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THE INFLUENCE OF NON-LINEAR HARVESTING ON A WEB OF TWO PREY WITH ONE PREDATOR, COMBINED WITH FEAR AND REFUGE STRATEGIES CONCERNING PREY POPULATIONS

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Abstract: Under the influence of various functional responses: Beddington–DeAngelis and Holling type IV via predator interaction with two different kinds of prey populations in a web containing all these creatures, tends for discussing two distinct types of investigations, analytical and numerical, the proposed model joining two defense strategies concerning the prey populations, fear effects and refuge respectively, with presentence of the force of nonlinear harvesting act as a pressure on both prey populations, the system's positivity and boundedness is the analytical already start with, seven acceptable equilibrium points find the existence, then the Jacobian and Lyapunov methods concerning these equilibria utilize to understand local and global stability conditions. Confirming analytical investigations via numerical analysis simulations, where the analytical results assert that fear and refuge stabilize population dynamics and reinforce coexistence. Still, while acting, the nonlinear harvesting can play a crucial role in determining the system's coexistence, extinction, or biodiversity loss. Sustaining predator–prey ecosystems confirms the Importance of Indirect strategies, as fear and refuge, and harvesting.

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1. INTRODUCTION

The necessary tools for explore the dynamics of ecological systems in the real lives in natural are the proposed mathematical models that simulate natural life focusing on understanding deep insight into the mechanisms controlling population persistence, extinction, and coexistence via the analysis of the presence of both natural and anthropogenic pressures. Predator–prey models, in private, have been modified via combining particular realistic logical features, like the effect of toxicant release as a defense attitude, fear effect prey from predation, prey refuge as a defense strategy, spread of diseases in the prey individual's population, nonlinear harvesting as a human behavior, and distinct complex functional responses, all of which strongly impact on the populations in ecological model long-term system results [1, 4]

The fear effect on prey of predation has arisen as a main key non-consumptive factor shaping prey attitude in the natural environment. It limits prey foraging competence, changes reproductive strategies, and yet alters the energy flow over the food web. Recently, various studies assert that fear can illustrate rich dynamical attitudes, like oscillations, bistability, and chaotic results, relying on its power and interaction with other ecological processes [5–7]. Strategies of prey refuge or other behavioral mechanisms that protect prey from predators have been present to stabilize the proposed systems, delay extinction events, and interact with fear in nontrivial ways [8, 9].

Functional responses, the predator's way of consuming prey at varying densities, remain the cornerstone of ecological modeling. The most widely applied of the classical Holling type II and III forms in various studies, while recent inquiries focus on the significance of more complex forms that simulate the realistic, like the Beddington–DeAngelis and Holling type IV functional responses. These capture illustrations of predator interference, prey satiation, and handling time more strictly, and have been shown to create new kinds of bifurcations and coexistence conditions [10, 11].

Adding human exploitation through nonlinear harvesting gets another extension to predator–prey dynamics. But linear harvesting catches basic extraction, nonlinear harvesting considers actual constraints such as saturation in fishing concerning the aquatic environment or hunting efforts. Leading to bifurcations, between coexistence and extinction, and even chaotic behavior according to recent studies [12–15], nonlinear harvesting can perform system dynamics. Merging nonlinear harvesting into ecological proposed models is therefore a necessary benefit for predicting long-term ecosystem sustainability.

Motivated by these developments, this study illustrates, develops, and analyzes a predator–two prey system, including the effect of fear on the first prey, the importance of the refuge strategy used by the second prey, and how the predator interacts with these two distinct types of prey populations through various functional responses—Beddington–DeAngelis and Holling type IV for the first and second prey respectevly. . Merging all these ecological features in the same model supply new premeditation into how different prey defense strategies, predator behavior, and harvesting together shape biodiversity, population stability, and persistence.

2. EXPLANATIONS OF THE PROPOSED ECOLOGICAL MODEL FORMULATION

This ecological model contains two prey and one predator living in one natural area, under the effect of fear in the first prey and refuge in the second prey, both of the prey populations under pressure of nonlinear harvesting, in the presence of competition among predators, predator consumption the first prey using Beddington-DeAngles functional response while using hollying type IV functional response with the second prey formulated mathematically based on densities of the three species with time $x(t), y(t)\ and\ z(t)$ respectively as the assumptions following:

1. No fear with respect of the first prey and no needs of refuge concerning the second prey when the absence of predators are grow logistic; otherwise, the fear that the predators generate in the first prey population have many effects, for the growth of the first prey with intrinsic growth rate (r) goes to affect fear function $\frac{1}{1+fz}$, where f fear parameter of the first prey, while the second prey grow logistic with intrinsic growth rate (s) and carrying capacity(k).

2. Two interactions between predators with the first prey are of Beddington-DeAngelis functional response $\frac{\alpha_1 xz}{\delta x+\beta y+g}$, where, α_1: raid rate of the predator on prey x. Biologically, predators face and capture prey, measured by (α_1) influentially, δ : the parameter of competition associated with prey x, (prey handling measure time). Biologically, whenever the values of (δ) increase, it implies that handling time or interference increases with first prey(x), conversely, leading to a lowering of consumption efficiency. β : due to second prey(y), it is the interference parameter Biologically, the presence of the other prey(y) refers to (prey switching, distraction, or competition), which implies a reduction in effective predation on prey(x). g: means (baseline interference of saturation constant).Biologically, act as setting effects such as predator cross-interference or minimal handling time with prey, and Hollying type IV functional responses for the second prey $\frac{\alpha_2(1-m)yz}{1+\gamma(1-m)y^2}$, where , α_2 : Rate of raid (extreme effort predation rate per predator). Biologically, it represents the amount of how effectively predators face and capture prey.m: the strategy of prey refuge (or defense strategy such that $(0 \leq m < 1)$. Biologically: the part of prey unobserved in refuge or away from predation. However, the remaining amount of the prey population $(1-m)$ proportion of prey is obtainable to predators .γ : involvement / restrained parameter. Biologically, due to predator distraction and prey interference, this parameter regulates the decline in predation at high prey densities. Hump _ shape of a Holling Type IV functional response comes from using the term (y^2).

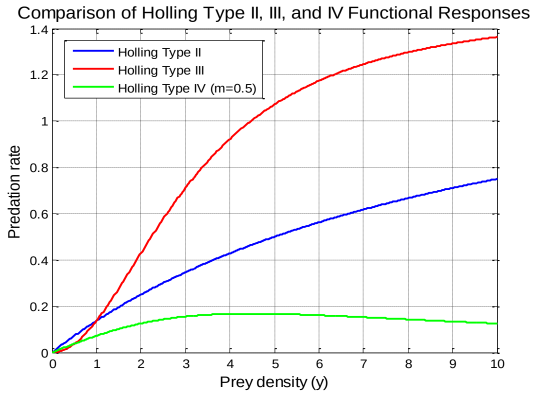

Figure 1: Various functional response functions illustrated as (blue) Holling type II, (red) type III, and (green, m=0.5) type IV.

