Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2025, 2025:122
https://doi.org/10.28919/cmbn/9515
ISSN: 2052-2541

OPTIMAL CONTROL STRATEGY FOR A DISCRETE-TIME MATHEMATICAL MODELING OF WATER POLLUTION

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Abstract. Water pollution is a major issue with serious consequences for human health and the environment, particularly in developing countries. It highlights that contaminated water is a source of waterborne diseases, such as cholera, and underscores the importance of integrating temperature variations into pollution management strategies. A discrete mathematical model is proposed, segmenting the problem into three compartments: at-risk water, polluted water, and the total sum of pollutants, accompanied by difference equations that represent their interactions. The challenge of optimal control aims to reduce pollutant concentrations through three approaches: awareness, purification, and source reduction. Numerical simulations conducted with MATLAB show that these interventions can significantly reduce water pollution. In conclusion, the article emphasizes that the application of mathematical modeling and optimal control strategies is crucial for mitigating the effects of pollution and proposing sustainable solutions for water management.

Keywords: water pollution; optimal control; mathematical model.

2020 AMS Subject Classification: 93A30.

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Received July 28, 2025

1. INTRODUCTION

Water pollution is a major issue in developing countries globally, due to its disastrous effects on the environment, human health, and the economy.Contaminated water is particularly responsible for the spread of waterborne diseases such as typhoid, cholera, diarrhea, hepatitis, and schistosomiasis [1], some of which can be fatal if untreated. Furthermore, water pollution can lead to long-term effects, such as chronic disorders, neurological problems related to chemical contaminants, and could even play a role in the development of cancer or cardiovascular diseases.

In Morocco, this issue is especially concerning. The excessive use of fertilizers and pesticides, the discharge of untreated wastewater, and the uncontrolled disposal of solid waste contribute to the degradation of water resources. Additionally, mining and rapid urbanization exacerbate this situation. These factors lead to an increase in water pollution, threatening not only public health but also the agricultural and industrial sectors that rely on this resource. Thus, the deterioration of water quality in Morocco represents a threat as severe as water scarcity, the urgency of adopting adequate protective measures. It is therefore imperative to implement strict controls to prevent this pollution and preserve both the environment and public health.

Numerous studies have also focused on this issue and related topics [2,3] Due to the complexity of real systems, mathematical modeling proves to be a valuable tool for analyzing and predicting the behavior of complex phenomena such as water pollution. Many researchers have studied this topic, including Guo and Cheng [4], who developed a mathematical model simulating the spatiotemporal variations of pollutants after water pollution incidents in Yuncheng, China. Their study revealed a decrease in pollutant concentrations as the affected area extended downstream. Furthermore, Issakhov et al [5] conducted a numerical modeling study focused on water pollution caused by chemical reactions in industrial plants, emphasizing the importance of considering environmental temperature variations in pollution management strategies. Finally, Asiyeh Ebrahimzadeh et al [6] proposed a fractional model to study water pollution, incorporating stability analysis, a numerical scheme, and a control variable to transform soluble pollutants into insoluble ones.Numerous studies have also focused on this issue and related topics [7,8,9,10].

In this article, we develop an in-depth mathematical model to analyze water pollution, focusing on the impact of various pollutants on water quality. The aim is to better understand the mechanisms of contamination and assess the consequences of these pollutants on aquatic ecosystems and human health.Based on this analysis, we propose tailored strategies to limit, control, or eliminate pollution sources, taking into account the specifics of each context. These approaches aim to provide sustainable and effective solutions for managing water pollution.

In this paper, Section 2 presents a discrete mathematical model illustrating water pollution. Section 3 is dedicated to formulating an optimal control problem associated with this model, where we demonstrate the existence of the optimal control and characterize it through Pontryagin's Maximum Principle in discrete time. Numerical simulation, performed using MATLAB, are provided in Section 4. Finally, the conclusion of the article is in Section 5.

2. Mathematical Model

in the discrete mathematical model EPE_P, for water pollution, we can conceptualize the pollution mechanism by breaking it down into three distinct compartments. These compartments represent different elements or stages within the system, each influenced by various environmental factors like contamination sources and water management practices.

