Available online at http://scik.org

Commun. Math. Biol. Neurosci. 2025, 2025:113

https://doi.org/10.28919/cmbn/9546

ISSN: 2052-2541

# STABILITY AND BIFURCATION ANALYSIS OF PREY-PREDATOR MODEL WITH SIS TYPE OF DISEASE AND ANTI-PREDATOR PROPERTY

AMEER M. SAHI, HUDA ABDUL SATAR*

Department of Mathematics, College of Science, University of Baghdad, Baghdad 10071, Iraq



**Abstract.** A prey-predator model involving anti-predator behavior and fear of predation with disease in a predator is proposed and studied. The main objective of this study is to find out the influence of anti-predator and fear on the population dynamics of the model in the presence of disease. It is assumed that the disease in the predator is of the SIS type. For this purpose, a mathematical model with a Holling- type II functional response was proposed and analyzed. The existence conditions and stability of different equilibrium points for the model were analyzed to determine the qualitative behavior of the model. Investigations of local bifurcations had been conducted. It can be shown that the results of numerical simulations are consistent with analytical results.

**Keywords:** eco-epidemiological; fear; anti-predator; bifurcation; bi-stability.

**2020 AMS Subject Classification:** 92D40, 92D30, 37G10.

## 1. INTRODUCTION

For many years, one of the main areas of ecology research has been the relationship between predators and their prey. A well-known and important topic in population dynamics and applied mathematical modeling is the prey-predator interaction. The behavior of the intricate ecological systems is largely determined by the various types of interspecies interactions, of which these relationships are only one [1]. Indirect effects, such as fear or panic, have been shown to play a

*Corresponding author

E-mail address: mislan.jos@fmipa.unmul.ac.id

Received July 13, 2025

crucial role in influencing the dynamics of prey-predator relationships and the ecosystem at large. Even though Cannon first proposed the idea of fear in 1915 [2], it is still a relatively new idea in mathematical modelling. After that, many researchers have proposed and studied extensively a prey-predator model with the impact of fear on the dynamics of the model; see [3-6]. Recently, various ecological models were introduced that have been used in investigating the role of fear on the dynamical behaviour of ecological systems, see for example [7-10].

Because there are several species in the ecosystem that interact with one another on a constant basis in various ways, there has been an increase in interest in studying illnesses in prey-predator models. This facilitates the quick spread of illness among species. In actuality, the ecological environment naturally contains disease in either the predator, the prey, or both populations. Understanding eco-epidemiological systems is essential to comprehending how diseases spread and how to prevent them. As a result, many researchers studied eco-epidemiology systems in which the disease affects either the prey, predator, or both populations [11–14]. While some researchers concentrated on studying disease in predators alone [10,15-18], several others only looked at disease in prey [19–24]. Nonetheless, significant research has been done on the illnesses that affect both predators and prey [25–27].

Prey often defend themselves by engaging in defensive and counterattacking actions. The final anti-predator tactic used by animals to defend themselves against predators is counterattack [10, 28–30]. Recently, a food chain model that included fear, foraging cooperation, and anti-predator behaviors was studied by Jabbar and Naji [31]. They observed that the system has rich dynamics, including periodic dynamics. Furthermore, Alwan and Satar [32] have developed and examined a three-dimensional system of ordinary differential equations that represents an eco-epidemiological model of a diseased predator with hunting cooperation and anti-predator properties. It has been noted that the system exhibits a variety of attractors, such as point and periodic attractors, because it is highly sensitive to changes in the majority of its parameters.

Keeping the above in mind, it is concluded the necessary to study a prey-predator model having the entire previous biological factor together to understand the influence of them on the dynamical behavior of the model. Therefore, a novel prey-predator system is proposed in this paper so that it has all these factors. This paper is organized in the following manner. Section 2 deals with model formulation. Section 3 existence conditions and stability of different equilibrium points for model were analyzed to determine the qualitative behavior of the model. The conditions that guarantee the occurrence of local bifurcation are determined in section 4. Section 5 deals with the numerical simulation of the system, and finally, the discussion and conclusion is addressed in Section 6.

## 2. MATHEMATICAL FORMULATION

In this section, the mathematical model used in this study is formulated, and the dynamics of the prey-predator system is described in a case with the existence of an infectious disease $SIS-$ type in the predator population, treatment, fear and anti-predator effect. Formulation of an eco-epidemiological system in terms of mathematical equations helps us to analyze the system dynamics and to extract the essential behaviors of the model. So, in order to make a simple model, the following system of nonlinear equations

$$\begin{aligned} \frac{dx}{dt} &= \frac{rx}{1+\alpha s} - d_1 x - bx^2 - \frac{(c_1 s + c_2 I)x}{a+x} \\ \frac{ds}{dt} &= b_1 \frac{c_1 sx}{a+x} - d_2 s - \delta s I - \frac{h_1 xs}{e_1 + s} + \beta I \\ \frac{dI}{dt} &= \delta s I + b_2 \frac{c_2 x I}{a+x} - (d_2 + d_3) I - \beta I - h_2 x I. \end{aligned} \tag{1}$$

Let the biomass of prey and susceptible predator, and infected predator populations at time $t$ be represented by $x, s,$ and $I$, respectively. with $x(0) = x_0 \geq 0, s(0) = s_0 \geq 0,$ and $I(0) = I_0 \geq 0$ representing the initial condition of the system (1), other parameters are shown in Table 1, and all the parameter values are regarded as nonnegative.

**Table 1. The description of the model parameters**

| Parameters | Description |
|---|---|
| $r$ | The prey's intrinsic growth rate. |
| $\alpha$ | Level of fear. |
| $b$ | The intraspecific competition. |
| $c_1, c_2$ | The attack rate of the susceptible predator and infected predator on the prey. |
| $m$ | The predator cooperation in hunting. |
| $\beta$ | The treatment rate. |

| | |
|---|---|
| $d_1$ | The death rate of the prey populations. |
| $b_1, b_2$ | The conversion efficiency from susceptible and infected prey biomass to predator biomass. |
| $d_2$ | The death rates of the susceptible predator populations. |
| $\delta$ | The infection rate. |
| $h_1,h_2$ | The anti-predator rate. |
| $d_3$ | The death rates of infected predator populations. |
| $e_1$ | Half saturation constant. |

As a result, the system (1) solution exists and is unique. In addition, the following theorem establishes that the solutions of the system (1) are uniformly bounded.

**Theorem 1.** Solutions of system (1) starting in $\mathbb{R}^3_+$, are uniformly bounded under the prey's survival condition

$$r > d_1. \tag{2}$$

**Proof.** From the first equation in system (1), we get $x \leq \frac{(r-d_1)}{b} = \Upsilon_1$.

It is sufficient to prove that the total population size $M = x + s + I$ is bounded for all $t$.

Then applying some calculations gives that

$$\frac{dM}{dt} = \frac{rx}{1+\alpha s} - d_1 x - bx^2 - \frac{c_1 xs}{a+x} - \frac{c_2 xI}{a+x} + b_1 \frac{c_1 xs}{a+x} - d_2 s - \delta sI$$
$$- \frac{h_1 xs}{e_1+s} + \beta I + \delta sI + b_2 \frac{c_2 xI}{a+x} - (d_2 + d_3)I - \beta I - h_2 xI.$$

Therefore, it is yield that

$$\frac{dM}{dt} \leq rx - d_1 x - d_2 s - (d_2 + d_3)I \ .$$

Thus, we obtain that $\frac{dM}{dt} \leq N - \Upsilon_2 M$, where $\Upsilon_2 = min\{d_1, d_2\}$, and $N = r\Upsilon_1$.Therefore, solving the differential inequality gives $M(t) \leq \frac{N}{\Upsilon_2} = \Upsilon_3 \ as \ t \to \infty$. Thus, every solution of system (1) is uniformly bounded in the region $\Lambda = \{(x, s, I) \in \mathbb{R}^3_+ : x(t) + s(t) + I(t)\} \leq \Upsilon_3\}$.

**Remark 1.** If the natural death rate exceeds the birth rate of the prey population, then the species ultimately goes extinct from the system with time.

## 3. EQUILIBRIUM POINTS AND STABILITY ANALYSIS

The equilibria and stability analysis of system (1) are examined in the following. The system (1) is shown to have the following equilibrium points

- The extinction equilibrium point $(EEP)$, $q_0 = (0,0,0)$, always exists.
- The axial equilibrium point $(AEP), q_1 = (\hat{x}, 0,0)$, where $\hat{x} = \frac{r-d_1}{b}$, which is exists under condition (2).
- The infected predator extinction equilibrium point $(IPEEP)$, $q_2 = (\bar{x}, \bar{s}, 0)$**,** where

$$\bar{s} = \frac{h_1\bar{x}^2 - \bar{x}\big((b_1c_1-d_2)e_1 - ah_1\big) + ad_2e_1}{\bar{x}(b_1c_1-d_2) - ad_2}\,. \tag{3a}$$

While $\bar{x}$ is a positive root of the following polynomial

$$A_5x^5 + A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 = 0, \tag{3b}$$

where

$A_5 = -b\alpha h_1(b_1c_1 - d_2).$

$$A_4 = -bb_1^2c_1^2 + 2bb_1c_1d_2 - bd_2^2 + b\alpha b_1^2c_1^2e_1 - 2b\alpha b_1c_1d_2e_1 + b\alpha d_2^2e_1 - 2ab\alpha b_1c_1h_1 - \alpha b_1c_1d_1h_1 + 3ab\alpha d_2h_1 + \alpha d_1d_2h_1 - \alpha c_1h_1^2$$

$$A_3 = -abb_1^2c_1^2 + rb_1^2c_1^2 - b_1^2c_1^2d_1 + 4abb_1c_1d_2 - 2rb_1c_1d_2 + 2b_1c_1d_1d_2 - 3abd_2^2 + rd_2^2 - d_1d_2^2 + ab\alpha b_1^2c_1^2e_1 + \alpha b_1^2c_1^2d_1e_1 - 4ab\alpha b_1c_1d_2e_1 - 2\alpha b_1c_1d_1d_2e_1 + 3ab\alpha d_2^2e_1 + \alpha d_1d_2^2e_1 - a^2b\alpha b_1c_1h_1 - b_1c_1^2h_1 - 2a\alpha b_1c_1d_1h_1 + 3a^2b\alpha d_2h_1 + c_1d_2h_1 + 3a\alpha d_1d_2h_1 + 2\alpha b_1c_1^2e_1h_1 - 2\alpha c_1d_2e_1h_1 - 2a\alpha c_1h_1^2.$$

$$A_2 = arb_1^2c_1^2 - ab_1^2c_1^2d_1 + 2a^2bb_1c_1d_2 - 4arb_1c_1d_2 + 4ab_1c_1d_1d_2 - 3a^2bd_2^2 + 3ard_2^2 - 3ad_1d_2^2 + b_1^2c_1^3e_1 + a\alpha b_1^2c_1^2d_1e_1 - 2a^2b\alpha b_1c_1d_2e_1 - 2b_1c_1^2d_2e_1 - 4a\alpha b_1c_1d_1d_2e_1 + 3a^2b\alpha d_2^2e_1 + c_1d_2^2e_1 + 3a\alpha d_1d_2^2e_1 - \alpha b_1^2c_1^3e_1^2 + 2\alpha b_1c_1^2d_2e_1^2 - \alpha c_1d_2^2e_1^2 - ab_1c_1^2h_1 - a^2\alpha b_1c_1d_1h_1 + a^3b\alpha d_2h_1 + 2ac_1d_2h_1 + 3a^2\alpha d_1d_2h_1 + 2a\alpha b_1c_1^2e_1h_1 - 4a\alpha c_1d_2e_1h_1 - a^2\alpha c_1h_1^2.$$

$$A_1 = -2a^2rb_1c_1d_2 + 2a^2b_1c_1d_1d_2 - a^3bd_2^2 + 3a^2rd_2^2 - 3a^2d_1d_2^2 - 2ab_1c_1^2d_2e_1 - 2a^2\alpha b_1c_1d_1d_2e_1 + a^3b\alpha d_2^2e_1 + 2ac_1d_2^2e_1 + 3a^2\alpha d_1d_2^2e_1 + 2a\alpha b_1c_1^2d_2e_1^2 - 2a\alpha c_1d_2^2e_1^2 + a^2c_1d_2h_1 + a^3\alpha d_1d_2h_1 - 2a^2\alpha c_1d_2e_1h_1.$$

