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THE ROLE OF PREY VIGILANCE, CANNIBALISM, AND HUNTING COOPERATION ON PREY-PREDATOR DYNAMICS

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Abstract: Examining a prey-predator system that includes intricate behaviors like cannibalism, prey vigilance, and predators' hunting coordination is necessary to comprehend the complex dynamics of natural systems. These factors significantly affect population stability, persistence, and environmental equilibrium. Consequently, a novel prey-predator model with these features is proposed for study in this paper. Using the Lyapunov method of equilibria, the requirements for global stability, local stability with the aid of the linearization approach, and the existence of equilibria were established. Local bifurcation types in the sense of Sotomayor and Hopf bifurcations were detected. Finally, numerical simulations are used to comprehend how changing the parameters affects the model's dynamics.

Keywords: cannibalism; vigilance; hunting cooperation; bifurcation; stability.

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1. INTRODUCTION

There are numerous ways in which different species interact with one another. While many of these exchanges take place indirectly and are more difficult to observe, some are straightforward and obvious. Other species and even the larger environment are frequently impacted when one

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species' population size, health, or habitat changes [1]. Predators and their prey engage in one of the most prevalent and significant forms of interaction. One species hunts and consumes another in this relationship. This is essential to maintaining population balance. For instance, prey populations typically increase while predator populations decline. More predators may eventually result from that. However, if there are too many predators, they may drastically cut down on prey, which could result in food shortages and ultimately a fall in the predator population as well [2].

One essential technique for comprehending the behavior of ecological systems and forecasting their future paths is mathematical modeling. It provides an exact analytical framework for analyzing the intricate relationships that exist between living things and their surroundings. Population dynamics, interspecies interactions, extinction threats, and the impact of behavioral or environmental changes on ecosystem stability are all commonly studied using these models [3]. These models have their roots in the groundbreaking work of Lotka and Volterra, who developed a mathematical framework to explain the interaction between two species, a predator and its prey, in the early 20th century. While Novak [4] applied concepts from evolutionary dynamics to biological systems, Freedman [5] made a significant contribution to the study of the boundaries of stability and growth using deterministic ecological models. Some scientists have expanded the scope of application of mathematical models to various biological fields, including population growth and disease transmission [6-8]. Others, who have studied the relationship between stability and complexity in ecosystems, have strengthened this [9-11].

As ecological knowledge grew, scientists began to incorporate intricate behavioral elements into these models, including cannibalism within species, cooperative hunting in predators, and vigilance in prey. These behaviors are now regarded as essential elements in ecological modeling, as they dramatically alter the results of classical models. Prey animals use vigilance as a defense strategy to lower the risk of predation by devoting time and energy to examining their surroundings rather than feeding or procreating [12]. The trait or condition of being watchful is called vigilance. The idea of vigilance is frequently used by ecologists to evaluate the nonlethal effects of predators on their prey [13]. Increased vigilance can decrease predator density, but it also lowers prey growth rates because of shorter feeding times, according to recent research. According to mathematical studies [14-16], adding vigilance could result in complex dynamics such as limit cycles and abrupt extinction or unstable equilibria.

Group-living predators such as wolves and lions exhibit cooperative hunting, a social activity that increases hunting efficiency, particularly when targeting larger or more vigilant prey [17-18]. Later, many researchers successively studied the impact of cooperation in hunting on the dynamics of communities in the environment [16, 19-21]. It frequently results in an Allee effect, which necessitates a minimum number of cooperative predators to guarantee hunting success. If their population declines below this level, predator extinction may result, see [22-25].

Intraspecific predation, or cannibalism, is frequently seen in situations where resources are few or crowded. As a survival tactic, it lessens competition among the population. Cannibalism can drastically change the dynamics of a system, resulting in decreased population densities, enhanced stability, or even system collapse, as shown by studies like [26-29]. Later on, other studies emphasized how cannibalism can be combined with stage structure, refuges or fear effects to produce ecological models that are more accurate [30-32].

Notably, rather than functioning alone, these three behaviors, cannibalism, cooperation, and vigilance, may interact in real ecological systems. To compensate for decreased success rates brought on by increased prey awareness, predators might, for instance, adopt cooperative hunting; if this doesn't work, they might resort to cannibalism. In addition to offering more accurate depictions of ecosystem dynamics, models that integrate all three behaviors are helpful for developing strategies for managing endangered species. Consequently, the model proposed by Eric et al. [16] has been modified in this research to incorporate cannibalism in the predator population.

The structure of this document is as follows: Section 2 presents the suggested modeling framework, together with the ecological presumptions that underlie it and the initial findings about the positivity and boundedness of solutions. Finding workable equilibria and doing stability analysis on the suggested model are the focus of Section 3. Section 4 discusses the analysis of global stability. The local bifurcation results are derived in Section 5. Section 6 discusses the occurrence of the Hopf bifurcation. However, section 7 offers numerical tests that support our theoretical conclusions. The paper's conclusion, which includes a discussion of our findings, is given in section 8.

2. MODEL FORMULATION

The following prey-predator model is proposed and studied to understand the true dynamic behavior in the environment and maintain the diversity and balance of the ecosystem. It takes into account many actual biological elements such as prey vigilance, cannibalism, and hunting cooperation. Let $X(t)$ and $Y(t)$ denote the prey and predator population densities at any time t. The dynamics of a prey-predator system that satisfies the following hypotheses can be represented in the model (1) below:

- In the absence of predators and vigilance behavior, the prey population increases logistically; the degree of vigilance is indicated by v, where $v \in [0,1]$. Furthermore, the lethality of predation in the absence of vigilance is $\frac{1}{l}$.
- Predators are thought to work together when hunting their prey. Additionally, the Holling type II functional response is used to explain the predation.
- The predator population is thought to exhibit cannibalism with the use of Holling type II functional response.
- Lastly, it is assumed that the predator has a natural death rate of δ_1 while the prey has a natural death rate of δ_0.

$$\begin{aligned} \frac{dX}{dt} &= rX\left[1 - v - \frac{X}{k}\right] - \delta_0 X - \frac{(q+cy)XY}{(b+X)(l+\mu v)} = F(X,Y), \\ \frac{dY}{dt} &= \frac{\gamma(q+cY)XY}{(b+X)(l+\mu v)} - \delta_1 Y + \beta Y - \frac{e(1-m)Y^2}{k_1+(1-m)Y} = G(X,Y), \end{aligned} \tag{1}$$

with a non-negative initial conditions $X(0) = x_0$ and $Y(0) = y_0$. It is assumed that all parameters involved are positive, and their descriptions are provided in Table 1. A prey-predator system that has hunting cooperation and antipredator behavior can be represented using the following set of differential equations

$$\begin{aligned} \frac{dX}{dt} &= X\left[r\left(1 - v - \frac{X}{k}\right) - \delta_0 - \frac{(q+cY)Y}{(b+X)(l+\mu v)}\right] = \mathrm{X}\, f(X,Y), \\ \frac{dY}{dt} &= Y\left[\frac{\gamma(q+cY)X}{(b+X)(l+\mu v)} - \delta_1 + \beta - \frac{e(1-m)Y}{k_1+(1-m)Y}\right] = \mathrm{Y}\, g(X,Y). \end{aligned} \tag{2}$$

The functions of system (2), $f(X,Y)$ and $g(X,Y)$, on the right-hand side, are Lipschitz continuous because they, along with their partial derivatives, are continuous. Therefore, system (2) with the initial conditions $X(0) \geq 0$ and $Y(0) \geq 0$ has a unique solution by the fundamental theorem of existence and uniqueness for initial value problems.

Table 1. Parameters description

Description	**Parameter**
r	The prey growth rate.
v	Level of prey vigilance.
k	The environment's carrying capacity.
δ_0	The prey's natural death rate.
q	The predator attack rate.
c	The hunting cooperation rate.
b	The half-saturation constant.
$1/l$	The predation lethality in the absence of prey vigilance.
µ	The prey's vigilance effectiveness.
γ	The energy gained by predation.
δ_1	The predator's natural death rate.
β	The self-predation's growth rate.
e	The cannibalism rate.
k_1	The cannibalism half-saturation constant.
m	The predator refuge rate.

Theorem 1. System (2) is a positively invariant system.

Proof of Theorem 1. Since system (2) is a Kolmogorov system, with the prey and predator growth rates $f(X,Y)$ and $g(X,Y)$, respectively. Therefore, we can solve it using the positive initial conditions $\big(X(0),Y(0)\big) = (x_0, y_0)$ to obtain:

$$X(\mathrm{t}) = x_0 e^{\int_0^t \left[r\left(1-v-\frac{X(s)}{k}\right)-\delta_0-\frac{(q+cY(s))Y(s)}{(b+X(s))(l+\mu v)}\right]ds}$$

$$Y(\mathrm{t}) = y_0 e^{\int_0^t \left[\frac{\gamma(q+cY(s))}{(b+X(s))(l+\mu v)}-\delta_1+\beta-\frac{e(1-m)Y(s)}{k_1+(1-m)Y(s)}\right]ds}.$$

Because of the exponential function's definition, any solution in the $int.\ \mathbb{R}_+^2 = \{(X,Y) \in \mathbb{R}^2 : X(T) > 0, Y(T) > 0\}$ remains there eternally, due to the previous two equations.

Theorem 2. In the region,

$$\varphi = \left\{(X,Y) \in \mathbb{R}_+^2 : 0 \leq X < \frac{r^2(1-v)}{k} = \varepsilon, 0 \leq X + \frac{1}{\gamma}Y < \frac{\varepsilon_1}{(\delta_1-\beta)}\right\}.$$

All of the system (2) solutions are uniformly bounded, provided that $\delta_1 - \beta > 0$.

Proof of Theorem 2. From the first equation of system (2), it is obtained that

$$\frac{dX}{dt} \leq X\left(r(1-v) - \frac{X}{\frac{k}{r}}\right).$$

Then, according to Lemma 2.2 [33], it is obtained that:

$$x(t) \leq \frac{r^2(1-v)}{k}\left[1 + \left(\frac{r^2(1-v)}{k}{x_0}^{-1} - 1\right)e^{-r(1-v)t}\right]^{-1}.$$

Then for $\rightarrow \infty$, then $X(t) \leq \frac{r^2(1-v)}{k} = \varepsilon > 0$.

Now, let $N = X + \frac{1}{\gamma}Y$, then it is obtained that:

$$\frac{dN}{dt} \leq [r(1-v) - \delta_0 + \delta_1 - \beta]X - (\delta_1 - \beta)\left[X + \frac{1}{\gamma}Y\right] \leq \varepsilon_1 - (\delta_1 - \beta)N.$$

where $\varepsilon_1 = r(1-v) - \delta_0 + \delta_1 - \beta$. Moreover, $r(1-v) - \delta_0 > 0$, which represents the prey's survival condition in the absence of a predator. Now, according to Lemma 2.1 [33], it is obtained that:

$$N(t) \leq \frac{\varepsilon_1}{(\delta_1-\beta)}\left[1 + \left(\frac{(\delta_1-\beta)}{\varepsilon_1}x(0) - 1\right)e^{-((\delta_1-\beta))t}\right].$$

Hence, as $t \rightarrow \infty$, then $N(t) \leq \frac{\varepsilon_1}{(\delta_1-\beta)}$. Thus, the proof is complete.

3. EQUILIBRIA AND THEIR STABILITY

The system (2) has four nonnegative equilibrium points. The entire extinction equilibrium point $s_0 = (0,0)$ always exists.