2.1. Description of the Model.

Compartment (E) refers to the section of water that is at risk of contamination through interaction with the pollutant. In this compartment, the volume of water generally increases due to external inflows, represented by Λ_1, which indicate a continuous influx of water into the system. However, this volume decreases due to two main factors: firstly, evaporation, which occurs at a rate denoted by μE_k, and secondly, the gradual contamination by the pollutant $\beta \frac{P_k E_K}{N}$, which degrades the water quality and may result in changes to the total available volume.

$$E_{k+1} = \Lambda_1 - \beta \frac{P_k E_K}{N} + (1-\mu)\, E_k \tag{1}$$

Compartment (E_p) monitors the amount of water affected by pollution. When the contaminant comes into contact with the water $\beta \frac{P_k E_K}{N}$, the volume of polluted water gradually increases. Conversely, this volume decreases over time due to evaporation μE_{pk} , which reduces the amount of water that can be contaminated.

(2) $$E_{p.k+1} = \beta \, \frac{P_k E_K}{N} + (1-\mu)\, E_{pk}$$

Compartment (P) represents the total amount of pollutants generated by the diversion process, including both direct emissions into the environment Λ_2 and the amount lost without interaction with water μP_k.

(3) $$P_{k+1} = \Lambda_2 + (1-\mu)\, P_k$$

The following diagram illustrates the flow directions of water pollution between different compartments. These directions will be represented by directed arrows in the compartment diagram in figure 1:

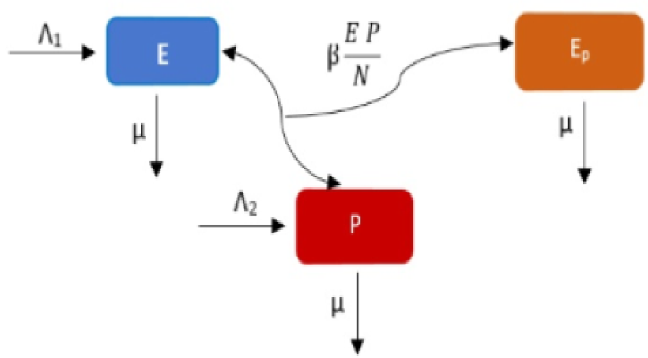

FIGURE 1. The flow between the three compartments

2.2. Model Equations.

Taking into account the rates of pollutant inflow and outflow in the specific compartment, as well as similar rates in the compartment containing the contaminated water, it is possible to formulate difference equations to model the evolution of concentrations in each compartment over discrete time. We therefore propose a water pollution model illustrated by the following system of difference equations:

(4) $$\begin{cases} E_{k+1} = \Lambda_1 - \beta \, \frac{P_k E_K}{N} + (1-\mu)\, E_k \\ E_{p.k+1} = \beta \, \frac{P_k E_K}{N} + (1-\mu)\, E_{pk} \\ P_{k+1} = \Lambda_2 + (1-\mu)\, P_k \end{cases}$$

And T : final time

with $E_0,\ E_{p;0},\ P_0 \geq 0$.

3. Formulation of the Model

The control strategy we are implementing aims to effectively reduce the concentration of pollutants in water. Our main goal is to limit contamination levels through targeted actions. In this model, we include three control u_k , v_k and w_k awareness-raising actions and educational programs designed to reduce pollution sources, water purification treatments to remove existing pollutants, and source reduction strategies to prevent their emergence with regular monitoring at time k. So the controlled mathematical system is given by the following system of difference equations :

$$\begin{cases} E_{k+1} = \Lambda_1 + (1-\mu)\ E_k - \beta\ (1-u_k)\ \frac{P_k E_K}{N} + v_k\ E_{pk} \\ E_{p.k+1} = \beta\ (1-u_k)\ \frac{P_k E_K}{N} + (1-\mu)\ E_{pk} - v_k\ E_{pk} \\ P_{k+1} = \Lambda_2 + (1-\mu\ )\ P_k - w_k\ P_k \end{cases} \tag{5}$$

And T : final time

with $E_0,\ E_{p.0},\ P_0 \geq 0$.