With prey density, the blue line in type II is saturated, due to prey switching. The red line in type III is sigmoidal, and the green line in type IV is hump-shaped; due to refuge and predator interference, it declines at high prey densities.

3. Both prey populations are under pressure from nonlinear harvesting $\frac{q_1\, E_1 x}{n_1 E_1+n_2 x}$ *and* $\frac{q_2\, E_2 y}{n_3 E_2+n_4 y}$. q_1 And q_2 represent the catchability degree for the first and second prey, respectively. E_1 And E_2 harvesting effort oriented at the first and second prey, respectively. $n1, n3$ Represent respectively the saturation constants for the first and second prey. (Effort-scaling). $n2, n4$ Saturation constant for the first and second prey, respectively. (prey-scaling)
4. Assuming competition between predators (cz^2) in the event of scarcity of the two types of prey and (d) is the natural death of the predator.

After reviewing all the hypotheses of the proposed model, the model will be modeled as a system of nonlinear differential equations shown in the following system:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{rx}{1+fz} - \frac{q_1\, E_1 x}{n_1 E_1+n_2 x} - \frac{\alpha_1 xz}{\delta x+\beta y+g},\\
\frac{dy}{dt} &= s\, y\left(1-\frac{y}{k}\right) - \frac{q_2\, E_2 y}{n_3 E_2+n_4 y} - \frac{\alpha_2(1-m)yz}{1+\gamma(1-m)y^2},\\
\frac{dz}{dt} &= \frac{e_1\alpha_1 xz}{\delta x+\beta y+g} + \frac{e_2\alpha_2(1-m)yz}{1+\gamma(1-m)y^2} - cz^2 - dz,
\end{aligned} \tag{1}$$

3. BOUNDED AND POSITIVITY OF THE SOLUTION

The model (1)'s parameters are considered to be positive and are illustrated above with initial conditions $X(0) \geq 0, Y(0) \geq 0$, and $Z(0) \geq 0$, where the domain of the model is $R_+^3 = \{(x,y,z) \in R^3, x(0) \geq 0, y(0) \geq 0, \text{and } z(0) \geq 0\}$.

The following theorem illustrates positivity and non-negativity of the solution of system (1).

Theorem 1: All solutions starting in R_+^3 in system (1) are uniformly bounded.

Proof: Initially with conditions$(x_0, y_0, z_0) \in R_+^3$, system (1) has non negative solution

From system (1) $\frac{dy}{dt} = sy\left(1-\frac{y}{k}\right) - \frac{q_2\, E_2 y}{n_3 E_2+n_4 y} - \frac{\alpha_2(1-m)yz}{1+\gamma(1-m)y^2}$,

After solving yield: $y(t) \leq \frac{k}{1+\left(\frac{k}{y}-1\right)e^{-st}}$ This implies $\lim_{t\to\infty} y(t) \leq k$

So, $y(t) \leq M_y = \max\{y(0), k\}$ for all$t \geq 0$, therefore $0 \leq y(t) \leq M_y$

In the same way about $\frac{dz}{dt} = \frac{e_1\alpha_1 xz}{\delta x+\beta y+g} + \frac{e_2\alpha_2(1-m)yz}{1+\gamma(1-m)y^2} - cz^2 - dz$

After solving yield: $\frac{dz}{dt} \leq Wz - cz^2$, where $W = \frac{e_1\alpha_1}{\delta} + e_2\alpha_2(1-m)M_y - d$

Then $\frac{dz}{dt} \leq M_z = \max\left\{z(0), \frac{C}{W}\right\}$ for all $t \geq 0$

So, $0 \leq z(t) \leq M_z$

Now to prove the bounded of $x(t)$ by solving $\frac{dx}{dt} = \frac{rx}{1+fz} - \frac{q_1 E_1 z}{n_1 E_1 + n_2 x} - \frac{\alpha_1 xz}{\delta x+\beta y+g}$

$$x(t) \leq x(t_0) \exp\left(\frac{r}{1+fM_z}(t - t_0)\right),$$

Therefore,$x(t), y(t)$ *and* $z(t)$ are all bounded, then proof is complete.

4. FINDING SUITABLE BIOLOGICAL POINTS OF EQUILIBRIUM

This section searches for the most suitable equilibrium points related to system (1). There are seven equilibrium points listed as follows:

$Q_0 = (0,0,0)$, trivial point always exists.

$Q_1 = (\bar{x}, 0, 0)$, where $\bar{x} = \frac{E_1(q_1 - rn_1)}{rn_2} > 0$ according the condition

$$q_1 > rn_1 \tag{2}$$

$Q_2 = (0, \bar{\bar{y}}, 0)$

Solving the following equation $Ay^2 + By - C = 0$

where $\bar{\bar{y}} = \frac{-B+\sqrt{B^2+4AC}}{2A} > 0$, according the condition

$$B < \sqrt{B^2 + 4AC} \tag{3}$$

Where: $A = sn_4$, $B = s(kn_4 + n_3)$, $C = k(sn_3 - q_2 E_2)$.

$Q_3 = (\breve{x}, \breve{y}, 0)$ where

$\breve{x} = \frac{E_1(q_1 - rn_1)}{rn_2}$ and $\breve{y} = \frac{-B+\sqrt{B^2+4AC}}{2A}$

$Q_4 = (\breve{x}, 0, \breve{z})$

Solving the following equation: $\frac{e_1\alpha_1 xz}{\delta x+\beta y+g} + \frac{e_2\alpha_2(1-m)yz}{1+\gamma(1-m)y^2} - cz^2 - dz = 0,$