The predator-free equilibrium point $s_1 = (\breve{X}, 0) = \left(k\left(1 - v - \frac{\delta_0}{r}\right), 0\right)$, that exists when the prey's survival condition holds

$$\delta_0 < r(1-v). \tag{3}$$

However, the prey-free equilibrium point $s_2 = (0, \hat{Y}) = \left(0, \frac{k_1(\delta_1-\beta)}{(1-m)(-\delta_1+\beta-e)}\right)$ that exists when the following condition holds:

$$\delta_1 < \beta < \delta_1 + e. \tag{4}$$

Finally, the co-existing equilibrium point $s_3 = (\tilde{X}, \tilde{Y})$ is the intersection point for the non-trivial prey and predator nullclines $f(X,Y) = 0$ and $g(X,Y) = 0$, where $f(X,Y)$ and $g(X,Y)$ are given in system (2). Direct computation shows that:

$$\tilde{X} = \frac{b(l+v\mu)[(1-m)\tilde{Y}(e-\beta+\delta_1)-k_1(\beta-\delta_1)]}{[(q+c\tilde{Y})\gamma+(l+v\mu)(\beta-\delta_1)][(1-m)\tilde{Y}+k_1]-e(1-m)\tilde{Y}(l+v\mu)}, \tag{5}$$

which is positive under the following sufficient condition:

$$\left.\begin{array}{c} \beta < \delta_1 \\ \frac{e(1-m)\tilde{Y}(l+v\mu)}{[(1-m)\tilde{Y}+k_1]} + (l+v\mu)(\delta_1-\beta) < (q+c\tilde{Y})\gamma \end{array}\right\}. \tag{6}$$

While the $\tilde{Y}$ is the positive root of the polynomial equation:

$$\Delta_1 Y^5 + \Delta_2 Y^4 + \Delta_3 Y^3 + \Delta_4 Y^2 + \Delta_5 Y + \Delta_6 = 0, \tag{7}$$

where $\Delta_i, i = 1,2,\ldots.6$ are given in the Appendix. Straightforward computation shows that $\tilde{Y}$ exists uniquely under one set of the following sets of conditions:

$$\left.\begin{array}{l} \Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \Delta_4 > 0, \Delta_5 > 0, \Delta_6 < 0 \\ \Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \Delta_4 > 0, \Delta_5 < 0, \Delta_6 < 0 \\ \Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \Delta_4 < 0, \Delta_5 < 0, \Delta_6 < 0 \\ \Delta_1 > 0, \Delta_2 > 0, \Delta_3 < 0, \Delta_4 < 0, \Delta_5 < 0, \Delta_6 < 0 \\ \Delta_1 > 0, \Delta_2 < 0, \Delta_3 < 0, \Delta_4 < 0, \Delta_5 < 0, \Delta_6 < 0 \end{array}\right\}. \tag{8}$$

Now, the Jacobian matrix of system (2) at the point (X,Y) can be written as:

$$J = (a_{ij})_{2\times 2}, \tag{9}$$

where:

$$a_{11} = X\left(-\frac{r}{k} + \frac{Y(q+cY)}{(b+X)^2(l+v\mu)}\right) + f(X,Y),$$

$$a_{12} = -X\left(\frac{q+2cY}{(b+X)(l+v\mu)}\right),$$

$$a_{21} = Y\left(\frac{b\gamma(q+cY)}{(b+X)^2(l+v\mu)}\right),$$

$$a_{22} = Y\left(\frac{c\gamma X}{(b+X)(l+v\mu)} - \frac{k_1 e(1-m)}{(k_1+(1-m)Y)^2}\right) + g(X,Y).$$

Therefore, the Jacobian matrix at the entire extinction point s_0 becomes:

$$J_{s_0} = \begin{pmatrix} r - rv - \delta_0 & 0 \\ 0 & -\delta_1 + \beta \end{pmatrix}. \tag{10}$$

Consequently, the eigenvalues are given by $\lambda_{01} = r(1-v) - \delta_0$, and $\lambda_{02} = -\delta_1 + \beta$. Thus, the point s_0 is a locally asymptotically stable or sink if the following condition holds.

$$\left.\begin{matrix} r(1-v) < \delta_0 \\ \beta < \delta_1 \end{matrix}\right\}. \tag{11}$$

The Jacobian matrix (9) at the predator-free equilibrium point s_1 becomes:

$$J_{s_1} = \begin{pmatrix} -\frac{r\breve{X}}{k} & -\frac{q\breve{X}}{(b+\breve{X})(l+v\mu)} \\ 0 & \beta + \frac{q\gamma\breve{X}}{(b+\breve{X})(l+v\mu)} - \delta_1 \end{pmatrix}. \tag{12}$$

Hence, the eigenvalues are given by $\lambda_{11} = -\frac{r\breve{X}}{k}$, and $\lambda_{12} = \beta + \frac{q\gamma\breve{X}}{(b+\breve{X})(l+v\mu)} - \delta_1$. Therefore, the point s_1 is locally asymptotically stable if and only if the following condition holds.

$$\beta + \frac{q\gamma\breve{X}}{(b+\breve{X})(l+v\mu)} < \delta_1. \tag{13}$$

However, if the condition (13) is reflected, then it becomes a saddle point.

The Jacobian matrix (9) at the prey-free equilibrium point s_2 turns into the form:

$$J_{s_2} = \begin{pmatrix} r(1-v) - \frac{\hat{Y}(q+c\hat{Y})}{b(l+v\mu)} - \delta_0 & 0 \\ \frac{\gamma\hat{Y}(q+c\hat{Y})}{b(l+v\mu)} & -\frac{k_1 e(1-m)\hat{Y}}{((1-m)\hat{Y}+k_1)^2} \end{pmatrix}. \tag{14}$$

Hence, the eigenvalues are given by $\lambda_{21} = r(1-v) - \frac{\hat{Y}(q+c\hat{Y})}{b(l+v\mu)} - \delta_0$, and $\lambda_{22} = -\frac{k_1 e(1-m)\hat{Y}}{((1-m)\hat{Y}+k_1)^2}$. Therefore, the point s_2 is locally asymptotically stable if and only if the following condition holds.

$$r(1-v) < \frac{\hat{Y}(q+c\hat{Y})}{b(l+v\mu)} + \delta_0. \tag{15}$$

However, if the condition (15) is reflected, then it becomes a saddle point.

Finally, the Jacobian matrix (9) at the co-existing point $s_3 = (\tilde{X}, \tilde{Y})$ turns into the form:

$$J_{s_3} = \begin{pmatrix} -\frac{r\tilde{X}}{k} + \frac{\tilde{X}\tilde{Y}(q+c\tilde{Y})}{(b+\tilde{X})^2(l+v\mu)} & -\frac{\tilde{X}(q+2c\tilde{Y})}{(b+\tilde{X})(l+v\mu)} \\ \frac{b\gamma(q+c\tilde{Y})\tilde{Y}}{(b+\tilde{X})^2(l+v\mu)} & \frac{c\gamma\tilde{X}\tilde{Y}}{(b+\tilde{X})(l+v\mu)} - \frac{k_1 e(1-m)\tilde{Y}}{(k_1+(1-m)\tilde{Y})^2} \end{pmatrix} = (\tilde{a}_{ij}). \tag{16}$$

Consequently, the local stability conditions for the point s_3 are contracted in the following theorem.

Theorem 3. The co-existing point s_3 is a sink if the next sufficient conditions are satisfied.

$$\frac{\tilde{Y}(q+c\tilde{Y})}{(b+\tilde{X})^2(l+v\mu)} < \frac{r}{k}. \tag{17}$$

$$\frac{c\gamma\tilde{X}}{(b+\tilde{X})(l+v\mu)} < \frac{k_1 e(1-m)}{(k_1+(1-m)\tilde{Y})^2}. \tag{18}$$

Proof of Theorem 3. The characteristic polynomial of the matrix (16) can be written in the form:

$$\lambda^2 - T_r\lambda + D_e = 0, \tag{19}$$

where $T_r = \tilde{a}_{11} + \tilde{a}_{22}$ and $D_e = \tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}\tilde{a}_{21}$. According to the Routh-Hurwitz criterion, the equation (19) has two roots with negative real parts if and only if $T_r < 0$ and $D_e > 0$. Direct calculations indicate that the above conditions (17)-(18) satisfy the requirements of the Routh-Hurwitz criterion. Therefore, the co-existing point is a sink.

4. GLOBAL STABILITY ANALYSIS

The trait of a dynamical system where all trajectories, independent of their initial circumstances, converge to a particular equilibrium point over time is known as global stability of equilibria. Stated differently, an equilibrium point is considered globally stable if it draws in all potential system solutions, not simply those that begin near the equilibrium (local stability). Therefore, in the next theorems, the global stability conditions of the equilibria are established.

Theorem 4. The extinction state s_0 is globally stable under the condition (11).

Proof of Theorem 4. Consider the Lyapunov function $V_0(X,Y) = \gamma\, X(t) + Y(t)$. Then it is noted that $V_0(s_0) = 0$ and $V_0(X,Y) > 0$ elsewhere in the domain $\mathbb{R}^2_+$. Moreover,

$$\begin{aligned}\frac{dV_0}{dt} = \gamma\left[rX\left(1 - v - \frac{X}{k}\right) - \delta_0 X - \frac{(q+cY)XY}{(b+X)(l+\mu v)}\right] + \frac{\gamma(q+cY)XY}{(b+X)(l+\mu v)} \\ -\delta_1 Y + \beta Y - \frac{e(1-m)Y^2}{k_1+(1-m)Y}\end{aligned}.$$

Then, direct computation leads to the following:

$$\frac{dV_0}{dt} = \gamma X(r(1-v) - \delta_0) - \frac{\gamma r X^2}{k} - \delta_1 Y + \beta Y - \frac{e(1-m)Y^2}{k_1+(1-m)Y}.$$

Furthermore, it yields:

$$\frac{dV_0}{dt} \le \gamma X(r(1-v) - \delta_0) - (\delta_1 - \beta)Y.$$

Since the derivative of the function V_0 is a negative definite function, then s_0 is globally stable. This completes the proof.

Theorem 5. The predator-free state s_1 is globally stable if the following conditions hold.

$$\frac{c\hat{X}}{b(l+\mu v)} < \frac{e(1-m)}{\gamma(k_1+(1-m)Y_{max})}, \tag{20}$$

$$\left(\frac{\beta}{\gamma} + \frac{q\hat{X}}{b(l+\mu v)}\right) < \frac{\delta_1}{\gamma}. \tag{21}$$

Proof of Theorem 5. Consider the Lyapunov function $V_1(X,Y) = \left[X - \breve{X} - \breve{X}\ln\left(\frac{X}{\breve{X}}\right) + \frac{Y}{\gamma}\right]$, which satisfies the positive definite property. Then the derivative of V_1 can be written as:

$$\frac{dV_1}{dt} = \left(X - \breve{X}\right)\left[r\left(1 - v - \frac{X}{k}\right) - \delta_0 - \frac{(q+cY)Y}{(b+X)(l+\mu v)}\right] + \frac{1}{\gamma}\left[\frac{\gamma(q+cY)XY}{(b+X)(l+\mu v)} - \delta_1 Y + \beta Y - \frac{e(1-m)Y^2}{k_1+(1-m)Y}\right].$$

By substituting $r(1-v) - \delta_0 = \frac{r}{k}\breve{X}$, it is obtained that:

$$\frac{dV_1}{dt} = -\frac{r(X-\breve{X})^2}{k} + \frac{q\breve{X}Y}{b(l+\mu v)} + \frac{c\breve{X}Y^2}{b(l+\mu v)} - \frac{1}{\gamma}(\delta_1 - \beta)Y - \frac{e(1-m)Y^2}{\gamma(k_1+(1-m)Y)}.$$

Furthermore, using the condition (20) yields:

$$\frac{dV_1}{dt} \le -\frac{r(X-\breve{X})^2}{k} - \left[\frac{\delta_1}{\gamma} - \frac{\beta}{\gamma} - \frac{q\breve{X}}{b(l+\mu v)}\right]Y.$$

Since the derivative of the function V_1 is a negative definite function due to condition (21), then s_1 is globally stable. This completes the proof.

Theorem 6. The prey-free state s_2 is globally stable if the following condition holds:

$$r(1-v) < \delta_0. \tag{22}$$

Proof of Theorem 6. Consider the Lyapunov function $V_2(t) = X + Y - \hat{Y} - \hat{Y}\ln\left(\frac{Y}{\hat{Y}}\right)$, which satisfies the positive definite property. The derivative of V_2 can be written as:

$$\frac{dV_2}{dt} = \frac{dX}{dt} + \left(\frac{Y-\hat{Y}}{Y}\right)\frac{dY}{dt}.$$

Then, it is obtained that:

$$\frac{dV_2}{dt} = rX\left(1 - v - \frac{X}{k}\right) - \delta_0 X - \frac{(q+cY)XY}{(b+X)(l+\mu v)} + \left(Y - \hat{Y}\right)\left[\frac{\gamma(q+cY)X}{(b+X)(l+\mu v)} - \frac{e(1-m)k_1(Y-\hat{Y})}{[k_1+(1-m)Y][k_1+(1-m)\hat{Y}]}\right].$$

Moreover, this yields that:

$$\frac{dV_2}{dt} \le -[\delta_0 - r(1-v)]X - \frac{e(1-m)k_1(Y-\hat{Y})^2}{[k_1+(1-m)Y][k_1+(1-m)\hat{Y}]}.$$

Since the derivative of the function V_2 is a negative definite function due to condition (22), then s_2 is globally stable. This completes the proof.

Theorem 7. The co-existence state s_3 has a basin of attraction that satisfies the following conditions.

$$\frac{M(q+c\,\tilde{Y}\,)\tilde{Y}}{A\tilde{A}} < \frac{r}{k}. \tag{23}$$

$$\frac{e(1-m)k_1}{B\tilde{B}} < \frac{\gamma c(b+\tilde{X})MX}{A\tilde{A}}. \tag{24}$$

$$\left[\frac{M_1}{A\tilde{A}}\right]^2 < 4\left[\frac{r}{k} - \frac{M(q+c\,\tilde{Y}\,)\tilde{Y}}{A\tilde{A}}\right]\left[\frac{\gamma c(b+\tilde{X})MX}{A\tilde{A}} - \frac{e(1-m)k_1}{B\tilde{B}}\right]. \tag{25}$$

All the new symbols are defined in the proof.

