frac{\Psi_1(P_1,V_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)}{\Psi_1(P,V)}\right)d\varphi+\frac{\Psi_2(P_1,T_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)}{\Psi_2(P,T)}\right)d\varphi\\
&+\frac{\Psi_{.3}(P_1,E_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)}{\Psi_3(P,E)}\right)d\varphi+\frac{\mu_2\Psi_3(P_1,E_1)}{\xi_E\aleph_2(E_1)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1(T)d\varphi\\
&+\frac{\mu_2\Psi_3(P_1,E_1)\aleph_1(T_1)}{\xi_E\aleph_2(E_1)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi+\frac{\beta\Psi_1(P_1,V_1)}{\xi_V\aleph_3(V_1)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1(T)d\varphi\\
&+\frac{\beta\Psi_1(P_1,V_1)\aleph_1(T_1)}{\xi_V\aleph_3(V_1)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi.
\end{aligned}
$$

Using the following conditions for equilibrium $Đ_1$:

$$
\begin{aligned}
&\yen(P_1)=\Psi_1(P_1,V_1)+\Psi_2(P_1,T_1)+\Psi_3(P_1,E_1)=\frac{(\mu_1+\xi_T)}{\mathscr{B}_1}\aleph_1(T_1),\\
&\aleph_2(E_1)=\frac{\mu_2\mathscr{B}_2\aleph_1(T_1)}{\xi_E},\quad \aleph_3(V_1)=\frac{\beta\mathscr{B}_3\aleph_1(T_1)}{\xi_V}.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\frac{d\Phi_1}{dt} &= (¥(P) - ¥(P_1))\left(1 - \frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}\right) + (\Psi_1(P_1,V_1) + \Psi_2(P_1,T_1) + \Psi_3(P_1,E_1)) \\
&\times \left(1 - \frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}\right) + \Psi_1(P_1,V_1)\frac{\Psi_1(P,V)}{\Psi_1(P,V_1)} + \Psi_2(P_1,T_1)\frac{\Psi_2(P,T)\Psi_1(P_1,V_1)}{\Psi_2(P_1,T_1)\Psi_1(P,V_1)} \\
&+ \Psi_3(P_1,E_1)\frac{\Psi_3(P,E)\Psi_1(P_1,V_1)}{\Psi_3(P_1,E_1)\Psi_1(P,V_1)} - \frac{\Psi_1(P_1,V_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_1(P_\varphi,V_\varphi)\,\aleph_1(T_1)}{\Psi_1(P_1,V_1)\,\aleph_1(T)}d\varphi \\
&- \frac{\Psi_2(P_1,T_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_2(P_\varphi,T_\varphi)\,\aleph_1(T_1)}{\Psi_2(P_1,T_1)\,\aleph_1(T)}d\varphi - \frac{\Psi_3(P_1,E_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi) \\
&\times \frac{\Psi_3(P_\varphi,E_\varphi)\,\aleph_1(T_1)}{\Psi_3(P_1,E_1)\,\aleph_1(T)}d\varphi - \Psi_2(P_1,T_1)\frac{\aleph_1(T)}{\aleph_1(T_1)} + \Psi_1(P_1,V_1) + \Psi_2(P_1,T_1) \\
&+ \Psi_3(P_1,E_1) + \frac{\lambda}{\mathscr{B}_1}\aleph_1(T_1)\,\aleph_4\left(Z^T\right) - \frac{\Psi_3(P_1,E_1)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\frac{\aleph_1(T_\varphi)\,\aleph_2(E_1)}{\aleph_1(T_1)\,\aleph_2(E)}d\varphi \\
&- \Psi_3(P_1,E_1)\frac{\aleph_2(E)}{\aleph_2(E_1)} + \Psi_3(P_1,E_1) - \frac{\Psi_1(P_1,V_1)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\frac{\aleph_1(T_\varphi)\,\aleph_3(V_1)}{\aleph_1(T_1)\,\aleph_3(V)}d\varphi \\
&- \Psi_1(P_1,V_1)\frac{\aleph_3(V)}{\aleph_3(V_1)} + \Psi_1(P_1,V_1) + \frac{\Psi_1(P_1,V_1)}{\xi_V}\psi\,\aleph_5\left(Z^V\right) - \frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\,\aleph_4\left(Z^T\right) \\
&- \frac{\psi\Psi_1(P_1,V_1)}{\varpi\xi_V\,\aleph_3(V_1)}\xi_{Z^V}\,\aleph_5\left(Z^V\right) + \frac{\Psi_1(P_1,V_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)}{\Psi_1(P,V)}\right)d\varphi \\
&+ \frac{\Psi_2(P_1,T_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)}{\Psi_2(P,T)}\right)d\varphi + \frac{\Psi_{.3}(P_1,E_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi) \\
&\times \ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)}{\Psi_3(P,E)}\right)d\varphi + \frac{\Psi_3(P_1,E_1)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi \\
&+ \frac{\Psi_1(P_1,V_1)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi.
\end{aligned}
$$

Using the equalities represented in Eqs. (32) in case of $i = 1$, we get

$$
\begin{aligned}
\frac{d\Phi_1}{dt} &= (¥(P) - ¥(P_1))\left(1 - \frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}\right) - (\Psi_1(P_1,V_1) + \Psi_2(P_1,T_1) + \Psi_3(P_1,E_1)) \\
&\times \left(\frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)} - 1 - \ln\left(\frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}\right)\right) \\
&- \frac{\Psi_1(P_1,V_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P_\varphi,V_\varphi)\,\aleph_1(T_1)}{\Psi_1(P_1,V_1)\,\aleph_1(T)} - 1 - \ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)\,\aleph_1(T_1)}{\Psi_1(P_1,V_1)\,\aleph_1(T)}\right)\right)d\varphi \\
&- \frac{\Psi_2(P_1,T_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_2(P_\varphi,T_\varphi)\,\aleph_1(T_1)}{\Psi_2(P_1,T_1)\,\aleph_1(T)} - 1 - \ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)\,\aleph_1(T_1)}{\Psi_2(P_1,T_1)\,\aleph_1(T)}\right)\right)d\varphi \\
&- \frac{\Psi_3(P_1,E_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_3(P_\varphi,E_\varphi)\,\aleph_1(T_1)}{\Psi_3(P_1,E_1)\,\aleph_1(T)} - 1 - \ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)\,\aleph_1(T_1)}{\Psi_3(P_1,E_1)\,\aleph_1(T)}\right)\right)d\varphi
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Psi_1(P_1,V_1)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_1)}{\aleph_1(T_1)\aleph_3(V)}-1-\ln\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_1)}{\aleph_1(T_1)\aleph_3(V)}\right)\right)d\varphi\\
&-\frac{\Psi_3(P_1,E_1)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\left(\frac{\aleph_1(T_\varphi)\aleph_2(E_1)}{\aleph_1(T_1)\aleph_2(E)}-1-\ln\left(\frac{\aleph_1(T_\varphi)\aleph_2(E_1)}{\aleph_1(T_1)\aleph_2(E)}\right)\right)d\varphi\\
&-\frac{\Psi_1(P_1,V_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_1)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_1)}-1-\ln\left(\frac{\Psi_1(P,V_1)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_1)}\right)\right)d\varphi\\
&-\frac{\Psi_2(P_1,T_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_1)\Psi_2(P_1,T_1)\aleph_1(T)}{\Psi_1(P_1,V_1)\Psi_2(P,T)\aleph_1(T_1)}-1-\ln\left(\frac{\Psi_1(P,V_1)\Psi_2(P_1,T_1)\aleph_1(T)}{\Psi_1(P_1,V_1)\Psi_2(P,T)\aleph_1(T_1)}\right)\right)d\varphi\\
&-\frac{\Psi_{.3}(P_1,E_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_1)\Psi_3(P_1,E_1)\aleph_2(E)}{\Psi_1(P_1,V_1)\Psi_3(P,E)\aleph_2(E_1)}-1-\ln\left(\frac{\Psi_1(P,V_1)\Psi_3(P_1,E_1)\aleph_2(E)}{\Psi_1(P_1,V_1)\Psi_3(P,E)\aleph_2(E_1)}\right)\right)d\varphi\\
&+\Psi_1(P_1,V_1)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_1)}-\frac{\aleph_3(V)}{\aleph_3(V_1)}-1+\frac{\Psi_1(P,V_1)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_1)}\right)\\
&+\Psi_2(P_1,T_1)\left(\frac{\Psi_2(P,T)\Psi_1(P_1,V_1)}{\Psi_2(P_1,T_1)\Psi_1(P,V_1)}-\frac{\aleph_1(T)}{\aleph_1(T_1)}-1+\frac{\Psi_1(P,V_1)\Psi_2(P_1,T_1)\aleph_1(T)}{\Psi_1(P_1,V_1)\Psi_2(P,T)\aleph_1(T_1)}\right)\\
&+\Psi_3(P_1,E_1)\left(\frac{\Psi_3(P,E)\Psi_1(P_1,V_1)}{\Psi_3(P_1,E_1)\Psi_1(P,V_1)}-\frac{\aleph_2(E)}{\aleph_2(E_1)}-1+\frac{\Psi_1(P,V_1)\Psi_3(P_1,E_1)\aleph_2(E)}{\Psi_1(P_1,V_1)\Psi_3(P,E)\aleph_2(E_1)}\right)\\
&+\frac{\lambda}{\mathscr{B}_1}\left(\aleph_1(T_1)-\frac{\xi_{Z^T}}{\sigma}\right)\aleph_4(Z^T)+\frac{\Psi_1(P_1,V_1)\psi}{\xi_V\aleph_3(V_1)}\left(\aleph_3(V_1)-\frac{\xi_{Z^V}}{\varpi}\right)\aleph_5(Z^V).
\end{aligned}
$$

Using the definition of $\mathfrak{F}_i^T(P,T)$ and $\mathfrak{F}_i^E(P,E)$ given in condition **(H6)** and **Remark 1** from Eqs. (30)-(31) in case of $i=1$, we obtain

$$
\begin{aligned}
\frac{d\Phi_1}{dt}&=(¥(P)-¥(P_1))\left(1-\frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}\right)-(\Psi_1(P_1,V_1)+\Psi_2(P_1,T_1)+\Psi_3(P_1,E_1))\\
&\times\chi\left(\frac{\Psi_1(P_1,V_1)}{\Psi_1(P,V_1)}\right)-\frac{\Psi_1(P_1,V_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_1)}{\Psi_1(P_1,V_1)\aleph_1(T)}\right)d\varphi\\
&-\frac{\Psi_2(P_1,T_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_1)}{\Psi_2(P_1,T_1)\aleph_1(T)}\right)d\varphi-\frac{\Psi_3(P_1,E_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\\
&\times\chi\left(\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_1)}{\Psi_3(P_1,E_1)\aleph_1(T)}\right)d\varphi-\frac{\Psi_1(P_1,V_1)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\chi\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_1)}{\aleph_1(T_1)\aleph_3(V)}\right)d\varphi\\
&-\frac{\Psi_3(P_1,E_1)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\chi\left(\frac{\aleph_1(T_\varphi)\aleph_2(E_1)}{\aleph_1(T_1)\aleph_2(E)}\right)d\varphi-\frac{\Psi_1(P_1,V_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\\
&\times\chi\left(\frac{\Psi_1(P,V_1)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_1)}\right)d\varphi-\frac{\Psi_2(P_1,T_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_1)\Psi_2(P_1,T_1)\aleph_1(T)}{\Psi_1(P_1,V_1)\Psi_2(P,T)\aleph_1(T_1)}\right)d\varphi\\
&-\frac{\Psi_{.3}(P_1,E_1)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_1)\Psi_3(P_1,E_1)\aleph_2(E)}{\Psi_1(P_1,V_1)\Psi_3(P,E)\aleph_2(E_1)}\right)d\varphi\\
&+\Psi_1(P_1,V_1)\left(1-\frac{\Psi_1(P,V_1)}{\Psi_1(P,V)}\right)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_1)}-\frac{\aleph_3(V)}{\aleph_3(V_1)}\right)\\
&+\Psi_2(P_1,T_1)\left(1-\frac{\mathfrak{F}_1^T(P_1,T_1)}{\mathfrak{F}_1^T(P,T)}\right)\left(\frac{\mathfrak{F}_1^T(P,T)}{\mathfrak{F}_1^T(P_1,T_1)}-\frac{\aleph_1(T)}{\aleph_1(T_1)}\right)
\end{aligned}
$$

$$+\Psi_3(P_1,E_1)\left(1-\frac{\mathfrak{F}_1^E(P_1,E_1)}{\mathfrak{F}_1^E(P,E)}\right)\left(\frac{\mathfrak{F}_1^E(P,E)}{\mathfrak{F}_1^E(P_1,E_1)}-\frac{\aleph_2(E)}{\aleph_2(E_1)}\right)$$
$$+\frac{\lambda}{\mathscr{B}_1}(\aleph_1(T_1)-\aleph_1(T_2))\aleph_4\left(Z^T\right)+\frac{\Psi_1(P_1,V_1)\psi}{\xi_V\aleph_3(V_1)}(\aleph_3(V_1)-\aleph_3(V_3))\aleph_5\left(Z^V\right).$$

If $\Re_1\le 0$ and $\Re_2\le 1$ then Đ$_1$ does not exist because $Z_2^T=\aleph_4^{-1}\left(\frac{(\mu_1+\xi_T)}{\lambda\mathscr{B}_1}[\Re_1-1]\right)\le 0$ and $Z_3^V=\aleph_5^{-1}\left(\frac{\xi_V}{\psi}[\Re_2-1]\right)\le 0$. Thus

$$\dot{Z}^T=\sigma\left(\aleph_1(T)-\frac{\xi_{Z^T}}{\sigma}\right)\aleph_4\left(Z^T\right)=\sigma\left(\aleph_1(T)-\aleph_1(T_2)\right)\aleph_4\left(Z^T\right)\text{ for all }Z^T>0,$$
$$\dot{Z}^V=\varpi\left(\aleph_3(V)-\frac{\xi_{Z^V}}{\varpi}\right)\aleph_5\left(Z^V\right)=\varpi\left(\aleph_3(V)-\aleph_3(V_3)\right)\aleph_5\left(Z^V\right)\text{ for all }Z^V>0,$$

which implies that $\aleph_1(T)\le\aleph_1(T_2)$ and $\aleph_3(V)\le\aleph_3(V_3)$. Therefore, $\frac{d\Phi_1}{dt}\le 0$ for all $P,T,E,V,Z^T,Z^V>0$. Moreover, $\frac{d\Phi_1}{dt}=0$ when $P=P_1$, $T(t)=T_1$, $E(t)=E_1$, $V(t)=V_1$, $\chi(.)=0$, $Z^T=0$ and $Z^V=0$. The solutions of model (3)-(8) converge to Γ_1', where $P(t)=P_1$, $T(t)=T_1$, $E(t)=E_1$, $V(t)=V_1$, $Z^T=0=Z^V=0$ and this yields that Γ_1' ={Đ$_1$}, and from L.I.P. we obtain that Đ$_1$ is G.A.S.□

Theorem 3. If $\Re_1>1$ and $\Re_4\le 0$ conditions **(H1)-(H4), (H6)** Đ$_2$ be satisfied, then is (G.A.S).