Then $\breve{z} = \frac{\breve{x}(e_1\alpha_1-\delta d)-dg}{s(\delta\breve{x}+g)} > 0$ according to the next conditions

$$e_1\alpha_1 > \delta d \tag{4}$$

$$\breve{x}(e_1\alpha_1 - \delta d) > dg \tag{5}$$

Now, substituted $(\breve{z})$, in $\frac{rx}{1+fz} - \frac{q_1\,E_1 z}{n_1 E_1 + n_2 x} - \frac{\alpha_1 xz}{\delta x + \beta y + g} = 0$ and making some algebraic simplifications yield with $\breve{x}$ is the positive solution to the following algebraic equation

$$G_1 x^4 + G_2 x^3 + G_3 x^2 + G_4 x + G_5 = 0$$

$$G_1 = r n_2 \delta^3 s^2,$$

$$G_2 = 2 r n_2 \delta^2 s^2 g + \delta^2 s^2\{r(n_1 E_1 \delta + n_2 g) - q_1 E_1 \delta\} - s\delta(e_1\alpha_1 - \delta d)(q_1 E_1 \delta f + \alpha_1 n_2) - \alpha_1 n_2 f (e_1\alpha_1 - \delta d)^2,$$

$$G_3 = r\delta n_2 s^2 g^2 + 2 s^2 \delta g\{r(n_1 E_1 \delta + n_2 g) - q_1 E_1 \delta\} + \delta^2 s^2 q_1 E_1 g - s\delta E_1(e_1\alpha_1 - \delta d)(q_1 \delta f + \alpha_1 n_1) + sg\{2\delta d - e_1\alpha_1\}[q_1 E_1 \delta f + \alpha_1 n_2] - (e_1\alpha_1 - \delta d)^2 \alpha_1 E_1 n_1 f + 2(e_1\alpha_1 - \delta d) g d \alpha_1 n_2 f,$$

$$G_4 = s^2 g^2\{r(n_1 E_1 \delta + n_2 g) - q_1 E_1 \delta\} + 2 s^2 \delta g^2 q_1 E_1 + sgE_1\{\delta d - (e_1\alpha_1 - \delta d)\}\{q_1 g f + \alpha_1 n_1\} + s g^2 d\{q_1 E_1 \delta f + \alpha_1 n_2\} + 2(e_1\alpha_1 - \delta d) g d \alpha_1 n_1 E_1 f - (gd)^2 \alpha_1 n_2 f,$$

$$G_5 = s^2 g^3 q_1 E_1 + s g^2 d E_1 (q_1 g f + \alpha_1 n_1) - (gd)^2 \alpha_1 n_1 E_1 f,$$

Exists provided that:

$$r(n_1 E_1 \delta + n_2 g) > q_1 E_1 \delta \tag{6}$$

$$e_1\alpha_1 > \delta d \tag{7}$$

$$2 r n_2 \delta^2 s^2 g + \delta^2 s^2\{r(n_1 E_1 \delta + n_2 g) - q_1 E_1 \delta\} > s\delta(e_1\alpha_1 - \delta d)(q_1 E_1 \delta f + \alpha_1 n_2) + \alpha_1 n_2 f (e_1\alpha_1 - \delta d)^2 \tag{8}$$

$$r\delta n_2 s^2 g^2 + 2 s^2 \delta g\{r(n_1 E_1 \delta + n_2 g) - q_1 E_1 \delta\} + \delta^2 s^2 q_1 E_1 g + sg\{2\delta d - e_1\alpha_1\}[q_1 E_1 \delta f + \alpha_1 n_2] + 2(e_1\alpha_1 - \delta d) g d \alpha_1 n_2 f > s\delta E_1(e_1\alpha_1 - \delta d)(q_1 \delta f + \alpha_1 n_1) + (e_1\alpha_1 - \delta d)^2 \alpha_1 E_1 n_1 f \tag{9}$$

$$s^2 g^3 q_1 E_1 + s g^2 d E_1 (q_1 g f + \alpha_1 n_1) < (gd)^2 \alpha_1 n_1 E_1 f \tag{10}$$

$$Q_5 = (0, \hat{y}, \hat{z})$$

Solving the following equation: $\frac{e_1\alpha_1 xz}{\delta x + \beta y + g} + \frac{e_2\alpha_2(1-m)yz}{1+\gamma(1-m)y^2} - cz^2 - dz = 0$

Then $\hat{z} = \frac{e_2\alpha_2(1-m)y - ds\{1+\gamma(1-m)y^2\}}{s(1+\gamma(1-m)y^2)} > 0$ according to the next conditions

$$e_2\alpha_2(1-m)y > ds\{1+\gamma(1-m)y^2\} \tag{11}$$

$$1 > m \tag{12}$$

substituted $(\hat{z})$, in $sy\left(1-\frac{y}{k}\right) - \frac{q_2\,E_2 y}{n_3E_2+n_4y} - \frac{\alpha_2(1-m)yz}{1+\gamma(1-m)y^2} = 0$,

And making some algebraic simplifications yield with $\breve{x}$ is the positive solution to the following algebraic equation

$$H_1y^6 + H_2y^5 + H_3y^4 + H_4y^3 + H_5y^2 + H_6y + H_7 = 0$$

Where,

$H_1 = s^2n_4\gamma^2(1-m)^2$,

$H_2 = -s^2\gamma^2(1-m)^2(kn_4 - n_3)$,

$H_3 = 2s^2n_4\gamma(1-m) - \gamma^2(1-m)^2sk(sn_3 - q_2E_2)$,

$H_4 = kn_4\alpha_2(1-m)^2ds\gamma - 2\gamma(1-m)s^2(kn_4 - n_3)$,

$H_5 = s^2n_4 - 2\gamma sk(1-m)(sn_3 + q_2E_2) + 2kn_3\alpha_2\gamma(1-m)^2ds - e_2kn_4\alpha_2^2(1-m)^2$,

$H_6 = kn_4\alpha_2^2(1-m)^2e_2ds - skq_2E_2e_2\alpha_2(1-m) - s^2(kn_4 - n_3)$,

$H_7 = -s^2kn_3 - skq_2E_2 + kn_3\alpha_2(1-m)ds$,

Exists provided that:

$$kn_4 > n_3 \tag{13}$$

$$2s^2n_4\gamma(1-m) < \gamma^2(1-m)^2sk(sn_3 - q_2E_2) \tag{14}$$

$$sn_3 > q_2E_2 \tag{15}$$

$$kn_4\alpha_2(1-m)^2ds\gamma < 2\gamma(1-m)s^2(kn_4 - n_3) \tag{16}$$

$$s^2n_4 + 2kn_3\alpha_2\gamma(1-m)^2ds < 2\gamma sk(1-m)(sn_3 + q_2E_2) + e_2kn_4\alpha_2^2(1-m)^2 \tag{17}$$

$$kn_4\alpha_2^2(1-m)^2e_2ds < skq_2E_2e_2\alpha_2(1-m) + s^2(kn_4 - n_3) \tag{18}$$

$$s^2kn_3 + skq_2E_2 > kn_3\alpha_2(1-m)ds \tag{19}$$

The positive equilibrium point $Q_7 = (\tilde{x}, \tilde{y}, \tilde{z})$, as:

$\tilde{z} = \frac{1}{s}\left\{\frac{e_1\alpha_1 x}{\delta x+\beta y+g} + \frac{e_2\alpha_2(1-m)y}{1+\gamma(1-m)y^2} - d\right\} > 0$ Under the following condition

$$\frac{c_1\alpha_1 x}{\delta x+\beta y+g} + \frac{c_2\alpha_2(1-m)y}{1+\gamma(1-m)y^2} > d \tag{20}$$

Setting only their first two equations of system (1) equal to zero and putting the value of $(\tilde{z})$, then $(\tilde{x},\tilde{y})$ represented two isoclines intersecting positively.

$$\left.\begin{array}{l} f_1(x,y) = v_1x^5 + v_2x^4 + v_3x^3 + v_4x^2 + v_5x + l_{30}x^3 + l_{31}x^2 + l_{32}x + l_{33}xy = 0 \\ f_2(x,y) = a_1y^8 + a_2y^7 + a_3y^6 + a_4y^5 + a_5y^4 + a_6y^3 + a_7y^2 + a_8y + a_{17}x + a_{18} = 0 \end{array}\right\} \tag{21}$$

Where, $v_1 = l_1y^4 + l_2x^3 + l_3x^2 + l_4x + l_5$

$$v_2 = l_6y^5 + l_7y^4 + l_8y^3 + l_9y^2 + l_{10}y + l_{11}$$

$$v_3 = l_{12}y^6 + l_{13}y^5 + l_{14}y^4 + l_{15}y^3 + l_{16}y^2 + l_{17}y$$

$$v_4 = l_{18}y^5 + l_{19}y^4 + l_{20}y^3 + l_{21}y^2 + l_{22}y + l_{23}$$

$$v_5 = l_{24}y^6 + l_{25}y^5 + l_{26}y^4 + l_{27}y^3 + l_{28}y^2 + l_{29}y$$

$$a_{17} = a_9y^7 + a_{10}y^6 + a_{11}y^5 + a_{12}y^4 + a_{13}y^3 + a_{14}y^2 + a_{15}y + a_{16}$$

Squarely to their enormous and complicated terms corresponding to l_i v_j and a_k for all, $= 1, \ldots, 33$, $j = 1, \ldots, 5$ and $k = 1, \ldots, 16$ are calculated simplifications, and it will not give

When $y \rightarrow 0$ then equation (20) yield as follow:

$$\left.\begin{array}{c} f_1(x,y) = l_5x^5 + l_{11}x^4 + l_{30}x^3 + (l_{23} + l_{31})x^2 + l_{32}x = 0 \\ f_2(x,y) = a_{16}x + a_{17} = 0 \end{array}\right\} \tag{22}$$

Due to the discarding rule of signs, each equation in equation (22) may have a unique positive root for the isocline given by x_1 and x_2, respectively, when

The leading and free coefficients have opposite signs.

Then system (22) has a unique intersection point given by $(\tilde{x},\tilde{y})$ under the following conditions:

$$\left.\begin{array}{c} \frac{\partial f_1}{\partial x} > 0 \;\; and \;\; \frac{\partial f_1}{\partial y} < 0 \\ or \\ \frac{\partial f_1}{\partial x} < 0 \; and \;\; \frac{\partial f_1}{\partial y} > 0 \end{array}\right], \tag{23}$$

$$\left.\begin{array}{c} \frac{\partial f_2}{\partial x} > 0 \;\; and \;\; \frac{\partial f_2}{\partial y} < 0 \\ or \\ \frac{\partial f_2}{\partial x} < 0 \; and \;\; \frac{\partial f_2}{\partial y} > 0 \end{array}\right], \tag{24}$$

Hence, there is only the positive point of equilibrium $Q_7 = (\tilde{x}, \tilde{y}, \tilde{z})$, exists in R^3_+ .

5. LOCAL STABILITY ANALYSIS

Via linearizing the proposed model with the Jacobian matrix, at each equilibrium point of model (1), the local stability can be investigated:

$$\mathrm{J} = \left[\mathrm{b_{ij}}\right]_{3\times 3} \tag{25}$$

Where:

$b_{11} = \frac{r}{1+fz} - \frac{n_1E_1^2q_1}{(n_1E_1+n_2x)^2} - \frac{\alpha_1 z(\delta x+\beta y+g)-\alpha_1\delta xz}{(\delta x+\beta y+g)^2}$, $b_{12} = \frac{\alpha_1\beta xz}{(\delta x+\beta y+g)^2}$, $b_{13} = \frac{-rfx}{(1+fz)^2}$,

$b_{22} = s - \frac{2sy}{k} - \frac{q_2E_2(n_3E_2+n_4y)-n_4q_2E_2y}{(n_3E_2+n_4y)^2} - \frac{\alpha_2(1-m)z\left(1+\gamma(1-m)y^2\right)-2\alpha_2^2\,\gamma(1-m)^2y^2z}{(1+\gamma(1-m)y^2)^2}$, $b_{23} = -\frac{\alpha_2(1-m)y}{1+\gamma(1-m)y^2}$, $b_{31} = \frac{e_1\alpha_1 z(\delta x+\beta y+g)-e_1\alpha_1\delta xz}{(\delta x+\beta y+g)^2}$,

$b_{32} = \frac{-e_1\alpha_1\beta xz}{(\delta x+\beta y+g)^2} + \frac{e_2\alpha_2(1-m)z\left(1+\gamma(1-m)y^2\right)-2e_2\alpha_2\gamma(1-m)^2y^2z}{(1+\gamma(1-m)y^2)^2}$ $b_{33} = \frac{e_1\alpha_1 x}{\delta x+\beta y+g} + \frac{e_2\alpha_2 y}{1+\gamma(1-m)y^2} - 2c\,z - d.$

The following theorems can be used to investigate the local asymptotic stability at each equilibrium point whenever all the eigenvalues are negative in the Jacobian matrix, illustrated as follows:

Theorem 5.1: If the following conditions are satisfied guarantee that $Q_0 = (0,0,0)$ is locally asymptotically stable

$$r < \frac{q_1}{n_1} \tag{26}$$

$$s < \frac{q_2}{n_2} \tag{27}$$

Proof: Substituting $Q_0 = (0,0,0)$ into the Jacobian matrix of model (1), $J_0(Q_0)$ yields:

$b_{11} = r - \frac{q_1}{n_1}$, $b_{12} = 0$, $b_{13} = 0$, $b_{21} = 0$

$b_{22} = s - \frac{q_2}{n_2}$, $b_{23} = 0$, $b_{31} = 0,$ $b_{32} = 0,$ $b_{33} = -d.$

Then the eigenvalues of the jacobian matrix $J_0(Q_0)$ are

$\lambda_1 = r - \frac{q_1}{n_1}$, $\lambda_2 = s - \frac{q_2}{n_2}$, $\lambda_3 = -d.$

Whenever the conditions (26) and (27) are satisfied, it tends to imply that Q_0 is asymptotically locally stable.