4. The Optimal Control Problem

There are three controls $u = (u_0, u_1, \cdots, u_{T-1})$, $v = (v_0, v_1, \cdots, v_{T-1})$ and $w = (w_0, w_1, \cdots, w_{T-1})$. The first control pertains to efforts in raising awareness and implementing educational programs designed to reduce water pollution sources. we note that $u\frac{P_k E_K}{N}$ is the proportion of reduction in pollution sources due to these awareness and educational initiatives at time step k. The second control addresses the efforts involved in water purification treatments to eliminate existing pollutants. So,we note that vE_{pk} is the proportion of purification actions that help decrease pollutant levels in the water at time step k. The third control relates to management strategies aimed at reducing pollutant generation at the source and preventing future pollution. In this case, wP_k reflects the proportion of measures taken to limit the creation of new pollutants at time step k.

The problem is to minimize the objective functional.

$$J(u,v,w) = P_T + E_{P\ T} + \sum_{k=0}^{T-1}\left[P_k + E_{p.k} + \frac{A_k}{2}u_k^2 + \frac{B_k}{2}v_k^2 + \frac{C_k}{2}w_k^2\right] \tag{6}$$

Where A_k, B_k and C_k are the cost coefficients. They are selected to weigh the relative importance of u_k ,$v_k$and w_k at time k, T is the final time.

In other words, we seek the optimal controls u^*, v^* and w^* such that

$$J(u^*,v^*,w^*) = \min_{u,v,w\in U} J(u,v,w) \tag{7}$$

Where U is the set of admissible controls defined by

$U = \{(u,v,w)/\ 0 \leq u_{\min} \leq u_k \leq u_{\max} \leq 1, 0 \leq v_{\min} \leq v_k \leq v_{\max} \leq 1$ and $0 \leq w_{\min} \leq w_k \leq w_{\max} \leq 1\ , k \in \{0,1,...,T-1\}\}$

In order to derive the necessary condition for optimal control, the pontryagin's maximum principle, in discrete time, given in was used. This principle converts into a problem of minimizing a Hamiltonian, H_k at time step k defined by

$$H_k = P_k + E_{p.k} + \frac{A_k}{2}u_k^2 + \frac{B_k}{2}v_k^2 + \frac{C_k}{2}w_k^2 \sum_{i=1}^{4} \lambda_{i,k+1} f_{i,k+1}(E_k, E_{pk}, P_k) \tag{8}$$

where $f_{i,k+1}$ is the right side of the difference equation of the i^{th} state variable at time step $k+1$.

5. The Optimal Control: Existence

We first show the existence of solutions of the system, after that we will prove the existence of optimal control $([8],[9])$.

Theorem 1. *Consider the control problem with the system. There are three optimal controls* $(u^*,v^*,w^*) \in U^3$ *such that*

$$J(u^*,v^*,w^*) = \min_{u,v,w\in U} J(u,v,w) \tag{9}$$

Given the optimal controls (u^*,v^*,w^*)*and the solutions* E^*, E_p^* *and* P^* *of the corresponding state system* (4)*, there exists adjoint variables* $\lambda_{1,k}$, $\lambda_{2,k}$ *, and* $\lambda_{3,k}$

$$\lambda_{1,k} = \frac{\partial H_k}{\partial E_k} = 1 + \lambda_{1,k+1}\left[(1-\mu) - \beta\ \frac{P_k}{N}(1-u_k)\right] + \lambda_{2,k+1}\left[\beta\ \frac{P_k}{N}(1-u_k)\right]$$

$$\lambda_{2,k} = \frac{\partial H_k}{\partial E_{p.k}} = \lambda_{1,k+1} v_k \; + \lambda_{2,k+1}(1-\mu - v_k)$$

$$\lambda_{3,k} = \frac{\partial H_k}{\partial P_k} = 1 - \lambda_{1,k+1}\, \beta\, \frac{E_k}{N}(1-u_k) + \lambda_{2,k+1} \beta\, \frac{E_k}{N}(1-u_k) \; + \lambda_{3,k+1}(1-\mu - w_k)$$