Proof. Construct a Lyapunov functional as:

$$\Phi_2=P-P_2-\int_{P_2}^{P}\frac{\Psi_1(P_2,V_2)}{\Psi_1(\theta,V_2)}d\theta+\frac{1}{\mathscr{B}_1}\left(T-T_2-\int_{T_2}^{T}\frac{\aleph_1(T_2)}{\aleph_1(\theta)}d\theta\right)$$
$$+\frac{\Psi_3(P_2,E_2)}{\xi_E\aleph_2(E_2)}\left(E-E_2-\int_{E_2}^{E}\frac{\aleph_2(E_2)}{\aleph_2(\theta)}d\theta\right)+\frac{\Psi_1(P_2,V_2)}{\xi_V\aleph_3(V_2)}\left(V-V_2-\int_{V_2}^{V}\frac{\aleph_3(V_2)}{\aleph_3(\theta)}d\theta\right)$$
$$+\frac{\lambda}{\sigma\mathscr{B}_1}\left(Z^T-Z_2^T-\int_{Z_2^T}^{Z^T}\frac{\aleph_4\left(Z_2^T\right)}{\aleph_4(\theta)}d\theta\right)+\frac{\psi\Psi_1(P_2,V_2)}{\varpi\xi_V\aleph_3(V_2)}Z^V$$
$$+\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_1(P(\phi),V(\phi))}{\Psi_1(P_2,V_2)}\right]d\phi d\varphi$$
$$+\frac{\Psi_2(P_2,T_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_2(P(\phi),T(\phi))}{\Psi_2(P_2,T_2)}\right]d\phi d\varphi$$
$$+\frac{\Psi_{.3}(P_2,E_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_3(P(\phi),E(\phi))}{\Psi_3(P_2,E_2)}\right]d\phi d\varphi$$
$$+\frac{\mu_2\Psi_3(P_2,E_2)\aleph_1(T_2)}{\xi_E\aleph_2(E_2)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\aleph_1(T(\phi))}{\aleph_1(T_2)}\right]d\phi d\varphi$$
$$+\frac{\beta\Psi_1(P_2,V_2)\aleph_1(T_2)}{\xi_V\aleph_3(V_2)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\aleph_1(T(\phi))}{\aleph_1(T_2)}\right]d\phi d\varphi.$$

We calculate $\frac{d\Phi_2}{dt}$ and collecting the terms, we obtain

$$\frac{d\Phi_2}{dt}=¥(P)\left(1-\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}\right)+\Psi_1(P,V)\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}+\Psi_2(P,T)\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}$$

$$
\begin{aligned}
&+\Psi_3(P,E)\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}-\frac{1}{\mathscr{B}_1}\frac{\aleph_1(T_2)}{\aleph_1(T)}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left[\Psi_1\left(P_\varphi,V_\varphi\right)+\Psi_2\left(P_\varphi,T_\varphi\right)+\Psi_3\left(P_\varphi,E_\varphi\right)\right]d\varphi\\
&-\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T)+\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T_2)+\frac{1}{\mathscr{B}_1}\lambda\aleph_4\left(Z^T\right)\aleph_1(T_2)\\
&-\frac{\Psi_3(P_2,E_2)}{\xi_E\aleph_2(E_2)}\frac{\aleph_2(E_2)}{\aleph_2(E)}\mu_2\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1\left(T_\varphi\right)d\varphi-\Psi_3(P_2,E_2)\frac{\aleph_2(E)}{\aleph_2(E_2)}+\Psi_3(P_2,E_2)\\
&-\frac{\Psi_1(P_2,V_2)}{\xi_V\aleph_3(V_2)}\frac{\aleph_3(V_2)}{\aleph_3(V)}\beta\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1\left(T_\varphi\right)d\varphi-\Psi_1(P_2,V_2)\frac{\aleph_3(V)}{\aleph_3(V_2)}+\Psi_1(P_2,V_2)\\
&+\frac{\Psi_1(P_2,V_2)}{\xi_V}\psi\aleph_5\left(Z^V\right)-\frac{\lambda}{\mathscr{B}_1}\aleph_1(T)\aleph_4\left(Z_2^T\right)-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right)+\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z_2^T\right)\\
&-\frac{\psi\Psi_1(P_2,V_2)}{\varpi\xi_V\aleph_3(V_2)}\xi_{Z^V}\aleph_5\left(Z^V\right)+\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_1\left(P_\varphi,V_\varphi\right)}{\Psi_1(P,V)}\right)d\varphi\\
&+\frac{\Psi_2(P_2,T_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_2\left(P_\varphi,T_\varphi\right)}{\Psi_2(P,T)}\right)d\varphi+\frac{\Psi_{.3}(P_2,E_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_3\left(P_\varphi,E_\varphi\right)}{\Psi_3(P,E)}\right)d\varphi\\
&+\frac{\mu_2\Psi_3(P_2,E_2)}{\xi_E\aleph_2(E_2)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1(T)d\varphi+\frac{\mu_2\Psi_3(P_2,E_2)\aleph_1(T_2)}{\xi_E\aleph_2(E_2)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\ln\left(\frac{\aleph_1\left(T_\varphi\right)}{\aleph_1(T)}\right)d\varphi\\
&+\frac{\beta\Psi_1(P_2,V_2)}{\xi_V\aleph_3(V_2)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1(T)d\varphi+\frac{\beta\Psi_1(P_2,V_2)\aleph_1(T_2)}{\xi_V\aleph_3(V_2)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\ln\left(\frac{\aleph_1\left(T_\varphi\right)}{\aleph_1(T)}\right)d\varphi.
\end{aligned}
$$

Using the following conditions for steady state $Đ_2$:

$$
\begin{gathered}
¥(P_2)=\Psi_1(P_2,V_2)+\Psi_2(P_2,T_2)+\Psi_3(P_2,E_2),\\
(\mu_1+\xi_T)\aleph_1(T_2)=\mathscr{B}_1[\Psi_1(P_2,V_2)+\Psi_2(P_2,T_2)+\Psi_3(P_2,E_2)]-\lambda\aleph_1(T_2)\aleph_4\left(Z_2^T\right),\\
\frac{\aleph_2(E_2)}{\aleph_1(T_2)}=\mathscr{B}_2\frac{\mu_2}{\xi_E},\quad\frac{\aleph_3(V_2)}{\aleph_1(T_2)}=\mathscr{B}_3\frac{\beta}{\xi_V},\quad\aleph_1(T_2)=\frac{\xi_{Z^T}}{\sigma},
\end{gathered}
$$

then, we obtain

$$
\begin{aligned}
\frac{d\Phi_2}{dt}&=(¥(P)-¥(P_2))\left(1-\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}\right)+[\Psi_1(P_2,V_2)+\Psi_2(P_2,T_2)+\Psi_3(P_2,E_2)]\\
&\times\left(1-\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}\right)+\Psi_1(P_2,V_2)\frac{\Psi_1(P,V)}{\Psi_1(P,V_2)}+\Psi_2(P_2,T_2)\frac{\Psi_2(P,T)\Psi_1(P_2,V_2)}{\Psi_2(P_2,T_2)\Psi_1(P,V_2)}\\
&+\Psi_3(P_2,E_2)\frac{\Psi_3(P,E)\Psi_1(P_2,V_2)}{\Psi_3(P_2,E_2)\Psi_1(P,V_2)}-\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_1\left(P_\varphi,V_\varphi\right)\aleph_1(T_2)}{\Psi_1(P_2,V_2)\aleph_1(T)}d\varphi\\
&-\frac{\Psi_2(P_2,T_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_2\left(P_\varphi,T_\varphi\right)\aleph_1(T_2)}{\Psi_2(P_2,T_2)\aleph_1(T)}d\varphi-\frac{\Psi_3(P_2,E_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\\
&\times\frac{\Psi_3\left(P_\varphi,E_\varphi\right)\aleph_1(T_2)}{\Psi_3(P_2,E_2)\aleph_1(T)}d\varphi-\Psi_2(P_2,V_2)\frac{\aleph_1(T)}{\aleph_1(T_2)}+\Psi_1(P_2,V_2)+\Psi_2(P_2,T_2)\\
&+\Psi_3(P_2,E_2)-\frac{\Psi_3(P_2,E_2)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\frac{\aleph_1\left(T_\varphi\right)\aleph_2(E_2)}{\aleph_1(T_2)\aleph_2(E)}d\varphi-\Psi_3(P_2,E_2)\frac{\aleph_2(E)}{\aleph_2(E_2)}
\end{aligned}
$$

$$+\Psi_3(P_2,E_2)-\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\frac{\aleph_1(T_\varphi)\aleph_3(V_2)}{\aleph_1(T_2)\aleph_3(V)}d\varphi-\Psi_1(P_2,V_2)\frac{\aleph_3(V)}{\aleph_3(V_2)}$$
$$+\Psi_1(P_2,V_2)+\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)}{\Psi_1(P,V)}\right)d\varphi+\frac{\Psi_2(P_2,T_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)$$
$$\times\ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)}{\Psi_2(P,T)}\right)d\varphi+\frac{\Psi_{.3}(P_2,E_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)}{\Psi_3(P,E)}\right)d\varphi$$
$$+\frac{\Psi_3(P_2,E_2)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi+\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)$$
$$\times\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi+\frac{\psi\Psi_1(P_2,V_2)}{\xi_V\aleph_3(V_2)}\left(\aleph_3(V_2)-\frac{\xi_{Z^V}}{\varpi}\right)\aleph_5\left(Z^V\right).$$

Utilizing Equalities (32) for $i=2$, we obtain

$$\frac{d\Phi_2}{dt}=(¥(P)-¥(P_2))\left(1-\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}\right)-\left[\Psi_1(P_2,V_2)+\Psi_2(P_2,T_2)+\Psi_3(P_2,E_2)\right]$$
$$\times\left(\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}-1-\ln\left(\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}\right)\right)$$
$$-\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_2)}{\Psi_1(P_2,V_2)\aleph_1(T)}-1-\ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_2)}{\Psi_1(P_2,V_2)\aleph_1(T)}\right)\right)d\varphi$$
$$-\frac{\Psi_2(P_2,T_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_2)}{\Psi_2(P_2,T_2)\aleph_1(T)}-1-\ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_2)}{\Psi_2(P_2,T_2)\aleph_1(T)}\right)\right)d\varphi$$
$$-\frac{\Psi_3(P_2,E_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_2)}{\Psi_3(P_2,E_2)\aleph_1(T)}-1-\ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_2)}{\Psi_3(P_2,E_2)\aleph_1(T)}\right)\right)d\varphi$$
$$-\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_2)}{\aleph_1(T_2)\aleph_3(V)}-1-\ln\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_2)}{\aleph_1(T_2)\aleph_3(V)}\right)\right)d\varphi$$
$$-\frac{\Psi_3(P_2,E_2)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\left(\frac{\aleph_1(T_\varphi)\aleph_2(E_2)}{\aleph_1(T_2)\aleph_2(E)}-1-\ln\frac{\aleph_1(T_\varphi)\aleph_2(E_2)}{\aleph_1(T_2)\aleph_2(E)}\right)d\varphi$$
$$-\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_2)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_2)}-1-\ln\left(\frac{\Psi_1(P,V_2)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_2)}\right)\right)d\varphi$$
$$-\frac{\Psi_2(P_2,T_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_2)\Psi_2(P_2,T_2)\aleph_1(T)}{\Psi_1(P_2,V_2)\Psi_2(P,T)\aleph_1(T_2)}-1-\ln\left(\frac{\Psi_1(P,V_2)\Psi_2(P_2,T_2)\aleph_1(T)}{\Psi_1(P_2,V_2)\Psi_2(P,T)\aleph_1(T_2)}\right)\right)d\varphi$$
$$-\frac{\Psi_{.3}(P_2,E_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_2)\Psi_3(P_2,E_2)\aleph_2(E)}{\Psi_1(P_2,V_2)\Psi_3(P,E)\aleph_2(E_2)}-1-\ln\left(\frac{\Psi_1(P,V_2)\Psi_3(P_2,E_2)\aleph_2(E)}{\Psi_1(P_2,V_2)\Psi_3(P,E)\aleph_2(E_2)}\right)\right)d\varphi$$
$$+\Psi_1(P_2,V_2)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_2)}-\frac{\aleph_3(V)}{\aleph_3(V_2)}-1+\frac{\Psi_1(P,V_2)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_2)}\right)$$
$$+\Psi_2(P_2,T_2)\left(\frac{\Psi_2(P,T)\Psi_1(P_2,V_2)}{\Psi_2(P_2,T_2)\Psi_1(P,V_2)}-\frac{\aleph_1(T)}{\aleph_1(T_2)}-1+\frac{\Psi_1(P,V_2)\Psi_2(P_2,T_2)\aleph_1(T)}{\Psi_1(P_2,V_2)\Psi_2(P,T)\aleph_1(T_2)}\right)$$
$$+\Psi_3(P_2,E_2)\left(\frac{\Psi_3(P,E)\Psi_1(P_2,V_2)}{\Psi_3(P_2,E_2)\Psi_1(P,V_2)}-\frac{\aleph_2(E)}{\aleph_2(E_2)}-1+\frac{\Psi_1(P,V_2)\Psi_3(P_2,E_2)\aleph_2(E)}{\Psi_1(P_2,V_2)\Psi_3(P,E)\aleph_2(E_2)}\right)$$
$$+\frac{\psi\Psi_1(P_2,V_2)}{\xi_V\aleph_3(V_2)}\left(\aleph_3(V_2)-\frac{\xi_{Z^V}}{\varpi}\right)\aleph_5\left(Z^V\right).$$

Using the definition of $\mathfrak{F}_i^T(P,T)$ and $\mathfrak{F}_i^E(P,E)$ given in condition (**H6**) and **Remark 1** from Eqs. (30)-(31) in case of $i=2$, we obtain

$$\begin{aligned}
\frac{d\Phi_2}{dt} &= (\yen(P)-\yen(P_2))\left(1-\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}\right)-[\Psi_1(P_2,V_2)+\Psi_2(P_2,T_2)+\Psi_3(P_2,E_2)] \\
&\times\chi\left(\frac{\Psi_1(P_2,V_2)}{\Psi_1(P,V_2)}\right)-\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_2)}{\Psi_1(P_2,V_2)\aleph_1(T)}\right)d\varphi \\
&-\frac{\Psi_2(P_2,T_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_2)}{\Psi_2(P_2,T_2)\aleph_1(T)}\right)d\varphi-\frac{\Psi_3(P_2,E_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi) \\
&\times\chi\left(\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_2)}{\Psi_3(P_2,E_2)\aleph_1(T)}\right)d\varphi-\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\chi\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_2)}{\aleph_1(T_2)\aleph_3(V)}\right)d\varphi \\
&-\frac{\Psi_3(P_2,E_2)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\chi\left(\frac{\aleph_1(T_\varphi)\aleph_2(E_2)}{\aleph_1(T_2)\aleph_2(E)}\right)d\varphi-\frac{\Psi_1(P_2,V_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi) \\
&\times\chi\left(\frac{\Psi_1(P,V_2)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_2)}\right)d\varphi-\frac{\Psi_2(P_2,T_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_2)\Psi_2(P_2,T_2)\aleph_1(T)}{\Psi_1(P_2,V_2)\Psi_2(P,T)\aleph_1(T_2)}\right)d\varphi \\
&-\frac{\Psi_{.3}(P_2,E_2)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_2)\Psi_3(P_2,E_2)\aleph_2(E)}{\Psi_1(P_2,V_2)\Psi_3(P,E)\aleph_2(E_2)}\right)d\varphi \\
&+\Psi_1(P_2,V_2)\left(1-\frac{\Psi_1(P,V_2)}{\Psi_1(P,V)}\right)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_2)}-\frac{\aleph_3(V)}{\aleph_3(V_2)}\right) \\
&+\Psi_2(P_2,T_2)\left(1-\frac{\mathfrak{F}_2^T(P_2,T_2)}{\mathfrak{F}_2^T(P,T)}\right)\left(\frac{\mathfrak{F}_2^T(P,T)}{\mathfrak{F}_2^T(P_2,T_2)}-\frac{\aleph_1(T)}{\aleph_1(T_2)}\right) \\
&+\Psi_3(P_2,E_2)\left(1-\frac{\mathfrak{F}_2^E(P_2,E_2)}{\mathfrak{F}_2^E(P,E)}\right)\left(\frac{\mathfrak{F}_2^E(P,E)}{\mathfrak{F}_2^E(P_2,E_2)}-\frac{\aleph_2(E)}{\aleph_2(E_2)}\right) \\
&+\frac{\psi\Psi_1(P_2,V_2)}{\xi_V\aleph_3(V_2)}(\aleph_3(V_2)-\aleph_3(V_4))\aleph_5(Z^V).
\end{aligned}$$

If $\Re_1>0$ and $\Re_4\leq 1$ then Đ$_2$ does not exist because and $Z_4^V=\aleph_5^{-1}\left(\frac{\xi_V}{\psi}[\Re_4-1]\right)\leq 0$.Thus

$$\dot{Z}^V=\varpi\left(\aleph_3(V)-\frac{\xi_{Z^V}}{\varpi}\right)\aleph_5(Z^V)=\varpi(\aleph_3(V)-\aleph_3(V_4))\aleph_5(Z^V)\text{ for all }Z^V>0,$$

which implies that $\aleph_3(V_2)\leq\aleph_3(V_4)$. Therefore, $\frac{d\Phi_2}{dt}\leq 0$ for all $P,T,E,V,Z^T,Z^V>0$. Moreover, $\frac{d\Phi_2}{dt}=0$ when $P=P_2$, $T(t)=T_2$, $E(t)=E_2$, $V(t)=V_2$, $\chi(.)=0$, $Z^T(t)=Z_2^T$ and $Z^V=0$. The solutions of model (3)-(8) converge to Γ_2', where $P=P_2$, $T(t)=T_2$, $E(t)=E_2$, $V(t)=V_2$, $\chi(.)=0$, $Z^T(t)=Z_2^T$, $Z^V=0$ and this yields that Γ_2' ={Đ$_2$}, and from L.I.P. we obtain that Đ$_2$ is G.A.S.□

Theorem 4. If $\Re_2>1$ and $\Re_3\leq 0$ conditions **(H1)-(H4), (H6)** Đ$_3$ be satisfied, then is (G.A.S).