Theorem 5.2: If the following conditions are satisfied guarantee that $Q_1 = (\bar{x}, 0, 0)$ is locally asymptotically stable

$$r < \frac{n_1 q_1 E_1^2}{\left(n_1 E_1 + \frac{E_1(q_1 - n_1 r)}{r}\right)^2} \tag{28}$$

$$\frac{e_1 \alpha_1 E_1 (q_1 - n_1 r)}{\delta E_1 (q_1 - n_1 r) + g r n_2} < d \tag{29}$$

Proof: Substituting $Q_1 = (\bar{x}, 0, 0)$ into the Jacobian matrix of model (1), $J_1(Q_1)$ yields:

$$\bar{b}_{11} = r - \frac{n_1 q_1 E_1^2}{\left(n_1 E_1 + \frac{E_1(q_1 - n_1 r)}{r}\right)^2}, \ \bar{b}_{12} = 0, \ \bar{b}_{13} = \frac{-r f E_1 (q_1 - n_1 r)}{n_2 r} - \frac{\alpha_1 E_1 (q_1 - n_1 r)}{\delta E_1 (q_1 - n_1 r) + g r n_2},$$

$$\bar{b}_{21} = 0$$

$$\bar{b}_{22} = s - \frac{q_2}{n_2}, \ \bar{b}_{23} = 0, \ \bar{b}_{31} = 0, \ \bar{b}_{32} = 0, \ \bar{b}_{33} = \frac{e_1 \alpha_1 E_1 (q_1 - n_1 r)}{\delta E_1 (q_1 - n_1 r) + g r n_2} - d.$$

Then the eigenvalues of the jacobian matrix $J_1(Q_1)$ are

$$\lambda_1 = r - \frac{n_1 q_1 E_1^2}{\left(n_1 E_1 + \frac{E_1(q_1 - n_1 r)}{r}\right)^2}, \ \lambda_2 = s - \frac{q_2}{n_2}, \ \lambda_3 = \frac{e_1 \alpha_1 E_1 (q_1 - n_1 r)}{\delta E_1 (q_1 - n_1 r) + g r n_2} - d.$$

Whenever the condition (27) in theorem 5.1, with the conditions (28) and (29) are satisfied, it tends to imply that Q_1 is asymptotically locally stable.

Theorem 5.3: If the following conditions are satisfied guarantee that $Q_2 = (0, \bar{\bar{y}}, 0)$ is locally asymptotically stable

$$s < \frac{2cy}{k} + \frac{q_2 E_2 (n_3 E_2 + n_4 \bar{\bar{y}}) - n_4 q_2 E_2 \bar{\bar{y}}}{(n_3 E_2 + n_4 \bar{\bar{y}})^2} \tag{30}$$

$$\frac{e_2 \alpha_2 \bar{\bar{y}}}{1 + \gamma (1 - m) \bar{\bar{y}}^2} < d \tag{31}$$

Proof: Substituting $Q_2 = (0, \bar{\bar{y}}, 0)$ into the Jacobian matrix of model (1), $J_2(Q_2)$ yields:

$$\bar{\bar{b}}_{11} = r - \frac{q_1}{n_1}, \ \bar{\bar{b}}_{12} = 0, \ \bar{\bar{b}}_{13} = 0,$$

$$\bar{\bar{b}}_{22} = s - \frac{2sy}{k} - \frac{q_2 E_2 (n_3 E_2 + n_4 \bar{\bar{y}}) - n_4 q_2 E_2 \bar{\bar{y}}}{(n_3 E_2 + n_4 \bar{\bar{y}})^2}, \ \bar{\bar{b}}_{23} = -\frac{\alpha_2 (1 - m) \bar{\bar{y}}}{1 + \gamma (1 - m) \bar{\bar{y}}^2}, \ \bar{\bar{b}}_{31} = 0,$$

$$\bar{\bar{b}}_{32} = 0 \ \bar{\bar{b}}_{33} = \frac{e_2 \alpha_2 \bar{\bar{y}}}{1 + \gamma (1 - m) \bar{\bar{y}}^2} - d.$$

Then the eigenvalues of the jacobian matrix $J_2(Q_2)$ are

$$\lambda_1 = r - \frac{q_1}{n_1}\ ,\ \lambda_2 = s - \frac{2sy}{k} - \frac{q_2E_2(n_3E_2+n_4\bar{\bar{y}})-n_4q_2E_2\bar{\bar{y}}}{(n_3E_2+n_4\bar{\bar{y}})^2}\ \ ,\ \lambda_3 = \frac{c_2\alpha_2\bar{\bar{y}}}{1+\gamma(1-m)\bar{\bar{y}}^2} - d.$$

Whenever the condition (26) in theorem 5.1, with the conditions (30) and (31) are satisfied, it tends to imply that Q_2 is asymptotically locally stable.

Theorem 5.4: If the following conditions are satisfied guarantee that $Q_3 = (\check{x}, \check{y}, 0)$ is locally asymptotically stable.

$$r < \frac{q_1E_1(n_1E_1+n_2\tilde{x})-n_2q_1E_1\tilde{x}}{(n_1E_1+n_2\tilde{x})^2} \tag{32}$$

$$s < \frac{2s\tilde{y}}{k} + \frac{q_2E_2(n_3E_2+n_4\tilde{y})-n_4q_2E_2\tilde{y}}{(n_3E_2+n_4\tilde{y})^2} \tag{33}$$

$$\frac{e_1\alpha_1\tilde{x}}{\delta\tilde{x}+\beta\tilde{y}+g} + \frac{e_2\alpha_2\tilde{y}}{1+\gamma(1-m)\tilde{y}^2} < d \tag{34}$$

Proof: Substituting $Q_3 = (\check{x}, \check{y}, 0)$ into the Jacobian matrix of model (1), $J_3(Q_3)$ yields:

$$\check{b}_{11} = r - \frac{q_1E_1(n_1E_1+n_2\tilde{x})-n_2q_1E_1\tilde{x}}{(n_1E_1+n_2\tilde{x})^2}\ ,\ \check{b}_{12} = 0\ ,\ \check{b}_{13} = -rf\tilde{x} - \frac{\alpha_1\tilde{x}}{\delta x+\beta y+g}\ \ ,\ \check{b}_{21} = 0$$

$$\check{b}_{22} = s - \frac{2s\tilde{y}}{k} - \frac{q_2E_2(n_3E_2+n_4\tilde{y})-n_4q_2E_2\tilde{y}}{(n_3E_2+n_4\tilde{y})^2}\ \ ,\ \check{b}_{23} = -\frac{\alpha_2(1-m)\tilde{y}}{1+\gamma(1-m)\tilde{y}^2}\ ,\ \check{b}_{31} = 0,$$

$$\check{b}_{32} = 0\ \ \check{b}_{33} = \frac{e_1\alpha_1\tilde{x}}{\delta\tilde{x}+\beta\tilde{y}+g} + \frac{e_2\alpha_2\tilde{y}}{1+\gamma(1-m)\tilde{y}^2} - d.$$