With the transversality conditions at time T : $\lambda_1,_T = 0$, $\lambda_{2,T} = 1$ and $\lambda_{3,T} = 1$

Furthermore, for $k = 0,1,2...T-1$, the optimal controls u^*, v^* and w^* are given by

$$u^* = \min\left(1, \max\left(0, (\frac{\lambda_{2,k+1} \; - \lambda_{1,k+1}}{A_k}\right) \times \beta\, \frac{E_k P_k}{N}\right)$$

$$v^* = \min\left(1, \max\left(0, \frac{(\lambda_{2,k+1} - \lambda_{1,k+1})}{B_k} \times E_{pk}\right)\right)$$

$$w^* = \min\left(1, \max\left(0, \frac{(P_k\; \lambda_{3,k+1})}{C_k}\right)\right)$$

Proof. Since the coefficients of the state equations are bounded and there are a finite number of time steps, $E = (E_0, E_1, ..., E_T)$, $E_P = (E_{P0}, E_{P1}, ..., E_{PT})$, and $P = (P_0, P_1, ..., P_T)$ are uniformly bounded for all (u, v, w) in the control set U, thus $J(u, v, w)$ is bounded for all (u, v, w) $\in U$. Since $J(u, v, w)$ is bounded, $\inf_{(u,v,w)\in U}\; J(u, v, w)$ is finite, and there exists a sequence $(u^i, v^i, w^i) \in U$ such that $\lim_{i\to+\infty} J(u^i, v^i, w^i) = \inf_{(u,v,w)\in U} J(u, v, w)$ and corresponding sequences of states E^i, E^i_p, and P^i. Since there is a finite number of uniformly bounded sequences, there exist $(u^*, v^*, w^*) \in U$ and E^*, E^*_P, and $P^* \in \mathbb{R}^{T+1}$ such that, on a subsequence, $(u^i, v^i, w^i) \mapsto (u^*, v^*, w^*)$, $E^i \mapsto E^*$, $E^i_P \mapsto E^*_P$, and $p^i \mapsto p^*$. Finally, due to the finite dimensional structure of system (2) and the objective function $J(u, v, w)$, and (u^*, v^*, w^*) is an optimal control with corresponding states E^*, E^*_p, and P^*. Therefore $\inf_{(u,v,w)\in U} J(u, v, w)$ is achieved.

The Hamiltonian at time step k is given by

$$(10) \qquad H_k = E_{pk} + P_k + \frac{A_k}{2}u_k^2 + \frac{B_k}{2}v_k^2 + \frac{C_k}{2}w_k^2 + \lambda_{1,k+1}\left[\Lambda_1 + (1-\mu)\, E_k - \beta\, \frac{P_k E_k}{N}(1-u_k) + v_k E_{pk}\right]$$
$$+\lambda_{2,k+1}\left[\beta\, \frac{P_k E_k}{N}(1-u_k) + (1-\mu)\, E_{pk} \; - v_k E_{pk}\right]$$
$$+\lambda_{3,k+1}\left[\Lambda_2 + (1-\mu)P_k - w_k P_k\right]$$

For, $k = 0,1...,T-1$ the optimal controls u_k, v_k and w_k can be solved from the optimality condition,

$$\frac{\partial H_k}{\partial u_k} = 0$$
$$\frac{\partial H_k}{\partial v_k} = 0$$

$$\frac{\partial H_k}{\partial w_k}=0$$

That are

$$\begin{aligned}\frac{\partial H_k}{\partial u_k} &= A_k u_k+(\lambda_{1,k+1}-\lambda_{2,k+1})\,\beta\,\frac{P_k E_k}{N}=0\\ \frac{\partial H_k}{\partial v_k} &= B_k v_k+(\lambda_{1,k+1}-\lambda_{2,k+1})E_{pk}=0\\ \frac{\partial H_k}{\partial w_k} &= C_k w_k-\lambda_{3,k+1}P_k=0\end{aligned}$$

we have

$$u_k=\frac{(\lambda_{2,k+1}-\lambda_{1,k+1})}{A_k}\times\beta\,\frac{P_k E_k}{N}$$

$$v_k=\frac{(\lambda_{2,k+1}-\lambda_{1,k+1})}{B_k}\times E_{Pk}$$

$$w_k=\frac{\lambda_{3,k+1}\,P_k}{C_k}$$

By the bounds in U of the controls, it is easy to obtain u_k^*, v_k^* and w_k^* in the form of system.