Proof of Theorem 7: Consider the Lyapunov function $V_3 = \left[X - \tilde{X} - \tilde{X}\ln\left(\frac{X}{\tilde{X}}\right)\right] + \left[Y - \tilde{Y} - \tilde{Y}\ln\left(\frac{Y}{\tilde{Y}}\right)\right]$ that satisfies the positive definite property. The derivative of V_3 can be written as:

$$\begin{aligned}\frac{dV_3}{dt} = \left(X - \tilde{X}\right)&\left[-\frac{r}{k}\left(X - \tilde{X}\right) - \frac{(b+\tilde{X})Mq(Y-\tilde{Y})}{A\tilde{A}} + \frac{M(q+c\,\tilde{Y}\,)\tilde{Y}(X-\tilde{X})}{A\tilde{A}} - \frac{c(b+\tilde{X})M(Y+\tilde{Y})(Y-\tilde{Y})}{A\tilde{A}}\right] \\ &- \left(Y - \tilde{Y}\right)\left[\frac{\gamma bM(q+c\,\tilde{Y}\,)(X-\tilde{X})}{A\tilde{A}} + \frac{\gamma c(b+\tilde{X})MX(Y-\tilde{Y})}{A\tilde{A}} - \frac{e(1-m)k_1(Y-\tilde{Y})}{B\tilde{B}}\right]\end{aligned},$$

where $M = (l + \mu v)$, $A = (b + X)(l + \mu v)$, $\tilde{A} = \left(b + \tilde{X}\right)(l + \mu v)$, $B = k_1 + (1 - m)Y$, and $\tilde{B} = k_1 + (1 - m)\tilde{Y}$. Moreover, this yields that:

$$\begin{aligned}\frac{dV_3}{dt} = -&\left[\frac{r}{k} - \frac{M(q+c\,\tilde{Y}\,)\tilde{Y}}{A\tilde{A}}\right]\left(X - \tilde{X}\right)^2 - \frac{M_1}{A\tilde{A}}\left(X - \tilde{X}\right)\left(Y - \tilde{Y}\right) \\ &- \left[\frac{\gamma c(b+\tilde{X})MX}{A\tilde{A}} - \frac{e(1-m)k_1}{B\tilde{B}}\right](Y - \tilde{Y})^2\end{aligned},$$

where $M_1 = M\left[\left(b + \tilde{X}\right)\left(q + cY + c\tilde{Y}\right) + \gamma b\left(q + c\tilde{Y}\right)\right]$.

Therefore, due to the definition of the negative definite quadratic form in two variables, $\frac{dV_3}{dt}$ is a negative definite function under the given conditions (23)-(25). Hence, the proof is done.

5. LOCAL BIFURCATION

Changes in the qualitative structure of sets of curves, such as the integral curves of vector fields or the solutions to differential equations, are studied by bifurcation theory. When a system's behavior is drastically changed by a slight, gradual change in its parameter values, this is known as a bifurcation. The main use of this idea is in the mathematical study of dynamical systems. There are two types of bifurcations: local and global. When a parameter reaches critical thresholds in the first category, the local dynamics undergo substantial modifications. The second kind occurs when the system's bigger invariant sets interact with one another or with the system's equilibrium; these cannot be identified merely by looking at the equilibrium's stability, which is outside the purview of this work.

Let us consider system (2) in the following form:

$$\frac{dN}{dt} = \boldsymbol{F}(\boldsymbol{N}), \text{ with } \boldsymbol{N} = \begin{pmatrix} X \\ Y \end{pmatrix} \text{ and } \boldsymbol{F} = \begin{pmatrix} X\,f(X,Y,\mu) \\ Y\,g(X,Y,\mu) \end{pmatrix}, \tag{26}$$

where $\mu \in \mathbb{R}$ is any parameter. Consequently, the second and third directional derivative of $\boldsymbol{F}$, where $\boldsymbol{V} = (v_1, v_2)^T$ is any vector, can be written using direct computation as:

$$D^2\boldsymbol{F}(\boldsymbol{N},\mu)(V,V) = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}, \tag{27}$$

where:

$$c_{11} = -\frac{2[(b+X)^3 r(l+v\mu) - bkY(q+cY)]v_1^2}{k(b+X)^3(l+v\mu)} - \frac{2b(q+2cY)v_1v_2}{(b+X)^2(l+v\mu)} - \frac{2cXv_2^2}{(b+X)(l+v\mu)}.$$

$$\begin{aligned} c_{21} = &-\frac{2bY(q+cY)\gamma v_1^2}{(b+X)^3(l+v\mu)} + \frac{2b(q+2cY)\gamma v_1v_2}{(b+X)^2(l+v\mu)} + \frac{2c(1-m)^3\gamma XY^3 + 6c(1-m)^2\gamma k_1XY^2}{(b+X)(l+v\mu)\big((1-m)Y+k_1\big)^3} v_2^2 \\ &+ \frac{2c\gamma k_1^3X - 2(1-m)(bel+elX-3c\gamma XY+ev(b+X)\mu)k_1^2}{(b+X)(l+v\mu)\big((1-m)Y+k_1\big)^3} v_2^2 \end{aligned}.$$

While:

$$D^3\boldsymbol{F}(\boldsymbol{N},\mu)(V,V,V) = \begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix}, \tag{28}$$

where:

$$d_{11} = -\frac{6bv_1(Yv_1-(b+X)v_2)((q+cY)v_1-c(b+X)v_2)}{(b+X)^4(l+v\mu)}.$$

$$d_{21} = 6\left(\frac{bY(q+cY)\gamma v_1^3}{(b+X)^4(l+v\mu)} - \frac{b(q+2cY)\gamma v_1^2v_2}{(b+X)^3(l+v\mu)} + \frac{bc\gamma v_1v_2^2}{(b+X)^2(l+v\mu)} + \frac{e(-1+m)^2k_1^2v_2^3}{(Y-mY+k_1)^4}\right)$$

Theorem 8. At $\beta = \delta_1 (:\equiv \beta^*)$, system (2) enters into a transcritical bifurcation around the entire extinction point s_0 provided that $r(1-v) \neq \delta_0$.

Proof of Theorem 8: At the $\beta = \beta^*$, the matrix (10) becomes:

$$J_{s_0}(\beta^*) = \begin{pmatrix} r(1-v)-\delta_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, the resulting eigenvalues are $\lambda_{01}(\beta^*) = r(1-v) - \delta_0 \neq 0$ and $\lambda_{02}(\beta^*) = 0$. Therefore, the eigenvectors of $J_{s_0}(\beta^*)$ and their transpose $\left[J_{s_0}(\beta^*)\right]^T$, will be $E_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $U_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. Moreover, it is obtained that:

$$U_0{}^T[\boldsymbol{F}_\beta(s_0, \beta^*)] = 0.$$

$$U_0{}^T[D\boldsymbol{F}_\beta(s_0, \beta^*)E_0] = 1.$$

Furthermore, by using Eq. (27) at (s_0, β^*), and the eigenvectors E_0 and U_0 gives:

$$U_0{}^T[D^2\boldsymbol{F}(s_0, \beta^*)(E_0, E_0)] = \frac{-2(1-m)e}{k_1} \neq 0.$$

Accordingly, all Sotomayor's theorem [34] requirements for having a transcritical bifurcation around s_0 are satisfied. Hence, the proof is done.

Theorem 9. At $\delta_1 = \beta + \frac{q\gamma\breve{X}}{(b+\breve{X})(l+v\mu)} (:\equiv \delta_1^*)$, system (2) enters into a transcritical bifurcation around s_1 provided that:

$$-\frac{2bq^2\gamma k}{r(b+\breve{X})^3(l+v\mu)^2} + \frac{2c\gamma\breve{X}}{(b+\breve{X})(l+v\mu)} - \frac{2(1-m)e}{k_1} \neq 0. \tag{29}$$

Otherwise, it has a pitchfork bifurcation around s_1 provided that:

$$-\frac{bq^3\gamma k^2}{r^2(b+\breve{X})^5(l+v\mu)^3} - \frac{bc\gamma qk}{r(b+\breve{X})^3(l+v\mu)^2} + \frac{e(1-m)^2}{k_1^2} \neq 0. \tag{30}$$

Proof. From the JM that is given in equation (12), it is observed that, for $\delta_1 = \delta_1^*$, it becomes

$$J_{s_1}(\delta_1^*) = \begin{pmatrix} -\frac{r\breve{X}}{k} & -\frac{q\breve{X}}{(b+\breve{X})(l+v\mu)} \\ 0 & 0 \end{pmatrix}$$

Hence the eigenvalues are $\lambda_{11}(\delta_1^*) = -\frac{r\breve{X}}{k}$ and $\lambda_{12}(\delta_1^*) = 0$ .Thus, s_1 becomes a non-hyperbolic point. Therefore, the eigenvectors of $J_{s_1}(\delta_1^*)$ and their transpose $\left[J_{s_1}(\delta_1^*)\right]^T$, will be $E_1 = \begin{pmatrix} -\frac{qk}{r(b+\breve{X})(l+v\mu)} \\ 1 \end{pmatrix}$ and $U_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. Moreover, it is obtained that:

$$U_1{}^T[\boldsymbol{F}_{\delta_1}(s_1, \delta_1^*)] = 0.$$

$$U_1{}^T[D\boldsymbol{F}_{\delta_1}(s_1, \delta_1^*)E_1] = -1.$$

Furthermore, by using Eq. (27) at (s_1, δ_1^*), and the eigenvectors E_1 and U_1 gives:

$$U_1^{\,T}[D^2\boldsymbol{F}(s_1,\delta_1^*)(E_1,E_1)] = -\frac{2bq^2\gamma k}{r(b+\breve{X})^3(l+v\mu)^2} + \frac{2c\gamma\breve{X}}{(b+\breve{X})(l+v\mu)} - \frac{2(1-m)e}{k_1}.$$

Hence, the first part of the proof is done, provided that condition (29) holds. Otherwise, using Eq. (28) leads to:

$$U_1^{\,T}[D^3\boldsymbol{F}(s_1,\delta_1^*)(E_1,E_1,E_1)] = 6\left(-\frac{bq^3\gamma k^2}{r^2(b+\breve{X})^5(l+v\mu)^3} - \frac{bc\gamma qk}{r(b+\breve{X})^3(l+v\mu)^2} + \frac{e(1-m)^2}{k_1^2}\right).$$

Therefore, the second part of the proof follows when the condition (30) holds.

Theorem 10. At $r = \frac{1}{(1-v)}\left(\frac{\hat{Y}(q+c\hat{Y})}{b(l+v\mu)} + \delta_0\right) (:\equiv r^*)$, system (2) enters into a transcritical bifurcation around s_2 provided that:

$$-\frac{2r^*}{k} + \frac{2\hat{Y}(q+c\hat{Y})}{b^2(l+v\mu)} - \frac{2k_1e(1-m)(q+2c\hat{Y})}{\gamma(q+c\hat{Y})((1-m)\hat{Y}+k_1)^2} \neq 0. \tag{31}$$

Otherwise, it has a pitchfork bifurcation around s_2 provided that:

$$(q+c\hat{Y})\hat{Y} - \frac{k_1e(1-m)b^2(l+v\mu)}{\gamma((1-m)\hat{Y}+k_1)^2}\left(\frac{c\hat{Y}}{(q+c\hat{Y})} + 1\right) + cb^2\left(\frac{k_1e(1-m)b(l+v\mu)}{\gamma(q+c\hat{Y})((1-m)\hat{Y}+k_1)^2}\right)^2 \neq 0. \tag{32}$$

Proof of Theorem 10. From the JM that is given by Eq. (14), it is observed that for $r = r^*$ it becomes:

$$J_{s_2}(r^*) = \begin{pmatrix} 0 & 0 \\ \frac{\gamma\hat{Y}(q+c\hat{Y})}{b(l+v\mu)} & -\frac{k_1e(1-m)\hat{Y}}{((1-m)\hat{Y}+k_1)^2} \end{pmatrix}.$$

Hence, the eigenvalues are given by $\lambda_{21} = 0$, $\lambda_{22} = -\frac{k_1e(1-m)\hat{Y}}{((1-m)\hat{Y}+k_1)^2}$. Thus, the prey-free equilibrium point s_2 becomes a non-hyperbolic point. Therefore, the eigenvectors of $J_{s_2}(r^*)$ and their transpose $\left[J_{s_2}(r^*)\right]^T$ will be $E_2 = \begin{pmatrix} 1 \\ \frac{k_1e(1-m)b(l+v\mu)}{\gamma(q+c\hat{Y})((1-m)\hat{Y}+k_1)^2} \end{pmatrix}$ and $U_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, respectively.