$A_0 = a^3 r d_2^2 - a^3 d_1 d_2^2 + a^2 c_1 d_2^2 e_1 + a^3 \alpha d_1 d_2^2 e_1 - a^2 \alpha c_1 d_2^2 e_1^2.$

Straightforward computation shows that there is at least one when $A_5$ and $A_0$ have opposite signs and the following condition holds:

$$\left.\begin{array}{c} \bar{x}(b_1 c_1 - d_2) - a d_2 > 0 \\ h_1 \bar{x}^2 - \bar{x}\big((b_1 c_1 - d_2)e_1 - a h_1\big) + a d_2 e_1 > 0 \end{array}\right\}. \tag{3c}$$

- The positive equilibrium point ($PEP$), denoted by $q_3 = (x^*, s^*, I^*)$, where

$$s^* = \frac{\beta(a+x^*) - b_2 c_2 x^* + (d_2 + d_3 + h_2 x^*)(a+x^*)}{\delta(a+x^*)}. \tag{4a}$$

$$I^* = \frac{(a+x^*)(-b_2 c_2 x^* + (a+x^*)(\beta + d_2 + d_3 + h_2 x^*))}{b_2 c_2 x^* - (a+x^*)(d_2 + d_3 + h_2 x^*)} \left[ -\frac{b_1 c_1 x^*}{(a+x^*)^2 \delta} + \frac{d_2}{(a+x^*)\delta} + \frac{h_1 x^*}{-b_2 c_2 x^* + (a+x^*)(\beta + d_2 + d_3 + \delta e_1 + h_2 x^*)} \right]. \tag{4b}$$

While $x^*$ is a positive root of the following equation:

$$\frac{r}{1+\alpha s^*} - d_1 - bx - \frac{(c_1 s^* + c_2 I^*)}{a+x} = 0 \tag{4c}$$

Direct computation shows that $q_3$ exists provided that:

$$\left.\begin{array}{c} (a+x^*)(d_2 + d_3 + h_2 x^*) < b_2 c_2 x^* < (a+x^*)(\beta + d_2 + d_3 + h_2 x^*) \\ \frac{b_1 c_1 x^*}{(a+x^*)^2 \delta} < \frac{d_2}{(a+x^*)\delta} + \frac{h_1 x^*}{-b_2 c_2 x^* + (a+x^*)(\beta + d_2 + d_3 + \delta e_1 + h_2 x^*)} \\ or \\ b_2 c_2 x^* < (a+x^*)(d_2 + d_3 + h_2 x^*) \\ \frac{d_2}{(a+x^*)\delta} + \frac{h_1 x^*}{-b_2 c_2 x^* + (a+x^*)(\beta + d_2 + d_3 + \delta e_1 + h_2 x^*)} < \frac{b_1 c_1 x^*}{(a+x^*)^2 \delta} \end{array}\right\}. \tag{4d}$$

The local stability (LS) of the equilibrium points (EPs) of system (1) is investigated by computing the Jacobian matrix (JM) of the system at each of these points and then computing their eigenvalues. Now, the JM of system (1)

$$J = [J_{ij}^*]_{3\times 3} \tag{5}$$

where

$$J_{11} = -bx - \frac{c_1 s + c_2 I}{a+x} + \frac{r}{1+\alpha s} + x\left(-b + \frac{c_1 s + c_2 I}{(a+x)^2}\right) - d_1;\ J_{12} = x\left(-\frac{r\alpha}{(1+\alpha s)^2} - \frac{c_1}{a+x}\right);$$

$$J_{13} = -\frac{c_2 x}{a+x};\ J_{21} = \frac{a b_1 c_1 s}{(a+x)^2} - \frac{h_1 s}{e_1 + s};J_{22} = \frac{b_1 c_1 x}{a+x} - \delta I - d_2 + \frac{h_1 e_1 x}{(e_1+s)^2};J_{23} = \beta - \delta s$$

$$J_{31} = \frac{a b_2 c_2 I}{(a+x)^2} - h_2 I\ ;J_{32} = \delta I;\ J_{33} = \delta s + \frac{b_2 c_2 x}{a+x} - \beta - (d_2 + d_3) - h_2 x.$$

The JM at $q_0 = (0,0,0)$ is given by

$$J(q_0) = \begin{pmatrix} r - d_1 & 0 & 0 \\ 0 & -d_2 & \beta \\ 0 & 0 & -(d_2 + d_3) - \beta \end{pmatrix} \tag{6}$$

However, $\lambda_{01} = r - d_1, \lambda_{02} = -d_2,$ and $\lambda_{03} = -(d_2 + d_3) - \beta$ , then $q_0$ is locally asymptotically stable (L.AS) if $r < d_1$.

The JM at $q_1 = (\hat{x}, 0,0)$ can be calculated as

$$J(q_1) = \begin{pmatrix} -b\hat{x} & -r\alpha\hat{x} - \frac{c_1\hat{x}}{(a+\hat{x})} & -\frac{c_2\hat{x}}{(a+\hat{x})} \\ 0 & \frac{b_1c_1\hat{x}}{(a+\hat{x})} - d_2 - \frac{h_1\hat{x}}{e_1} & \beta \\ 0 & 0 & \frac{b_2c_2\hat{x}}{(a+\hat{x})} - (d_2 + d_3) - \beta - h_2\hat{x} \end{pmatrix} \tag{7a}$$

implies, $\lambda_{11} = -b\hat{x}$, $\lambda_{12} = \frac{b_1c_1\hat{x}}{(a+\ddot{x})} - d_2 - \frac{h_1\hat{x}}{e_1}$ and $\lambda_{13} = \frac{b_2c_2\hat{x}}{(a+\hat{x})} - (d_2 + d_3) - \beta - h_2\hat{x}$, then $q_1$ is L.AS provided that:

$$\frac{b_1c_1\hat{x}}{(a+\hat{x})} < d_2 + \frac{h_1\hat{x}}{e_1}, \tag{7b}$$

$$\frac{b_2c_2\hat{x}}{(a+\hat{x})} < (d_2 + d_3) + \beta + h_2\hat{x} \tag{7c}$$

The JM at $q_2 = (\bar{x}, \bar{s}, 0)$ can be calculated as:

$$J(q_2) = \begin{pmatrix} -b\bar{x} + \frac{c_1\bar{x}\bar{s}}{(a+\bar{x})^2} & -\frac{r\alpha\bar{x}}{(1+\alpha\bar{s})^2} - \frac{c_1\bar{x}}{(a+\bar{x})} & -\frac{c_2\bar{x}}{(a+\bar{x})} \\ \frac{ab_1c_1\bar{s}}{(a+\bar{x})^2} - \frac{h_1\bar{s}}{(e_1+\bar{s})} & \frac{c_1b_1\bar{x}}{(a+\bar{x})} - d_2 - \frac{e_1h_1\bar{x}}{(e_1+\bar{s})^2} & -\delta\bar{s} + \beta \\ 0 & 0 & \delta\bar{s} + \frac{b_2c_2\bar{x}}{(a+\bar{x})} - (d_2 + d_3) - \beta - h_2\bar{x} \end{pmatrix} \tag{8a}$$

Obviously, the eigenvalue $\lambda_{23} = \delta\bar{s} + \frac{b_2c_2\bar{x}}{(a+\bar{x})} - (d_2 + d_3) - \beta - h_2\bar{x}$ , and the other two eigenvalues are given

$$\lambda_{21} = \frac{\tilde{T}}{2} + \frac{1}{2}\sqrt{\tilde{T}^2 - 4\tilde{D}};\ \lambda_{22} = \frac{\tilde{T}}{2} - \frac{1}{2}\sqrt{\tilde{T}^2 - 4\tilde{D}},$$

where $\tilde{T} = -b\bar{x} + \frac{c_1\bar{x}\bar{s}}{(a+\bar{x})^2} + \frac{c_1b_1\bar{x}}{(a+\bar{x})} - d_2 - \frac{e_1h_1\bar{x}}{(e_1+\bar{s})^2}$, and $\tilde{D} = (a_{11}a_{22} - a_{12}a_{21})$. The eigenvalues $\lambda_{21}$, $\lambda_{22}$ have negative real parts with the following conditions

$$\frac{c_1\bar{x}\bar{s}}{(a+\bar{x})^2} + \frac{c_1b_1\bar{x}}{(a+\bar{x})} < b\bar{x} + d_2 + \frac{e_1h_1\bar{x}}{(e_1+\bar{s})^2}, \tag{8b}$$

$$\left(-b\bar{x}+\frac{c_1\,\bar{x}\bar{s}}{(a+\bar{x})^2}\right)\left(\frac{c_1b_1\bar{x}}{(a+\bar{x})}-d_2-\frac{e_1h_1\bar{x}}{(e_1+\bar{s})^2}\right)+\left(\frac{r\alpha\bar{x}}{(1+\alpha\bar{s})^2}+\frac{c_1\bar{x}}{(a+\bar{x})}\right)\left(\frac{ab_1c_1\bar{s}}{(a+\bar{x})^2}-\frac{h_1\bar{s}}{(e_1+\bar{s})}\right)>0. \quad (8c)$$

Thus $q_2$ is L.AS if, in addition to the above last conditions, the following condition holds:

$$\delta\bar{s}+\frac{b_2c_2\bar{x}}{(a+\bar{x})}<(d_2+d_3)+\beta+h_2\bar{x}. \quad (8d)$$

Finally, the JM evaluated at the $PEP, q_3=(s^*,I^*,y^*)$, is given by

$$J(q_3)=(w_{ij})_{3\times 3} \quad (9a)$$

$$w_{11}=-bx^*+\frac{(c_1s+c_2I)x^*}{(a+x)^2}, w_{12}=-\frac{r\alpha x^*}{(1+s\alpha)^2}-\frac{c_1x^*}{a+x^*}, w_{13}=-\frac{c_2x^*}{(a+x^*)},$$

$$w_{21}=\frac{ab_1c_1s}{(a+x^*)^2}-\frac{h_1s}{e_1+s}, w_{22}=\frac{b_1c_1x^*}{a+x^*}-\delta I-d_2+\frac{e_1h_1x^*}{(s+e_1)^2}, \quad w_{23}=-\delta s^*+\beta\ ,$$

$$w_{31}=\frac{ab_2c_2I^*}{(a+x^*)^2}-h_2I^*,\ w_{32}=\delta I^*,\ w_{33}=0.$$

The corresponding characteristic equation is

$$\lambda_3^3+\rho_1\lambda_3^2+\rho_2\lambda_3+\rho_3=0 \quad (9b)$$

where,

$$\rho_1=-(w_{11}+w_{22})$$

$$\rho_2=(w_{11}w_{22}-w_{12}w_{21})-w_{13}w_{31}-w_{23}w_{32}.$$

$$\rho_3=-[w_{23}(w_{12}w_{31}-w_{11}w_{32})+w_{13}(w_{21}w_{32}-w_{22}w_{31})].$$