6. Numerical Simulation

Algorithm. In this section, we present the results obtained by solving numerically the optimality system.

This system consists of the state system, adjoint system, initial and final time conditions, and the controls characterization. So, the optimality system is given by the following :

Step 1: $E_0=e_0,\ P_0=p_0,\ Ep_0=ep_0,\ \lambda_1,_T=0,\ \lambda_{2,T}=1,\ \lambda_{3,T}=1$ and given $u^*_{k,0}$, $v^*_{k,0}$ and $w^*_{k,0}$

Step 2. For $k=0;1;...;T-1$ do:

$$\begin{aligned}&E_{k+1}=\Lambda_1+(1-\mu)\,E_k-\beta\,(1-u_k)\,\tfrac{P_k E_K}{N}+v_k\,E_{pk}\\ &E_{p.k+1}=\beta\,(1-u_k)\,\tfrac{P_k E_K}{N}+(1-\mu)\,E_{pk}-v_k\,E_{pk}\\ &P_{k+1}=\Lambda_2+(1-\mu\,)\,P_k-w_k\,P_k\end{aligned}\tag{11}$$

and

$$\lambda_{1,T\ k}=1+\lambda_{1,T\ k+1}\left[(1-\mu)-\beta\,\frac{P_k}{N}(1-u_k)\right]+\lambda_{2,T\ k+1}\left[\beta\,\frac{P_k}{N}(1-u_k)\right]$$

$$\lambda_{2,T\ k}=\lambda_{1,T\ k+1}v_k\ \ +\lambda_{2,T\ k+1}(1-\mu-v_k)$$

$$\lambda_{3,T\ k} = 1 - \lambda_{1,T\ k+1}\ \beta\ \frac{E_k}{N}(1-u_k) + \lambda_{2,T\ k+1}\beta\ \frac{E_k}{N}(1-u_k)\ + \lambda_{3,T\ k+1}(1-\mu-w_k)$$

and

$$u^* = \min\left(1, \max\left(0, (\frac{\lambda_{2,T\ k+1}\ - \lambda_{1,T\ k+1}}{A_k}\right) \times \beta\ \frac{E_k P_k}{N}\right)$$

$$v^* = \min\left(1, \max\left(0, \frac{(\lambda_{2,T\ k+1} - \lambda_{1,T\ k+1})}{B_k} \times E_{pk}\right)\right)$$

$$w^* = \min\left(1, \max\left(0, \frac{(P_k\ \lambda_{3,T\ k+1})}{C_k}\right)\right)$$

end for

Step 3.For $k = 0; 1; ...; T$ write:

$E_k^* = E_k^i$, $Ep_k^* = Ep_k^i$ and $P_k^* = P_k^i$

$u_k^* = u_k^i$, $v_k^* = v_k^i$ and $w_k^* = w_k^i$

Different simulations can be carried out using various values of parameters. In the present numerical approach, we use the following parameters values taken from [2]:

paramter	value in mth^{-1}
μ_1	0.05
μ_2	0.1
β	0.2
Λ_1	2000
Λ_2	1000

Table 1: Parameter values used in numerical simulation

$E(0) = 10000,\ Ep(0) = 6000, P(0) = 6000,\ n = 100,\ \lambda_1(n) = 0,\ \lambda_2(n) = 1$, and $\lambda_3(n) = 1$.

Since control and state functions are on different scales, the weight constant value is chosen as follows: $A = 10000$, $B = 20000$ and $C = 20000$.