Proof. Construct a Lyapunov functional as:

$$\Phi_3=P-P_3-\int_{P_3}^{P}\frac{\Psi_1(P_3,V_3)}{\Psi_1(\theta,V_3)}d\theta+\frac{1}{\mathscr{B}_1}\left(T-T_3-\int_{T_3}^{T}\frac{\aleph_1(T_3)}{\aleph_1(\theta)}d\theta\right)$$

$$
\begin{aligned}
&+\frac{\Psi_3(P_3,E_3)}{\xi_E\aleph_2(E_3)}\left(E-E_3-\int_{E_3}^{E}\frac{\aleph_2(E_3)}{\aleph_2(\theta)}d\theta\right)\\
&+\frac{\Psi_1(P_3,V_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}\left(V-V_3-\int_{V_3}^{V}\frac{\aleph_3(V_3)}{\aleph_3(\theta)}d\theta\right)+\frac{\lambda}{\sigma\mathscr{B}_1}Z^T\\
&+\frac{\psi\Psi_1(P_3,V_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\varpi\aleph_3(V_3)}\left(Z^V-Z_3^V-\int_{Z_3^V}^{Z^V}\frac{\aleph_4\left(Z_3^V\right)}{\aleph_4(\theta)}d\theta\right)\\
&+\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_1(P(\phi),V(\phi))}{\Psi_1(P_3,V_3)}\right]d\phi d\varphi\\
&+\frac{\Psi_2(P_3,T_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_2(P(\phi),T(\phi))}{\Psi_2(P_3,T_3)}\right]d\phi d\varphi\\
&+\frac{\Psi_{.3}(P_3,E_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_3(P(\phi),E(\phi))}{\Psi_3(P_3,E_3)}\right]d\phi d\varphi\\
&+\frac{\mu_2\Psi_3(P_3,E_3)\aleph_1(T_3)}{\xi_E\aleph_2(E_3)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\aleph_1(T(\phi))}{\aleph_1(T_3)}\right]d\phi d\varphi\\
&+\frac{\beta\Psi_1(P_3,V_3)\aleph_1(T_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\aleph_1(T(\phi))}{\aleph_1(T_3)}\right]d\phi d\varphi.
\end{aligned}
$$

We calculate $\frac{d\Phi_3}{dt}$ and collecting the terms, we obtain

$$
\begin{aligned}
\frac{d\Phi_3}{dt}&=¥(P)\left(1-\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}\right)+\Psi_1(P,V)\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}+\Psi_2(P,T)\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}\\
&+\Psi_3(P,E)\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}-\frac{1}{\mathscr{B}_1}\frac{\aleph_1(T_3)}{\aleph_1(T)}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left[\Psi_1\left(P_\varphi,V_\varphi\right)+\Psi_2\left(P_\varphi,T_\varphi\right)+\Psi_3\left(P_\varphi,E_\varphi\right)\right]d\varphi\\
&-\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T)+\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T_3)+\frac{1}{\mathscr{B}_1}\lambda\aleph_1(T_3)\aleph_4\left(Z^T\right)\\
&-\frac{\Psi_3(P_3,E_3)}{\xi_E\aleph_2(E_3)}\frac{\aleph_2(E_3)}{\aleph_2(E)}\mu_2\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1\left(T_\varphi\right)d\varphi-\Psi_3(P_3,E_3)\frac{\aleph_2(E)}{\aleph_2(E_3)}\\
&+\Psi_3(P_3,E_3)-\frac{\Psi_1(P_3,V_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}\frac{\aleph_3(V_3)}{\aleph_3(V)}\beta\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1\left(T_\varphi\right)d\varphi\\
&-\frac{\Psi_1(P_3,V_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}\xi_V\aleph_3(V)+\frac{\Psi_1(P_3,V_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)}\xi_V\\
&+\frac{\Psi_1(P_3,V_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)}\psi\aleph_5\left(Z^V\right)-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right)-\frac{\psi\Psi_1(P_3,V_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}\aleph_3(V)\aleph_5\left(Z_3^V\right)\\
&-\frac{\xi_{Z^V}\psi\Psi_1(P_3,V_3)}{\varpi\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}\aleph_5\left(Z^V\right)+\frac{\psi\Psi_1(P_3,V_3)}{\varpi\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}\xi_{Z^V}\aleph_5\left(Z_3^V\right)\\
&+\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_1\left(P_\varphi,V_\varphi\right)}{\Psi_1(P,V)}\right)d\varphi+\frac{\Psi_2(P_3,T_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_2\left(P_\varphi,T_\varphi\right)}{\Psi_2(P,T)}\right)d\varphi\\
&+\frac{\Psi_{.3}(P_3,E_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_3\left(P_\varphi,E_\varphi\right)}{\Psi_3(P,E)}\right)d\varphi+\frac{\mu_2\Psi_3(P_3,E_3)}{\xi_E\aleph_2(E_3)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1(T)d\varphi
\end{aligned}
$$

$$+\frac{\mu_2\Psi_3(P_3,E_3)\aleph_1(T_3)}{\xi_E\aleph_2(E_3)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi+\frac{\beta\Psi_1(P_3,V_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}$$
$$\times\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1(T)\,d\varphi+\frac{\beta\Psi_1(P_3,V_3)\aleph_1(T_3)}{\left(\xi_V+\psi\aleph_5\left(Z_3^V\right)\right)\aleph_3(V_3)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi.$$

Using the following conditions for steady state Đ$_3$:

$$Ұ(P_3)=\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3),$$
$$(\mu_1+\xi_T)\aleph_1(T_3)=\mathscr{B}_1\left[\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3)\right],$$
$$\frac{\aleph_2(E_3)}{\aleph_1(T_3)}=\mathscr{B}_2\frac{\mu_2}{\xi_E},\xi_V+\psi\aleph_5\left(Z_3^V\right)=\frac{\beta\mathscr{B}_3\aleph_1(T_3)}{\aleph_3(V_3)},\aleph_1(V_3)=\frac{\xi_{Z^V}}{\varpi},$$

then, we obtain

$$\frac{d\Phi_3}{dt}=(Ұ(P)-Ұ(P_3))\left(1-\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}\right)+\left[\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3)\right]$$
$$\times\left(1-\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}\right)+\Psi_1(P_3,V_3)\frac{\Psi_1(P,V)}{\Psi_1(P,V_3)}+\Psi_2(P_3,T_3)\frac{\Psi_2(P,T)\Psi_1(P_3,V_3)}{\Psi_2(P_3,T_3)\Psi_1(P,V_3)}$$
$$+\Psi_3(P_3,E_3)\frac{\Psi_3(P,E)\Psi_1(P_3,V_3)}{\Psi_3(P_3,E_3)\Psi_1(P,V_3)}-\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_1\left(P_\varphi,V_\varphi\right)\aleph_1(T_3)}{\Psi_1(P_3,V_3)\aleph_1(T)}d\varphi$$
$$-\frac{\Psi_2(P_3,T_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_2\left(P_\varphi,T_\varphi\right)\aleph_1(T_3)}{\Psi_2(P_3,T_3)\aleph_1(T)}d\varphi-\frac{\Psi_3(P_3,E_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_3\left(P_\varphi,E_\varphi\right)\aleph_1(T_3)}{\Psi_3(P_3,E_3)\aleph_1(T)}d\varphi$$
$$-\Psi_2(P_3,T_3)\frac{\aleph_1(T)}{\aleph_1(T_3)}+\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3)+\frac{\lambda}{\mathscr{B}_1}\aleph_1(T_3)\aleph_4\left(Z^T\right)$$
$$-\frac{\Psi_3(P_3,E_3)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\frac{\aleph_1\left(T_\varphi\right)\aleph_2(E_3)}{\aleph_1(T_3)\aleph_2(E)}d\varphi-\Psi_3(P_3,E_3)\frac{\aleph_2(E)}{\aleph_2(E_3)}+\Psi_3(P_3,E_3)$$
$$-\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\frac{\aleph_1\left(T_\varphi\right)\aleph_3(V_3)}{\aleph_1(T_3)\aleph_3(V)}d\varphi-\Psi_1(P_3,V_3)\frac{\aleph_3(V)}{\aleph_3(V_3)}+\Psi_1(P_3,V_3)$$
$$-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right)+\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_1\left(P_\varphi,V_\varphi\right)}{\Psi_1(P,V)}\right)d\varphi$$
$$+\frac{\Psi_2(P_3,T_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_2\left(P_\varphi,T_\varphi\right)}{\Psi_2(P,T)}\right)d\varphi+\frac{\Psi_{.3}(P_3,E_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_3\left(P_\varphi,E_\varphi\right)}{\Psi_3(P,E)}\right)d\varphi$$
$$+\frac{\Psi_3(P_3,E_3)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\ln\left(\frac{\aleph_1\left(T_\varphi\right)}{\aleph_1(T)}\right)d\varphi+\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\ln\left(\frac{\aleph_1\left(T_\varphi\right)}{\aleph_1(T)}\right)d\varphi.$$

Utilizing Equalities (32) for $i=3$, we obtain

$$\frac{d\Phi_3}{dt}=(Ұ(P)-Ұ(P_3))\left(1-\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}\right)-\left[\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3)\right]$$
$$\times\left(\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}-1-\ln\left(\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}\right)\right)$$

$$
\begin{aligned}
&-\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P_\varphi,V_\varphi)\,\aleph_1(T_3)}{\Psi_1(P_3,V_3)\,\aleph_1(T)}-1-\ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)\,\aleph_1(T_3)}{\Psi_1(P_3,V_3)\,\aleph_1(T)}\right)\right)d\varphi\\
&-\frac{\Psi_2(P_3,T_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_2(P_\varphi,T_\varphi)\,\aleph_1(T_3)}{\Psi_2(P_3,T_3)\,\aleph_1(T)}-1-\ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)\,\aleph_1(T_3)}{\Psi_2(P_3,T_3)\,\aleph_1(T)}\right)\right)d\varphi\\
&-\frac{\Psi_3(P_3,E_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_3(P_\varphi,E_\varphi)\,\aleph_1(T_3)}{\Psi_3(P_3,E_3)\,\aleph_1(T)}-1-\ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)\,\aleph_1(T_3)}{\Psi_3(P_3,E_3)\,\aleph_1(T)}\right)\right)d\varphi\\
&-\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\left(\frac{\aleph_1(T_\varphi)\,\aleph_3(V_3)}{\aleph_1(T_3)\,\aleph_3(V)}-1-\ln\left(\frac{\aleph_1(T_\varphi)\,\aleph_3(V_3)}{\aleph_1(T_3)\,\aleph_3(V)}\right)\right)d\varphi\\
&-\frac{\Psi_3(P_3,E_3)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\left(\frac{\aleph_1(T_\varphi)\,\aleph_2(E_3)}{\aleph_1(T_3)\,\aleph_2(E)}-1-\ln\left(\frac{\aleph_1(T_\varphi)\,\aleph_2(E_3)}{\aleph_1(T_3)\,\aleph_2(E)}\right)\right)d\varphi\\
&-\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_3)\,\aleph_3(V)}{\Psi_1(P,V)\,\aleph_3(V_3)}-1-\ln\left(\frac{\Psi_1(P,V_3)\,\aleph_3(V)}{\Psi_1(P,V)\,\aleph_3(V_3)}\right)\right)d\varphi\\
&-\frac{\Psi_2(P_3,T_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_3)\Psi_2(P_3,T_3)\,\aleph_1(T)}{\Psi_1(P_3,V_3)\Psi_2(P,T)\,\aleph_1(T_3)}-1-\ln\left(\frac{\Psi_1(P,V_3)\Psi_2(P_3,T_3)\,\aleph_1(T)}{\Psi_1(P_3,V_3)\Psi_2(P,T)\,\aleph_1(T_3)}\right)\right)d\varphi\\
&-\frac{\Psi_{.3}(P_3,E_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_3)\Psi_3(P_3,E_3)\,\aleph_2(E)}{\Psi_1(P_3,V_3)\Psi_3(P,E)\,\aleph_2(E_3)}-1-\ln\left(\frac{\Psi_1(P,V_3)\Psi_3(P_3,E_3)\,\aleph_2(E)}{\Psi_1(P_3,V_3)\Psi_3(P,E)\,\aleph_2(E_3)}\right)\right)d\varphi\\
&+\Psi_1(P_3,V_3)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_3)}-\frac{\aleph_3(V)}{\aleph_3(V_3)}-1+\frac{\Psi_1(P,V_3)\,\aleph_3(V)}{\Psi_1(P,V)\,\aleph_3(V_3)}\right)\\
&+\Psi_2(P_3,T_3)\left(\frac{\Psi_2(P,T)\Psi_1(P_3,V_3)}{\Psi_2(P_3,T_3)\Psi_1(P,V_3)}-1-\frac{\aleph_1(T)}{\aleph_1(T_3)}+\frac{\Psi_1(P,V_3)\Psi_2(P_3,T_3)\,\aleph_1(T)}{\Psi_1(P_3,V_3)\Psi_2(P,T)\,\aleph_1(T_3)}\right)\\
&+\Psi_3(P_3,E_3)\left(\frac{\Psi_3(P,E)\Psi_1(P_3,V_3)}{\Psi_3(P_3,E_3)\Psi_1(P,V_3)}-1-\frac{\aleph_2(E)}{\aleph_2(E_3)}+\frac{\Psi_1(P,V_3)\Psi_3(P_3,E_3)\,\aleph_2(E)}{\Psi_1(P_3,V_3)\Psi_3(P,E)\,\aleph_2(E_3)}\right)\\
&+\frac{\lambda}{\mathscr{B}_1}\left(\aleph_1(T_3)-\frac{\xi_{Z^T}}{\sigma}\right)\aleph_4\left(Z^T\right).
\end{aligned}
$$

Using the definition of $\mathfrak{F}_i^T(P,T)$ and $\mathfrak{F}_i^E(P,E)$ given in condition (**H6**) and **Remark 1** from Eqs. (30)-(31) in case of $i=3$, we obtain

$$
\begin{aligned}
\frac{d\Phi_3}{dt}&=(¥(P)-¥(P_3))\left(1-\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}\right)-\left[\Psi_1(P_3,V_3)+\Psi_2(P_3,T_3)+\Psi_3(P_3,E_3)\right]\\
&\times\chi\left(\frac{\Psi_1(P_3,V_3)}{\Psi_1(P,V_3)}\right)-\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P_\varphi,V_\varphi)\,\aleph_1(T_3)}{\Psi_1(P_3,V_3)\,\aleph_1(T)}\right)d\varphi\\
&-\frac{\Psi_2(P_3,T_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_2(P_\varphi,T_\varphi)\,\aleph_1(T_3)}{\Psi_2(P_3,T_3)\,\aleph_1(T)}\right)d\varphi\\
&-\frac{\Psi_3(P_3,E_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_3(P_\varphi,E_\varphi)\,\aleph_1(T_3)}{\Psi_3(P_3,E_3)\,\aleph_1(T)}\right)d\varphi\\
&-\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\chi\left(\frac{\aleph_1(T_\varphi)\,\aleph_3(V_3)}{\aleph_1(T_3)\,\aleph_3(V)}\right)d\varphi\\
&-\frac{\Psi_3(P_3,E_3)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\chi\left(\frac{\aleph_1(T_\varphi)\,\aleph_2(E_3)}{\aleph_1(T_3)\,\aleph_2(E)}\right)d\varphi
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Psi_1(P_3,V_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_3)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_3)}\right)d\varphi\\
&-\frac{\Psi_2(P_3,T_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_3)\Psi_2(P_3,T_3)\aleph_1(T)}{\Psi_1(P_3,V_3)\Psi_2(P,T)\aleph_1(T_3)}\right)d\varphi\\
&-\frac{\Psi_{.3}(P_3,E_3)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_3)\Psi_3(P_3,E_3)\aleph_2(E)}{\Psi_1(P_3,V_3)\Psi_3(P,E)\aleph_2(E_3)}\right)d\varphi\\
&+\Psi_1(P_3,V_3)\left(1-\frac{\Psi_1(P,V_3)}{\Psi_1(P,V)}\right)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_3)}-\frac{\aleph_3(V)}{\aleph_3(V_3)}\right)\\
&+\Psi_2(P_3,T_3)\left(1-\frac{\mathfrak{F}_3^T(P_3,T_3)}{\mathfrak{F}_3^T(P,T)}\right)\left(\frac{\mathfrak{F}_3^T(P,T)}{\mathfrak{F}_3^T(P_3,T_3)}-\frac{\aleph_1(T)}{\aleph_1(T_3)}\right)\\
&+\Psi_3(P_3,E_3)\left(1-\frac{\mathfrak{F}_3^E(P_3,E_3)}{\mathfrak{F}_3^E(P,E)}\right)\left(\frac{\mathfrak{F}_3^E(P,E)}{\mathfrak{F}_3^E(P_3,E_3)}-\frac{\aleph_2(E)}{\aleph_2(E_3)}\right)\\
&+\frac{\lambda}{\mathscr{B}_1}\left(\aleph_1(T_3)-\aleph_1(T_4)\right)\aleph_4\left(Z^T\right).
\end{aligned}
$$

If $\Re_2 > 0$ and $\Re_3 \le 1$ then $Đ_3$ does not exist because $Z_4^T = \aleph_4^{-1}\left(\frac{(\mu_1+\xi_T)\mathscr{B}_1}{\lambda}[\Re_3-1]\right) \le 0$.Thus

$$
\dot{Z}^T=\sigma\left(\aleph_1(T)-\frac{\xi_{Z^T}}{\sigma}\right)\aleph_4\left(Z^T\right)=\sigma\left(\aleph_1(T)-\aleph_1(T_4)\right)\aleph_4\left(Z^T\right)\text{ for all }Z^T>0,
$$

which implies that $\aleph_1(T_3) \le \aleph_1(T_4)$. Therefore, $\frac{d\Phi_3}{dt} \le 0$ for all $P,T,E,V,Z^T,Z^V > 0$. Moreover, $\frac{d\Phi_3}{dt} = 0$ when $P = P_3$, $T(t) = T_3$, $E(t) = E_3$, $V(t) = V_3$, $\chi(.) = 0, Z^T = 0$ and $Z^V = 0$. The solutions of model (3)-(8) converge to Γ_3', where $P(t) = P_3$, $T(t) = T_3$, $E(t) = E_3$, $V(t) = V_3$, $Z^T = 0$, $Z^V = 0$ and this yields that $\Gamma_3' = \{Đ_3\}$, and from L.I.P. we obtain that $Đ_3$ is G.A.S.□

Theorem 5. If $\Re_3 > 1$ and $\Re_4 > 0$ conditions **(H1)-(H4), (H6)** $Đ_4$ be satisfied, then is (G.A.S).