After the simple calculating, we obtain that $\Delta=\rho_1\rho_2-\rho_3$

$$=-(w_{11}+w_{22})(w_{11}w_{22}-w_{12}w_{21})+w_{13}(w_{11}w_{31}+w_{32}w_{21})+w_{23}(w_{22}w_{32}+w_{12}w_{31}).$$

Based on the criterion of Routh -Hawirtiz, whole eigenvalues of $J(q_3)$ possess roots with negative real portions if $\rho_i,(i=1,3)>0$ and $\Delta=\rho_1\rho_2-\rho_3>0$. Then, $q_3$ is L.AS if

$$\frac{(c_1s+c_2I^*)x^*}{(a+x)^2}<bx^* \quad (9c)$$

$$\frac{b_1c_1x^*}{a+x^*}+\frac{e_1h_1x^*}{(s^*+e_1)^2}<\delta I^*+d_2 \quad (9d)$$

$$\frac{h_1s}{e_1+s}<\frac{ab_1c_1s}{(a+x^*)^2} \quad (9e)$$

$$w_{11}w_{22}-w_{12}w_{21}>0 \quad (9f)$$

$$w_{13}(w_{11}w_{31}+w_{32}w_{21})+w_{23}(w_{22}w_{32}+w_{12}w_{31})>0 \quad (9h)$$

In the following theorems, the globally asymptotically stable G.AS of all the locally stable equilibrium points is studied with the help of the Lyapunov method.

**Theorem 2.** The $EEP$, $q_0 = (0,0,0)$ is G.AS whenever its L.AS.

**Proof.** We select an appropriate positive definite function about $q_0$ as

$$P_0 = x + s + I$$

Then, the derivative $\frac{dP_0}{dt}$ can be determined as

$$\frac{dP_0}{dt} = \left[\frac{rx}{1+\alpha s} - d_1 x - bx^2 - \frac{(c_1 s + c_2 I)x}{a+x}\right] + \left[b_1 \frac{c_1 sx}{a+x} - d_2 s - \delta sI - \frac{h_1 xs}{e_1+s} + \beta I\right]$$
$$+[\delta sI + b_2 \frac{c_2 Ix}{a+x} - (d_2 + d_3)I - \beta I - h_2 xI \ ].$$

By using system (1) with some algebraic manipulations, we obtain that

$$\frac{dP_0}{dt} \leq (r - d_1)x - \ d_2 s - q_2 y - (d_2 + d_3)I.$$

So, the function $\frac{dP_0}{dt}$ is negative definite under the L.AS condition. Thus $q_0$ is G.AS.

**Theorem 3.** The $AEP$ given by $q_1 = (\hat{x}, 0,0)$ is L.AS, then it is GAS if the following condition met.

$$r\alpha\hat{x} + \frac{c_1\hat{x}}{a} < d_2 \tag{10a}$$

$$\frac{c_2\hat{x}}{a} < (d_2 + d_3) \tag{10b}$$

**Proof.** We select an appropriate positive definite function about $q_1$ as

$$P_1 = \left[x - \hat{x} - \hat{x}\ln\left(\frac{x}{\hat{x}}\right)\right] + s + I$$

Now, it is clear to see that $\frac{dP_1}{dt}$ as follows

$$\frac{dP_1}{dt} = (x - \hat{x})\left[\frac{r}{1+\alpha s} - d_1 - bx - \frac{(c_1 s + c_2 I)}{a+x}\right] + \left[b_1 \frac{c_1 sx}{a+x} - d_2 s - \delta sI - \frac{h_1 xs}{e_1+s} + \beta I\right] + [\delta sI +$$
$$b_2 \frac{c_2 Ix}{a+x} - (d_2 + d_3)I - \beta I - h_2 xI \ ].$$

By using system (1) with some algebraic manipulations, we obtain that

$$\frac{dP_1}{dt} \leq -b(x - \hat{x})^2 - \left[d_2 - r\alpha\hat{x} - \frac{c_1\hat{x}}{a}\right]s - \left[(d_2 + d_3) - \frac{c_2\hat{x}}{a}\right]I.$$

Therefore, $q_1$ is G.AS if the conditions (10a) and (10b) hold.

**Theorem 4.** The $IPEEP$ given by $q_2 = (\bar{x}, \bar{s}, 0)$ is L.AS then it is G.AS if the following condition met.

$$\frac{c_1\bar{s}}{a\bar{B}} < b + \frac{L_1}{2} \tag{11a}$$

$$\frac{b_1 c_1 x}{a} < \frac{h_1 e_1 x}{(e_1+\Upsilon_3)\bar{C}} + \frac{L_1}{2} + d_2 \tag{11b}$$

$$\frac{c_2\bar{x}}{a} < \beta + (d_2 + d_3) + \beta\bar{s} \tag{11c}$$

**Proof**. We select an appropriate positive definite function about $q_1$ as

$$P_2 = \left[x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right] + \frac{(s-\hat{s})^2}{2} + I$$

Now, it is clear to see that $\frac{dP_2}{dt}$ as follows

$$\frac{dP_2}{dt} = \left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt} + (s-\hat{s})\frac{ds}{dt} + \frac{dI}{dt}$$

and,

$$\frac{dP_2}{dt} = (x-\bar{x})\left[\frac{r}{1+\alpha s} - d_1 - bx - \frac{(c_1 s + c_2 I)}{a+x}\right] + (s-\bar{s})\left[b_1\frac{c_1 s x}{a+x} - d_2 s - \delta s I - \frac{h_1 x s}{e_1+s} + \beta I\right] + [\delta s I + b_2\frac{c_2 I x}{a+x} - (d_2+d_3)I - \beta I - h_2 x I\ ]$$

By using system (1) with some algebraic manipulations, we obtain that

$$\frac{dP_2}{dt} \leq -\left[b - \frac{c_1\bar{s}}{B\bar{B}}\right](x-\bar{x})^2 - \left[\frac{r\alpha}{A\bar{A}} + \frac{c_1}{B} + \frac{h_1\bar{s}}{\bar{C}} - \frac{b_1 c_1 a\bar{s}}{B\bar{B}}\right](x-\bar{x})(s-\bar{s}) - \left[d_2 - \frac{b_1 c_1 x}{B} + \frac{h_1 e_1 x}{C\bar{C}}\right](s-\bar{s})^2 - \left[\beta + (d_2 - d_3) + \beta\bar{s} - \frac{c_2\bar{x}}{B}\right]I$$

Hence,

$$\frac{dP_2}{dt} \leq -\left[b - \frac{c_1\bar{s}}{B\bar{B}} + \frac{L_1}{2}\right](x-\bar{x})^2 - \left[d_2 - \frac{b_1 c_1 x}{B} + \frac{h_1 e_1 x}{C\bar{C}} + \frac{L_1}{2}\right](s-\bar{s})^2 - \left[\beta + (d_2 + d_3) + \beta\bar{s} - \frac{c_2\bar{x}}{B}\right]I$$

where $A = 1 + \alpha s, \bar{A} = 1 + \alpha\bar{s}, B = a + x, \bar{B} = a + \bar{x}, C = e_1 + s,$ and $\bar{C} = e_1 + \bar{s}$, and $L_1 = \frac{r\alpha}{(1+\alpha\Upsilon_3)\bar{A}} + \frac{c_1}{(a+\Upsilon_1)} + \frac{h_1\bar{s}}{\bar{C}} - \frac{b_1 c_1 a\bar{s}}{a\bar{B}}$

Therefore, $q_2$ is G.AS if the conditions (11a) and (11c) hold**.**

**Theorem5.** The $PEP$ given by $q_3 = (x^*, s^*, I^*)$ is L.AS then it is G.AS if the following condition met

$$\frac{(c_1 s^* + c_2 I^*)}{BB^*} < b + \frac{(L_2 + L_3)}{2} \quad (12a)$$

$$\frac{b_1 c_1 x}{B} < d_2 + \delta I^* + \frac{h_1 e_1 x}{CC^*} + \frac{(L_2 + L_4)}{2} \quad (12b)$$

**Proof.** We select an appropriate positive definite function about $q_3$ as

$$P_3 = \left[x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right] + \frac{(s - s^*)^2}{2} + \left[I - I^* - I^* \ln\left(\frac{I}{I^*}\right)\right]$$

Then, the derivative $\frac{dP_0}{dt}$ can be determined as

$$\frac{dP_3}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + (s - s^*)\frac{ds}{dt} + \left(\frac{I - I^*}{I}\right)\frac{dI}{dt}$$

$$\frac{dP_3}{dt} = (x - x^*)\left[\frac{r}{1 + \alpha s} - d_1 - bx - \frac{(c_1 s + c_2 I)}{a + x}\right] + (s - s^*)\left[b_1 \frac{c_1 s x}{a + x} - d_2 s - \delta s I - \frac{h_1 x s}{e_1 + s} + \beta I\right] + (I - I^*)\left[\delta s + b_2 \frac{c_2 x}{a + x} - (d_2 + d_3) - \beta - h_2 x\right]$$

By using system (1) with some algebraic manipulations, we obtain that

$$\frac{dP_3}{dt} \le -\left[b - \frac{(c_1 s^* + c_2 I^*)}{BB^*}\right](x - x^*)^2 - \left[\frac{r\alpha}{AA^*} + \frac{c_1}{B} - \frac{b_1 c_1 a}{BB^*} + \frac{h_1 s^*}{C^*}\right](x - x^*)(s - s^*) - \left[\frac{c_2}{B} - \frac{b_2 c_2 a}{BB^*} + h_2\right](x - x^*)(I - I^*) - \left[d_2 - \frac{b_1 c_1 x}{B} + \delta I^* + \frac{h_1 e_1 x}{CC^*}\right](s - s^*)^2 + [\delta(S - 1) - \beta](s - s^*)(I - I^*)$$

Hence

$$\frac{dP_3}{dt} \le -\left[b - \frac{(c_1 s^* + c_2 I^*)}{BB^*} + \frac{(L_2 + L_3)}{2}\right](x - x^*)^2 - [\frac{(L_3 + L_4)}{2}](I - I^*)^2 - \left[d_2 - \frac{b_1 c_1 x}{B} + \delta I^* + \frac{h_1 e_1 x}{CC^*} + \frac{(L_2 + L_4)}{2}\right](s - s^*)^2$$

where

$$A^* = 1 + \alpha s^*, B^* = a + x^*, C^* = e_1 + s^*, L_2 = \frac{r\alpha}{(1 + \alpha \Upsilon_3)A^*} + \frac{c_1}{(a + \Upsilon_1)} - \frac{b_1 c_1}{B^*} + \frac{h_1 s^*}{C^*},\ L_3 = \frac{c_2}{(a + \Upsilon_1)} - \frac{b_2 c_2}{B^*} + h_2, \text{and } L_4 = \delta(S - 1) - \beta.$$

Therefore, the derivative $\frac{dP_3}{dt}$ is negative definite under the conditions (12a) and (12b) then $q_3$ is G.AS.

## 4. LOCAL BIFURCATION

In this section, the possibility of the occurrence of (L.B) near the equilibrium points of system (1) is investigated, and to understand how the system behavior varies when the model's parameters change, local bifurcation is used by applying the Sotomayor's theorem [33], and obtained results are summarized in the next theorems.