Moreover, it is obtained that:

$$U_2^{\,T}[\boldsymbol{F}_r(s_2,r^*)] = 0.$$

$$U_2^{\,T}[D\boldsymbol{F}_r(s_2,r^*)E_2] = 1 - v \neq 0.$$

Furthermore, by using Eq. (27) at (s_2,r^*), and the eigenvectors E_2 and U_2 gives:

$$U_2^{\,T}[D^2\boldsymbol{F}(s_2,r^*)(E_2,E_2)] = -\frac{2r^*}{k} + \frac{2\hat{Y}(q+c\hat{Y})}{b^2(l+v\mu)} - \frac{2k_1e(1-m)(q+2c\hat{Y})}{\gamma(q+c\hat{Y})((1-m)\hat{Y}+k_1)^2}.$$

Hence, the first part of the proof is done, provided that condition (31) holds. Otherwise, using Eq.

(28) leads to:

$$U_2{}^T[D^3\boldsymbol{F}(s_2,r^*)(E_2,E_2,E_2)] = -\frac{6}{b^3(l+v\mu)}\left[(q+c\hat{Y})\hat{Y} - \frac{k_1 e(1-m)b^2(l+v\mu)}{\gamma((1-m)\hat{Y}+k_1)^2}\left(\frac{c\hat{Y}}{(q+c\hat{Y})}+1\right) + cb^2\left(\frac{k_1 e(1-m)b(l+v\mu)}{\gamma(q+c\hat{Y})((1-m)\hat{Y}+k_1)^2}\right)^2\right].$$

Therefore, the second part of the proof follows when the condition (32) holds.

Theorem 11. Let $e^* = \frac{(k_1+(1-m)\tilde{Y})^2}{k_1(1-m)\tilde{Y}\tilde{a}_{11}}\left[\frac{\tilde{a}_{11}c\gamma\tilde{X}\tilde{Y}}{(b+\tilde{X})(l+v\mu)} - \tilde{a}_{12}\tilde{a}_{21}\right]$, then if $e = e^*$, system (2) possesses a saddle-node bifurcation at co-existing point s_3, if the following condition holds.

$$-\frac{\tilde{a}_{21}}{\tilde{a}_{11}}\, c_{11}(s_3,e^*) + c_{21}(s_3,e^*) \neq 0. \qquad (33)$$

Proof of Theorem 11. According to the JM that is given by Eq. (16), it is observed that, for $e = e^*$, it can be represented as:

$$J_3 = J_{s_3,e^*} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22}^* \end{bmatrix},$$

where $\tilde{a}_{22}^* = \tilde{a}_{22}(e^*)$.

Straightforward computation shows that the determinant of J_3 at $e = e^*$, which is given by $D_e = \tilde{a}_{11}\tilde{a}_{22}^* - \tilde{a}_{12}\tilde{a}_{21}$, is zero. Hence, the characteristic polynomial of J_3 has a zero root (eigenvalue) that is denoted by $\lambda_{31}^* = 0$, with the second eigenvalue $\lambda_{32}^* = T_r(e^*)$.

Thus, s_3 is a non-hyperbolic point when $e = e^*$. Therefore, the eigenvectors of J_3 and their transpose $[J_3]^T$ will be $E_3 = \begin{pmatrix} -\frac{\tilde{a}_{12}}{\tilde{a}_{11}} \\ 1 \end{pmatrix}$ and $U_3 = \begin{pmatrix} -\frac{\tilde{a}_{21}}{\tilde{a}_{11}} \\ 1 \end{pmatrix}$, respectively. Moreover, it is obtained that:

$$U_3{}^T[\boldsymbol{F}_e(s_3,e^*)] = -\frac{(1-m)\tilde{Y}^2}{k_1+(1-m)\tilde{Y}} \neq 0.$$

Furthermore, by using Eq. (27) at (s_3,e^*), and the eigenvectors E_3 and U_3 gives:

$$U_3{}^T[D^2\boldsymbol{F}(s_3,e^*)(E_3,E_3)] = -\frac{\tilde{a}_{21}}{\tilde{a}_{11}}\, c_{11}(s_3,e^*) + c_{21}(s_3,e^*),$$

with:

$$c_{11}(s_3,e^*) = -\frac{2\left[(b+\tilde{X})^3 r(l+v\mu) - bk\tilde{Y}(q+c\tilde{Y})\right]}{k(b+\tilde{X})^3(l+v\mu)}\left(\frac{\tilde{a}_{21}}{\tilde{a}_{11}}\right)^2 - \frac{2b(q+2c\tilde{Y})}{(b+\tilde{X})^2(l+v\mu)}\left(\frac{\tilde{a}_{21}}{\tilde{a}_{11}}\right) - \frac{2c\tilde{X}}{(b+\tilde{X})(l+v\mu)}.$$

$$\begin{aligned} c_{21}(s_3,e^*) = &-\frac{2b\tilde{Y}(q+c\tilde{Y})\gamma}{(b+\tilde{X})^3(l+v\mu)}\left(\frac{\tilde{a}_{21}}{\tilde{a}_{11}}\right)^2+\frac{2b(q+2c\tilde{Y})\gamma}{(b+\tilde{X})^2(l+v\mu)}\left(\frac{\tilde{a}_{21}}{\tilde{a}_{11}}\right) \\ &+\frac{2c(1-m)^3\gamma\tilde{X}\tilde{Y}^3+6c(1-m)^2\gamma k_1\tilde{X}\tilde{Y}^2}{(b+\tilde{X})(l+v\mu)\big((1-m)\tilde{Y}+k_1\big)^3} \\ &+\frac{2c\gamma k_1^3\tilde{X}-2(1-m)(be^*l+e^*l\tilde{X}-3c\gamma\tilde{X}\tilde{Y}+e^*v(b+\tilde{X})\mu)k_1^2}{(b+\tilde{X})(l+v\mu)\big((1-m)\tilde{Y}+k_1\big)^3}. \end{aligned}$$

Hence, the proof is done provided that condition (33) holds.

6. HOPF BIFURCATION

In dynamical systems, a Hopf bifurcation happens when two complex conjugate eigenvalues of the linearized system cross the imaginary axis, causing an equilibrium to lose stability. Thus, the system begins to exhibit periodic oscillations, or a limit cycle. This indicates that, in a prey-predator system, once a critical parameter value is crossed, the populations of prey and predator begin to oscillate repeatedly around an equilibrium point. In order to comprehend the dynamics of ecological interactions, particularly how populations change over time and the circumstances under which stable behavior gives way to oscillatory or cyclic behavior, it is imperative to investigate a Hopf bifurcation in a prey-predator system. The Hopf bifurcation occurrence at the co-existence point of system (2) is analyzed by selecting r as the bifurcation parameter. According to the JM at s_3 given in equation (16), the characteristic equation was determined in equation (19) depending on JM's trace $(T_r = \tilde{a}_{11}+\tilde{a}_{22})$ and determinant $(D_e = \tilde{a}_{11}\tilde{a}_{22}-\tilde{a}_{12}\tilde{a}_{21})$, where $\tilde{a}_{ij}, i,j = 1,2$ are the JM elements. Therefore, the following theorem provides the necessary and sufficient condition for having a Hopf bifurcation.

Theorem 12. If the parameter r passes through a positive value $r = r^*$ that is given below. Then system (2) undergoes a Hopf bifurcation around the co-existence point s_3 provided the following condition is met.

$$D_e(s_3,r^*) = \tilde{a}_{11}(r^*)\tilde{a}_{22}(r^*)-\tilde{a}_{12}(r^*)\tilde{a}_{21}(r^*) > 0. \tag{34}$$

where $\tilde{a}_{ij}(r^*)$ are JM (16) elements at the value of r^* that is given by:

$$r^* = \frac{k}{\tilde{x}}\left[\frac{(q+c\tilde{y})\tilde{y}\tilde{x}}{(b+\tilde{x})^2(l+\mu v)}+\frac{\gamma c\tilde{x}\tilde{y}}{(b+\tilde{x})(l+\mu v)}-\frac{k_1 e(1-m)\tilde{y}}{(k_1+(1-m)\tilde{y})^2}\right].$$

Proof of Theorem 12. At $r = r^*$, the $T_r(s_3,r^*) = \tilde{a}_{11}(r^*)+\tilde{a}_{22}(r^*) = 0$, and hence the characteristic equation (19) becomes

$$\lambda^2 + D_e(s_3, r^*) = 0, \tag{35}$$

where $D_e(s_3, r^*) = \tilde{a}_{11}(r^*)\tilde{a}_{22}(r^*) - \tilde{a}_{12}(r^*)\tilde{a}_{21}(r^*)$. Moreover, since $D_e(s_3, r^*) > 0$ under the condition (34).

It is easy to verify that equation (35) has roots $\lambda_1 = i\sqrt{D_e(s_3, r^*)}$ and $\lambda_2 = -i\sqrt{D_e(s_3, r^*)}$. Thus, JM of the system (2) has two purely imaginary eigenvalues at (s_3, r^*).

Not that $T_r(s_3, r)$ and $D_e(s_3, r)$ are smooth functions of r. Therefore, in the neighbourhood of r^*, the characteristic equation (19)'s roots are written in the form

$$\lambda_1(r) = \sigma_1(r) + i\sigma_2(r) = \frac{T_r(s_3,r)+\sqrt{(T_r(s_3,r))^2-4D_e(s_3,r)}}{2},$$
$$\lambda_2(r) = \sigma_1(r) - i\sigma_2(r) = \frac{T_r(s_3,r)-\sqrt{(T_r(s_3,r))^2-4D_e(s_3,r)}}{2},$$

where σ_i, $i = 1,2$ are real functions.

Now, due to the Hopf bifurcation theorem [34], the proof follows if the transversality condition $\frac{d}{dr} Re\, \lambda_1(r) \left.\right|_{r=r^*} \neq 0$ is satisfied. Since $Re\, \lambda_1(r) = \sigma_1(r) = \frac{T_r(s_3,r)}{2} = \frac{\tilde{a}_{11}+\tilde{a}_{22}}{2}$, then direct computation gives that:

$$\frac{d}{dr} Re\ \lambda_i(r) = \frac{-\tilde{x}}{2k} \neq 0.$$

Hence, the system (2) undergoes a Hopf bifurcation at s_3, when $r = r^*$.

7. NUMERICAL SIMULATION

In the study of prey-predator systems, which are usually described using nonlinear differential equations such as the suggested system (2), numerical simulation is essential. Analyzing parameter sensitivity, validating theoretical models, and comprehending complex dynamics all highlight the value of performing numerical simulations. Therefore, for the following set of hypothetical parameter values:

$$\begin{gathered} r = 1, v = 0.1, k = 20, \delta_0 = 0.01, q = 0.2, c = 0.1, b = 2, l = 0.5, \\ \mu = 0.5, \gamma = 0.15, \delta_1 = 0.15, \beta = 0.16, e = 0.5, m = 0.5, k_1 = 5. \end{gathered} \tag{36}$$

System (2) is solved numerically starting from different initial points, and then the obtained results are drawn in the form of a phase portrait and time series as represented in Figure 1.

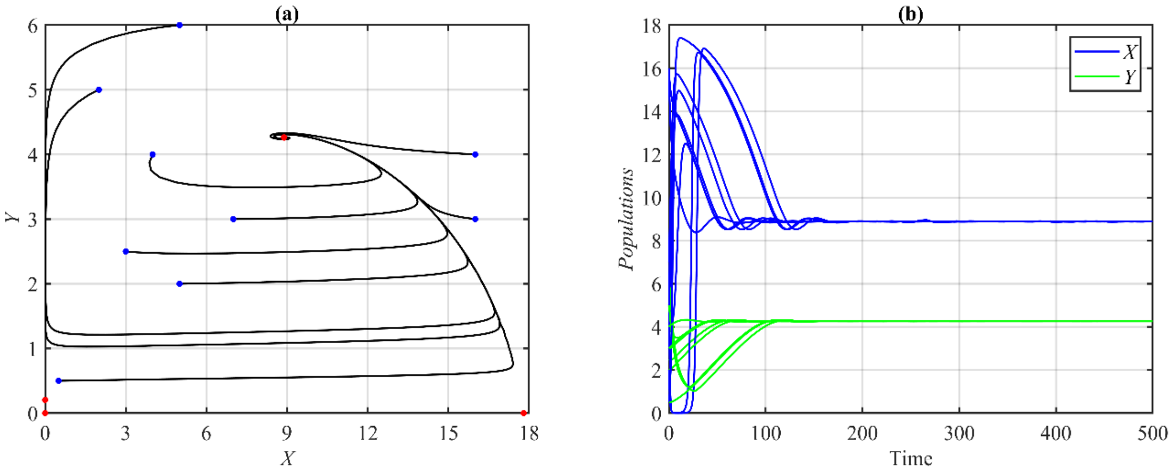


Figure 1: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach $s_3 = (8.88,4.26)$. (b) Trajectories as a function of time.