After analyzing the data (Table 1), it is evident that the volume of polluted water has significantly increased after 100 days, rising from $0.5.10^4\ to\ 2.6.10^4$ (Figure 2). This increase can be attributed to two main factors: pollution sources and the influence of human and industrial activities on the environment. Furthermore, it is noteworthy that the pollution level stabilizes

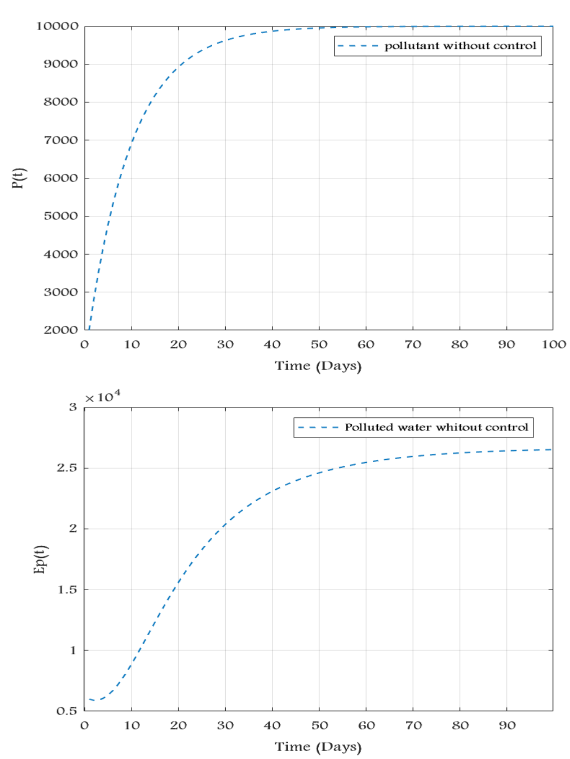

FIGURE 2. The evolution of the amount of contaminated water and that of the pollutant without control

after some time, suggesting that it reaches a steady state. This trend reflects a continuous increase in water pollution, alongside changes in water management, which are shaped by various environmental and social factors.

Regarding the pollutant curve, there is an initial sharp increase in the pollutant. However, after about 30 days, this rise begins to stabilize. After 80 days, the pollutant stabilizes around 10000, indicating a high level of pollution. It is essential to implement measures to reduce this pollution before it becomes dangerous.

In this formulation, there are initial conditions for the state variables and terminal conditions for the adjoints. In other words, the optimality system is a two-point boundary problem with boundary conditions separated at the time steps $k = 0$ and $k = T$. We solve the optimality system using an iterative method, with a direct solution of the state system followed by an inverse solution of the adjoint system. We begin with an initial estimate of the controls during the first iteration, and before the next iteration, we update the controls using the characterization.

We continue until convergence of the successive iterations is achieved.

The proposed control strategy in this work helps to achieve several objectives.

6.1. Strategy A: Control through awareness actions, educational programs.

Strategy with control u_k

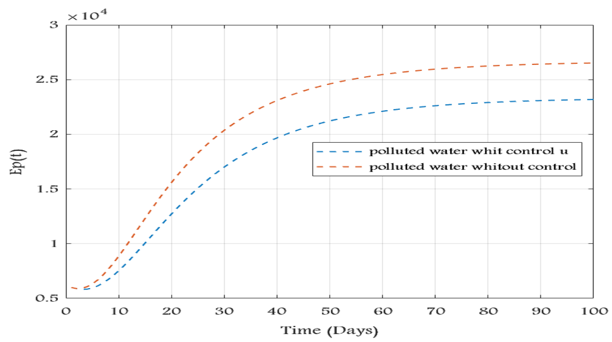

FIGURE 3. The evolution of the amount of polluted water with one controls u_k

This strategy involves implementing a unique optimal control (Figure 3) to minimize polluted water. It includes launching an awareness program through education, media, campaigns, and business partnerships.

After applying the strategy u_k it is clear that the concentration of polluted water decreased from $2.6.10^4$ *to* $2.3.10^4$ over the course of 100 days. This demonstrates that these interventions have truly contributed to reducing the impact of pollution

6.2. Strategy B: Control through awareness actions, educational programs, and water purification treatments.

Strategy with two controls u_k and v_k

In this strategy, we integrate an additional optimal control v_k (Figure 4), which we combine with previous approaches to achieve better results. This includes the application of water purification treatments aimed at removing existing pollutants, using methods such as mechanical and physical filtration, activated carbon filtration, chemical treatment, UV disinfection, and biological treatment, with weekly monitoring of the processes.