Proof. Construct a Lyapunov functional as:

$$
\begin{aligned}
\Phi_4 = {} & P-P_4-\int_{P_4}^{P}\frac{\Psi_1(P_4,V_4)}{\Psi_1(\theta,V_4)}d\theta+\frac{1}{\mathscr{B}_1}\left(T-T_4-\int_{T_4}^{T}\frac{\aleph_1(T_4)}{\aleph_1(\theta)}d\theta\right)\\
&+\frac{\Psi_3(P_4,E_4)}{\xi_E\aleph_2(E_4)}\left(E-E_4-\int_{E_4}^{E}\frac{\aleph_2(E_4)}{\aleph_2(\theta)}d\theta\right)+\frac{\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\aleph_3(V_4)}\\
&\times\left(V-V_4-\int_{V_4}^{V}\frac{\aleph_3(V_4)}{\aleph_3(\theta)}d\theta\right)+\frac{\lambda}{\sigma\mathscr{B}_1}\left(Z^T-Z_4^T-\int_{Z_4^T}^{Z^T}\frac{\aleph_4\left(Z_4^T\right)}{\aleph_4(\theta)}d\theta\right)\\
&+\frac{\psi\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\varpi\aleph_3(V_4)}\left(Z^V-Z_4^V-\int_{Z_4^V}^{Z^V}\frac{\aleph_5\left(Z_4^V\right)}{\aleph_5(\theta)}d\theta\right)\\
&+\frac{\Psi_1(P_4,V_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_1(P(\phi),V(\phi))}{\Psi_1(P_4,V_4)}\right]d\phi d\varphi\\
&+\frac{\Psi_2(P_4,T_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_2(P(\phi),T(\phi))}{\Psi_2(P_4,T_4)}\right]d\phi d\varphi\\
&+\frac{\Psi_{.3}(P_4,E_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\Psi_3(P(\phi),E(\phi))}{\Psi_3(P_4,E_4)}\right]d\phi d\varphi
\end{aligned}
$$

$$+\frac{\mu_2\Psi_3(P_4,E_4)\aleph_1(T_4)}{\xi_E\aleph_2(E_4)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\aleph_1(T(\phi))}{\aleph_1(T_4)}\right]d\phi d\varphi$$
$$+\frac{\beta\Psi_1(P_4,V_4)\aleph_1(T_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\aleph_3(V_4)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\int_{t-\varphi}^{t}\chi\left[\frac{\aleph_1(T(\phi))}{\aleph_1(T_4)}\right]d\phi d\varphi.$$

We calculate $\frac{d\Phi_4}{dt}$ and collecting terms, we get

$$\begin{aligned}
\frac{d\Phi_4}{dt}&=\yen(P)\left(1-\frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}\right)+\Psi_1(P,V)\frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}+\Psi_2(P,T)\frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}\\
&+\Psi_3(P,E)\frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}-\frac{1}{\mathscr{B}_1}\frac{\aleph_1(T_4)}{\aleph_1(T)}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left[\Psi_1\left(P_\varphi,V_\varphi\right)+\Psi_2\left(P_\varphi,T_\varphi\right)+\Psi_3\left(P_\varphi,E_\varphi\right)\right]d\varphi\\
&-\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T)+\frac{1}{\mathscr{B}_1}(\mu_1+\xi_T)\aleph_1(T_4)+\frac{1}{\mathscr{B}_1}\lambda\aleph_1(T_4)\aleph_4\left(Z^T\right)\\
&-\frac{\Psi_3(P_4,E_4)}{\xi_E\aleph_2(E_4)}\frac{\aleph_2(E_4)}{\aleph_2(E)}\mu_2\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\aleph_1\left(T_\varphi\right)d\varphi-\frac{\Psi_3(P_4,E_4)}{\aleph_2(E_4)}\aleph_2(E)+\Psi_3(P_4,E_4)\\
&-\frac{\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\aleph_3(V_4)}\frac{\aleph_3(V_4)}{\aleph_3(V)}\beta\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\aleph_1\left(T_\varphi\right)d\varphi-\frac{\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\aleph_3(V_4)}\xi_V\aleph_3(V)\\
&+\frac{\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)}\xi_V+\frac{\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\aleph_3(V_4)}\frac{\aleph_3(V_4)}{\aleph_3(V)}\psi\aleph_3(V)\aleph_5\left(Z^V\right)\\
&-\frac{\lambda}{\mathscr{B}_1}\aleph_1(T)\aleph_4\left(Z_4^T\right)-\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z^T\right)+\frac{\lambda}{\sigma\mathscr{B}_1}\xi_{Z^T}\aleph_4\left(Z_4^T\right)\\
&-\frac{\psi\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\aleph_3(V_4)}\aleph_3(V)\aleph_5\left(Z_4^V\right)-\frac{\psi\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\varpi\aleph_3(V_4)}\xi_{Z^V}\aleph_5\left(Z^V\right)\\
&+\frac{\psi\Psi_1(P_4,V_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\varpi\aleph_3(V_4)}\xi_{Z^V}\aleph_5\left(Z_4^V\right)+\frac{\Psi_1(P_4,V_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_1\left(P_\varphi,V_\varphi\right)}{\Psi_1(P,V)}\right)d\varphi\\
&+\frac{\Psi_2(P_4,T_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_2\left(P_\varphi,T_\varphi\right)}{\Psi_2(P,T)}\right)d\varphi+\frac{\Psi_{.3}(P_4,E_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_3\left(P_\varphi,E_\varphi\right)}{\Psi_3(P,E)}\right)d\varphi\\
&+\frac{\mu_2\Psi_3(P_4,E_4)\aleph_1(T_4)}{\xi_E\aleph_2(E_4)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\frac{\aleph_1(T)}{\aleph_1(T_4)}d\varphi+\frac{\mu_2\Psi_3(P_4,E_4)\aleph_1(T_4)}{\xi_E\aleph_2(E_4)}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\ln\left(\frac{\aleph_1\left(T_\varphi\right)}{\aleph_1(T)}\right)d\varphi\\
&+\frac{\beta\Psi_1(P_4,V_4)\aleph_1(T_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\aleph_3(V_4)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\frac{\aleph_1(T)}{\aleph_1(T_4)}d\varphi\\
&+\frac{\beta\Psi_1(P_4,V_4)\aleph_1(T_4)}{\left(\xi_V+\psi\aleph_5\left(Z_4^V\right)\right)\aleph_3(V_4)}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\ln\left(\frac{\aleph_1\left(T_\varphi\right)}{\aleph_1(T)}\right)d\varphi.
\end{aligned}$$

Using the following conditions for steady state Đ_4:

$$\yen(P_4)=\Psi_1(P_4,V_4)+\Psi_2(P_4,T_4)+\Psi_3(P_4,E_4),$$
$$\frac{(\mu_1+\xi_T)\aleph_1(T_4)}{\mathscr{B}_1}=\left[\Psi_1(P_4,V_4)+\Psi_2(P_4,T_4)+\Psi_3(P_4,E_4)\right]-\frac{\lambda}{\mathscr{B}_1}\aleph_1(T_4)\aleph_4\left(Z_4^T\right),$$
$$\frac{\aleph_2(E_4)}{\aleph_1(T_4)}=\mathscr{B}_2\frac{\mu_2}{\xi_E},\ \xi_V+\psi\aleph_5\left(Z_4^V\right)=\frac{\beta\mathscr{B}_3\aleph_1(T_4)}{\aleph_3(V_4)},\ \aleph_1(T_4)=\frac{\xi_{Z^T}}{\sigma},\ \aleph_3(V_4)=\frac{\xi_{Z^V}}{\varpi},$$

then, we obtain

$$
\begin{aligned}
\frac{d\Phi_4}{dt} &= (¥(P) - ¥(P_4))\left(1 - \frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}\right) + [\Psi_1(P_4,V_4) + \Psi_2(P_4,T_4) + \Psi_3(P_4,E_4)]\left(1 - \frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}\right) \\
&+ \Psi_1(P_4,V_4)\frac{\Psi_1(P,V)}{\Psi_1(P,V_4)} + \Psi_2(P_4,T_4)\frac{\Psi_2(P,T)\Psi_1(P_4,V_4)}{\Psi_2(P_4,T_4)\Psi_1(P,V_4)} + \Psi_{.3}(P_4,E_4)\frac{\Psi_3(P,E)\Psi_1(P_4,V_4)}{\Psi_{.3}(P_4,E_4)\Psi_1(P,V_4)} \\
&- \frac{\Psi_1(P_4,V_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_4)}{\Psi_1(P_4,V_4)\aleph_1(T)}d\varphi - \frac{\Psi_2(P_4,T_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_4)}{\Psi_2(P_4,T_4)\aleph_1(T)}d\varphi \\
&- \frac{\Psi_{.3}(P_4,E_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_4)}{\Psi_{.3}(P_4,E_4)\aleph_1(T)}d\varphi - \Psi_2(P_4,T_4)\frac{\aleph_1(T)}{\aleph_1(T_4)} \\
&+ [\Psi_1(P_4,V_4) + \Psi_2(P_4,T_4) + \Psi_3(P_4,E_4)] - \frac{\Psi_3(P_4,E_4)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\frac{\aleph_1(T_\varphi)\aleph_2(E_4)}{\aleph_1(T_4)\aleph_2(E)}d\varphi \\
&- \Psi_3(P_4,E_4)\frac{\aleph_2(E)}{\aleph_2(E_4)} + \Psi_3(P_4,E_4) - \frac{\Psi_1(P_4,V_4)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\frac{\aleph_1(T_\varphi)\aleph_3(V_4)}{\aleph_1(T_4)\aleph_3(V)}d\varphi \\
&- \Psi_1(P_4,V_4)\frac{\aleph_3(V)}{\aleph_3(V_4)} + \Psi_1(P_4,V_4) + \frac{\Psi_1(P_4,V_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)}{\Psi_1(P,V)}\right)d\varphi \\
&+ \frac{\Psi_2(P_4,T_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)}{\Psi_2(P,T)}\right)d\varphi + \frac{\Psi_{.3}(P_4,E_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)}{\Psi_3(P,E)}\right)d\varphi \\
&+ \frac{\Psi_3(P_4,E_4)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi + \frac{\Psi_1(P_4,V_4)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\ln\left(\frac{\aleph_1(T_\varphi)}{\aleph_1(T)}\right)d\varphi.
\end{aligned}
$$

Utilizing Equalities (32) for $i = 4$, we obtain

$$
\begin{aligned}
\frac{d\Phi_4}{dt} &= (¥(P) - ¥(P_4))\left(1 - \frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}\right) - [\Psi_1(P_4,V_4) + \Psi_2(P_4,T_4) + \Psi_3(P_4,E_4)] \\
&\times\left(\frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)} - 1 - \ln\left(\frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}\right)\right) \\
&- \frac{\Psi_1(P_4,V_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_4)}{\Psi_1(P_4,V_4)\aleph_1(T)} - 1 - \ln\left(\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_4)}{\Psi_1(P_4,V_4)\aleph_1(T)}\right)\right)d\varphi \\
&- \frac{\Psi_2(P_4,T_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_4)}{\Psi_2(P_4,T_4)\aleph_1(T)} - 1 - \ln\left(\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_4)}{\Psi_2(P_4,T_4)\aleph_1(T)}\right)\right)d\varphi \\
&- \frac{\Psi_{.3}(P_4,E_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_4)}{\Psi_{.3}(P_4,E_4)\aleph_1(T)} - 1 - \ln\left(\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_4)}{\Psi_3(P_4,E_4)\aleph_1(T)}\right)\right)d\varphi \\
&- \frac{\Psi_1(P_4,V_4)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_4)}{\aleph_1(T_4)\aleph_3(V)} - 1 - \ln\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_4)}{\aleph_1(T_4)\aleph_3(V)}\right)\right)d\varphi \\
&- \frac{\Psi_3(P_4,E_4)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\left(\frac{\aleph_1(T_\varphi)\aleph_2(E_4)}{\aleph_1(T_4)\aleph_2(E)} - 1 - \ln\frac{\aleph_1(T_\varphi)\aleph_2(E_4)}{\aleph_1(T_4)\aleph_2(E)}\right)d\varphi \\
&- \frac{\Psi_1(P_4,V_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_4)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_4)} - 1 - \ln\left(\frac{\Psi_1(P,V_4)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_4)}\right)\right)d\varphi \\
&- \frac{\Psi_2(P_4,T_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_4)\Psi_2(P_4,T_4)\aleph_1(T)}{\Psi_1(P_4,V_4)\Psi_2(P,T)\aleph_1(T_4)} - 1 - \ln\left(\frac{\Psi_1(P,V_4)\Psi_2(P_4,T_4)\aleph_1(T)}{\Psi_1(P_4,V_4)\Psi_2(P,T)\aleph_1(T_4)}\right)\right)d\varphi
\end{aligned}
$$

$$+\frac{\Psi_{.3}(P_4,E_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\left(\frac{\Psi_1(P,V_4)\Psi_3(P_4,E_4)\aleph_2(E)}{\Psi_1(P_4,V_4)\Psi_3(P,E)\aleph_2(E_4)}-1-\ln\left(\frac{\Psi_1(P,V_4)\Psi_3(P_4,E_4)\aleph_2(E)}{\Psi_1(P_4,V_4)\Psi_3(P,E)\aleph_2(E_4)}\right)\right)d\varphi$$
$$+\Psi_1(P_4,V_4)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_4)}-1-\frac{\aleph_3(V)}{\aleph_3(V_4)}+\frac{\Psi_1(P,V_4)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_4)}\right)$$
$$+\Psi_2(P_4,T_4)\left(\frac{\Psi_2(P,T)\Psi_1(P_4,V_4)}{\Psi_2(P_4,T_4)\Psi_1(P,V_4)}-1-\frac{\aleph_1(T)}{\aleph_1(T_4)}+\frac{\Psi_1(P,V_4)\Psi_2(P_4,T_4)\aleph_1(T)}{\Psi_1(P_4,V_4)\Psi_2(P,T)\aleph_1(T_4)}\right)$$
$$+\Psi_{.3}(P_4,E_4)\left(\frac{\Psi_3(P,E)\Psi_1(P_4,V_4)}{\Psi_{.3}(P_4,E_4)\Psi_1(P,V_4)}-1-\frac{\aleph_2(E)}{\aleph_2(E_4)}+\frac{\Psi_1(P,V_4)\Psi_3(P_4,E_4)\aleph_2(E)}{\Psi_1(P_4,V_4)\Psi_3(P,E)\aleph_2(E_4)}\right).$$