Now for simplifying the notations rewrite the system(1) in the vector form as follows

$$\frac{d\boldsymbol{X}}{dT} = F(X), \text{ with } \boldsymbol{X} = (x, s, I)^t \text{ and } F = (xf_1, s, If_3)^t \tag{13}$$

So, according to the JM of system (1) at $(x, s, I)$, it is easy to verify that for any vector $V = (v_1, v_2, v_3)^t$, we have that

$$D^2\boldsymbol{F}(\boldsymbol{X})(V, V) = \left[g_{ij}\right]_{3\times1} \tag{14}$$

where

$$g_{11} = 2(-b + \frac{a(c_1 s + c_2 I)}{(a+x)^3})v_1^2 + \frac{2r\alpha^2 x v_2^2}{(1+\alpha s)^3} - \frac{2v_1((r(a+x)^2\alpha + a(1+\alpha s)^2 c_1)v_2 + a(1+\alpha s)^2 c_2 v_3)}{(a+x)^2(1+\alpha s)^2}$$

$$g_{21} = \frac{2ab_1 c_1 v_1(-sv_1 + (a+x)v_2)}{(a+x)^3} + 2v_2(-\frac{e_1 h_1((s+e_1)v_1 - xv_2)}{(s+e_1)^3} - \delta v_3)$$

$$g_{31} = 2(-h_2 v_1 + \delta v_2)v_3 + \frac{2ab_2 c_2 v_1(-Iv_1 + (a+x)v_3)}{(a+x)^3}.$$

On other hand we have also

$$D^3F(X, \emptyset)(V, V, V) = [n_{i1}], \tag{15}$$

where:

$$n_{11} = 6(\frac{ac_1 v_1^2(-sv_1 + (a+x)v_2)}{(a+x)^4} + \frac{r\alpha^2 v_2^2(v_1 + s\alpha v_1 - x\alpha v_2)}{(1+s\alpha)^4} + \frac{ac_2 v_1^2(-Jv_1 + (a+x)v_3)}{(a+x)^4}).$$

$$n_{21} = \frac{6e_1 h_1 v_2^2((s+e_1)v_1 - xv_2)}{(s+e_1)^4} - \frac{6ab_1 c_1 v_1^2(-sv_1 + (a+x)v_2)}{(a+x)^4}.$$

$$n_{31} = -\frac{6ab_2 c_2 v_1^2(-v_1 I + (a+x)v_3)}{(a+x)^4}$$

The following theorems investigate the possibility of LB in the system (1).

**Theorem 6.** The system (1) at the $EEP$ undergoes a T.B when the parameter $r^*$ passes the value $r^* = r$.

**Proof.** It is clear to verify that as $r^* = r$, the JM in Eq. (6) at the EP. $q_0$ has zero eigenvalue with two negative eigenvalues, then

$$J_0 = J_{(q_0, r^*)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & -(d_2 + d_3) - \beta \end{bmatrix}$$

Let $T_0 = (\nu_{01}, \nu_{02}, \nu_{03})^T$ be the eigenvectors of $\lambda_{01} = 0$. Then simple computation gives that $T_0 = (v_{01}, 0,0)^T$, where $v_{01} \neq 0$,then $J_0 T_0 = 0$

Now, let $\pi_0 = (\pi_{01}, \pi_{02}, \pi_{03})^T$ represents the eigenvectors $J_0{}^T$with the eigenvalue $\hat{\lambda}_{01} = 0$, Then $J_0{}^T \pi_0 = \pi_0 = (\pi_{01}, 0, 0)^T$, with ($\pi_{01} \neq 0$ ).

Since $F_r = (\frac{x}{1+\alpha s}, 0, 0)^T$. Hence $F_r(q_0, r^*) = (0, 0, 0)^T$

Therefore, $\pi_0{}^T[DF_r(q_0, r^*)] = 0$.

Thus system (1) at $q_0$ with $r = r^*$ does not experience S.NB. Moreover, since $\pi_0{}^T[DF_r(q_0, r^*)T_0] = \nu_{01}\pi_{01} \neq 0$.

Also, $\pi_0{}^T\ [D^2F(q_0, r^*)(T_0, T_0)] = -2b\nu_{01}{}^2\pi_{01}{}^2 \neq 0$

Accordingly, by Sotomoyr's theorem, system (1) near the EP, $q_0$, with $r = r^*$ possesses a T.B.

T**heorem 7.** The system (1) at the AEP undergoes T.B when $h_1{}^* = h_1 = \frac{e_1 b_1 c_1}{(a+\hat{x})} - \frac{e_1 d_2}{\hat{x}}$ under the condition

$$\frac{ab_1c_1\sigma_1}{(a+\hat{x})^2} \neq \frac{h_1 e_1^2 \sigma_1 + \hat{x}}{e_1{}^3} \tag{16a}$$

otherwise it has a P.B provided that the following

$$\frac{(h_1\sigma_1 - \hat{x})}{e_1{}^2} \neq \frac{ab_1c_1\sigma_2^2}{(a+x)^3} \tag{16b}$$

**Proof.** It is clear to verify that as $h_1{}^* = h_1$, the JM in Eq. (7a) at the EP, $q_1$ has zero eigenvalue with two negative eigenvalues,

$$J_1 = J_{(q_1, h_1^*)} = \begin{pmatrix} -b\hat{x} & -r\alpha\hat{x} - \frac{c_1\hat{x}}{(a+\hat{x})} & -\frac{c_2\hat{x}}{(a+\hat{x})} \\ 0 & 0 & \beta \\ 0 & 0 & \frac{b_2c_2\hat{x}}{(a+\hat{x})} - (d_2 + d_3) - \beta - h_2\hat{x} \end{pmatrix}$$

Let $T_1 = (\nu_{11}, \nu_{12}, \nu_{13})^T$ be the eigenvectors of $\lambda_{12} = 0$. Then simple computation gives that $T_1 = (\sigma_1\, \nu_{12}, \nu_{12}, 0)^T$, where $\sigma_1 = \frac{-\hat{a}_{11}}{\hat{a}_{12}} < 0, (\nu_{12} \neq 0\,)$. then $J_1 \pi_1 = 0$

Now, let $\pi_1 = (\pi_{11}, \pi_{12}, \pi_{13})^T$ be eigenvectors with the eigenvalue $\widehat{\lambda_{12}} = 0$, Then $J_1{}^T \pi_1 = 0$ with $\pi_1 = (0, \pi_{12}, \sigma_2 \pi_{12})^{\mathrm{T}}$ with $\sigma_2 = \frac{-\bar{a}_{23}}{\bar{a}_{33}}$, with $(\pi_{12} \neq 0\,)$.

Since $F_{h_1} = \left(0, \frac{-xs}{e_1+s}, 0\right)^T$. Hence $F_{h_1}(q_1,\ h_1^*) = (0,0,0)^T$ which yields

$\pi_1{}^T\left[F_{h_1}(q_1, h_1^*)\right] = 0$, Thus system (1) at $q_1$ with $h_1 = h_1{}^*$ does not experience S.NB.

Moreover, since $\pi_1{}^T\left[DF_{h_1}(q_1, h_1^*)T_1\right] = -\frac{\hat{x}}{e_1}\nu_{12}\pi_{12} \neq 0$.

Also, $\pi_1{}^T[D^2F(q_1, h_1^*)(T_1, T_1)] = 2[\frac{ab_1c_1\sigma_1}{(a+\hat{x})^2} - \frac{h_1e_1^2\sigma_1+\hat{x}}{e_1{}^3}]\nu_{12}{}^2\pi_{12}{}^2 \neq 0$

Then, system (1) near the EP, $q_1$, with $h_1 = h_1{}^*$ possesses a T.B. However violating condition (16a)

$$\pi_1{}^T[D^3F(q_1, h_1^*)(T_1, T_1, T_1)] = 6[\frac{(h_1\sigma_1 - \hat{x})}{e_1{}^2} - \frac{ab_1c_1\sigma_2^2}{(a+x)^3}]v_2^3\pi_{12}{}^3$$

Hence, system (1) undergoes P.B under the condition (16b).

**Theorem 8.** The system (1) undergoes a T.B near $IPEEP$ when $d_3 = d_3^* = \delta\bar{s} + \frac{b_2c_2\bar{x}}{(a+\bar{x})} - d_2 - \beta - h_2\bar{x}$ under the condition

$$\frac{ab_2c_2\sigma_3}{(a+\bar{x})^2} + \delta\sigma_4 \neq h_2\sigma_3 \tag{17a}$$

Otherwise, it has a P.B .

**Proof.** It is clear to verify that as $d_3{}^* = d_3$, the JM in Eq. (8a) at the EP, $q_2$ has zero eigenvalue with two negative eigenvalues,

$$J_2 = J_{(q_2, d_3^*)} = \begin{pmatrix} -b\bar{x} + \frac{c_1\,\bar{x}\bar{s}}{(a+\bar{x})^2} & \frac{r\alpha\bar{x}}{(1+\alpha\hat{s})^2} - \frac{c_1\bar{x}}{(a+\bar{x})} & -\frac{c_2\bar{x}}{(a+\bar{x})} \\ \frac{ab_1c_1\bar{s}}{(a+\bar{x})^2} - \frac{h_1\bar{s}}{(e_1+\bar{s})} & \frac{c_1b_1\bar{x}}{(a+\bar{x})} - d_2 - \frac{e_1h_1\bar{x}}{(e_1+\bar{s})^2} & -\delta\bar{s} + \beta \\ 0 & 0 & 0 \end{pmatrix}$$

Let $T_2 = (\nu_{21}, \nu_{22}, \nu_{23})^T$ be the eigenvectors of $\lambda_{23}^* = 0$. Then simple computation gives that $T_2 = (\sigma_3\, \nu_{23}, \sigma_4\, \nu_{23}, \nu_{23})^T$, where $\sigma_3 = \frac{a_{13}a_{22} - a_{12}a_{23}}{a_{11}a_{22} - a_{12}a_{21}}$ and $\sigma_4 = \frac{a_{12}a_{23} - a_{22}a_{13}}{a_{11}a_{22} - a_{12}a_{21}}, (\nu_{23} \neq 0\,)$. Then $J_2 \pi_2 = 0$

Now, let $\pi_2 = (\pi_{21}, \pi_{22}, \pi_{23})^T$ be eigenvectors with the eigenvalue $\widehat{\lambda_{23}} = 0$, Then $J_2{}^T \pi_2 = 0$ with $\pi_2 = (0,0,\pi_{23})^T$ with ($\pi_{23} \neq 0$ ).

since $F_{d_3} = (0,0,-I)^T$ Hence $F_{d_3}(q_2, d_3^*) = (0,0,0)^T$ which yields

$\pi_2{}^T F_{d_3}(q_2, d_3^*) = 0$. Thus system (1) at $q_2$ with $d_3 = d_3{}^*$ does not experience S.NB.

Moreover, since $\pi_2{}^T \left[DF_{d_3}(q_2, d_3^*)T_2\right] = -\nu_{23}\pi_{23} \neq 0$.

Also, $\pi_2{}^T [D^2 F(q_2, d_3^*)(T_2, T_2)] = 2\left(\frac{ab_2c_2\sigma_3}{(a+\bar{x})^2} - h_2\sigma_3 + \delta\sigma_4\right)\pi_{23}\nu_{23}^2$

Then, system (1) near the EP., $q_2$, with $d_3 = d_3{}^*$ possesses a T.B if the condition (17a) hold. However violating condition (17a) then

$$\pi_2{}^T [D^3 F(q_2, d_3^*)(T_2, T_2, T_2)] = -6\frac{ab_2c_2\sigma_3}{(a+\bar{x})^3}\nu_2^3 \pi_{23}{}^3 \neq 0 \qquad (17b)$$

Hence, system (1) undergoes P.B under the condition (17b).