Then the eigenvalues of the jacobian matrix $J_3(Q_3)$ are

$$\lambda_1 = r - \frac{q_1E_1(n_1E_1+n_2\tilde{x})-n_2q_1E_1\tilde{x}}{(n_1E_1+n_2\tilde{x})^2}\ ,\ \lambda_2 = s - \frac{2s\tilde{y}}{k} - \frac{q_2E_2(n_3E_2+n_4\tilde{y})-n_4q_2E_2\tilde{y}}{(n_3E_2+n_4\tilde{y})^2}\ \ ,\ \lambda_3 =$$

$$\frac{e_1\alpha_1\tilde{x}}{\delta\tilde{x}+\beta\tilde{y}+g} + \frac{e_2\alpha_2\tilde{y}}{1+\gamma(1-m)\tilde{y}^2} - d.$$

Whenever the conditions (32-34) are satisfied, it tends to imply that Q_3 is asymptotically locally stable.

Theorem 5.5: If the following conditions are satisfied guarantee that $Q_4 = (\breve{x}, 0, \breve{z})$ is locally asymptotically stable.

$$\frac{r}{1+f\breve{z}} < \frac{n_1E_1^2q_1}{(n_1E_1+n_2\breve{x})^2} + \frac{\alpha_1\breve{z}(\delta\breve{x}+g)-\alpha_1\delta\breve{x}\breve{z}}{(\delta\breve{x}+g)^2} \tag{35}$$

$$s < \frac{q_2}{n_3} + \alpha_2(1-m)\breve{z} \tag{36}$$

$$\frac{e_1\alpha_1\breve{x}}{\delta\breve{x}+g} < 2s\breve{z} + d \tag{37}$$

Proof: Substituting $Q_4 = (\breve{x}, 0, \breve{z})$ into the Jacobian matrix of model (1), $J_4(Q_4)$ yields:

$$J_4=J(Q_4)=\begin{bmatrix} \breve{b}_{11} & \breve{b}_{12} & \breve{b}_{13} \\ 0 & \breve{b}_{22} & 0 \\ \breve{b}_{31} & \breve{b}_{32} & \breve{b}_{33} \end{bmatrix},$$

$\breve{b}_{11} = \frac{r}{1+f\breve{z}} - \frac{n_1 E_1^2 q_1}{(n_1E_1+n_2\breve{x})^2} - \frac{\alpha_1\breve{z}(\delta\breve{x}+g)-\alpha_1\delta\breve{x}\breve{z}}{(\delta\breve{x}+g)^2}$, $\breve{b}_{12} = \frac{\alpha_1\beta\breve{x}\breve{z}}{(\delta\breve{x}+g)^2}$, $\breve{b}_{13} = -\left\{\frac{rf\breve{x}}{(1+f\breve{z})^2} + \frac{\alpha_1\breve{x}}{\delta\breve{x}+g}\right\}$, $\breve{b}_{21} = 0$, $\breve{a}_{22} = s - \frac{q_2}{n_3} - \alpha_2(1-m)\breve{z}$, $\breve{b}_{23} = 0$, $\breve{b}_{31} = \frac{e_1\alpha_1 z(\delta x+\beta y+g)-e_1\alpha_1\delta xz}{(\delta x+\beta y+g)^2}$, $\breve{b}_{32} = \frac{-e_1\alpha_1\beta\breve{x}\breve{z}}{(\delta\breve{x}+g)^2} + e_2\alpha_2(1-m)\breve{z}$

$\breve{b}_{33} = \frac{e_1\alpha_1\breve{x}}{\delta\breve{x}+g} - 2s\breve{z} - d.$

Then the characteristic equation of J_4 is:

$[\lambda^2 - B_1\lambda + B_2](\breve{b}_{22} - \lambda) = 0$

So, either

$(\breve{b}_{22} - \lambda)$ which gives $\lambda_2 = s - \frac{q_2}{n_3} - \alpha_2(1-m)\breve{z}.$

Or

$\lambda^2 - B_1\lambda + B_2 = 0$

Where:

$B_1 = \lambda_1 + \lambda_3 = \breve{b}_{11} + b_{33}$ *and* $B_2 = \breve{b}_{11}\breve{b}_{33} - \breve{b}_{13}\breve{b}_{31.}$

Whenever the conditions (35-37) are satisfied, it tends to imply that Q_4 is asymptotically locally stable.

Theorem 5.6: If the following conditions are satisfied guarantee that $Q_5 = (0, \hat{y}, \hat{z})$ is locally asymptotically stable.

$$\frac{r}{1+f\hat{z}} < \frac{q_1}{n_1} + \frac{\alpha_1\hat{z}(\beta\hat{y}+g)}{(\beta\hat{y}+g)^2} \tag{38}$$

$$s < \frac{2s\hat{y}}{k} + \frac{q_2E_2(n_3E_2+n_4\hat{y})-n_4q_2E_2\hat{y}}{(n_3E_2+n_4\hat{y})^2} + \frac{\alpha_2(1-m)\hat{z}\left(1+\gamma(1-m)\hat{y}^2\right)-2\alpha_2^2\,\gamma(1-m)^2\hat{y}^2\hat{z}}{(1+\gamma(1-m)\hat{y}^2)^2} \tag{39}$$

$$\frac{e_2\alpha_2\hat{y}}{1+\gamma(1-m)\hat{y}^2} < 2s\hat{z} + d. \tag{40}$$

Proof: Substituting $Q_5 = (0, \hat{y}, \hat{z}\,)$ into the Jacobian matrix of model (1), $J_5(Q_5)$ yields:

$$\mathrm{J}_5=\mathrm{J}(Q_5)=\begin{bmatrix} \hat{b}_{11} & 0 & 0 \\ 0 & \hat{b}_{22} & \hat{b}_{23} \\ \hat{b}_{31} & \hat{b}_{32} & \hat{b}_{33} \end{bmatrix},$$

$\hat{b}_{11} = \frac{r}{1+f\hat{z}} - \frac{q_1}{n_1} - \frac{\alpha_1\hat{z}(\beta\hat{y}+g)}{(\beta\hat{y}+g)^2}$,