Using the definition of $\mathfrak{F}_i^T(P,T)$ and $\mathfrak{F}_i^E(P,E)$ given in condition (**H6**) and **Remark 1** from Eqs. (30)-(31) in case of $i=4$, we obtain

$$\frac{d\Phi_4}{dt}=(\yen(P)-\yen(P_4))\left(1-\frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}\right)$$
$$-[\Psi_1(P_4,V_4)+\Psi_2(P_4,T_4)+\Psi_3(P_4,E_4)]\,\chi\left(\frac{\Psi_1(P_4,V_4)}{\Psi_1(P,V_4)}\right)$$
$$-\frac{\Psi_1(P_4,V_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P_\varphi,V_\varphi)\aleph_1(T_4)}{\Psi_1(P_4,V_4)\aleph_1(T)}\right)d\varphi$$
$$-\frac{\Psi_2(P_4,T_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_2(P_\varphi,T_\varphi)\aleph_1(T_4)}{\Psi_2(P_4,T_4)\aleph_1(T)}\right)d\varphi$$
$$-\frac{\Psi_{.3}(P_4,E_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_3(P_\varphi,E_\varphi)\aleph_1(T_4)}{\Psi_{.3}(P_4,E_4)\aleph_1(T)}\right)d\varphi$$
$$-\frac{\Psi_1(P_4,V_4)}{\mathscr{B}_3}\int_0^{\varkappa_3}\tilde{\mathscr{B}}_3(\varphi)\chi\left(\frac{\aleph_1(T_\varphi)\aleph_3(V_4)}{\aleph_1(T_4)\aleph_3(V)}\right)d\varphi$$
$$-\frac{\Psi_3(P_4,E_4)}{\mathscr{B}_2}\int_0^{\varkappa_2}\tilde{\mathscr{B}}_2(\varphi)\chi\left(\frac{\aleph_1(T_\varphi)\aleph_2(E_4)}{\aleph_1(T_4)\aleph_2(E)}\right)d\varphi$$
$$-\frac{\Psi_1(P_4,V_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_4)\aleph_3(V)}{\Psi_1(P,V)\aleph_3(V_4)}\right)d\varphi$$
$$-\frac{\Psi_2(P_4,T_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_4)\Psi_2(P_4,T_4)\aleph_1(T)}{\Psi_1(P_4,V_4)\Psi_2(P,T)\aleph_1(T_4)}\right)d\varphi$$
$$+\frac{\Psi_{.3}(P_4,E_4)}{\mathscr{B}_1}\int_0^{\varkappa_1}\tilde{\mathscr{B}}_1(\varphi)\chi\left(\frac{\Psi_1(P,V_4)\Psi_3(P_4,E_4)\aleph_2(E)}{\Psi_1(P_4,V_4)\Psi_3(P,E)\aleph_2(E_4)}\right)d\varphi$$
$$+\Psi_1(P_4,V_4)\left(1-\frac{\Psi_1(P,V_4)}{\Psi_1(P,V)}\right)\left(\frac{\Psi_1(P,V)}{\Psi_1(P,V_4)}-\frac{\aleph_3(V)}{\aleph_3(V_4)}\right)$$
$$+\Psi_2(P_4,T_4)\left(1-\frac{\mathfrak{F}_4^T(P_4,T_4)}{\mathfrak{F}_4^T(P,T)}\right)\left(\frac{\mathfrak{F}_4^T(P,T)}{\mathfrak{F}_4^T(P_4,T_4)}-\frac{\aleph_1(T)}{\aleph_1(T_4)}\right)$$
$$+\Psi_{.3}(P_4,E_4)\left(1-\frac{\mathfrak{F}_4^E(P_4,E_4)}{\mathfrak{F}_4^E(P,E)}\right)\left(\frac{\mathfrak{F}_4^E(P,E)}{\mathfrak{F}_4^E(P_4,E_4)}-\frac{\aleph_2(E)}{\aleph_2(E_4)}\right).$$

Hence, if $\Re_3>0$ and $\Re_4>1$ then Đ$_4$, $\frac{d\Phi_4}{dt}\le 0$ for all $P,T,E,V,Z^T,Z^V>0$. Moreover, $\frac{d\Phi_4}{dt}=0$ when $P=P_4$, $T(t)=T_4$, $E(t)=E_4$, $V(t)=V_4$, $\chi(.)=0$. The solutions of model (3)-(8) converge to Γ_4', where

$P(t)=P_4$, $T(t)=T_4$, $E(t)=E_4$, $V(t)=V_4$. From Eq. (4) and Eq. (6)

$$\dot{T}=\mathscr{B}_1\left[\Psi_1(P_4,V_4)+\Psi_2(P_4,T_4)+\Psi_3(P_4,E_4)\right]-(\mu_1+\xi_T)\aleph_1(T_4)-\lambda\aleph_1(T_4)\aleph_4\left(Z^T\right)\ \Rightarrow Z^T=Z_4^T \text{ for all } t,$$
$$\dot{V}=\beta\mathscr{B}_3\aleph_1(T_4)d\varphi-\xi_V\aleph_3(V_4)-\psi\aleph_3(V_4)\aleph_5\left(Z^V\right)\Rightarrow\ Z^V=Z_4^V \text{ for all } t,$$

this yields that $\Gamma_4'=\{Đ_4\}$, and from L.I.P. we obtain that $Đ_4$ is G.A.S. □

6. Example and Numerical Simulations

In this part, we demonstrate our theoretical results using an example and a few numerical simulations. We change the distributed-time delay model (3)-(8) to a discrete-time delay one for numerical reasons using a dirac delta function $\tilde{\mathscr{B}}(.)$ as a particular version of the kernel $\mathscr{B}_i(\eth)$ as [40]:

$$\mathscr{B}_i(\eth)=\tilde{\mathscr{B}}(\eth-\varphi_i),\quad \varphi_i\in[0,\varkappa_i],i=1,2,3.$$

The discrete-time delays represented by the constants $\varphi_i\in[0,\varkappa_i],i=1,2,3$ are special instances of the three distributed-time delays shown in models (3)-(8). Let $\varkappa_i\in 1$, then using the properties of Dirac delta function we get:

$$\int_0^\infty\mathscr{B}_j(\eth)d\eth=1,\quad \mathscr{B}_j=\int_0^\infty\tilde{\mathscr{B}}(\eth-\varphi_i)e^{-\kappa_j\eth}d\eth=e^{-\kappa_j\varphi_j},\quad j=1,2,3.$$

Furthermore, the related stability results of system (3)-(8) from Theorems 1-5 read as:

Corollary 1. Let $\Re_i$, $i=0,1,...4$ be defined as in (22). The following statements hold true.

(i) If $\Re_0\le 1$ and conditions **(H1)-(H5)** $Đ_0$ be satisfied, then is (G.A.S).

(ii) If $\Re_0>1$, $\Re_1\le 0$ and $\Re_2\le 0$ conditions **(H1)-(H4),(H6)** $Đ_1$ be satisfied, then is (G.A.S).

(iii) If $\Re_1>1$ and $\Re_4\le 0$ conditions **(H1)-(H4),(H6)** $Đ_2$ be satisfied, then is (G.A.S).

(iv) If $\Re_2>1$ and $\Re_3\le 0$ conditions **(H1)-(H4),(H6)** $Đ_3$ be satisfied, then is (G.A.S).

(v) If $\Re_3>1$ and $\Re_4>0$ conditions **(H1)-(H4),(H6)** $Đ_4$ be satisfied, then is (G.A.S).

Let us consider the following example:

$$\dot{P}=\alpha-\omega P+\zeta P\left(1-\frac{P}{P_{\max}}\right)-\frac{P^q}{1+\delta P^q}\left(\frac{\eta_1 V}{1+\gamma_1 V}+\frac{\eta_2 T}{1+\gamma_2 T}+\frac{\eta_3 E}{1+\gamma_3 E}\right),$$
$$\dot{T}=\frac{e^{-\kappa_1\varphi_1}P_{\varphi_1}^q}{1+\delta P_{\varphi_1}^q}\left(\frac{\eta_1 V_{\varphi_1}}{1+\gamma_1 V_{\varphi_1}}+\frac{\eta_2 T_{\varphi_1}}{1+\gamma_2 T_{\varphi_1}}+\frac{\eta_3 E_{\varphi_1}}{1+\gamma_3 E_{\varphi_1}}\right)-(\mu_1+\xi_T)T-\lambda TZ^T,$$
$$\dot{E}=\mu_2 e^{-\kappa_2\varphi_2}T_{\varphi_2}-\xi_E E, \tag{33}$$
$$\dot{V}=\beta e^{-\kappa_3\varphi_3}T_{\varphi_3}-\xi_V V-\psi VZ^V,$$
$$\dot{Z}^T=\sigma TZ^T-\xi_{Z^T}Z^T,$$

$$\dot{Z}^V = \varpi V Z^V - \xi_{Z^V} Z^V.$$

•The healthy CD4$^+$T cells' intrinsic growth rate is selected as

$$¥(P) = \alpha - \omega P + \zeta P\left(1 - \frac{P}{P_{\max}}\right)$$

Here, we examine the development of the body's natural healthy cells as an extra way to generate CD4$^+$T cells [11]. It is well recognised that the body includes a maximum number of healthy CD4$^+$T cells, as shown by the $P_{\max} > 0$ parameter. The maximum proliferation rate of healthy CD4$^+$T cells is provided by $\zeta > 0$. The concentration ought to drop if it hits $P_{\max}$. We assume that $\zeta < \omega$ [50]. It is clear that $¥(0) = \alpha > 0$ and $¥(P_0) = 0$,where

$$P_0 = \frac{P_{\max}}{2\zeta}\left(\zeta - \omega + \sqrt{(\zeta - \omega)^2 + \frac{4\alpha\zeta}{P_{\max}}}\right).$$

Furthermore, we have

$$¥'(P) = \zeta - \omega - \frac{2\zeta P}{P_{\max}} < 0. \tag{34}$$

Clearly, $¥(P) > 0$ whereas $¥'(P) < 0$for all $P \in [0, P_0)$. Hence, condition **H1** is hold true. The rates of infection caused by virus-to-cell, cell-to-cell and cell-inflammatory cytokines are

$$\Psi_1(P,V) = \frac{\eta_1 P^q V}{(1+\delta P^q)(1+\gamma_1 V)},$$
$$\Psi_2(P,T) = \frac{\eta_2 P^q T}{(1+\delta P^q)(1+\gamma_2 T)},$$
$$\Psi_3(P,E) = \frac{\eta_3 P^q E}{(1+\delta P^q)(1+\gamma_3 E)},$$

and then,

$$\Psi_1(P,V) > 0, \quad \Psi_2(P,T) > 0, \quad \Psi_3(P,E) > 0, \quad \text{for all } P,T,E,V > 0,$$
$$\Psi_1(P,0) = \Psi_1(0,V) = \Psi_2(P,0) = \Psi_2(0,T) = \Psi_3(P,0) = \Psi_3(0,E), \quad \text{for all } P,T,E,V > 0.$$

Moreover,

$$\frac{\partial \Psi_1(P,V)}{\partial P} = \frac{q\eta_1 P^{q-1} V}{(1+\delta P^q)^2(1+\gamma_1 V)} > 0, \frac{\partial \Psi_2(P,T)}{\partial P} = \frac{q\eta_2 P^{q-1} T}{(1+\delta P^q)^2(1+\gamma_2 T)} > 0,$$
$$\frac{\partial \Psi_3(P,E)}{\partial P} = \frac{q\eta_3 P^{q-1} E}{(1+\delta P^q)^2(1+\gamma_3 E)} > 0, \frac{\partial \Psi_1(P,V)}{\partial V} = \frac{\eta_1 P^q}{(1+\delta P^q)(1+\gamma_1 V)^2} > 0,$$
$$\frac{\partial \Psi_2(P,T)}{\partial T} = \frac{\eta_2 P^q}{(1+\delta P^q)(1+\gamma_2 T)^2} > 0, \frac{\partial \Psi_3(P,E)}{\partial E} = \frac{\eta_3 P^q}{(1+\delta P^q)(1+\gamma_3 E)^2} > 0,$$

$$\frac{\partial\Psi_1(P,0)}{\partial V}=\frac{\eta_1P^q}{1+\delta P^q}>0,\frac{\partial\Psi_2(P,0)}{\partial T}=\frac{\eta_2P^q}{1+\delta P^q}>0,\frac{\partial\Psi_3(P,0)}{\partial E}=\frac{\eta_3P^q}{1+\delta P^q}>0,$$

for all $P,T,E,V>0$. Furthermore, we have

$$\frac{d}{dP}\left(\frac{\partial\Psi_1(P,0)}{\partial V}\right)=\frac{q\eta_1P^{q-1}}{(1+\delta P^q)^2}>0,\frac{d}{dP}\left(\frac{\partial\Psi_2(P,0)}{\partial T}\right)=\frac{q\eta_2P^{q-1}}{(1+\delta P^q)^2}>0,$$
$$\frac{d}{dP}\left(\frac{\partial\Psi_3(P,0)}{\partial E}\right)=\frac{q\eta_3P^{q-1}}{(1+\delta P^q)^2}>0,\text{for all }P>0.$$

The discussion above guarantees that criterion **H2** is confirmed. The natural death rate of the active infection caused by virus-to-cell, cell-to-cell and cell-inflammatory cytokines are given by

$$\aleph_{\Bbbk}(\rho)=\rho,\qquad \Bbbk=1,2,3,4,5.$$

Obviously, condition **H3** is valid. In addition, we have

$$\frac{d}{dV}\left(\frac{\Psi_1(P,V)}{\aleph_3(V)}\right)=\frac{d}{dV}\left(\frac{\eta_1P^q}{(1+\delta P^q)(1+\gamma_1V)}\right)=-\frac{\eta_1\gamma_1P^q}{(1+\delta P^q)(1+\gamma_1V)^2}<0,$$
$$\frac{d}{dT}\left(\frac{\Psi_2(P,T)}{\aleph_1(T)}\right)=\frac{d}{dT}\left(\frac{\eta_2P^q}{(1+\delta P^q)(1+\gamma_2T)}\right)=-\frac{\eta_2\gamma_2P^q}{(1+\delta P^q)(1+\gamma_2T)^2}<0,$$
$$\frac{d}{dE}\left(\frac{\Psi_3(P,E)}{\aleph_2(E)}\right)=\frac{d}{dE}\left(\frac{\eta_3P^q}{(1+\delta P^q)(1+\gamma_3E)}\right)=-\frac{\eta_3\gamma_3P^q}{(1+\delta P^q)(1+\gamma_3E)^2}<0,$$

for all $P,T,E,V>0$. Therefore, condition **H4** is also verified. On the other hand, we have $\aleph'_{\Bbbk}(\rho)=1$ and then

$$\mathscr{F}_1(P)=\frac{\partial\Psi_1(P,0)}{\partial V}=\frac{\eta_1P^q}{1+\delta P^q},\;\;\mathscr{F}_2(P)=\frac{\partial\Psi_2(P,0)}{\partial T}=\frac{\eta_2P^q}{1+\delta P^q},\;\;\mathscr{F}_3(P)=\frac{\partial\Psi_3(P,0)}{\partial E}=\frac{\eta_3P^q}{1+\delta P^q}.$$

Clearly, $\frac{\mathscr{F}_2(P)}{\mathscr{F}_1(P)}=\frac{\eta_2}{\eta_1}$ and $\frac{\mathscr{F}_3(P)}{\mathscr{F}_1(P)}=\frac{\eta_3}{\eta_1}$, hence, condition **H5** is satisfied. In addition,

$$\mathfrak{F}_i^T(P,T)=\frac{\Psi_2(P,T)}{\Psi_1(P,V_i)}=\frac{\eta_2(1+\gamma_1V_i)T}{\eta_1(1+\gamma_2T)V_i},\;\mathfrak{F}_i^T(P_i,T_i)=\frac{\Psi_2(P_i,T_i)}{\Psi_1(P_i,V_i)}=\frac{\eta_2(1+\gamma_1V_i)T_i}{\eta_1(1+\gamma_2T_i)V_i},$$
$$\mathfrak{F}_i^E(P,E)=\frac{\Psi_3(P,E)}{\Psi_1(P,V_i)}=\frac{\eta_3(1+\gamma_1V_i)E}{\eta_1(1+\gamma_3E)V_i},\;\;\mathfrak{F}_i^E(P_i,E_i)=\frac{\Psi_3(P_i,E_i)}{\Psi_1(P_i,V_i)}=\frac{\eta_3(1+\gamma_1V_i)E_i}{\eta_1(1+\gamma_3E_i)V_i},$$

and

$$\left(1-\frac{\mathfrak{F}_i^T(P_i,T_i)}{\mathfrak{F}_i^T(P,T)}\right)\left(\frac{\mathfrak{F}_i^T(P,T)}{\mathfrak{F}_i^T(P_i,T_i)}-\frac{\aleph_1(T)}{\aleph_1(T_i)}\right)=\left(1-\frac{(1+\gamma_2T)T_i}{(1+\gamma_2T_i)T}\right)\left(\frac{(1+\gamma_2T_i)T}{(1+\gamma_2T)T_i}-\frac{T}{T_i}\right)$$
$$=\frac{-\gamma_2(T-T_i)^2}{T_i(1+\gamma_2T)(1+\gamma_2T_i)}\leq 0,$$
$$\left(1-\frac{\mathfrak{F}_i^E(P_i,E_i)}{\mathfrak{F}_i^T(P,E)}\right)\left(\frac{\mathfrak{F}_i^E(P,E)}{\mathfrak{F}_i^E(P_i,E_i)}-\frac{\aleph_2(E)}{\aleph_2(E_i)}\right)=\left(1-\frac{(1+\gamma_3E)E_i}{(1+\gamma_3E_i)E}\right)\left(\frac{(1+\gamma_3E_i)E}{(1+\gamma_3E)E_i}-\frac{E}{E_i}\right)$$
$$=\frac{-\gamma_3(E-E_i)^2}{E_i(1+\gamma_3E)(1+\gamma_3E_i)}\leq 0,$$

for all $T,E>0, P\in(0,P_0)$,where $i=1,2,3,4$. Hence, condition **H6** is ensured. The global stability results shown in Theorems 1-5 are therefore guaranteed to be valid for this example due to the validity of conditions **H1-H6**. Thus, the threshold parameters for system (33) are given by:

$$\Re_0=\frac{P_0^q e^{-\kappa_1\varphi_1}}{(\mu_1+\xi_T)\left(1+\delta P_0^q\right)}\left[\frac{\eta_1\beta e^{-\kappa_3\varphi_3}}{\xi_V}+\eta_2+\frac{\mu_2\eta_3 e^{-\kappa_2\varphi_2}}{\xi_E}\right],$$
$$\Re_1=\frac{P_2^q e^{-\kappa_1\varphi_1}}{(\mu_1+\xi_T)\left(1+\delta P_2^q\right)T_2}\left[\frac{\eta_1 V_2}{(1+\gamma_1 V_2)}+\frac{\eta_2 T_2}{(1+\gamma_2 T_2)}+\frac{\eta_3 E_2}{(1+\gamma_3 E_2)}\right],$$
$$\Re_2=\frac{\varpi\beta P_3^q e^{-(\kappa_1\varphi_1+\kappa_3\varphi_3)}}{\xi_V\xi_{Z^V}(\mu_1+\xi_T)\left(1+\delta P_3^q\right)}\left(\frac{\eta_1 V_3}{(1+\gamma_1 V_3)}+\frac{\eta_2 T_3}{(1+\gamma_2 T_3)}+\frac{\eta_3 E_3}{(1+\gamma_3 E_3)}\right),$$
$$\Re_3=\frac{\sigma P_4^q}{\xi_{Z^T}(\mu_1+\xi_T)\left(1+\delta P_4^q\right)}\left(\frac{\eta_1 V_4}{(1+\gamma_1 V_4)}+\frac{\eta_2 T_4}{(1+\gamma_2 T_4)}+\frac{\eta_3 E_4}{(1+\gamma_3 E_4)}\right),$$
$$\Re_4=\frac{\varpi\beta\xi_{Z^T}e^{-\kappa_3\varphi_3}}{\sigma\xi_V\xi_{Z^V}}.$$

TABLE 1. Model parameters

Parameter	Value	Source	Parameter	Value	Source
α	10	[42], [46]	ψ	0.3	[34], [36]
ω	0.01	[36], [43], [47]	ξ_T	0.5	[10], [11]
ζ	0.005	[34], [36]	ξ_E	0.1	[44]
δ	0.7	[34], [36]	ξ_V	2	[36], [48]
γ_1	0.1	[34], [36]	ξ_{Z^T}	0.1	[48]
γ_2	0.2	[34], [36]	ξ_{Z^V}	0.2	[36]
γ_3	0.3	[34], [36]	κ_1	0.1	[6], [44]
μ_1	0.2	[34]	κ_2	1	[49]
μ_2	0.07	[44]	κ_3	0.1	[44]
λ	0.001	[44],[45]	q	2	[34]
β	15	Assumed			

To solve system (33) numerically we fix the values of some parameters (see Table 1) and the others will be varied. In the coming subsections, we present some numerical simulations for model (33).