**Theorem 9.** The system (1) undergoes a S.NB near $PEP$, $q_3 = (s^*, I^*, y^*)$ when the parameter $h_2$ crosses the value $h_2 = h_2^* = \frac{w_{32}(w_{11}w_{23} - w_{13}w_{21})}{(w_{13}w_{22} - w_{12}w_{23})I^*} + \frac{ab_2c_2}{(a+x^*)^2}$ under the condition

$$g_{11} + g_{21} + g_{31} \neq 0 \qquad (18)$$

**Proof.** The JM of the system (1) at $(q_3, h_2^*)$ can be represented by

$$J_3 = J_{(q_3, h_2^*)} = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & 0 \end{pmatrix}$$

It is straightforward to check that the coefficient $\rho_3 = 0$ at $h_2 = h_2^*$ in equation (9b). Hence the characteristic equation has zero root.

Let $T_3 = (\nu_{31}, \nu_{32}, \nu_{33})^T$ be the eigenvectors corresponding to $\lambda_{31}^* = 0$. Thus $J_3 T_3 = 0$ gives that $T_3 = (\sigma_5\nu_{33}, \sigma_6\nu_{33}, \nu_{33})^T$ , with $\sigma_5 = \frac{w_{12}w_{23} - w_{22}w_{13}}{w_{11}w_{22} - w_{12}w_{21}}$ and $\sigma_6 = \frac{w_{21}w_{13} - w_{23}w_{11}}{w_{11}w_{22} - w_{12}w_{21}}$ and, ($\nu_{33} \neq 0$).

Now, let $\pi_3 = (\pi_{31}, \pi_{32}, \pi_{33})^T$ be eigenvectors $J^T$ with the eigenvalue $\lambda_{31}^* = 0$ then $J_3{}^T \; \pi_3 = 0$ with $\pi_3 = (\sigma_7\pi_{33}, \sigma_8\pi_{33}, \pi_{33})^T$ with ( $\pi_{33} \neq 0$)and $\sigma_7 = \frac{w_{22}w_{31} - w_{21}w_{32}}{w_{11}w_{22} - w_{12}w_{21}}$, $\sigma_8 = \frac{w_{12}w_{31} - w_{32}w_{11}}{w_{11}w_{22} - w_{12}w_{21}}$

since $F_{h_2} = (0,0,-xI)^T$ Hence $F_{h_2}(q_3, h_2^*) = (0,0\,, -x^*I^*)^T$ which yields

$\pi_3{}^T F_{h_2}(q_3, h_2^*) = -x^*I^*\pi_{33} \neq 0$.

Also, $\pi_3{}^T [D^2 F(q_3, h_2^*)(T_3, T_3)] = (g_{11} + g_{21} + g_{31})v_3^2 \pi_{33} \neq 0$

Hence, S.NB takes place near $q_3$.

## 5. Numerical Simulation

In this section, an investigation of the global dynamics of system (1) is carried out using numerical simulation. The objectives were to confirm our obtained findings and specify the role of each parameter in the dynamic system's behavior. All numerical results are given in the form of phase portraits and time series using MATLAB version R2021a. Phase portraits of the resulting trajectories and their direction fields are shown in Figure 1 using the following set of parameters, with varying initial values used.

$$r = 2, \alpha = 0.5, d_1 = 0.1, b = 0.25, c_1 = 0.75, c_2 = 0.6, a = 2, b_1 = 0.5, d_2 = 0.1, \delta = 0.3, h_1 = 0.01, e_1 = 2, \beta = 0.15, b_2 = 0.3, d_3 = 0.1, h_2 = 0.05. \quad (19)$$

It is observed, for this set of data, that the system (1) approaches asymptotically to the unique $PEP$, starting from three different initial values, as shown in the following figures (1).

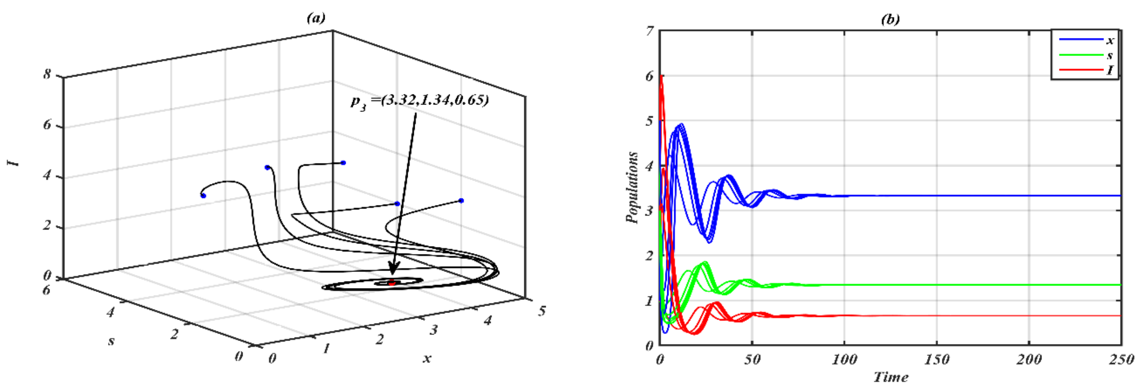


**Figure 1.** The trajectory of system (1) using data set (19) approaches asymptotically $q_3 = (3.33,1.34,0.64)$ starting from different initial points (IP) (a) G.AS of the $PEP$. (b) Trajectories of populations versus time.

Now, in order to discuss the effect of varying the parameters' values of system (1) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (19) and then the obtained solutions are drawn as shown below. It is observed that, for the values of parameter $r \in [0.3,0.93]$ the solution of system (1) to $IPEEP$, and the solution goes to $AEP$ when $r \in [0.1,0.29]$, while when $r \leq 0.09$ the solution go to $EEP$, as illustrated Fig. (2). It is noted that system (1) approaches to $PEP$ of the system (1) for $r \geq 0.94$, as illustrated in Fig. (1).

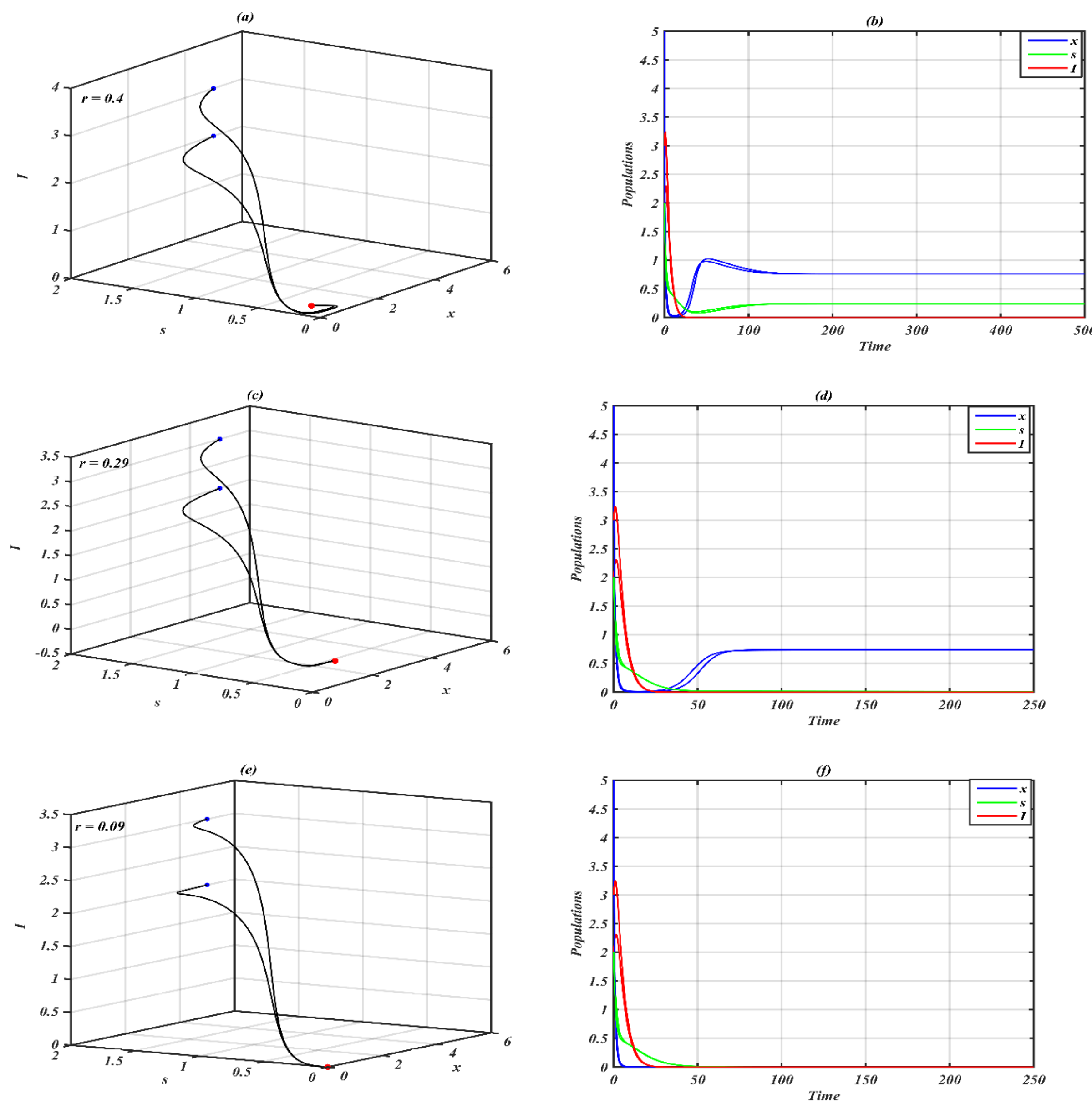


**Figure 2.** The trajectories of system (1) versus time for the data given by Eq.(19) with different values $r$. (a) Approaches to $q_2 = (0.76,0.24,0)$ for $r = 0.4$. (b) Time series for $r = 0.4$. (c) Approaches to $q_1 = (0.76,0,0)$ for $r = 0.29$. (d) Time series for $r = 0.29$. (e) Approaches to $q_0 = (0,0,0)$ for $r = 0.09$. (f) Time series for $r = 0.09$.

The influence of parameter $d_1$ on the dynamic of system (1) is studied numerically and it is observed that for $d_1 \geq 0.79$ the system approaches to to $IPEEP$, as illustrated in Fig.(3). Otherwise, it is noted that system (1) still to $PEP$.