Note that the red dots throughout the Figures represent the equilibrium points, while the blue dots represent the initial points.

Obviously, Figure 1 shows the existence of a symptomatically stable co-existence point for the system (2) using the given data set. Now, varying the parameter r shows that for the range $r \leq 2.12$, the system approaches s_3; however, it approaches periodic dynamics when $2.12 < r$, indicating to occurrence of a Hopf bifurcation, see Figure 2 for typical values.

The findings in Figure 2 show that when r is increased, the system persists until it destabilizes as a result of a Hopf bifurcation. Changing the parameters k, q, c, γ, and k_1 produced results that were comparable to those with altering values of r.

For the parameter v, it is observed that when $v \leq 0.08$, $0.08 < v < 0.963$, and $0.963 \leq v < 1$, the system (2) has a periodic dynamics, an asymptotically stable point s_3, and an asymptotically stable point s_2, respectively, see Figure 3.

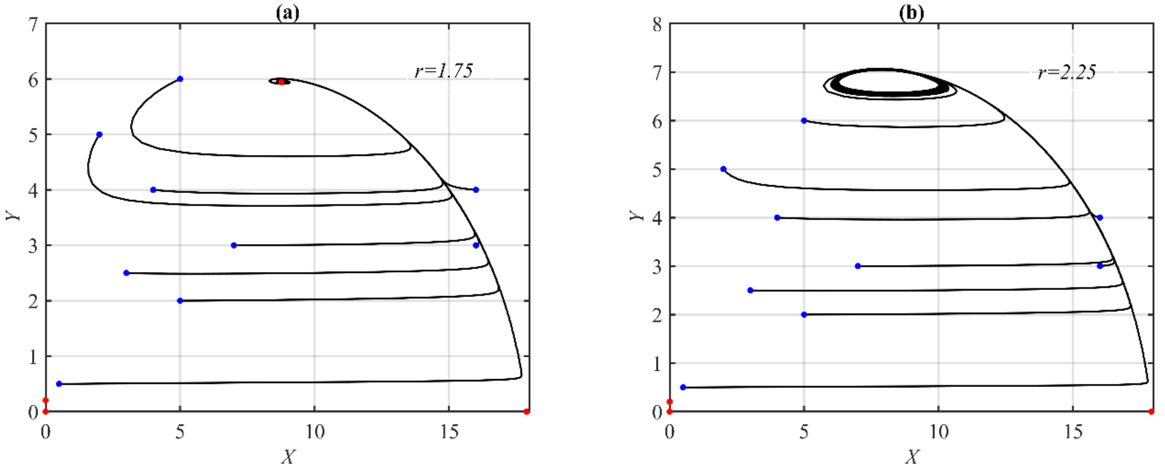


Figure 2: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach $s_3 = (8.76,5.94)$. (b) Trajectories approach periodic dynamics.

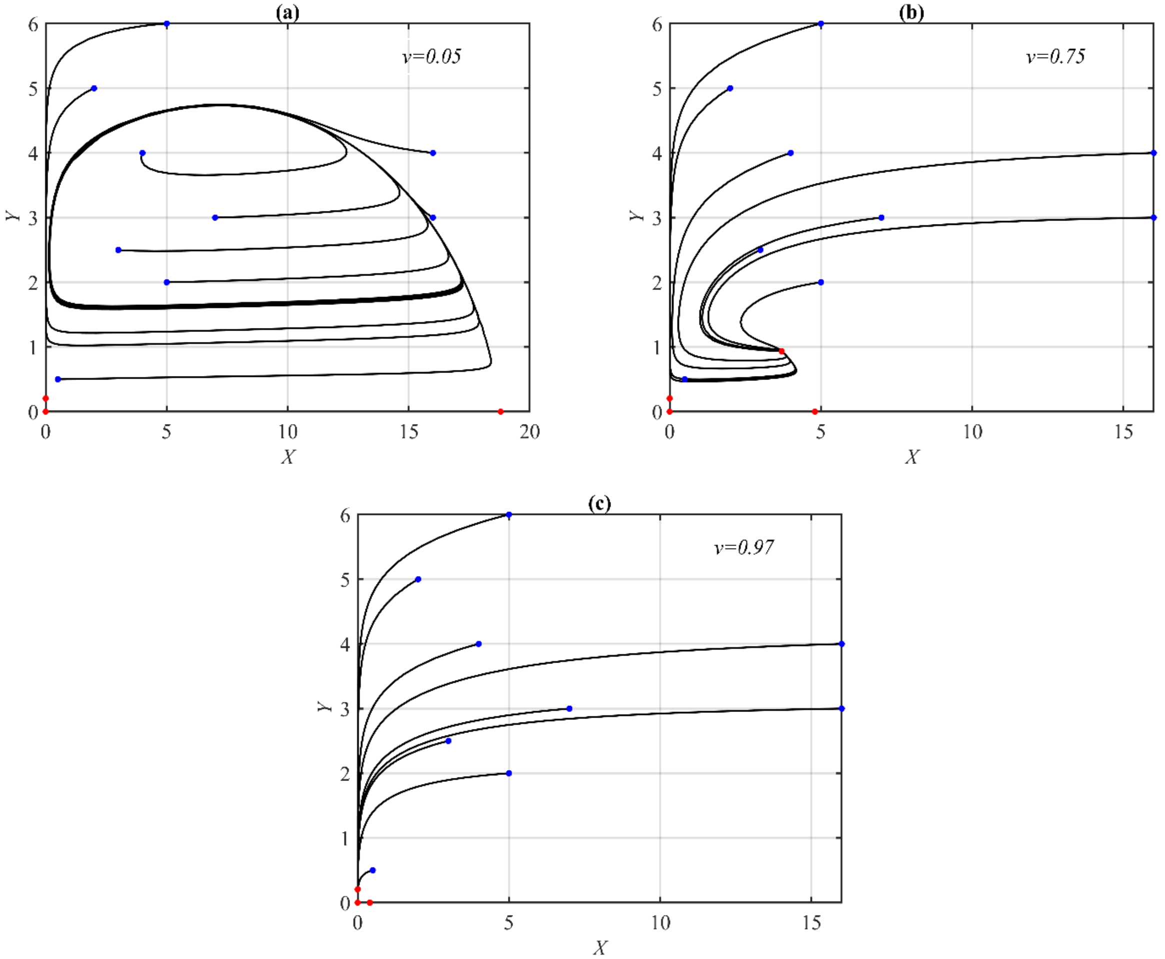


Figure 3: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach periodic dynamics. (b) Trajectories approach $s_3 = (3.7,0.93)$. (c) Trajectories approach $s_2 = (0,0.2)$.

It is observed from Figure 3 that decreasing the value of v below a specific point destabilizes the system (2) due to the occurrence of Hopf bifurcation while increasing it above a specific point leads to the extinction of prey.

For the ranges $b \leq 1.81$ and $b > 1.81$, it is observed that the system (3) approach asymptotically to periodic dynamics and s_3, respectively, see Figure 4.

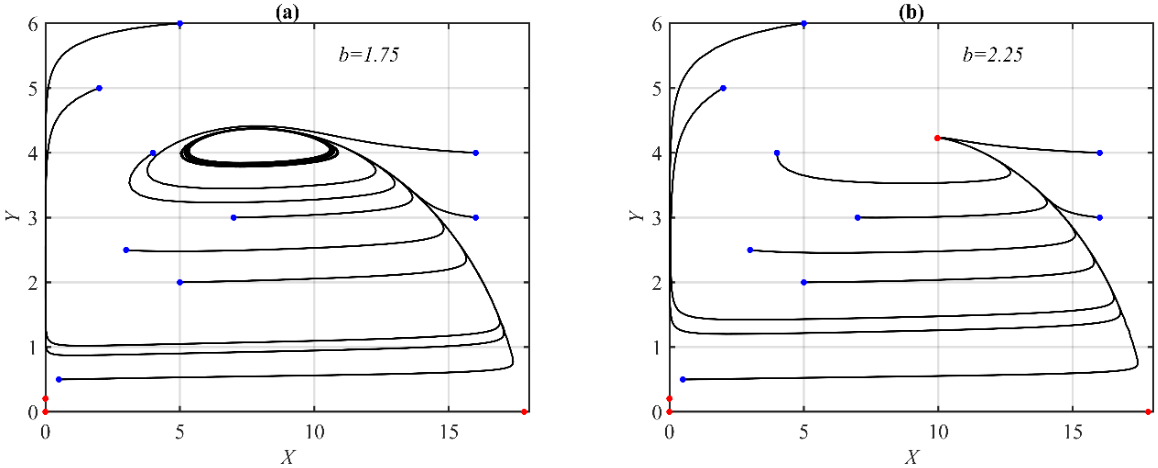


Figure 4: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach periodic dynamics. (b) Trajectories approach $s_3 = (9.96,4.22)$.

The findings in Figure 4 show that when b is decreased, the system persists until it destabilizes as a result of a Hopf bifurcation. However, it persists at s_3 for large values. Changing the parameters l and μ produced results that were comparable to those with altering values of b. Moreover, transferring the value of δ_0 from the range $\delta_0 < 0.85$ to the range $0.85 \leq \delta_0 < 1$ makes the system (2) change its stability from s_3 to s_2, see Figure 5.

Now, to study the influence of altering the value δ_1, the system (2) is solved numerically in the ranges $\delta_1 \leq 0.06$, $0.06 < \delta_1 \leq 0.073$, $0.073 < \delta_1 \leq 0.146$, $0.146 < \delta_1 \leq 0.21$, and $0.21 < \delta_1 < 1$, it is obtained that the system approaches asymptotically to s_2, s_3, periodic dynamics, s_3, and s_1, as shown in Figure 6.