Stability of the equilibria

In this subsection, we select the delay parameters as $\varphi_1=3$, $\varphi_2=0.5$, and $\varphi_3=3$. Besides, we choose the following three different initial conditions for system (33):

I.1: $(P(\varphi),T(\varphi),E(\varphi),V(\varphi),Z^T(\varphi),Z^V(\varphi)) = (800,1,1,5,20,1)$,

I.2: $(P(\varphi),T(\varphi),E(\varphi),V(\varphi),Z^T(\varphi),Z^V(\varphi)) = (400,5,3,15,50,3)$,

I.3: $(P(\varphi),T(\varphi),E(\varphi),V(\varphi),Z^T(\varphi),Z^V(\varphi)) = (100,10,6,30,80,5)$, where $\varphi \in [-3,0]$.

Under initial conditions we are choose different parameter values of η_1, η_2, η_3, σ and ϖ which gives the following scenarios:

Scenario 1 (Stability of Đ$_0$): $\eta_1 = 0.04$, $\eta_2 = 0.01$, $\eta_3 = 0.02$, $\sigma = 0.02$ and $\varpi = 0.001$. These values give $\Re_0 = 0.269 < 1$ with the fact that the equilibrium Đ$_0 = (1061.32,0,0,0,0,0,0)$ is G.A.S as shown in Figure 1. Theorem 1's study results are in line with the numerical outcomes shown in Figure 1. This implies the eventual eradication of HIV-1 particles.

Scenario 2 (Stability of Đ$_1$): $\eta_1 = 0.4$, $\eta_2 = 0.1$, $\eta_3 = 0.2$, $\sigma = 0.001$ and $\varpi = 0.001$. These choice give $\Re_0 = 3.63 > 1$, $\Re_1 = 0.99 \leq 1$ and $\Re_2 = 0.147 \leq 1$. Further, they ensure the existence of the equilibrium Đ$_1 = (646.991,5.31,2.256,29.52,0,0)$. As demonstrated in Figure 2, the concentrations of every compartment eventually trend to Đ$_1$ over time, as demonstrated by Theorem 2. Because immune cells are not being triggered to eliminate viruses and infected cells, this indicates that the infection will spread broadly.

Scenario 3 (Stability of Đ$_2$): $\eta_1 = 0.4$, $\eta_2 = 0.1$, $\eta_3 = 0.4$, $\sigma = 0.02$ and $\varpi = 0.001$. These choice give $\Re_1 = 1.12 > 1$ and $\Re_4 = 0.138 \leq 1$. Further, they ensure the existence of the equilibrium Đ$_2 = (619.83,5,2.12,27.78,85.26,0)$. Figure 3 shows that, regardless of starting point, all compartment concentrations progressively shift towards Đ$_2$ over time. Theorem 3 states that when CTLs immunity is present but the antibodies are not triggered to eradicate viruses, the illness will propagate throughout the population.

Scenario 4 (Stability of Đ$_3$): $\eta_1 = 0.4$, $\eta_2 = 0.1$, $\eta_3 = 0.4$, $\sigma = 0.002$ and $\varpi = 0.02$. These choice give $\Re_2 = 2.24 > 1$ and $\Re_3 = 0.1 \leq 1$. Further, they ensure the existence of the equilibrium Đ$_3 = (757.23,4.04,1.71,10,0,8.32)$. Figure 4 shows that, independent of starting point, all compartment concentrations gradually increase towards Đ$_3$ with time. Though the CTLs are not triggered to eradicate infected cells, Theorem 4 states that the presence of antibody immunity suggests the infection will become endemic.

Scenario 5 (Stability of Đ$_4$): $\eta_1 = 0.4$, $\eta_2 = 0.1$, $\eta_3 = 0.4$, $\sigma = 0.02$ and $\varpi = 0.01$. These choice give $\Re_3 = 1.4 > 1$ and $\Re_4 = 1.389 > 1$. Further, they ensure the existence of the equilibrium Đ$_4 = (657.83,5,2.12,20,27.149,2.59)$. Figure 5 shows that, regardless of starting point, all compartment concentrations gradually shift towards $Ð_4$ over time. The presence of both CTLs and antibody immunities will result in the sickness becoming endemic, according to Theorem 5.

Effect of time delays on the HIV-1 dynamics

In this part we vary the delay parameters φ_i, $i = 1,2,3$ and fix the parameters $\eta_1 = 0.4$, $\eta_2 = 0.1$, $\eta_3 = 0.4$, $\sigma = 0.02$ and $\varpi = 0.03$. Additionally, table 1 will be used to obtain the remaining parameters. For simplicity, let us take $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi$ as a result, the basic reproduction number $\Re_0$ turns into

$$\Re_0 = \frac{P_0^q e^{-\kappa_1\varphi}}{(\mu_1+\xi_T)\left(1+\delta P_0^q\right)}\left[\frac{\eta_1\beta e^{-\kappa_3\varphi}}{\xi_V}+\eta_2+\frac{\mu_2\eta_3 e^{-\kappa_2\varphi}}{\xi_E}\right].$$

We observed that $\Re_0$ is a decreasing function of φ. Consequently, when φ varies, so will the system's stability. Since the stabilisation of the uninfected equilibrium $Ð_0$ is of importance to us, we calculated the delay's critical value φ_{cr}, which makes

$$\Re_0 = \frac{P_0^q e^{-\kappa_1\varphi_{cr}}}{(\mu_1+\xi_T)\left(1+\delta P_0^q\right)}\left[\frac{\eta_1\beta e^{-\kappa_3\varphi_{cr}}}{\xi_V}+\eta_2+\frac{\mu_2\eta_3 e^{-\kappa_2\varphi_{cr}}}{\xi_E}\right] = 1. \tag{35}$$

By solving Equation (35) numerically, we obtain φ_{cr} =9.47217. Then, it is easily to notice that $\Re_0 \leq 1$ if $\varphi \geq$9.47217. This guarantees that the virus will be eliminated from the body and that $Ð_0$ is G.A.S. We now examine how the delay parameter φ affects system (33) solutions with initial values:

I.4: $(P(\varphi), T(\varphi), E(\varphi), V(\varphi), Z^T(\varphi), Z^V(\varphi)) = (500, 4, 1, 50, 150, 10)$,

where $\varphi \in [-\max\{\varphi_1, \varphi_2, \varphi_3\}, 0]$.

The effect of φ on the system's solutions is seen in figure 6 and table 2. We discovered that as rises φ, the quantity of healthy CD4$^+$T cells increases while the levels of other compartments decrease.

TABLE 2. Impact of delay parameters.

Delay parameters φ	Steady states	$\Re_0$
0	$Ð_4 = (775.344, 5, 3.5, 6.66667, 23.6917, 30.8333)$	6.89795
3	$Ð_3 = (862.17, 2.74, 0.09, 6.66, , 0, 8.57)$	3.53233
6	$Ð_1 = (820.94, 2.42, 0.004, 9.96, 0, 0)$	1.95
9.47212	$Ð_0 = (1061.32, 0, 0, 0, 0, 0)$	1
14	$Ð_0 = (1061.32, 0, 0, 0, 0, 0)$	0.42
20	$Ð_0 = (1061.32, 0, 0, 0, 0, 0)$	0.139

Time delays have a major impact on HIV-1 progression and provide valuable therapeutic insights. Sufficient intervals of time can stop HIV-1 from spreading, aid in its containment, and perhaps possibly result in its extinction. This suggests a workable strategy to create new HIV-1 treatments to extend these delay periods.

CTC transmission's influence on HIV-1 dynamics

In this part, we demonstrate how CTC transfer affects HIV-1 dynamics. We used the parameters given in table 1 and fixed the parameters $\eta_1 = 0.05$, $\eta_3 = 0.02$, $\sigma = 0.03$, $\varpi = 0.03$ $\varphi_1 = 3$, $\varphi_2 = 0.5$ and $\varphi_3 = 3$. We considered the following initial condition:

I.6: $(P(\varphi), T(\varphi), E(\varphi), V(\varphi), Z^T(\varphi), Z^V(\varphi)) = (300, 2, 1, 5, 200, 6)$, where $\varphi \in [-3, 0]$.

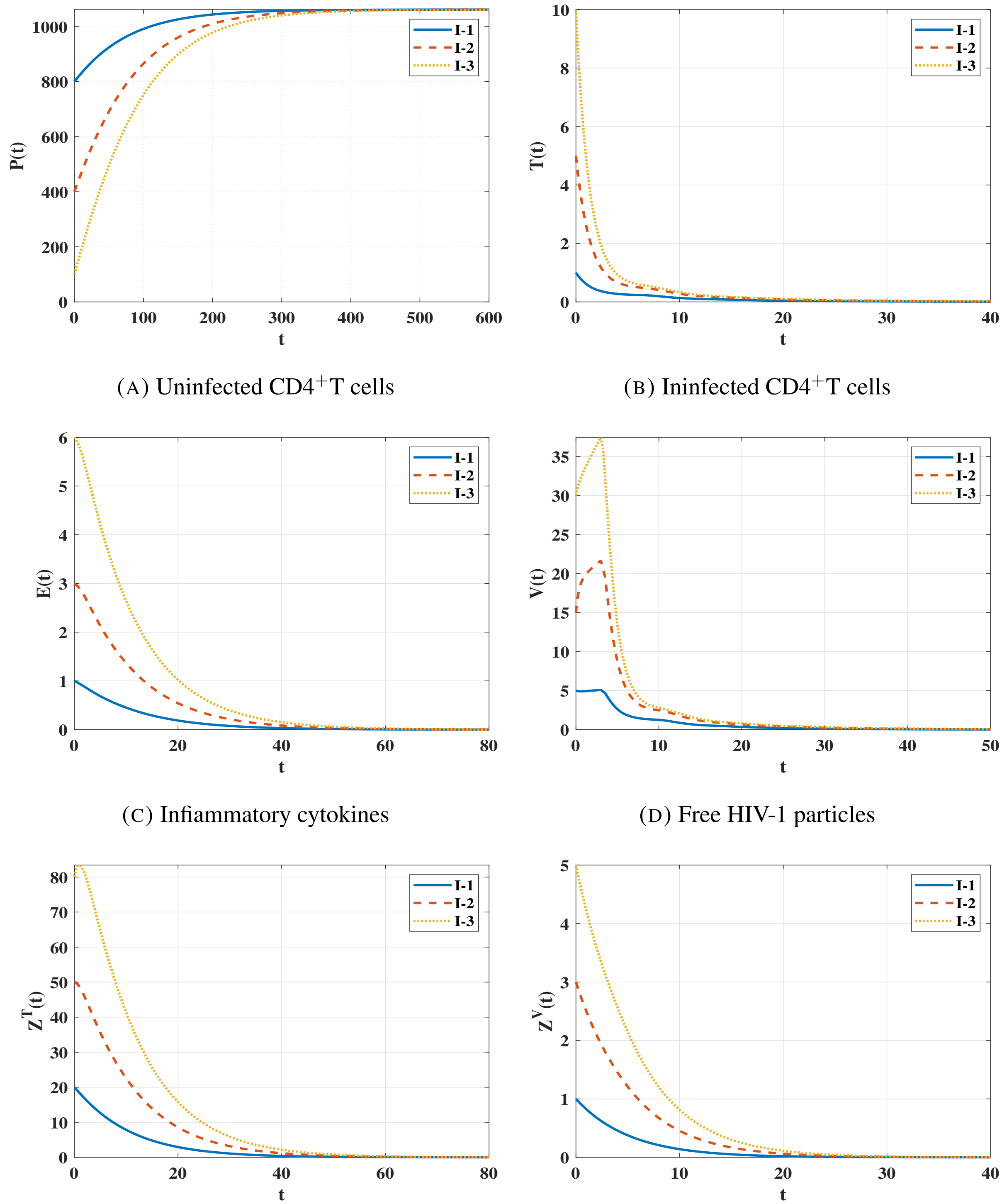
We varied the parameters η_2 as shown in figure 7 , which shows that the dynamic behaviour of the virus changes whenever the activity of the CTC transmissions changes. We observe that as the populations of infected cells, inflammatory cytokines, and free HIV-1 particles decrease, the numbers of uninfected $CD4^+$T cells and antibodies rise.

Influence of the antibody on the HIV-1 dynamics

In this part, we demonstrate how antibody response affects HIV-1 dynamics. We used the parameters given in table 1 and fixed the parameters $\eta_1 = 0.4$, $\eta_2 = 0.1$, $\eta_3 = 0.4$, $\varphi_1 = 3$, $\varphi_2 = 0.5$ and $\varphi_3 = 3$. We considered the following initial condition:

I.5: $(P(\varphi), T(\varphi), E(\varphi), V(\varphi), Z^T(\varphi), Z^V(\varphi)) = (700, 2, 1, 5, 30, 10)$, where $\varphi \in [-3, 0]$.

We varied the parameters σ and ϖ as shown in figure 8, which displays that, the dynamic behaviour of the virus changes whenever the activity of the antibody changes. We see that when σ and ϖ increase, the numbers of uninfected $CD4^+$T cells and antibodies increase while the populations of infected cells, inflammatory cytokines, and free HIV-1 particles decrease.



(A) Uninfected CD4$^+$T cells

(B) Ininfected CD4$^+$T cells

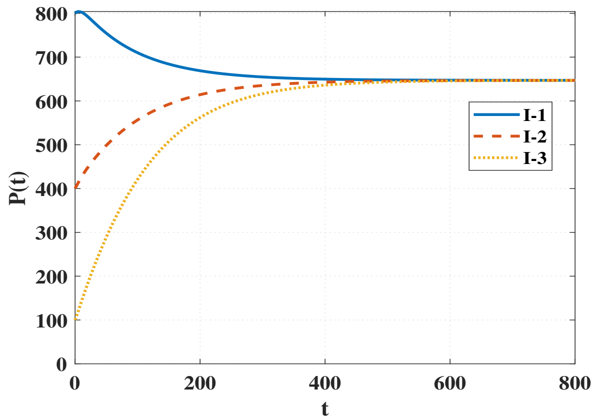
(C) Infiammatory cytokines

(D) Free HIV-1 particles

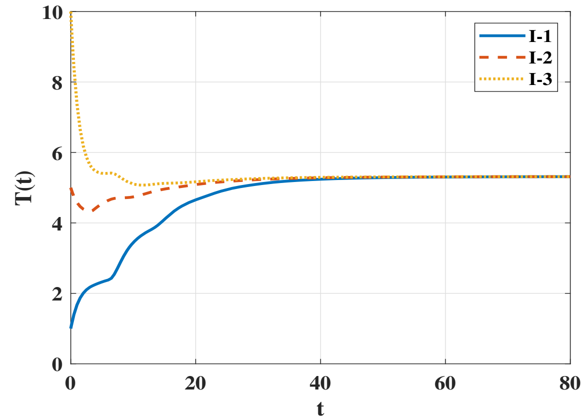
(E) CTLs

(F) Antibodies

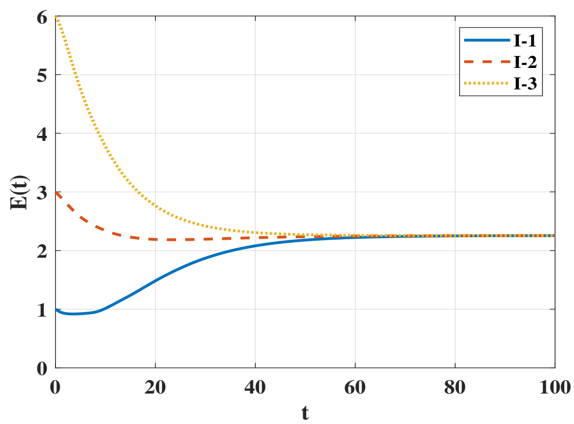
FIGURE 1. The equilibrium $Đ_0 = (1061.32, 0, 0, 0, 0, 0, 0)$ is G.A.S when ever $\Re_0 \leq 1$.