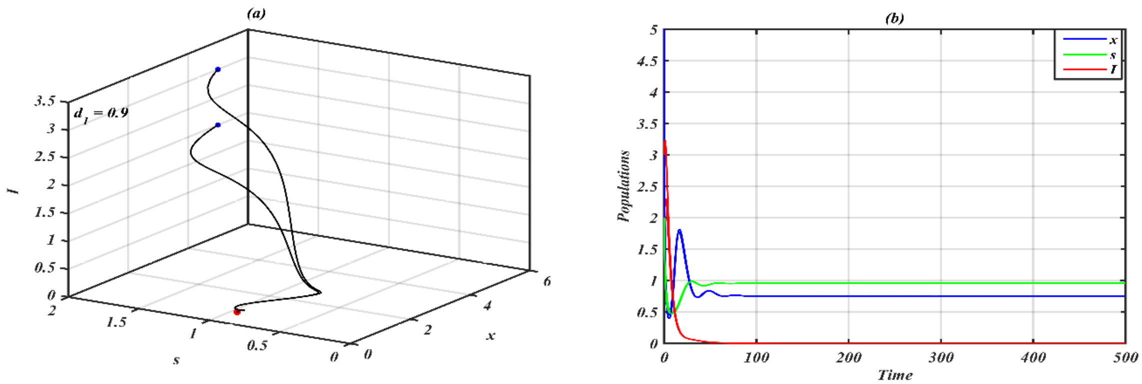


**Figure 3.** The trajectories of system (1) versus time for the data given by Eq.(19) with different values $d_1$. (a) Approaches to $q_2 = (0.75, 0.96, 0)$ for $d_1 = 0.9$. (b) Time series for $d_1 = 0.9$.

Further investigation of the remaining parameters, show that the parameters $\beta$ and $d_3$ have a similar influence as that shown for $d_1$on the dynamic of the system (1).

Now, adjusting the value of $b$ affects the dynamics of the system (1). when $b \leq 0.12$, the trajectories of system (1) approach to a stable limit cycle, as illustrated in Figure (4). Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig. (1).

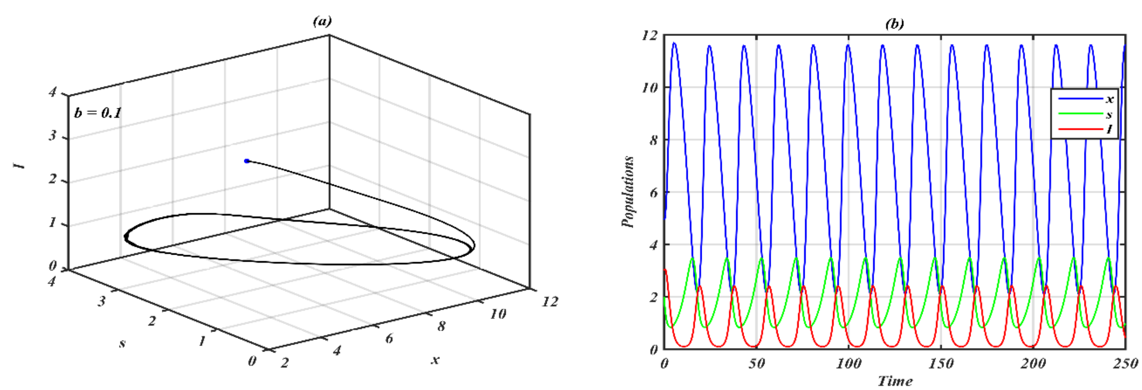


**Figure 4.** The trajectories of system (1) versus time for the data given by Eq.(19) with different values $b$. (a) Periodic dynamics in $\mathcal{R}_+^3$ for $b = 0.1$. (b) Time series for $b = 0.1$.

For the parameter $c_1$ in the range of $c_1 \leq 0.33$, the system (1) approaches $AEP$, as illustrated in Fig. (5). Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig. (1).

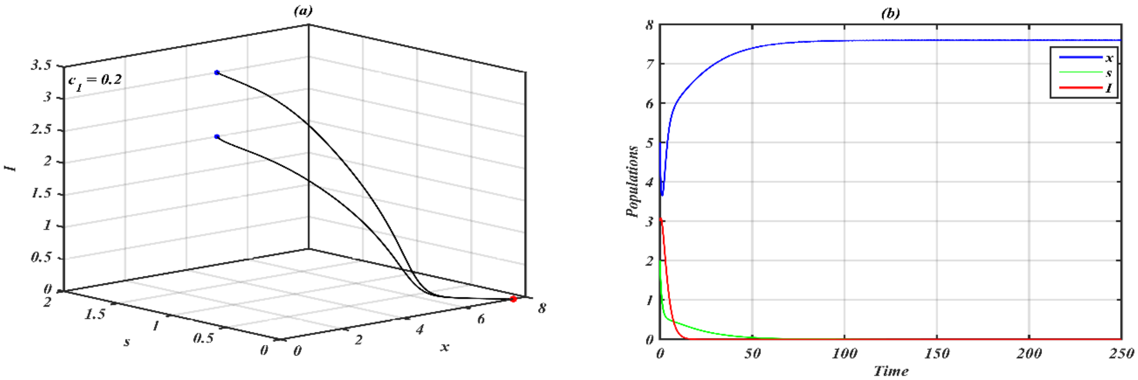


**Figure 5.** The trajectories of system (1) versus time for the data given by Eq.(19) with different values $c_1$. (a) Approach to $q_1 = (7.6,0,0)$ for $c_1 = 0.2$. (b) Time series for $c_1 = 0.2$.

the effect of changing the parameter $h_2$ on the dynamic of system (1) shows that the range $0.82 < a < 0.55$, system (1) undergoes a bi-stable between $PEP$and $3D$ periodic dynamics. While the system (1) has $2D$ periodic dynamics in $xs-$ plane when $a \leq 0.54$, as illustrated in Fig.(6). Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig. (1).

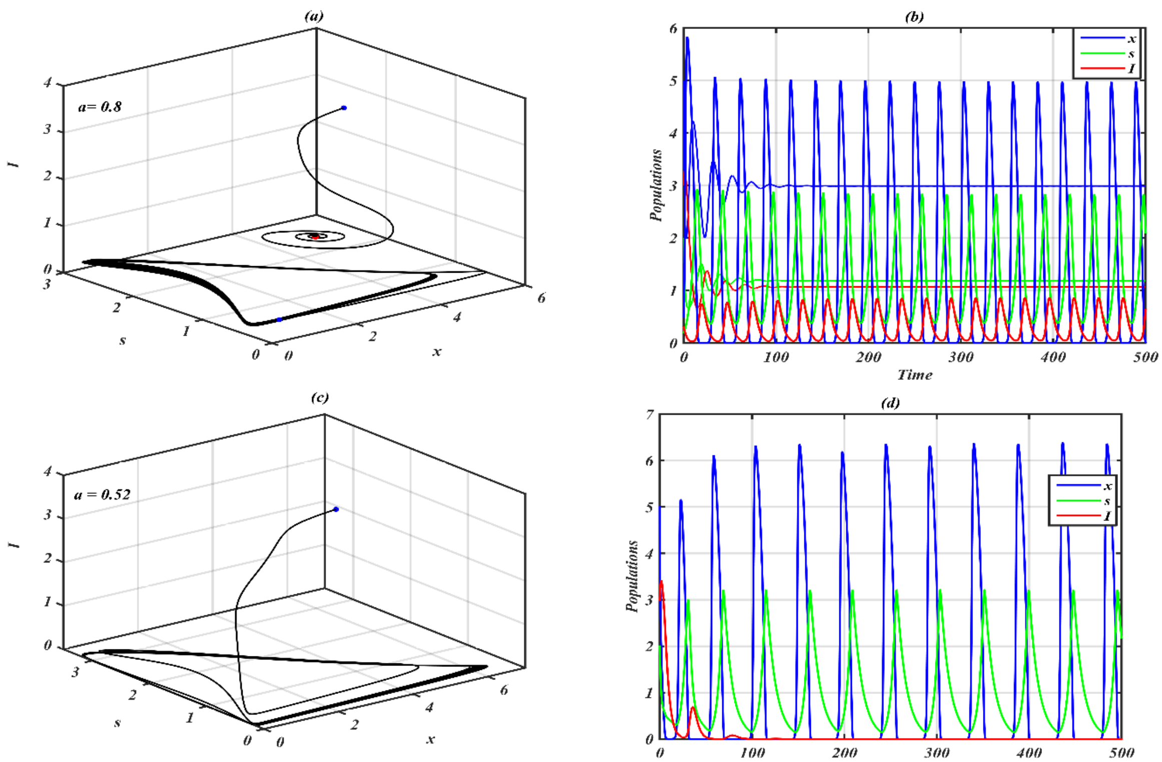


**Figure 6.** The bi-stable behaviour of the system (1) with data given by Eq.(19) with different values $a$. (a) $3D$ phase portrait approaches $q_3 = (0.03,2.41,0.27)$ and $3D$ periodic attractor for $a = 0.8$. (b) Time series for for $a = 0.8$. (c) (a) Periodic dynamics in $xs-$ plane for $a = 0.52$ (b) Time series for $a = 0.52$.

The study of the effect of changing the parameter $b_1$on the dynamic of system (1) shows that when $b_1 \leq 0.22$, the system approaches $AEP$, the system approaches $IPEEP$ when $b_1 \in (0.22,0.24)$, as illustrated in Fig.(7). Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig.(1).

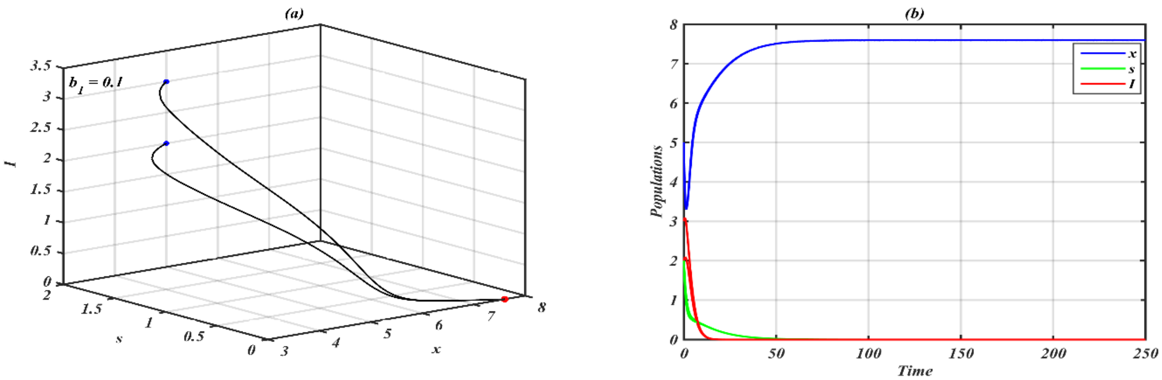


**Figure 7.** The trajectories of system (1) versus time for the data given by Eq.(19) with different values $b_1$. (a) Approaches to $q_2 = (7.60,0,0)$ for $b_1 = 0.1$. (b) Time series for for $b_1 = 0.1$. (c) Approaches to $q_2 = (3.59,1.38,0)$ for $b_1 = 0.23$.. (d) Time series for $b_1 = 0.23$.

The effect of changing the parameter $d_2$ on the dynamics of system (1) indicates that when $d_2 \in [0.22,0.35]$, the system approaches $IPEEP$, and the system approaches $AEP$ when $d_2 \geq 0.26$ , as illustrated in Fig.(8), Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig.(1).