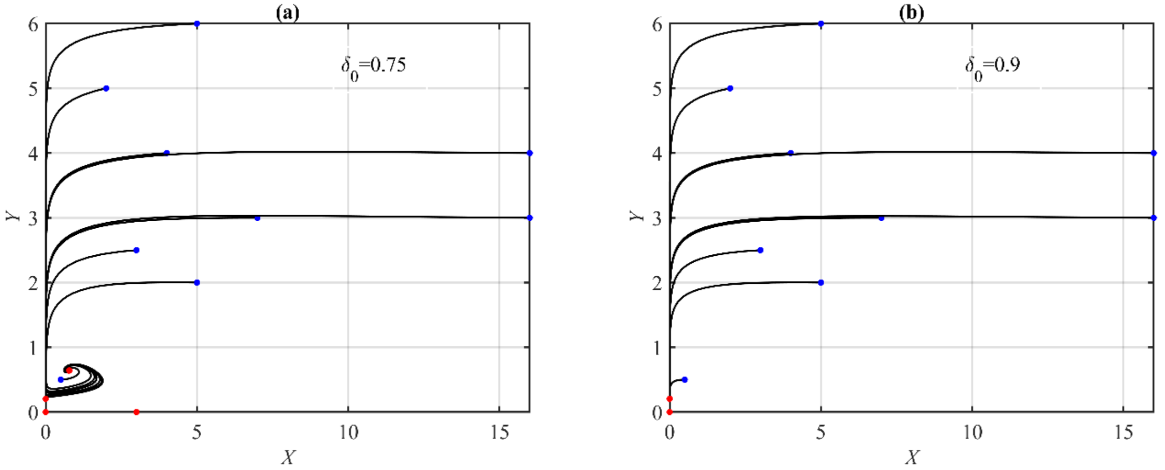
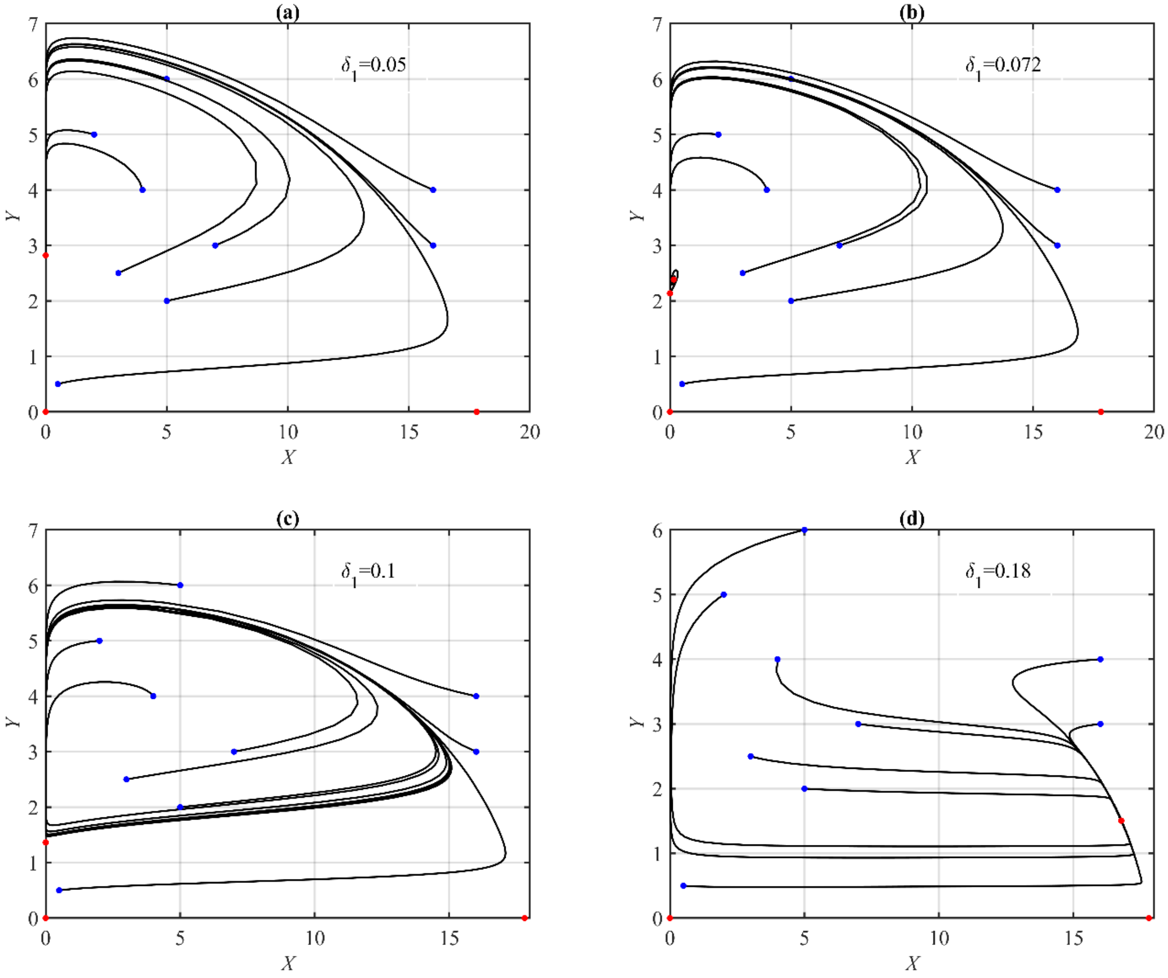


Figure 5: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach $s_3 = (0.78,0.64)$. (c) Trajectories approach $s_2 = (0,0.2)$.

The findings of Figure 5 explain the loss of persistence of system (2) as the parameter δ_0 exceeds a specific point.



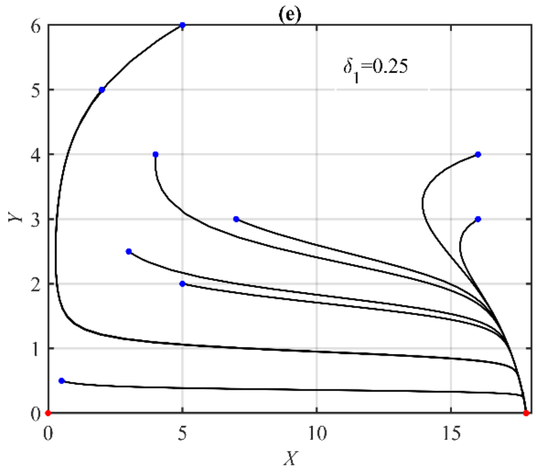
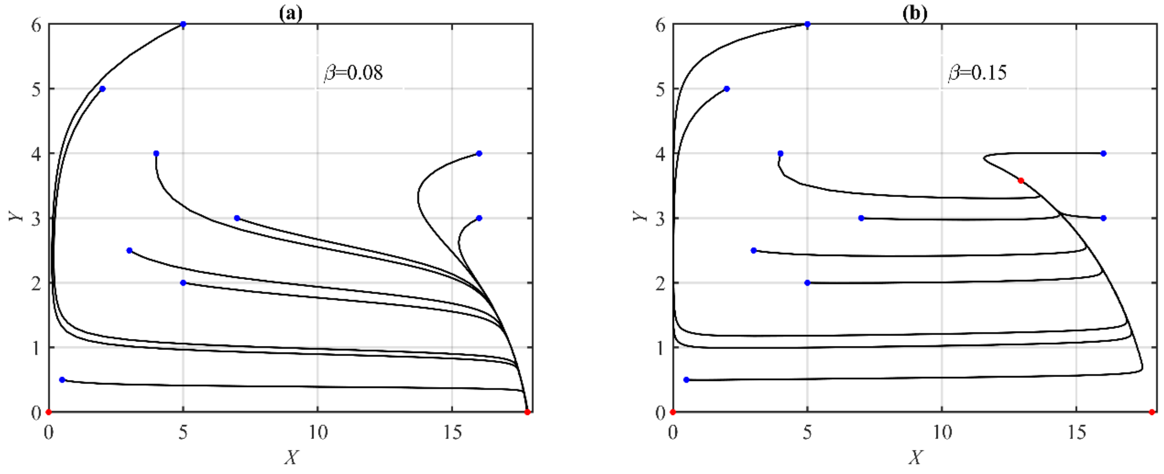


Figure 6: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach $s_2 = (0,2.82)$. (b) Trajectories approach $s_3 = (0.15,2.38)$. (c) Trajectories approach periodic dynamics. (d) Trajectories approach $s_3 = (16.77,1.5)$. (e) Trajectories approach $s_2 = (17.79,0)$.

The findings of Figure 6 explain the transfer of the attracting set of system (2) from s_2 to s_1, gradually passing through s_3, periodic, and s_3, when the value of the parameter δ_1 changed gradually from the lower value to the upper value. This indicates to sensitivity of the system (2) to changes in δ_1. On the other hand, the system (2) is solved numerically in the ranges $\beta \leq 0.1$, $0.1 < \beta \leq 0.163$, $0.163 < \beta \leq 0.236$, $0.236 < \beta < 0.243$, and $0.243 \leq \beta < 1$, it is obtained that the system approaches asymptotically to s_1, s_3, periodic dynamics, s_3, and s_2, as shown in Figure 7.



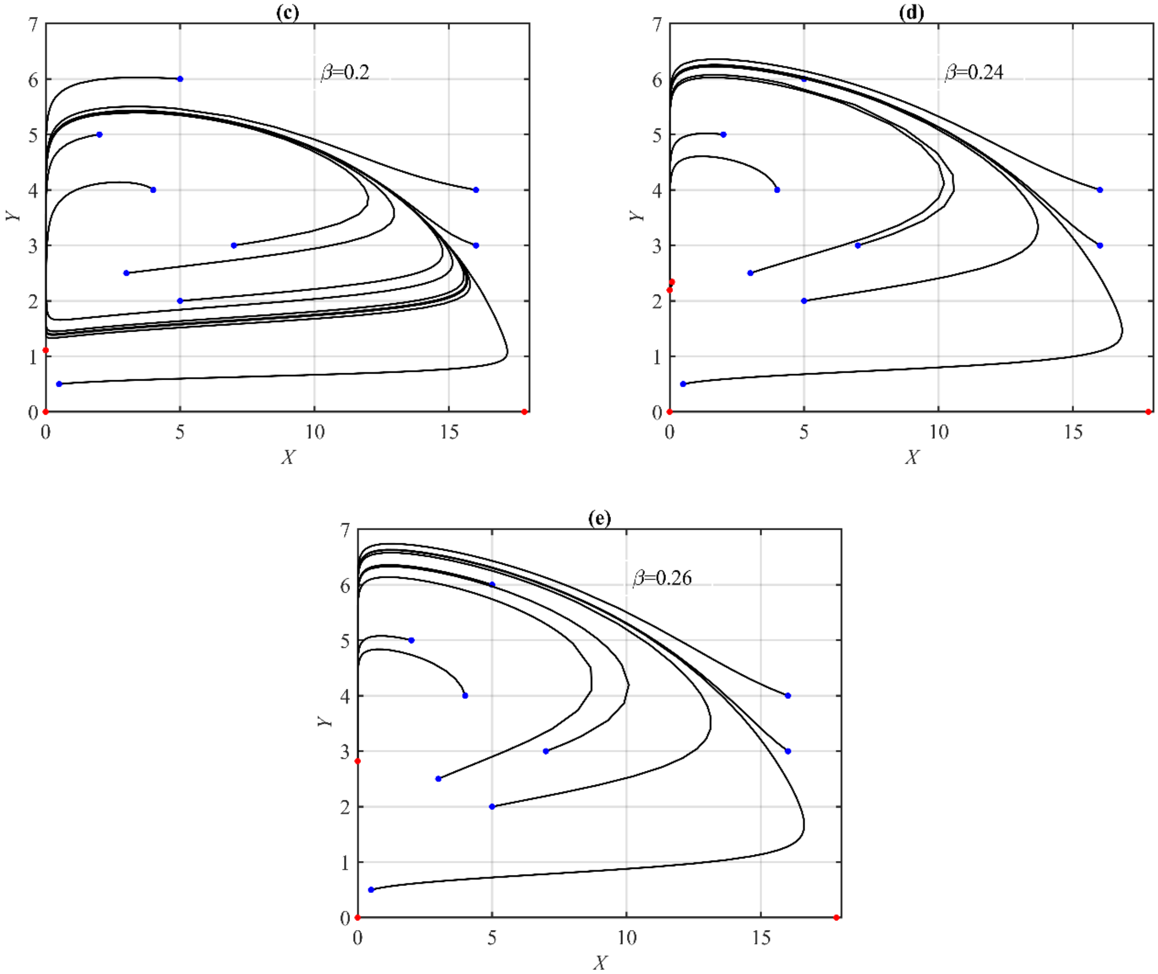


Figure 7: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach $s_1 = (17.79,0)$. (b) Trajectories approach $s_3 = (12.93,3.57)$. (c) Trajectories approach periodic dynamics. (d) Trajectories approach $s_3 = (0.08,2.34)$. (e) Trajectories approach $s_2 = (0,2.82)$.

The findings of Figure 7 explain the transfer of the attracting set of system (2) from s_1 to s_2, gradually passing through s_3, periodic, and s_3, when the value of the parameter β changed gradually from the lower value to the upper value. This indicates to sensitivity of the system (2) to changes in β. In addition to the above, the parameters δ_1 and β have an opposite effect on the dynamic behavior of the system (2).

Now, for $e \leq 0.05$, $0.05 < e < 0.49$, and $0.49 \leq e < 1$, it is obtained that the system (2) approaches asymptotically to s_2, periodic dynamics, and s_3, respectively, as shown in Figure 8. However, for $m \leq 0.51$, $0.51 < m \leq 0.94$, and $0.94 < m < 1$, it is found that the system (2) approaches asymptotically to s_3, periodic dynamics, and s_2, respectively, as shown in Figure 9.