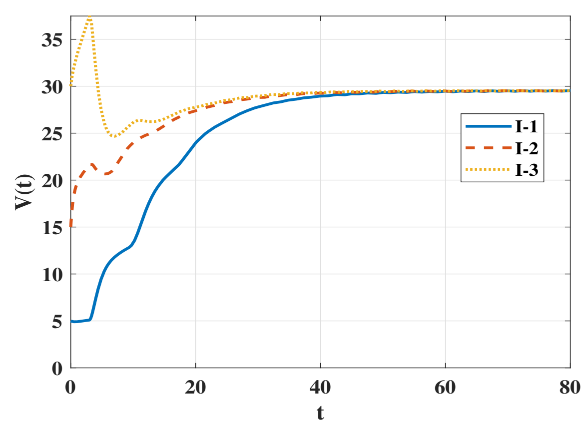
(A) Uninfected CD4$^+$T cells



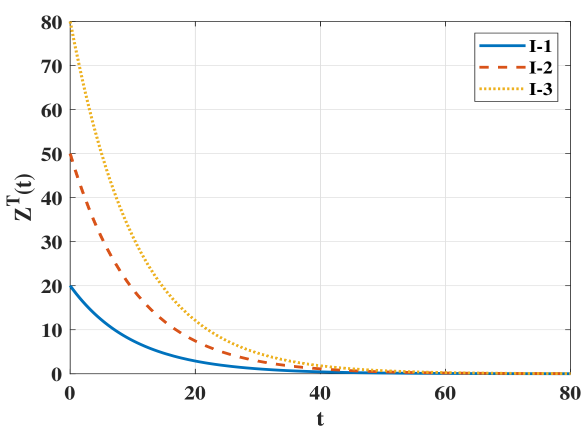
(B) Ininfected CD4$^+$T cells



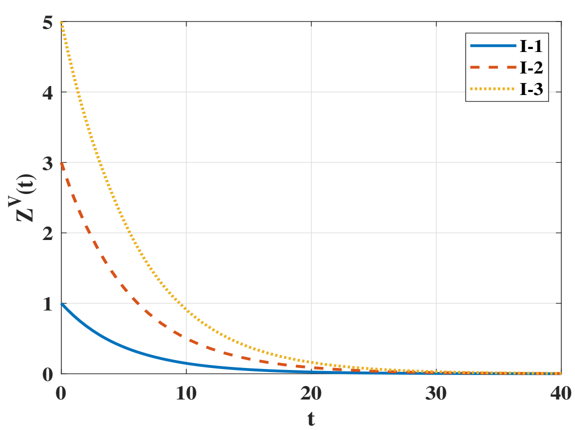
(C) Infiammatory cytokines



(D) Free HIV-1 particles

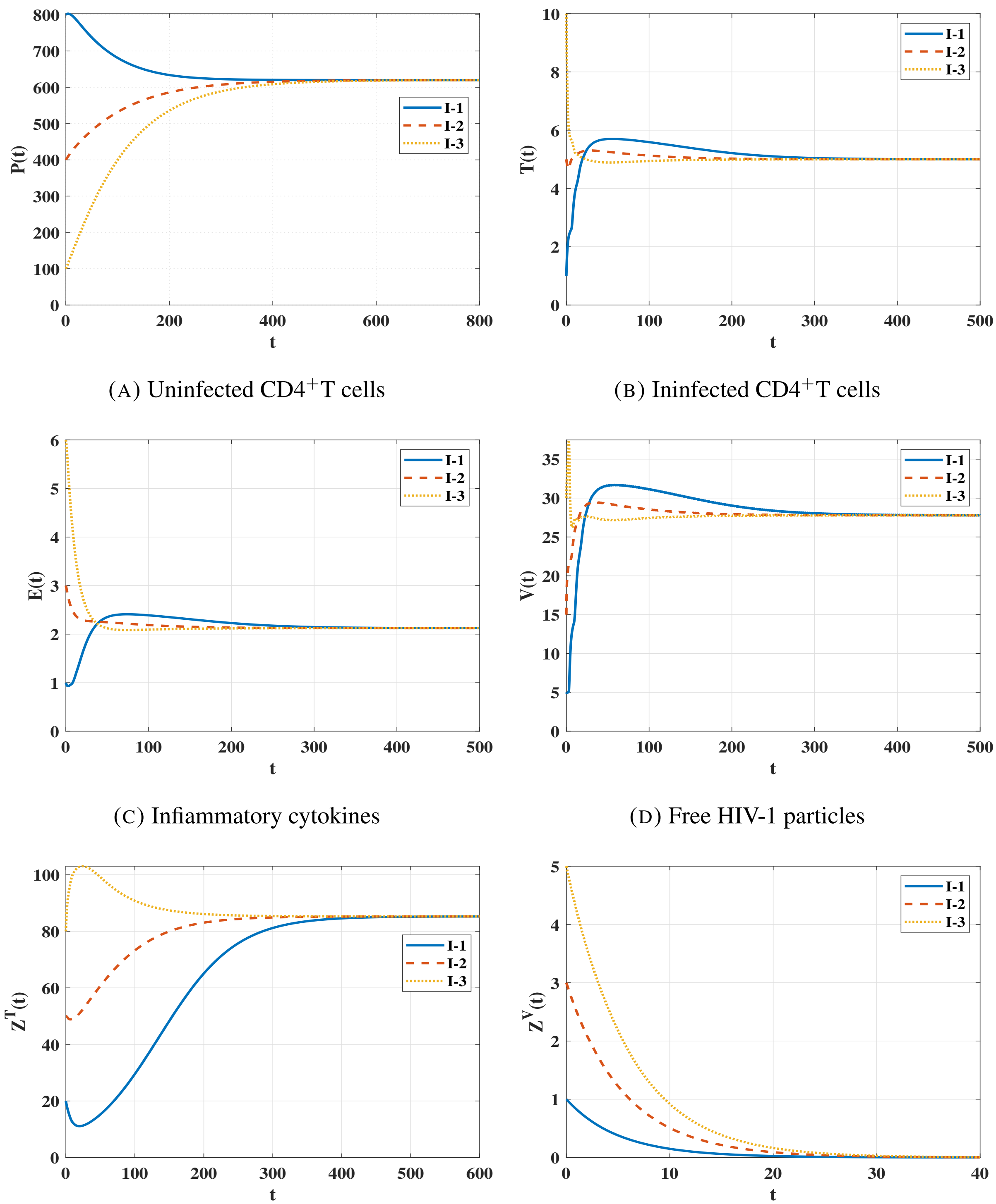


(E) CTLs



(F) Antibodies

FIGURE 2. The equilibrium $Đ_1 = (646.991, 5.31, 2.256, 29.52, 0, 0)$ is G.A.S whenever $\Re_0 > 1$, $\Re_1 \leq 1$ and $\Re_2 \leq 1$.



(A) Uninfected CD4$^+$T cells

(B) Ininfected CD4$^+$T cells

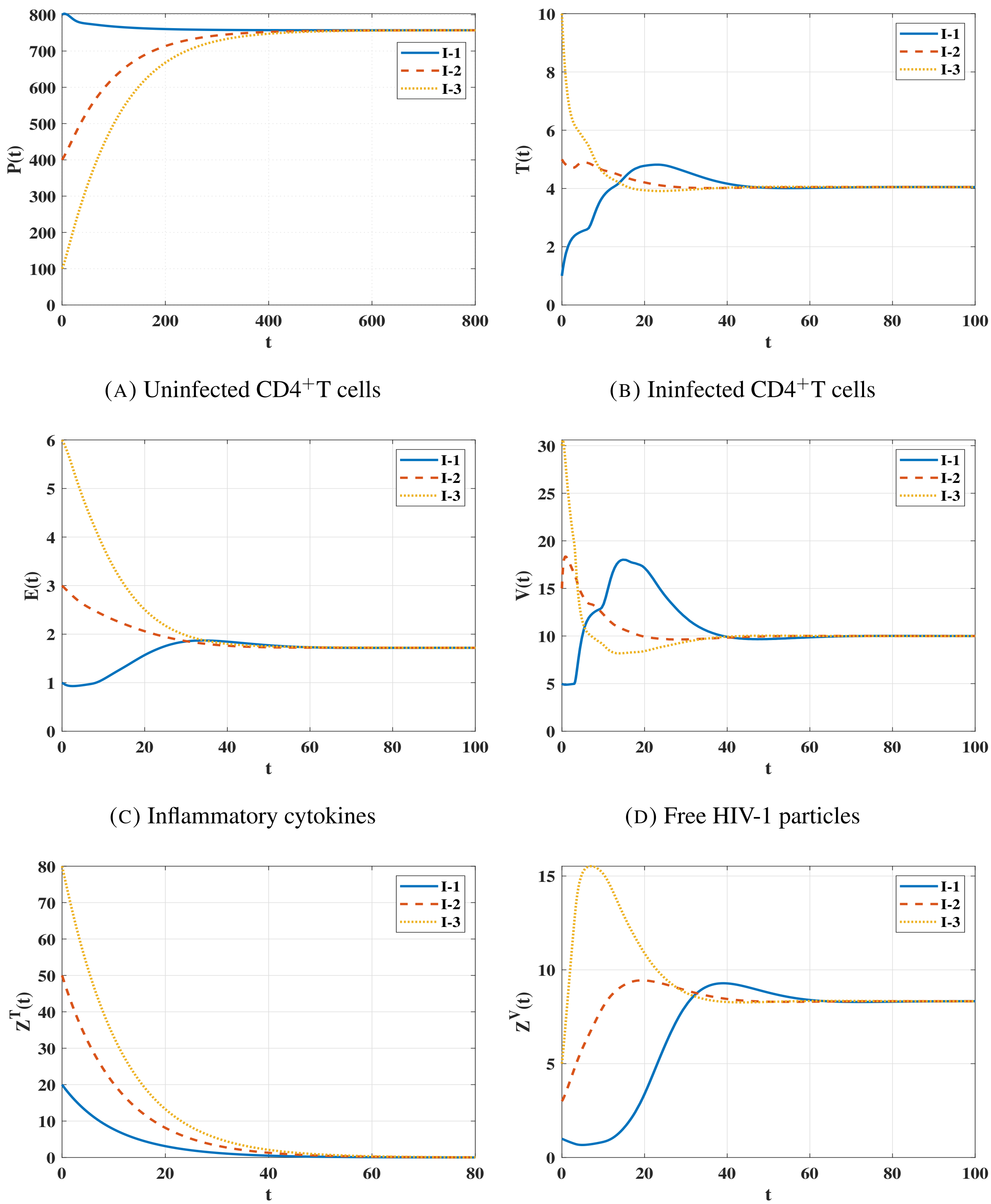
(C) Infiammatory cytokines

(D) Free HIV-1 particles

(E) CTLs

(F) Antibodies

FIGURE 3. The equilibrium $Đ_2 = (619.83, 5, 2.12, 27.78, 85.26, 0)$ is G.A.S when ever $\Re_1 > 1$ and $\Re_4 \leq 1$.



(A) Uninfected CD4$^+$T cells

(B) Ininfected CD4$^+$T cells

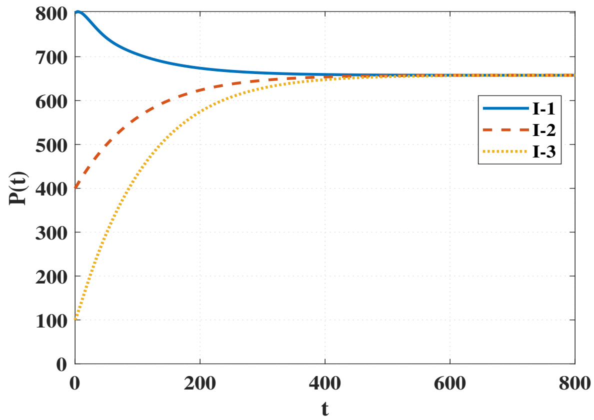
(C) Inflammatory cytokines

(D) Free HIV-1 particles

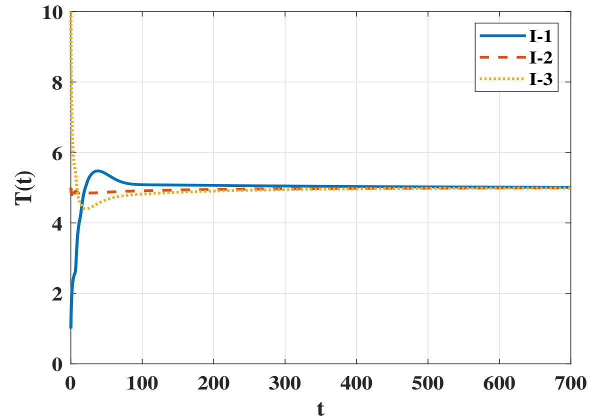
(E) CTLs

(F) Antibodies

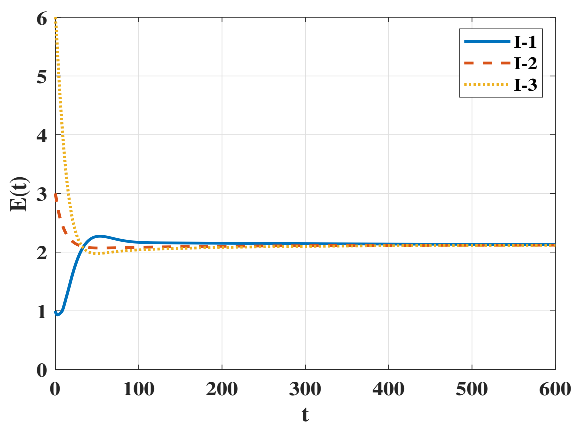
FIGURE 4. The equilibrium $Đ_3 = (757.23, 4.04, 1.71, 10, 0, 8.32)$ is G.A.S when ever $\Re_2 > 1$ and $\Re_3 \leq 1$.