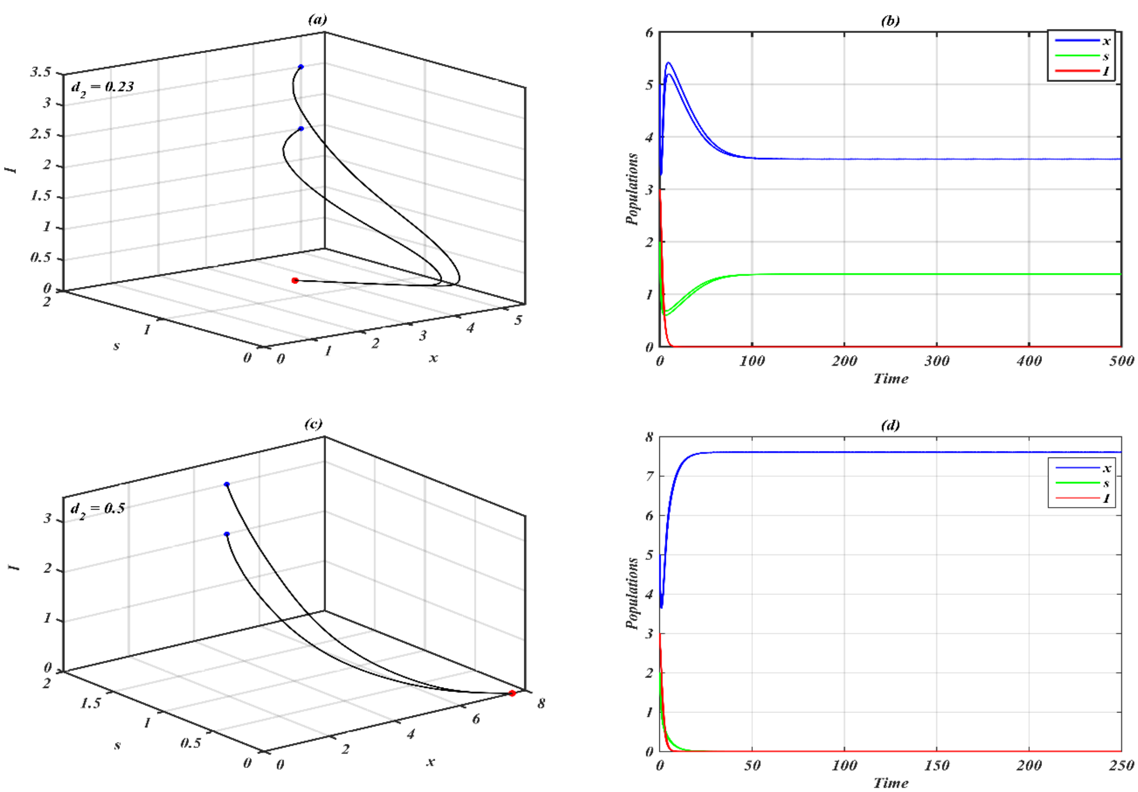


**Figure 8.** The trajectories of system (1) versus time for the data given by Eq.(19) with different values $d_2$. (a) Approaches to $q_2 = (3.75,1.38,0)$ for $d_2 = 0.23$. (b) Time series for for $d_2 = 0.23$ .(c) Approaches to $q_1 = (7.60,0,0)$ for $d_2 = 0.5$ (d) Time series for $d_2 = 0.5$.

Moreover, it is observed that for the parameter $\delta \leq 0.14$,the system (1) approaches $IPEEP$ as illustrated in Fig.(9), Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig.(1).

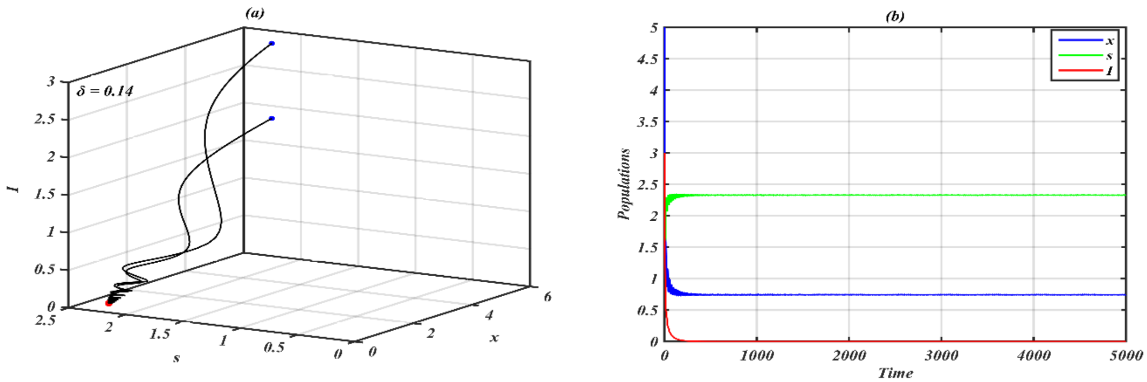


**Figure 9.** The trajectories of system (1) versus time for the data given by Eq.(19) with different values $\delta$. (a) Approach to $q_2 = (0.73,2.32,0)$ for $\delta = 0.14$. (b) Time series for $\delta = 0.14$.

Now, the effect of changing the parameter $h_2$ on the dynamic of system (1) shows that when $h_2 \geq 0.09$, the system approaches to $AEP$. While for the range $0.07 < h_2 < 0.09$ system (1) undergoes a bi-stable between $AEP$and $3D$ periodic dynamics, as illustrated in Fig.(10). Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig.(1).

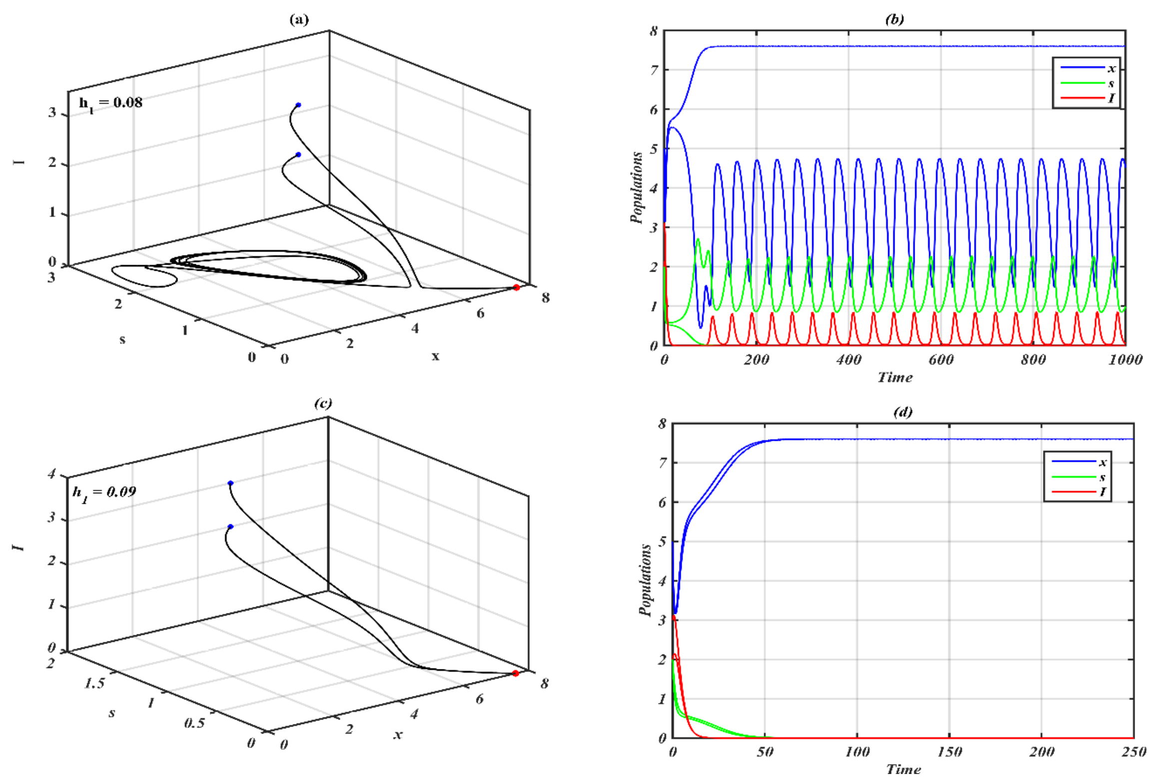


**Figure 10.** The bi-stable behaviour of the system (1) with data given by Eq.(19) with different values $h_1$.(a) $3D$ phase portrait approaches $q_1 = (7.60,0,0)$ and $3D$ periodic attractor for $h_1 = 0.08$. (b) Time series for for $h_1 = 0.08$. (c) Approaches to $q_1 = (7.60,0,0)$ for $h_1 = 0.09$. (d) Time series for $h_1 = 0.09$.

Now, the impact of varying the parameter $e_1$ on the system's dynamics shows the system (1) undergoes a bi-stable between $AEP$and $PEP$, when $e_1 \leq 0.08$, as illustrated in Fig.(11). Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig.(1).

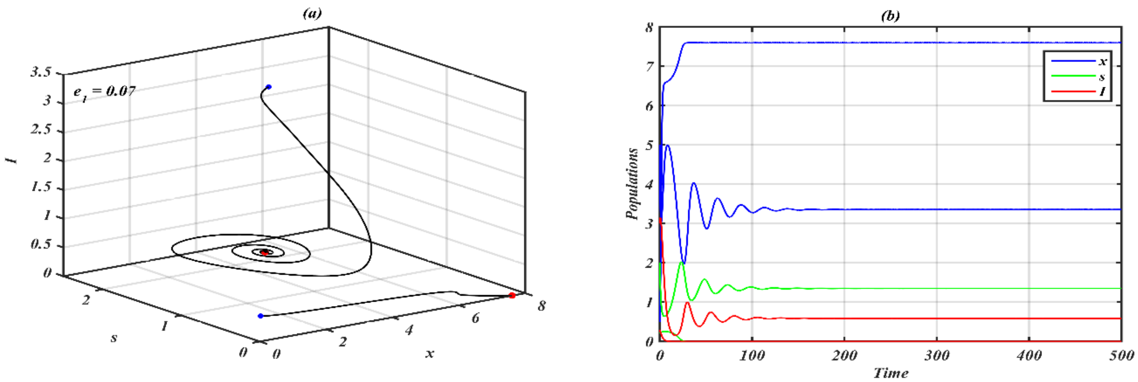


**Figure 11.** The bi-stable behaviour of the system (1) with data given by Eq.(19) with different values $e_1$. (a) $3D$ phase portrait approaches $q_1 = (7.60,0,0)$ and $q_3 = (3.35,1.35,0.58)$ for $e_1 = 0.08$. (b) Time series for $e_1 = 0.08$.

The effect of varying the parameters $\alpha, c_2,$ and $b_2$ has a quantitative impact on the position of $PEP$. Finally, The effect of varying the parameter $h_2$on the system's dynamic shows when $0.17 \leq h_2 \leq 0.53$, the system has $3D$ perodic dynamics, and when $h_2 \geq 0.54$, the system(1) approaches $IPEEP$. as illustrated in Fig.(12). Otherwise, the $PEP$ of the system (1) is a G.AS, as illustrated in Fig.(1).