$\hat{b}_{22} = s - \frac{2s\hat{y}}{k} - \frac{q_2E_2(n_3E_2+n_4\hat{y})-n_4q_2E_2\hat{y}}{(n_3E_2+n_4\hat{y})^2} - \frac{\alpha_2(1-m)\hat{z}\left(1+\gamma(1-m)\hat{y}^2\right)-2\alpha_2^2\,\gamma(1-m)^2\hat{y}^2\hat{z}}{(1+\gamma(1-m)\hat{y}^2)^2}$, $\hat{b}_{23} = -\frac{\alpha_2(1-m)\hat{y}}{1+\gamma(1-m)\hat{y}^2}$, $\hat{b}_{31} = \frac{e_1\alpha_1\hat{z}(\beta\hat{y}+g)}{(\beta\hat{y}+g)^2}$,

$\hat{b}_{32} = \frac{-e_1\alpha_1\beta\hat{x}\hat{z}}{(\beta\hat{y}+g)^2} + \frac{e_2\alpha_2(1-m)\hat{z}\left(1+\gamma(1-m)\hat{y}^2\right)-2e_2\alpha_2\gamma(1-m)^2\hat{y}^2\hat{z}}{(1+\gamma(1-m)y^2)^2}$,

$\hat{b}_{33} = \frac{e_2\alpha_2\hat{y}}{1+\gamma(1-m)\hat{y}^2} - 2s\hat{z} - d.$

Then the characteristic equation of J_5 is:

$$\left(\hat{b}_{11} - \lambda\right)\left(\hat{b}_{22} - \lambda\right)\left(\hat{b}_{33} - \lambda\right) = 0$$

So, $\lambda_1 = \frac{r}{1+f\hat{z}} - \frac{q_1}{n_1} - \frac{\alpha_1\hat{z}(\beta\hat{y}+g)}{(\beta\hat{y}+g)^2}$, $\lambda_2 = s - \frac{2sy}{k} - \frac{q_2E_2(n_3E_2+n_4\hat{y})-n_4q_2E_2\hat{y}}{(n_3E_2+n_4\hat{y})^2} - \frac{\alpha_2(1-m)\hat{z}\left(1+\gamma(1-m)\hat{y}^2\right)-2\alpha_2^2\,\gamma(1-m)^2\hat{y}^2\hat{z}}{(1+\gamma(1-m)\hat{y}^2)^2}$

$\lambda_3 = \frac{e_2\alpha_2\hat{y}}{1+\gamma(1-m)\hat{y}^2} - 2s\hat{z} - d.$

Whenever the conditions (38-40) are satisfied, it tends to imply that Q_5 is asymptotically locally stable.

Theorem 5.7: If the following conditions are satisfied guarantee that $Q_6 = (\tilde{x}, \tilde{y}, \tilde{z})$ is locally asymptotically stable.

$$\frac{r}{1+f\tilde{z}} < \frac{n_1E_1^2q_1}{(n_1E_1+n_2\tilde{x})^2} + \frac{\alpha_1\tilde{z}(\delta\tilde{x}+\beta\tilde{y}+g)-\alpha_1\delta\tilde{x}\tilde{z}}{(\delta\tilde{x}+\beta\tilde{y}+g)^2} \tag{41}$$

$$s < \frac{2s\tilde{y}}{k} + \frac{q_2E_2(n_3E_2+n_4\tilde{y})-n_4q_2E_2\tilde{y}}{(n_3E_2+n_4\tilde{y})^2} + \frac{\alpha_2(1-m)\tilde{z}\left(1+\gamma(1-m)\tilde{y}^2\right)-2\alpha_2^2\,\gamma(1-m)^2\tilde{y}^2\tilde{z}}{(1+\gamma(1-m)\tilde{y}^2)^2} \tag{42}$$

$$\frac{e_1\alpha_1\tilde{x}}{\delta\tilde{x}+\beta\tilde{y}+g} + \frac{e_2\alpha_2\tilde{y}}{1+\gamma(1-m)\tilde{y}^2} < 2s\tilde{z} + d. \tag{43}$$

$$\vartheta_1 > \vartheta_2 \tag{44}$$

Proof: Substituting $Q_6 = (\tilde{x}, \tilde{y}, \tilde{z})$ into the Jacobian matrix of model (1), $J_6(Q_6)$ yields:

$$J_6 = J(Q_6) = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ 0 & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{bmatrix},$$

$\tilde{b}_{11} = \frac{r}{1+f\tilde{z}} - \frac{n_1 E_1^2 q_1}{(n_1 E_1 + n_2 \tilde{x})^2} - \frac{\alpha_1 \tilde{z}(\delta\tilde{x}+\beta\tilde{y}+g) - \alpha_1 \delta \tilde{x}\tilde{z}}{(\delta\tilde{x}+\beta\tilde{y}+g)^2}$, $\tilde{b}_{12} = \frac{\alpha_1 \beta \tilde{x}\tilde{z}}{(\delta\tilde{x}+\beta\tilde{y}+g)^2}$, $\tilde{b}_{13} = \frac{-rf\tilde{x}}{(1+f\tilde{z})^2}$,

$\tilde{b}_{22} = s - \frac{2s\tilde{y}}{k} - \frac{q_2 E_2(n_3 E_2 + n_4 \tilde{y}) - n_4 q_2 E_2 \tilde{y}}{(n_3 E_2 + n_4 \tilde{y})^2} - \frac{\alpha_2 (1-m)\tilde{z}\left(1+\gamma(1-m)\tilde{y}^2\right) - 2\alpha_2^2\, \gamma(1-m)^2 \tilde{y}^2 \tilde{z}}{(1+\gamma(1-m)\tilde{y}^2)^2}$, $\tilde{b}_{23} = -\frac{\alpha_2(1-m)\tilde{y}}{1+\gamma(1-m)\tilde{\tilde{y}}^2}$, $\tilde{b}_{31} = \frac{c_1 \alpha_1 z(\delta\tilde{x}+\beta\tilde{y}+g) - c_1 \alpha_1 \delta \tilde{x}\tilde{z}}{(\delta\tilde{x}+\beta\tilde{y}+g)^2}$,

$\tilde{b}_{32} = \frac{-e_1 \alpha_1 \beta \tilde{x}\tilde{z}}{(\delta x+\beta y+g)^2} + \frac{e_2 \alpha_2 (1-m)\tilde{z}\left(1+\gamma(1-m)\tilde{y}^2\right) - 2e_2 \alpha_2 \gamma(1-m)^2 \tilde{y}^2 \tilde{z}}{(1+\gamma(1-m)\tilde{y}^2)^2}$ $\tilde{b}_{33} = \frac{e_1 \alpha_1 \tilde{x}}{\delta\tilde{x}+\beta\tilde{y}+g} + \frac{e_2 \alpha_2 \tilde{y}}{1+\gamma(1-m)\tilde{y}^2} - 2s\tilde{z} - d.$