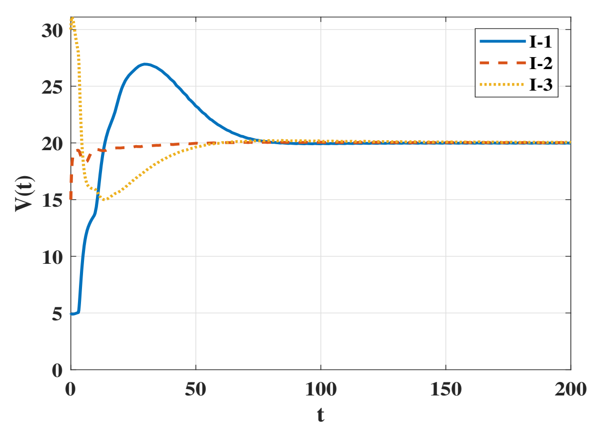
(A) Uninfected CD4$^+$T cells



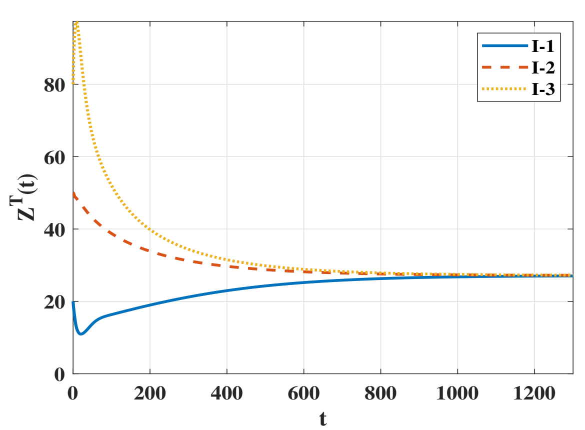
(B) Infected CD4$^+$T cells



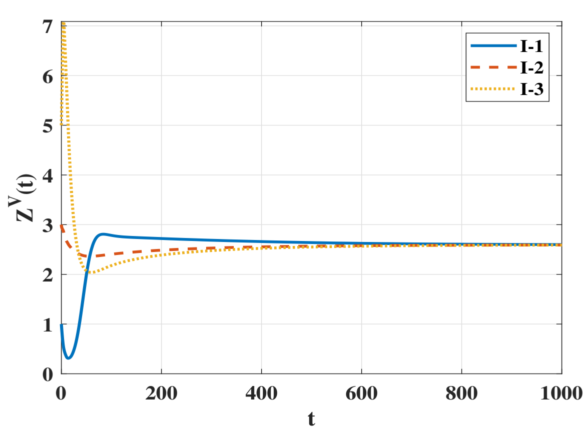
(C) Inflammatory cytokines



(D) Free HIV-1 particles

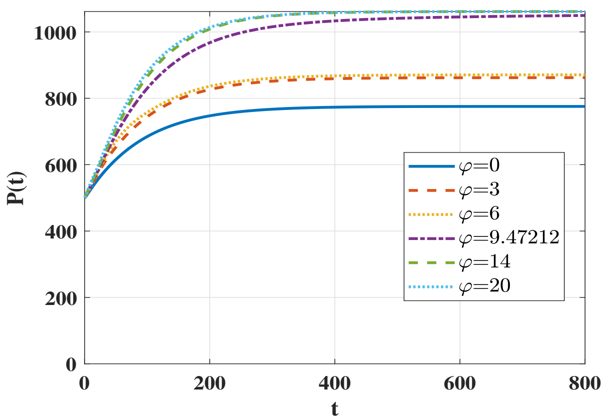


(E) CTLs

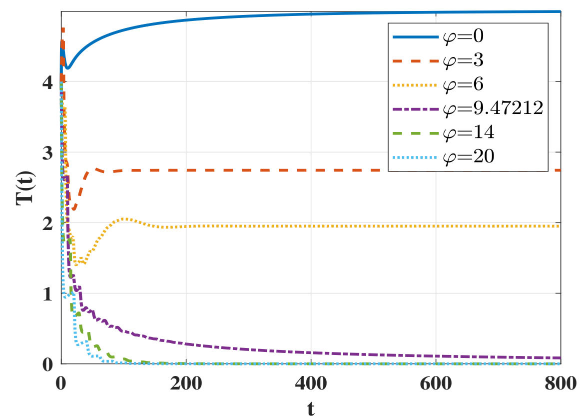


(F) Antibodies

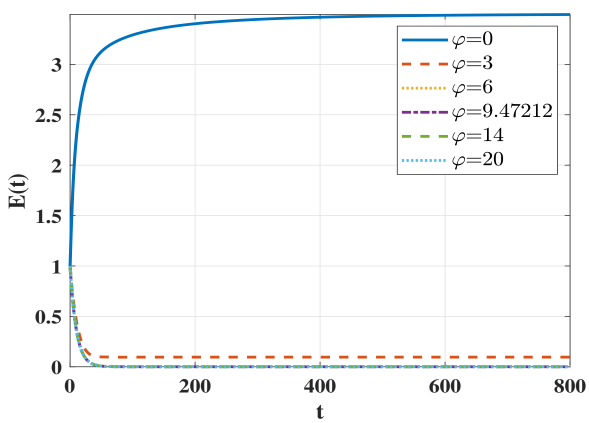
FIGURE 5. The equilibrium $Đ_4 = (657.83, 5, 2.12, 20, 27.149, 2.59)$ is G.A.S when ever $\Re_3 > 1$ and $\Re_4 > 1$.



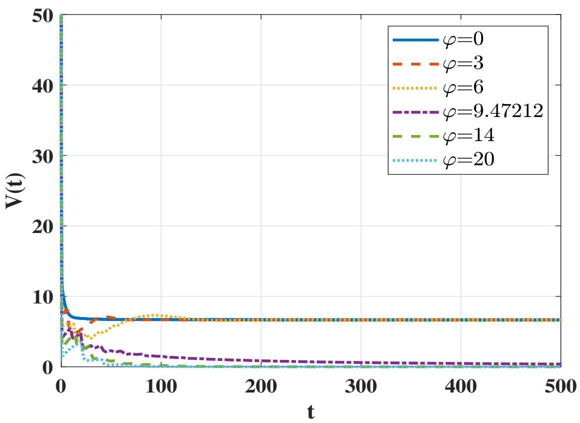
(A) Uninfected CD4$^+$T cells



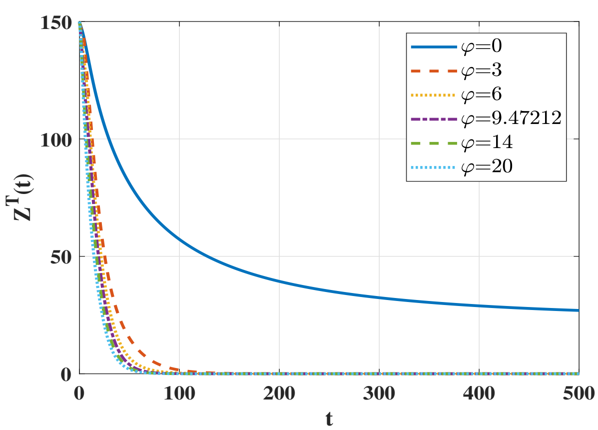
(B) Infected CD4$^+$T cells



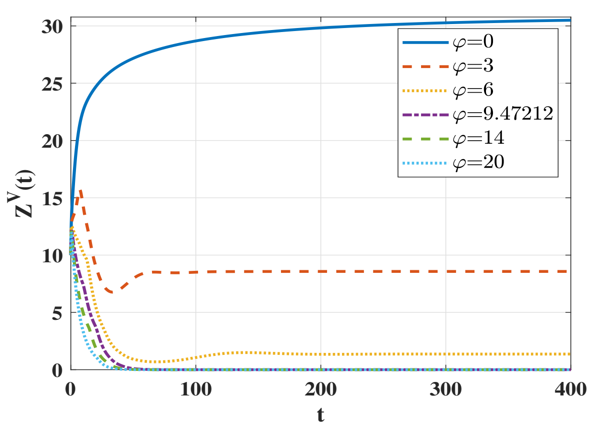
(C) Inflammatory cytokines



(D) Free HIV-1 particles

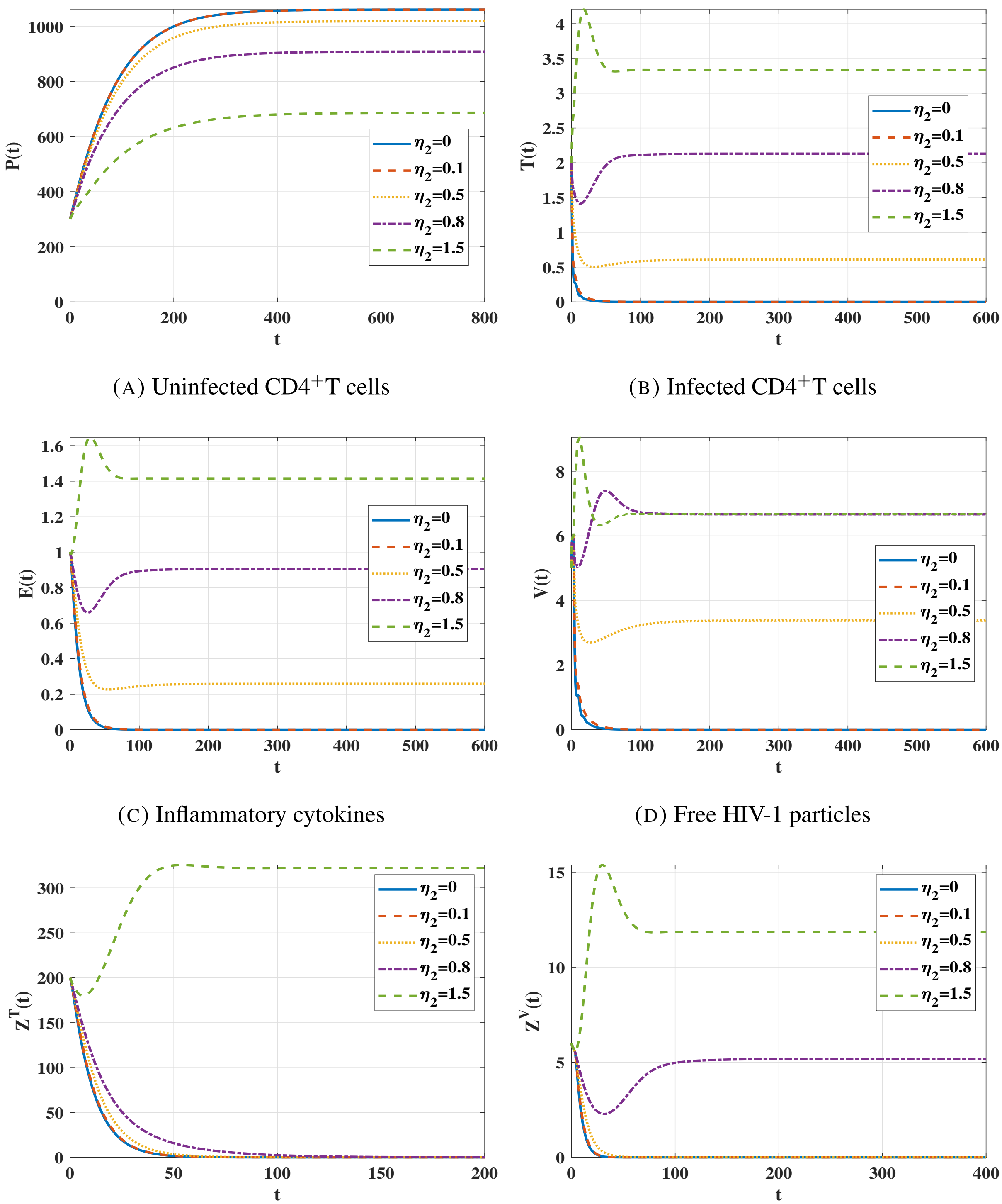


(E) CTLs



(F) Antibodies

FIGURE 6. Influence of the delay parameter.



(A) Uninfected CD4$^+$T cells

(B) Infected CD4$^+$T cells

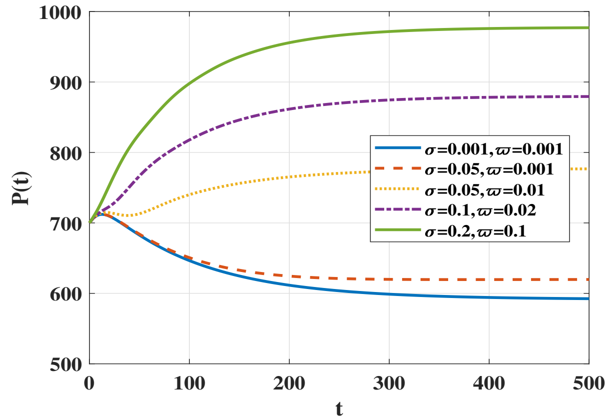
(C) Inflammatory cytokines

(D) Free HIV-1 particles

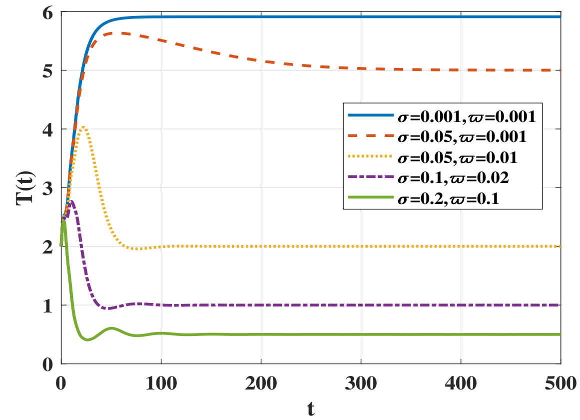
(E) CTLs

(F) Antibodies

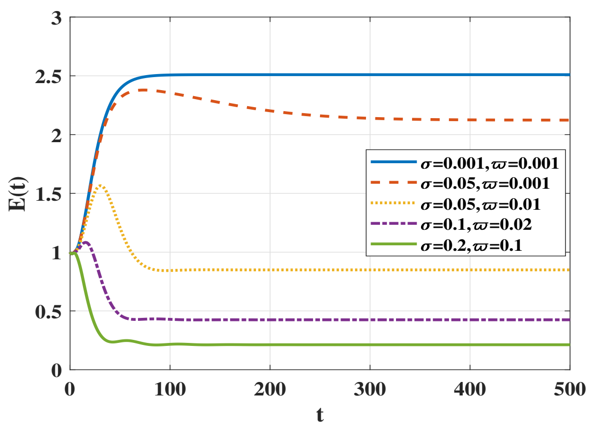
FIGURE 7. Influence of the C T C transmission.



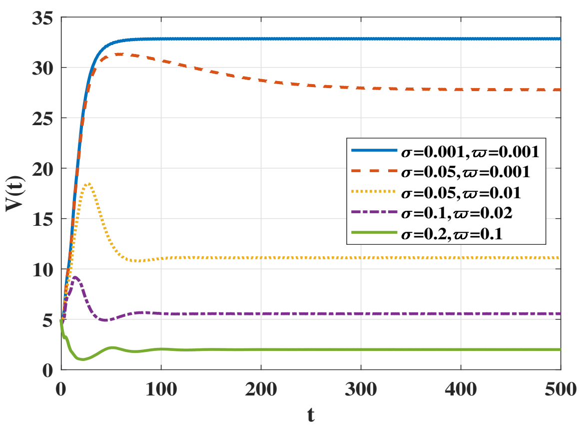
(A) Uninfected CD4$^+$T cells



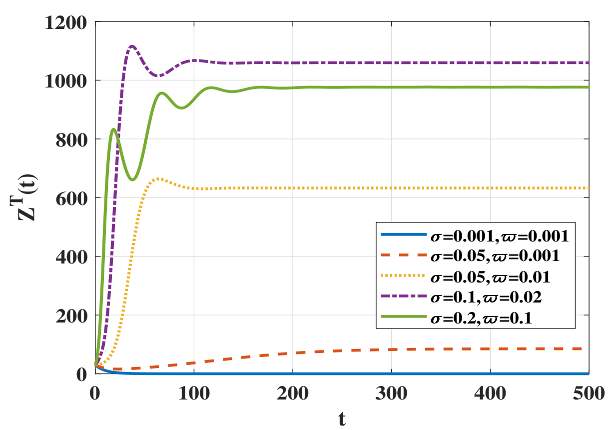
(B) Infected CD4$^+$T cells



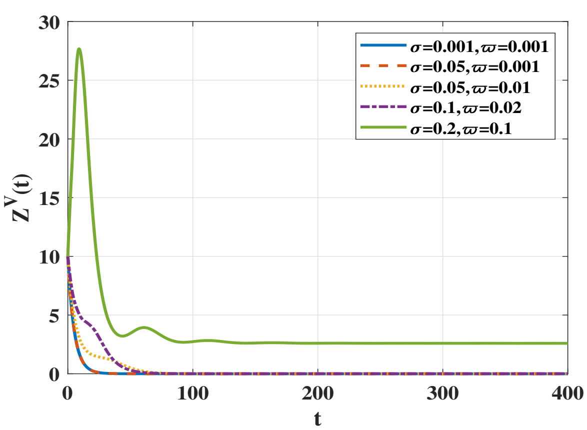
(C) Inflammatory cytokines



(D) Free HIV-1 particles



(E) CTLs



(F) Antibodies

FIGURE 8. Influence of the immune response parameters.

7. Conclusions

This study established a model for studying HIV-1 that includes cytokine-enhanced CTLs and cell-to-cell transmission infection. General functions provide the incidence rates of infected CD4$^+$T cells, inflammatory cytokines, and healthy CD4$^+$T cells with viruses. We established four distinct types of distributed-time to improve our comprehension of HIV-1 infection. The model included both VTC and CTC as transmission routes, CTC is caused by interaction between HIV-1-infected cells and healthy CD4$^+$T cells. We demonstrated that the model's solutions are limited and nonnegative. The presence and globally stability of these equilibria are represented by five threshold numbers, $\Re_0$, $\Re_1$, $\Re_2$, $\Re_3$ and $\Re_4$. Lyapunov functionals and LaSalle's invariance principle were used to demonstrate the global asymptotic stability for every equilibrium. The analytical results were validated by numerical simulations, which demonstrated a high degree of agreement with the theoretical predictions. The impact of inflammatory cytokines, time delays, and CTC transmission on the dynamics of HIV-1 were deliberated. CTC transmission and inflammatory cytokines both contribute to the number $\Re_0$, consequently, if any of They are neglected, $\Re_0$ will be underestimated. Additionally, it has been shown the extending time delays can successfully reduce $\Re_0$ and stop HIV-1 replication. That means if we can give the patient some treatments to delay of replication of HIV-1, that can be successfully stop the virus. This could indicate that new treatments are being created, which would lengthen the wait. According to our findings, CTC transmission, time delay and inflammatory cytokines are important elements of the HIV-1 model that cannot be disregarded. Finally, this study recommended giving the patients some medicine that can delay the appearance of the disease, which can help to stop HIV-1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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