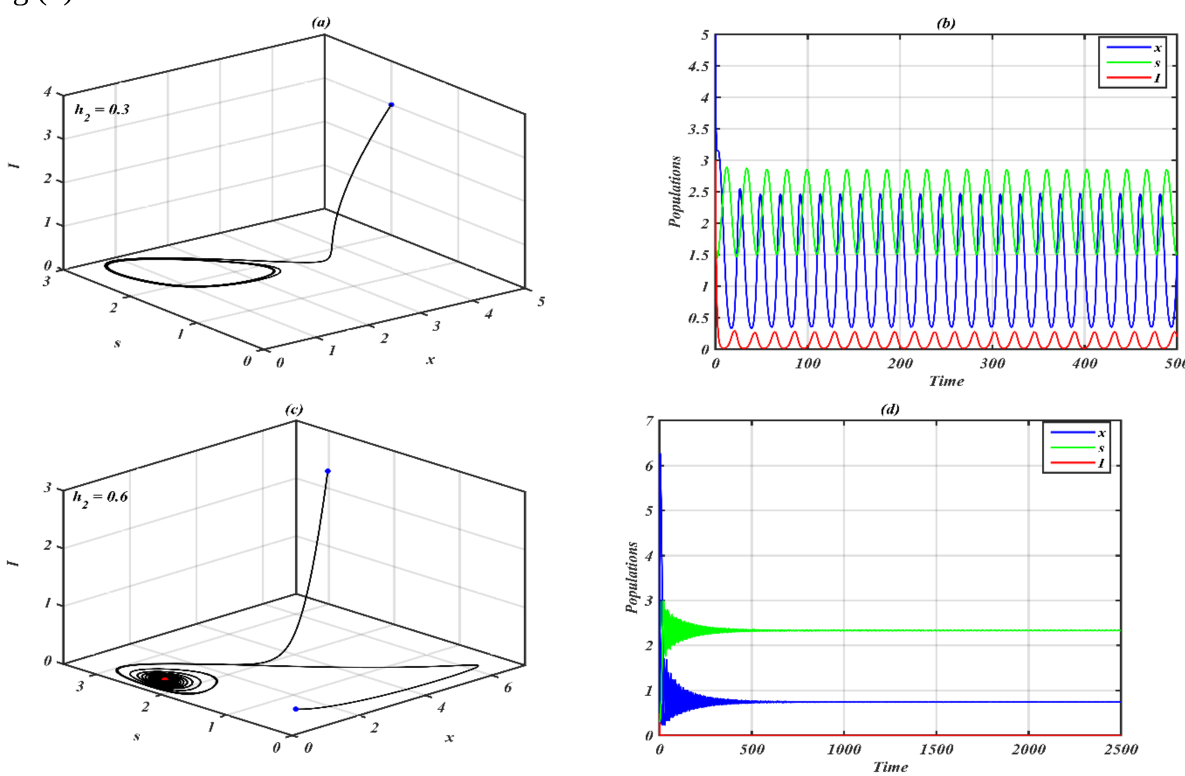


**Figure 12.** The trajectories of system (1) versus time for the data given by Eq.(19) with different values $h_2$. (a) Periodic dynamics in $\mathcal{R}_+^3$ for $h_2 = 0.6$. (b) Time series for for $h_2 = 0.6$ . (c) Approaches to $q_2 = (0.74,2.32,0)$ for $h_2 = 0.3$. (d) Time series for $h_2 = 0.3$.

## 6. CONCLUSION

In this paper, a mathematical model considering anti-predator on the prey-predator system in case of the existence of disease in the predator, with treatment in infected predator and fear of predation plays an important role in the growth of a prey species in a prey-predator system is proposed and studied. The properties of the solution of the mathematical model, including existence, boundedness, and uniqueness, are discussed. All the possible EPs are found. The local, as well as global, stability of these equilibrium points is investigated. The local bifurcation conditions around each EP are also established. It is observed that the proposed model has a rich dynamic, and many of its parameters represent bifurcation parameters for the system. Finally, the numerical simulations of the system, with the help of the hypothetical set of data given by Eq. (19) have been shown in different types of dynamics. It is observed numerically that the system includes stable points, bi-stable, and periodic behavior.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

## REFERENCES

[1] J.D. Murray, Mathematical Biology I. An Introduction, Springer, 2002. https://doi.org/10.1007/b98868.

[2] W.B. Cannon, Bodily Changes in Pain, Hunger, Fear and Rage: An Account of Recent Researches into the Function of Emotional Excitement, D Appleton & Company, New York, 1915. https://doi.org/10.1037/10013-000.

[3] L.Y. Zanette, A.F. White, M.C. Allen, M. Clinchy, Perceived Predation Risk Reduces the Number of Offspring Songbirds Produce Per Year, Science 334 (2011), 1398-1401. https://doi.org/10.1126/science.1210908.

[4] X. Wang, L. Zanette, X. Zou, Modelling the Fear Effect in Predator–prey Interactions, J. Math. Biol. 73 (2016), 1179-1204. https://doi.org/10.1007/s00285-016-0989-1.

[5] P. Panday, N. Pal, S. Samanta, J. Chattopadhyay, Stability and Bifurcation Analysis of a Three-Species Food Chain Model with Fear, Int. J. Bifurc. Chaos 28 (2018), 1850009. https://doi.org/10.1142/s0218127418500098.

[6] S. Pal, S. Majhi, S. Mandal, N. Pal, Role of Fear in a Predator–prey Model with Beddington–deangelis Functional Response, Z. Für Naturforschung 74 (2019), 581-595. https://doi.org/10.1515/zna-2018-0449.

[7] A.R.M. Jamil, R.K. Naji, Modeling and Analysis of the Influence of Fear on the Harvested Modified Leslie–gower Model Involving Nonlinear Prey Refuge, Mathematics 10 (2022), 2857. https://doi.org/10.3390/math10162857.

[8] F.H. Maghool, R.K. Naji, Chaos in the Three-Species Sokol-Howell Food Chain System with Fear, Commun. Math. Biol. Neurosci. 2022 (2022), 14. https://doi.org/10.28919/cmbn/7056.

[9] H.A. Ibrahim, R.K. Naji, The Impact of Fear on a Harvested Prey–predator System with Disease in a Prey, Mathematics 11 (2023), 2909. https://doi.org/10.3390/math11132909.

[10] A.M. Sahi, H.A. Satar, The Role of the Fear, Hunting Cooperation, and Anti-predator Behavior in the Prey-predator Model Having Disease in Predator, Commun. Math. Biol. Neurosci. 2024 (2024), 75. https://doi.org/10.28919/cmbn/8663.

[11] R.M. Anderson, R.M. May, Population Biology of Infectious Diseases: Part I, Nature 280 (1979), 361-367. https://doi.org/10.1038/280361a0.

[12] M. Saifuddin, S. Samanta, S. Biswas, J. Chattopadhyay, An Eco-Epidemiological Model with Different Competition Coefficients and Strong-Allee in the Prey, Int. J. Bifurc. Chaos 27 (2017), 1730027. https://doi.org/10.1142/s0218127417300270.

[13] L. Wang, Z. Qiu, T. Feng, Y. Kang, An Eco-Epidemiological Model with Social Predation Subject to a Component Allee Effect, Appl. Math. Model. 101 (2022), 111-131. https://doi.org/10.1016/j.apm.2021.07.037.

[14] H.A. Satar, R.K. Naji, A Mathematical Study for the Transmission of Coronavirus Disease, Mathematics 11 (2023), 2330. https://doi.org/10.3390/math11102330.

[15] K.P. Das, A Mathematical Study of a Predator-Prey Dynamics with Disease in Predator, ISRN Appl. Math. 2011 (2011), 807486. https://doi.org/10.5402/2011/807486.

[16] M.V.R. Murthy, D.K. Bahlool, Modeling and Analysis of a Prey-Predator System with Disease in Predator, IOSR J. Math. 12 (2016), 21-40.

[17] R.K. Naji, A.A. Thirthar, Stability and Bifurcation of an SIS Epidemic Model with Saturated Incidence Rate and Treatment Function, Iran. J. Math. Sci. Inform. 15 (2020), 129-146.

[18] D. Barman, J. Roy, S. Alam, Dynamical Behaviour of an Infected Predator-Prey Model with Fear Effect, Iran. J. Sci. Technol. Trans.: Science 45 (2020), 309-325. https://doi.org/10.1007/s40995-020-01014-y.

[19] S. Kant, V. Kumar, Dynamics of a Prey-Predator System with Infection in Prey, Electron. J. Differ. Equ. 2017 (2017), 209.

[20] H.A. Ibrahim, R.K. Naji, A Prey-Predator Model with Michael Mentence Type of Predator Harvesting and Infectious Disease in Prey, Iraqi J. Sci. 61 (2020), 1146-1163. https://doi.org/10.24996/ijs.2020.61.5.23.

[21] Z.K. Mahmood, H.A. Satar, The Influence of Fear on the Dynamic of an Eco-epidemiological System with Predator Subject to the Weak Allee Effect and Harvesting, Commun. Math. Biol. Neurosci., 2022 (2022), 90. https://doi.org/10.28919/cmbn/7638.

[22] A.M. Sahi, H.A. Satar, Stability and Bifurcation of Prey-predator Model with Disease in Prey Incorporating Anti-predator and Hunting Cooperation, Commun. Math. Biol. Neurosci. 2025 (2025), 35. https://doi.org/10.28919/cmbn/9162.

[23] W. Hussein, H. Abdul Satar, The Dynamics of a Prey-Predator Model with Infectious Disease in Prey: Role of Media Coverage, Iraqi J. Sci. 62 (2021), 4930-4952. https://doi.org/10.24996/ijs.2021.62.12.31.

[24] Walaa Madhat Alwan, Huda Abdul Satar, The Effects of Media Coverage on the Dynamics of Disease in Prey-Predator Model, Iraqi J. Sci. 62 (2021), 981-996. https://doi.org/10.24996/ijs.2021.62.3.28.

[25] K.P. Das, S.K. Sasmal, J. Chattopadhyay, Disease Control Through Harvesting Conclusion Drawn from a Mathematical Study of a Predator-prey Model with Disease in Both the Population, Int. J. Biomath. Syst. Biol. 1 (2014), 1-29.

[26] S. Kant, V. Kumar, Stability Analysis of Predator–prey System with Migrating Prey and Disease Infection in Both Species, Appl. Math. Model. 42 (2017), 509-539. https://doi.org/10.1016/j.apm.2016.10.003.

[27] A.A. Thirthar, R.K. Naji, F. Bozkurt, A. Yousef, Modeling and Analysis of an $SI_1I_2R$ Epidemic Model with Nonlinear Incidence and General Recovery Functions of $I_1$, Chaos Solitons Fractals 145 (2021), 110746. https://doi.org/10.1016/j.chaos.2021.110746.

[28] C. Li, Y. Zhang, J. Xu, Y. Zhou, Global Dynamics of a Prey-Predator Model with Antipredator Behavior and Two Predators, Discret. Dyn. Nat. Soc. 2019 (2019), 3586508. https://doi.org/10.1155/2019/3586508.

[29] P. Panday, N. Pal, S. Samanta, P. Tryjanowski, J. Chattopadhyay, Dynamics of a Stage-Structured Predator-Prey Model: Cost and Benefit of Fear-Induced Group Defense, J. Theor. Biol. 528 (2021), 110846. https://doi.org/10.1016/j.jtbi.2021.110846.

[30] M. He, Z. Li, Stability of a Fear Effect Predator–prey Model with Mutual Interference or Group Defense, J. Biol. Dyn. 16 (2022), 480-498. https://doi.org/10.1080/17513758.2022.2091800.

[31] M.S. Jabbar, R.K. Naji, The Consequences of Pollution on the Producer-Consumer-Predator Food Chain Dynamics When the Predator Has Access to Extra Food, Commun. Math. Biol. Neurosci. 2024 (2024), 107. https://doi.org/10.28919/cmbn/8862.

[32] W.M. Alwan, H.A. Satar, The Influence of Hunting Cooperation, and Anti-Predator Behavior on an Eco-Epidemiological Model with Harvest, Communications in Mathematical Biology and Neuroscience, 2024 (2024), 90. https://doi.org/10.28919/cmbn/8775.

[33] L. Perko, Differential Equations and Dynamical Systems, Springer, New York, 2001. https://doi.org/10.1007/978-1-4613-0003-8.