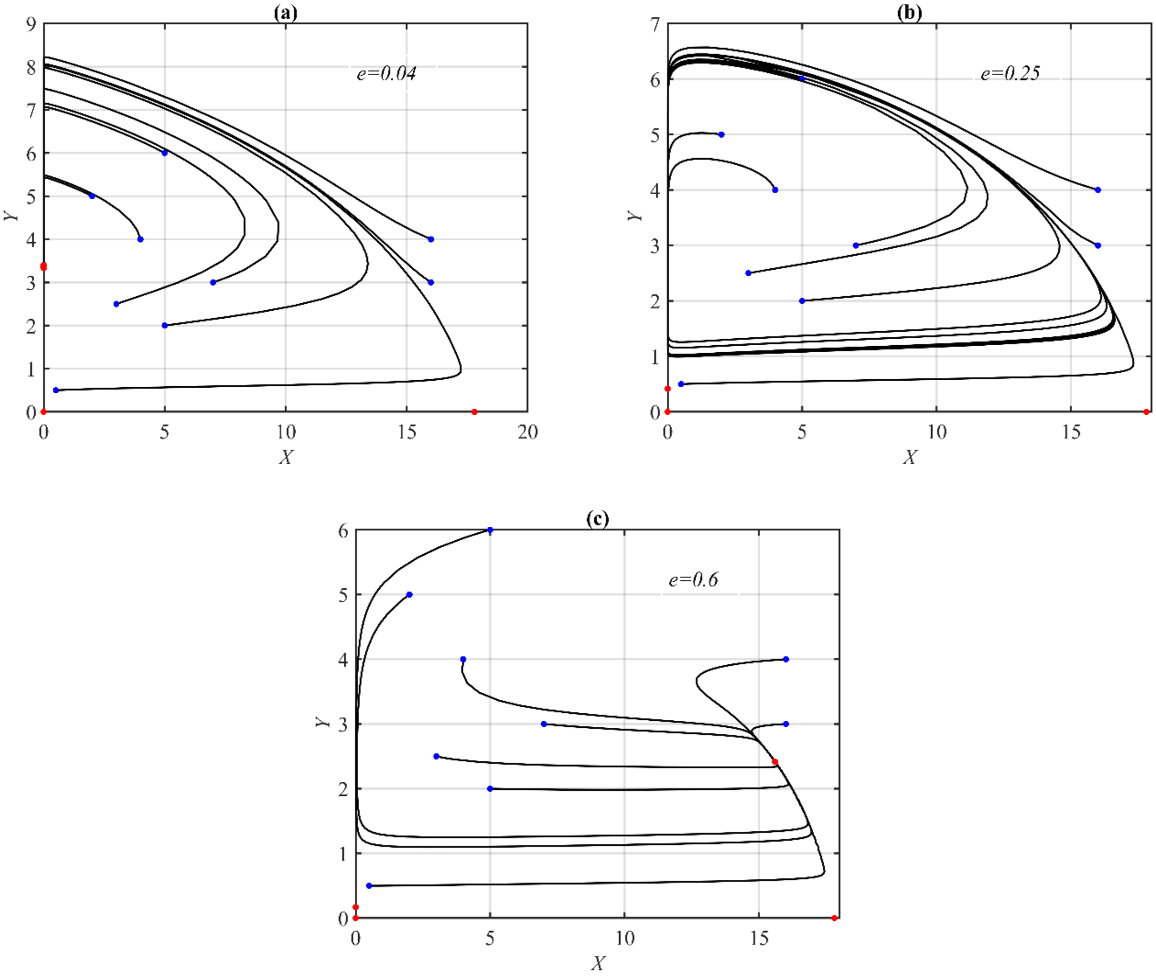


Figure 8: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach $s_2 = (0,3.39)$. (b) Trajectories approach periodic dynamics. (c) Trajectories approach $s_3 = (15.59,2.41)$.

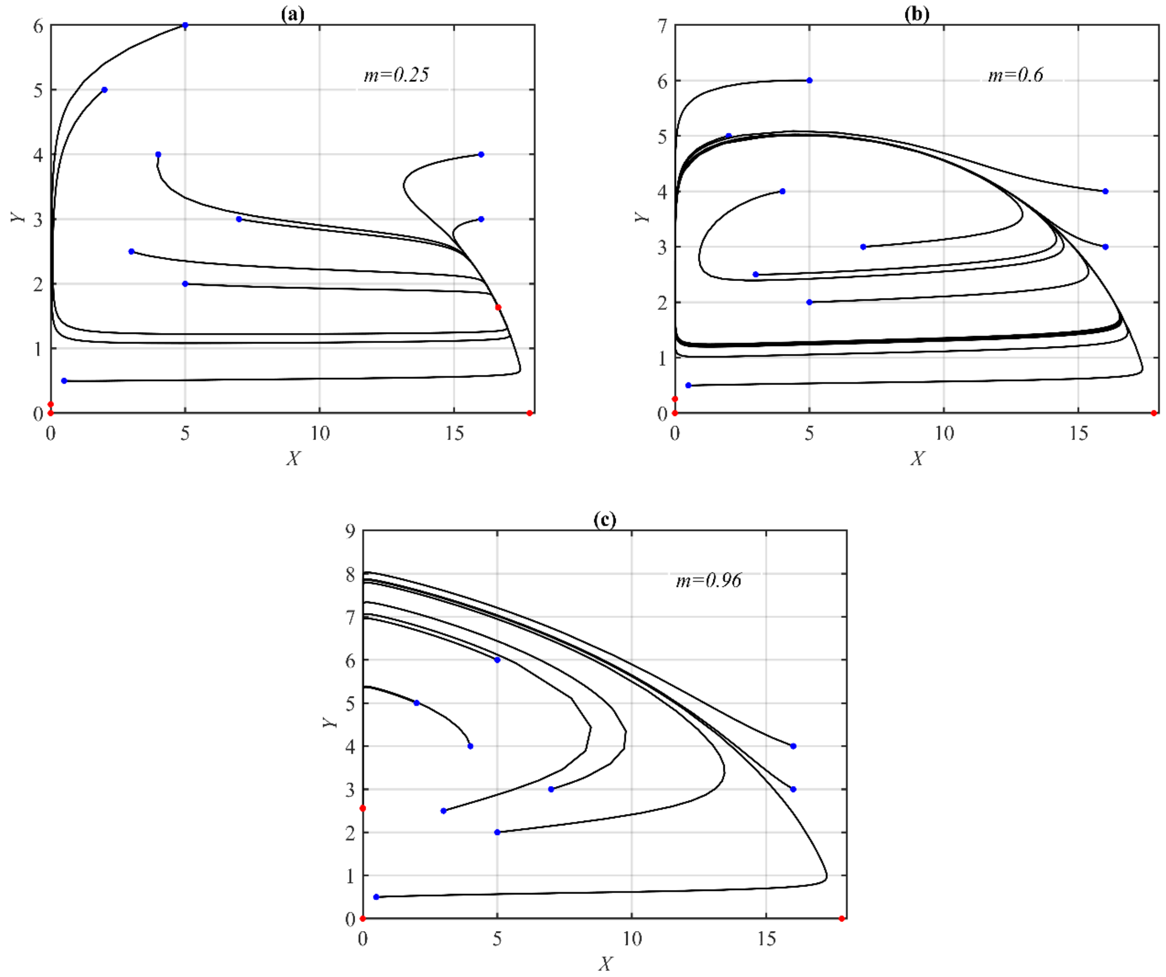


Figure 9: The trajectories of system (2) using parameter values (36) with various initial points. (a) Trajectories approach $s_3 = (16.63,1.63)$. (b) Trajectories approach periodic dynamics. (c) Trajectories approach $s_2 = (0,2.56)$.

According to Figures 8 and 9, the parameter e makes the dynamics of system (2) transfer from s_2 to s_3 passing through a periodic dynamics that indicates its increasing stabilizes the system (2). However, the parameter m makes the dynamics of system (2) transfer from s_3 to s_2 passing through a periodic dynamics that indicates its increasing extinctors the system (2).

In addition to the above, the parameters e and m have an opposite effect on the dynamic behavior of the system (2). Finally, Table 2 summarizes the rest of the parameters that influence the dynamical behavior of system (2).

Table 2: The dynamical behavior of system (2) as a function of parameter values.

Parameter range	**Dynamical Behavior of System (2)**
$k < 23$	Approaches to s_3
$23 \leq k$	periodic
$q \leq 0.21$	Approaches to s_3
$0.21 < q$	periodic
$c \leq 0.103$	Approaches to s_3
$0.103 < c$	periodic
$l < 0.49$	periodic
$0.49 \leq l$	Approaches to s_3
$\mu \leq 0.38$	periodic
$0.38 < \mu$	Approaches to s_3
$\gamma \leq 0.15$	Approaches to s_3
$\gamma > 0.15$	periodic
$k_1 \leq 5.1$	Approaches to s_3
$5.1 < k_1$	Periodic

8. CONCLUSION

A prey-predator model incorporating numerous real biological components, including prey vigilance (alertness), cannibalism, and hunting cooperation, was proposed and investigated to understand the actual dynamic behavior in the environment and maintain the diversity and equilibrium of the ecosystem. The majority of the suggested model's solution qualities have been examined. The suggested system contains four equilibrium points, all of which we were able to locate. We looked at each point's stability both locally and globally. It is found that, in some circumstances, all of the equilibrium points can be asymptotically stable both locally and globally. The system's persistence requirements are established. Lastly, the local bifurcations, including Hopf bifurcation, are examined in order to determine how altering parameter values affect the system's dynamic behavior. We have performed numerical simulations using biologically acceptable default parameter values in order to analytically validate the results and comprehend the influence of parameters on dynamic behavior. The following conclusions have been drawn from these simulations:

Increasing the prey growth rate, the environment's carrying capacity, predator attack rate, hunting cooperation rate, energy gained by predation, or cannibalism half-saturation constant destabilizes the system dynamics, due to the existence of a Hopf bifurcation, while maintaining the persistence of the system (2). On the other hand, increasing the half-saturation constant, prey's vigilance effectiveness, or decreasing predation lethality in the absence of prey vigilance $(1/l)$ stabilizes the system due to transfer from periodic dynamics to the co-existing equilibrium point, while maintaining the persistence of the system (2). Moreover, raising the value of the level of prey's vigilance, prey's natural death rate, or predator refuge rate makes the prey extinct, and hence the system (2) does not persist. While increasing the cannibalism rate works as a persistence and stabilizing factor due to transfer from the boundary point to periodic dynamics and then to co-existing points.

Finally, it was noted that the system (2) is very sensitive to changes in the values of both the predator's natural death rate and the self-predation's growth rate, as each of them contains four bifurcation values. It is observed that increasing the predator's natural death rate leads to a transfer from the prey-free point to the predator-free point. However, increasing the self-predator's growth rate leads to a transfer from the predator-free point to the prey-free point.

Appendix A

In the following, the coefficient of the polynomial equation represented by Eq. (17) can be written as:

$$\Delta_1 = c^2 k\gamma^2 (m-1)^2 > 0.$$

$$\Delta_2 = 2ck(1-m)^2[\gamma(\beta - e - \delta_1)(l + v\mu) + q\gamma^2] + 2c^2 k\gamma^2 k_1 (1-m).$$

$$\Delta_3 = (1-m)^2[e^2 kl^2 - 2ekl^2\beta + kl^2\beta^2 - 2eklq\gamma + 2klq\beta\gamma + kq^2\gamma^2 - bcklr\gamma^2 + bcklrv\gamma^2 + 2e^2 klv\mu - 4eklv\beta\mu + 2klv\beta^2\mu - 2ekqv\gamma\mu + 2kqv\beta\gamma\mu - bckrv\gamma^2\mu + bckrv^2\gamma^2\mu + e^2 kv^2\mu^2 - 2ekv^2\beta\mu^2 + kv^2\beta^2\mu^2 + bckl\gamma^2\delta_0 + bckv\gamma^2\mu\delta_0 + 2ekl^2\delta_1 - 2kl^2\beta\delta_1 - 2klq\gamma\delta_1 + 4eklv\mu\delta_1 - 4klv\beta\mu\delta_1 - 2kqv\gamma\mu\delta_1 + 2ekv^2\mu^2\delta_1 -$$

$$\begin{gathered}
2kv^2\beta\mu^2\delta_1 + kl^2\delta_1^2 + 2klv\mu\delta_1^2 + kv^2\mu^2\delta_1^2] - 2(1-m)[ceklγk_1 - 2ckl\beta\gamma k_1 - \\
2ckq\gamma^2k_1 + cekv\gamma\mu k_1 - 2ckv\beta\gamma\mu k_1 + 2ckl\gamma k_1\delta_1 + 2ckv\gamma\mu k_1\delta_1] + c^2k\gamma^2k_1^2
\end{gathered}$$