After applying various strategies u_k and v_k, it is clear that the level of polluted water decreased from $2.6.10^4$ *to* $0.4.10^4$ at the end of the 100 days, demonstrating that the actions effectively reduced the impact of the pollution.

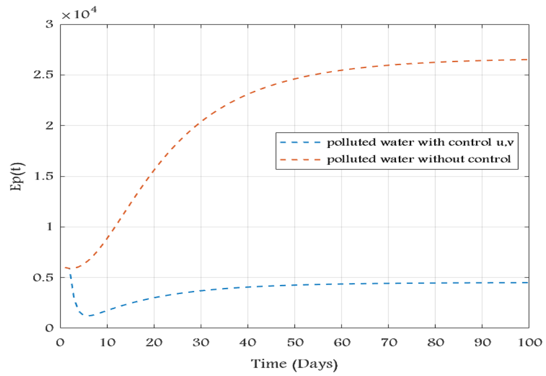

FIGURE 4. The evolution of the amount of polluted water with two controls u_k and v_k

6.3. Strategy C: Controls with strategies to reduce pollutants in order to prevent their emergence.

control with three controls u_k ,$v_k$and w_k

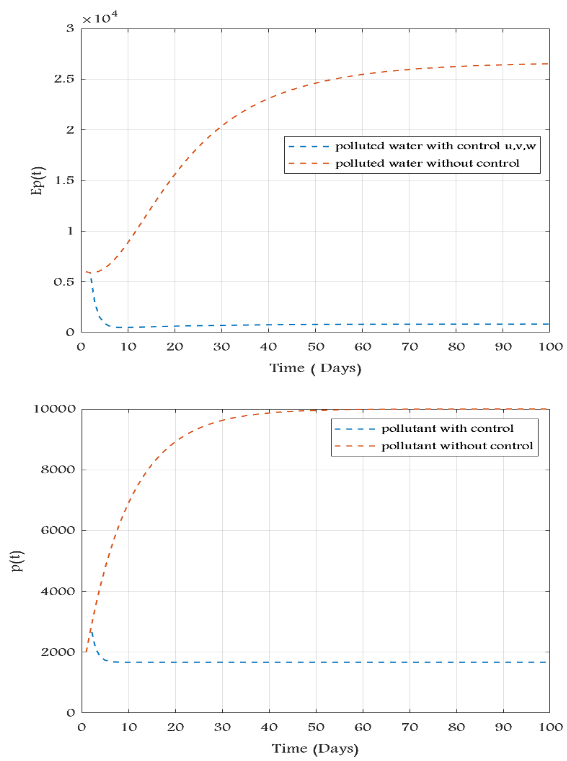

FIGURE 5. The evolution of the amount of polluted water with three controls u_k, v_k and w_k

In this approach, we use another optimal control w_k that we combine with previous strategies to achieve better results. This includes measures such as reducing pollutants at the source, managing urban storm water, treating solid waste, regulating household waste, preventing hydrocarbon pollution, and implementing awareness and education programs, all aimed at reducing pollutants. Figure:5 shows that the pollutant level decreases from $10.10^3 to\ 1.8.10^3$, indicating a significant improvement in the environment.

After applying the three strategies, it is clear that the concentration of polluted water decreased from $2.6.10^4$ *to* $0.1.10^4$ over the course of 100 days. This demonstrates that these interventions have truly contributed to reducing the impact of pollution

7. Conclusion

In this paper, we proposed a discrete modeling of polluted water and its impacts on the environment, aiming to reduce water pollution and mitigate the negative effects of contamination. We also defined three types of control measures: awareness actions and educational programs to reduce pollution sources, purification treatments to eliminate existing pollutants, and strategies to prevent the emergence of new pollution sources. We applied control theory to determine the characteristics of the optimal controls, and the numerical simulation of the results demonstrated the effectiveness of the proposed strategies.

Conflict of Interests

The authors declare that there is no conflict of interests.

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