$$\begin{gathered}
\Delta_4 = b^2el^2r\gamma + bekl^2r\gamma - 2b^2el^2mr\gamma - 2bekl^2mr\gamma + b^2el^2m^2r\gamma + bekl^2m^2r\gamma \\
-bekl^2rv\gamma + 2bekl^2mrv\gamma - bekl^2m^2rv\gamma - b^2l^2r\beta\gamma - bkl^2r\beta\gamma \\
+2b^2l^2mr\beta\gamma + 2bkl^2mr\beta\gamma - b^2l^2m^2r\beta\gamma - bkl^2m^2r\beta\gamma + bkl^2rv\beta\gamma \\
-2bkl^2mrv\beta\gamma + bkl^2m^2rv\beta\gamma - bklqr\gamma^2 + 2bklmqr\gamma^2 - bklm^2qr\gamma^2 \\
+bklqrv\gamma^2 - 2bklmqrv\gamma^2 + bklm^2qrv\gamma^2 + 2b^2elrv\gamma\mu + 2beklrv\gamma\mu \\
-4b^2elmrv\gamma\mu - 4beklmrv\gamma\mu + 2b^2elm^2rv\gamma\mu + 2beklm^2rv\gamma\mu - 2beklrv^2\gamma\mu \\
+4beklmrv^2\gamma\mu - 2beklm^2rv^2\gamma\mu - 2b^2lrv\beta\gamma\mu - 2bklrv\beta\gamma\mu + 4b^2lmrv\beta\gamma\mu \\
+4bklmrv\beta\gamma\mu - 2b^2lm^2rv\beta\gamma\mu - 2bklm^2rv\beta\gamma\mu + 2bklrv^2\beta\gamma\mu - 4bklmrv^2\beta\gamma\mu \\
+2bklm^2rv^2\beta\gamma\mu - bkqrv\gamma^2\mu + 2bkmqrv\gamma^2\mu - bkm^2qrv\gamma^2\mu + bkqrv^2\gamma^2\mu \\
-2bkmqrv^2\gamma^2\mu + bkm^2qrv^2\gamma^2\mu + b^2erv^2\gamma\mu^2 + bekrv^2\gamma\mu^2 - 2b^2emrv^2\gamma\mu^2 \\
-2bekmrv^2\gamma\mu^2 + b^2em^2rv^2\gamma\mu^2 + bekm^2rv^2\gamma\mu^2 - bekrv^3\gamma\mu^2 + 2bekmrv^3\gamma\mu^2 \\
-bekm^2rv^3\gamma\mu^2 - b^2rv^2\beta\gamma\mu^2 - bkrv^2\beta\gamma\mu^2 + 2b^2mrv^2\beta\gamma\mu^2 + 2bkmrv^2\beta\gamma\mu^2 \\
-b^2m^2rv^2\beta\gamma\mu^2 - bkm^2rv^2\beta\gamma\mu^2 + bkrv^3\beta\gamma\mu^2 - 2bkmrv^3\beta\gamma\mu^2 + bkm^2rv^3\beta\gamma\mu^2 \\
-2ekl^2\beta k_1 + 2ekl^2m\beta k_1 + 2kl^2\beta^2k_1 - 2kl^2m\beta^2k_1 - 2eklq\gamma k_1 \\
+2eklmq\gamma k_1 + 4klq\beta\gamma k_1 - 4klmq\beta\gamma k_1 + 2kq^2\gamma^2k_1 - 2kmq^2\gamma^2k_1 \\
-2bcklr\gamma^2k_1 + 2bcklmr\gamma^2k_1 + 2bcklrv\gamma^2k_1 - 2bcklmrv\gamma^2k_1 - 4eklv\beta\mu k_1 \\
+4eklmv\beta\mu k_1 + 4klv\beta^2\mu k_1 - 4klmv\beta^2\mu k_1 - 2ekqv\gamma\mu k_1 + 2ekmqv\gamma\mu k_1 \\
+4kqv\beta\gamma\mu k_1 - 4kmqv\beta\gamma\mu k_1 - 2bckrv\gamma^2\mu k_1 + 2bckmrv\gamma^2\mu k_1 + 2bckrv^2\gamma^2\mu k_1 \\
-2bckmrv^2\gamma^2\mu k_1 - 2ekv^2\beta\mu^2k_1 + 2ekmv^2\beta\mu^2k_1 + 2kv^2\beta^2\mu^2k_1 - 2kmv^2\beta^2\mu^2k_1 \\
+2ckl\beta\gamma k_1^2 + 2ckq\gamma^2k_1^2 + 2ckv\beta\gamma\mu k_1^2 - bekl^2\gamma\delta_0 + 2bekl^2m\gamma\delta_0 \\
-bekl^2m^2\gamma\delta_0 + bkl^2\beta\gamma\delta_0 - 2bkl^2m\beta\gamma\delta_0 + bkl^2m^2\beta\gamma\delta_0 + bklq\gamma^2\delta_0 \\
-2bklmq\gamma^2\delta_0 + bklm^2q\gamma^2\delta_0 - 2beklv\gamma\mu\delta_0 + 4beklmv\gamma\mu\delta_0 - 2beklm^2v\gamma\mu\delta_0 \\
+2bklv\beta\gamma\mu\delta_0 - 4bklmv\beta\gamma\mu\delta_0 + 2bklm^2v\beta\gamma\mu\delta_0 + bkqv\gamma^2\mu\delta_0 - 2bkmqv\gamma^2\mu\delta_0 \\
+bkm^2qv\gamma^2\mu\delta_0 - bekv^2\gamma\mu^2\delta_0 + 2bekmv^2\gamma\mu^2\delta_0 - bekm^2v^2\gamma\mu^2\delta_0 \\
+bkv^2\beta\gamma\mu^2\delta_0 - 2bkmv^2\beta\gamma\mu^2\delta_0 + bkm^2v^2\beta\gamma\mu^2\delta_0 + 2bckl\gamma^2k_1\delta_0 \\
-2bcklm\gamma^2k_1\delta_0 + 2bckv\gamma^2\mu k_1\delta_0 - 2bckmv\gamma^2\mu k_1\delta_0 + b^2l^2r\gamma\delta_1 + bkl^2r\gamma\delta_1 \\
-2b^2l^2mr\gamma\delta_1 - 2bkl^2mr\gamma\delta_1 + b^2l^2m^2r\gamma\delta_1 + bkl^2m^2r\gamma\delta_1 - bkl^2rv\gamma\delta_1 \\
+2bkl^2mrv\gamma\delta_1 - bkl^2m^2rv\gamma\delta_1 + 2b^2lrv\gamma\mu\delta_1 + 2bklrv\gamma\mu\delta_1 - 4b^2lmrv\gamma\mu\delta_1 \\
-4bklmrv\gamma\mu\delta_1 + 2b^2lm^2rv\gamma\mu\delta_1 + 2bklm^2rv\gamma\mu\delta_1 - 2bklrv^2\gamma\mu\delta_1 \\
+4bklmrv^2\gamma\mu\delta_1 - 2bklm^2rv^2\gamma\mu\delta_1 + b^2rv^2\gamma\mu^2\delta_1 + bkrv^2\gamma\mu^2\delta_1 \\
-2b^2mrv^2\gamma\mu^2\delta_1 - 2bkmrv^2\gamma\mu^2\delta_1 + b^2m^2rv^2\gamma\mu^2\delta_1 + bkm^2rv^2\gamma\mu^2\delta_1 \\
-bkrv^3\gamma\mu^2\delta_1 + 2bkmrv^3\gamma\mu^2\delta_1 - bkm^2rv^3\gamma\mu^2\delta_1 + 2ekl^2k_1\delta_1 \\
-2ekl^2mk_1\delta_1 - 4kl^2\beta k_1\delta_1 + 4kl^2m\beta k_1\delta_1 - 4klq\gamma k_1\delta_1 + 4klmq\gamma k_1\delta_1
\end{gathered}$$

$$+4eklv\mu k_1\delta_1 - 4eklmv\mu k_1\delta_1 - 8klv\beta\mu k_1\delta_1 + 8klmv\beta\mu k_1\delta_1 - 4kqv\gamma\mu k_1\delta_1$$
$$+4kmqv\gamma\mu k_1\delta_1 + 2ekv^2\mu^2k_1\delta_1 - 2ekmv^2\mu^2k_1\delta_1 - 4kv^2\beta\mu^2k_1\delta_1$$
$$+4kmv^2\beta\mu^2k_1\delta_1 - 2ckl\gamma k_1^2\delta_1 - 2ckv\gamma\mu k_1^2\delta_1 - bkl^2\gamma\delta_0\delta_1 + 2bkl^2m\gamma\delta_0\delta_1$$
$$-bkl^2m^2\gamma\delta_0\delta_1 - 2bklv\gamma\mu\delta_0\delta_1 + 4bklmv\gamma\mu\delta_0\delta_1 - 2bklm^2v\gamma\mu\delta_0\delta_1$$
$$-bkv^2\gamma\mu^2\delta_0\delta_1 + 2bkmv^2\gamma\mu^2\delta_0\delta_1 - bkm^2v^2\gamma\mu^2\delta_0\delta_1 + 2kl^2k_1\delta_1^2$$
$$-2kl^2mk_1\delta_1^2 + 4klv\mu k_1\delta_1^2 - 4klmv\mu k_1\delta_1^2 + 2kv^2\mu^2k_1\delta_1^2 - 2kmv^2\mu^2k_1\delta_1^2$$

$$\Delta_5 = (b+k)[bel^2r\gamma k_1 - bel^2mr\gamma k_1 - 2bl^2r\beta\gamma k_1 + 2\mathrm{b}l^2mr\beta\gamma k_1 + 2belrv\gamma\mu k_1$$
$$- 2belmrv\gamma\mu k_1 - 4blrv\beta\gamma\mu k_1 + 4blmrv\beta\gamma\mu k_1 + berv^2\gamma\mu^2k_1$$
$$- bemrv^2\gamma\mu^2k_1 - 2brv^2\beta\gamma\mu^2k_1 + 2bmrv^2\beta\gamma\mu^2k_1 + 2bl^2r\gamma k_1\delta_1$$
$$- 2bl^2mr\gamma k_1\delta_1 + 4blrv\gamma\mu k_1\delta_1 - 4blmrv\gamma\mu k_1\delta_1 + 2brv^2\gamma\mu^2k_1\delta_1$$
$$- 2bmrv^2\gamma\mu^2k_1\delta_1] + (1$$
$$- m)[2bkl^2rv\beta\gamma k_1 - bekl^2rv\gamma k_1 - 2bklqr\gamma^2k_1 + 2bklqrv\gamma^2k_1$$
$$- 2beklrv^2\gamma\mu k_1 + 4bklrv^2\beta\gamma\mu k_1 - 2bkqrv\gamma^2\mu k_1 + 2bkqrv^2\gamma^2\mu k_1$$
$$- bekrv^3\gamma\mu^2k_1 + 2bkrv^3\beta\gamma\mu^2k_1 - bekl^2\gamma k_1\delta_0 + 2bkl^2\beta\gamma k_1\delta_0$$
$$+ 2bklq\gamma^2k_1\delta_0 - 2beklv\gamma\mu k_1\delta_0 + 4bklv\beta\gamma\mu k_1\delta_0 + 2bkqv\gamma^2\mu k_1\delta_0$$
$$- bekv^2\gamma\mu^2k_1\delta_0 + 2bkv^2\beta\gamma\mu^2k_1\delta_0 - 2bkl^2rv\gamma k_1\delta_1$$
$$- 4bklrv^2\gamma\mu k_1\delta_1 - 2bkrv^3\gamma\mu^2k_1\delta_1 - 2bkl^2\gamma k_1\delta_0\delta_1$$
$$- 4bklv\gamma\mu k_1\delta_0\delta_1 - 2bkv^2\gamma\mu^2k_1\delta_0\delta_1] + kk_1^2(l\beta + q\gamma)^2$$
$$- bcklr\gamma^2k_1^2(1-v) + 2klv\beta^2\mu k_1^2(l\beta + q\gamma) - bckrv\gamma^2\mu k_1^2(1-v)$$
$$+ kv^2\beta^2\mu^2k_1^2 + bck\gamma^2k_1^2\delta_0(l + v\mu) - 2klk_1^2\delta_1(l\beta + q\gamma)$$
$$- 2v\mu kk_1^2\delta_1(2l\beta + q\gamma + v\mu\beta) + kk_1^2\delta_1^2(l + v\mu)^2$$

$$\Delta_6 = -bl^2r\beta\gamma k_1^2(b+k) + bkl^2rv\beta\gamma k_1^2 - bklqr\gamma^2k_1^2(1-v) - 2blrv\beta\gamma\mu k_1^2(b+k)$$
$$+ 2bklrv^2\beta\gamma\mu k_1^2 - bkqrv\gamma^2\mu k_1^2 + bkqrv^2\gamma^2\mu k_1^2 - b^2rv^2\beta\gamma\mu^2k_1^2$$
$$- bkrv^2\beta\gamma\mu^2k_1^2 + bkrv^3\beta\gamma\mu^2k_1^2 + bkl^2\beta\gamma k_1^2\delta_0 + bklq\gamma^2k_1^2\delta_0$$
$$+ 2bklv\beta\gamma\mu k_1^2\delta_0 + bkqv\gamma^2\mu k_1^2\delta_0 + bkv^2\beta\gamma\mu^2k_1^2\delta_0 + b^2l^2r\gamma k_1^2\delta_1$$
$$+ bkl^2r\gamma k_1^2\delta_1 - bkl^2rv\gamma k_1^2\delta_1 + 2b^2lrv\gamma\mu k_1^2\delta_1 + 2bklrv\gamma\mu k_1^2\delta_1$$
$$- 2bklrv^2\gamma\mu k_1^2\delta_1 + b^2rv^2\gamma\mu^2k_1^2\delta_1 + bkrv^2\gamma\mu^2k_1^2\delta_1 - bkrv^3\gamma\mu^2k_1^2\delta_1$$
$$- bkl^2\gamma k_1^2\delta_0\delta_1 - 2bklv\gamma\mu k_1^2\delta_0\delta_1 - bkv^2\gamma\mu^2k_1^2\delta_0\delta_1$$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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