Then the characteristic equation of J_6 is:

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \qquad (45)$$

Where,

$$A_1 = -\left(\tilde{b}_{11} + \tilde{b}_{22} + \tilde{b}_{33}\right)$$

$$A_2 = \tilde{b}_{11}\tilde{b}_{22} + \tilde{b}_{33}\left(\tilde{b}_{11} + \tilde{b}_{22}\right) - \tilde{b}_{13}\tilde{b}_{31} - \tilde{b}_{23}\tilde{b}_{32}$$

$$A_3 = \tilde{b}_{11}\tilde{b}_{23}\tilde{b}_{32} + \tilde{b}_{13}\tilde{b}_{22}\tilde{b}_{31} - \tilde{b}_{12}\tilde{b}_{23}\tilde{b}_{31} - \tilde{b}_{11}\tilde{b}_{22}\tilde{b}_{33}$$

equation (45) has(eigenvalues) accompanied by negative real parts by assist Routh-Hurwitz criterion if and only if:: $A_1 > 0$, $A_3 > 0$ $and\, \Delta = (A_1\, A_2 - A_3)A_3 > 0$.

Now, $A_i > 0\,, i = 1{,}3$, as long as the conditions (41-43) satisfied

Straightforward computation shows that:

$\Delta = \vartheta_1 - \vartheta_2$, Where

$\vartheta_1 = -\left(\tilde{b}_{11} + \tilde{b}_{22} + \tilde{b}_{33}\right)[\tilde{b}_{11}\tilde{b}_{22} + \tilde{b}_{33}\left(\tilde{b}_{11} + \tilde{b}_{22}\right) - \tilde{b}_{13}\tilde{b}_{31} - \tilde{b}_{23}\tilde{b}_{32}]\tilde{b}_{11}\tilde{b}_{23}\tilde{b}_{32} + \tilde{b}_{13}\tilde{b}_{22}\tilde{b}_{31} - \tilde{b}_{12}\tilde{b}_{23}\tilde{b}_{31} - \tilde{b}_{11}\tilde{b}_{22}\tilde{b}_{33}$,

$\vartheta_2 = \left(\tilde{b}_{11}\tilde{b}_{23}\tilde{b}_{32} + \tilde{b}_{13}\tilde{b}_{22}\tilde{b}_{31} - \tilde{b}_{12}\tilde{b}_{23}\tilde{b}_{31} - \tilde{b}_{11}\tilde{b}_{22}\tilde{b}_{33}\right)^2$

So, $\Delta > 0$ presuming both of conditions (41-44) and as long as the condition satisfied below:

Therefore Q_6 is stable under the above conditions and, apart from that, it is unstable.

6. GLOBAL STABILITY ANALYSIS

Along wishful the basin of attraction using t. Via the Lyapunov function technique, the following detection theorems explore exhibition conditions of the global dynamic of all possible states of the proposed biological equilibrium model (1), including $Q_i,\ i = 0, 1, \dots, 6$,

Theorem 6.1: As stated earlier by Theorem 5.1, the point Q_0 is asymptotically locally. Then, for the guaranteed conditions below, Q_0 is asymptotically globally stable in R^3_+:

$$q_1 E_1 > r \tag{46}$$

$$q_2 E_2 > s \tag{47}$$

Proof: presume the Lyapunov function is a positive real value:

$M_0 = x + y + z$, where $M_0(t): R^3_+ \to R$ is C^1 and satisfying that $M_0(0,0,0) = 0$,

And for all (x, y, z) then $M_0(x, y, z)$ greater than zero,

After some algebraic steps to yield:

$$\frac{dM_0}{dt} < -e_1(q_1E_1 - r)x - e_2(q_2E_2 - s)y - dz,$$

Consequently, conditions (46-47) tend to $\frac{dM_0}{dt} < 0$, in R^3_+. Also, Q_0 is asymptotically globally stable.

Theorem 6.2: As stated earlier by Theorem 5.2, the point Q_1 is asymptotically locally. Then, for the guaranteed condition below, Q_1 is asymptotically globally stable in R^3_+:

$$x > \bar{x} \tag{48}$$

Proof: let $M_1 = \left(x - \bar{x}\, ln\frac{x}{\bar{x}}\right) + y + z$**,** is a positive real value Lyapunov function where $M_1(t): R^3_+ \to R$ is C^1 and satisfying that $M_1(\bar{x}, 0, 0) = 0$,

And for all (x, y, z) then $M_1(x, y, z) > 0$,

$$\frac{dM_1}{dt} < -q_1 n_2 E_1 (x - \bar{x})^2 - (q_2E_2 - s)y - (rf\bar{x} + \alpha_1\bar{x} + d)z$$

Consequently, condition (48) with condition (47) in theorem 6.1, tend to $\frac{dM_1}{dt} < 0$, in R_+^3. Also, Q_1 is asymptotically globally stable.

Theorem 6.3: As stated earlier by Theorem 5.3, the point Q_2 is asymptotically locally. Then, for the guaranteed condition below, Q_2 is asymptotically globally stable in R_+^3:

$$y > \bar{\bar{y}} \tag{49}$$

Proof: let $M_2 = x + \left(y - \bar{\bar{y}} ln \frac{y}{\bar{\bar{y}}}\right) + z$**,** is a positive real value Lyapunov function

where $M_2(t): R_+^3 \rightarrow R$ is C^1 and satisfying that $M_2(0, \bar{\bar{y}}, 0) = 0$,

And for all (x, y, z) then $M_2(x, y, z) > 0$,

$$\frac{dM_2}{dt} < -(q_1E_1 + r)x - s(y - \bar{\bar{y}})^2 - (\alpha_2(1 - m)\bar{\bar{y}} + d)z$$

Consequently, condition (49) with biological fact $1 > m$, tend to $\frac{dM_2}{dt} < 0$, in R_+^3.

Also, Q_2 is asymptotically globally stable.

Theorem 6.4: As stated earlier by Theorem 5.4, the point Q_3 is asymptotically locally. Then, for the guaranteed condition below, Q_3 is asymptotically globally stable in R_+^3:

$$x > \tilde{x} \tag{50}$$

$$y - \tilde{y} \tag{51}$$

Proof: let $M_3 = \left(x - \tilde{x}\, ln \frac{x}{\tilde{x}}\right) + \left(y - \tilde{y} ln \frac{y}{\tilde{y}}\right) + z$**,** is a positive real value Lyapunov function

where $M_3(t): R_+^3 \rightarrow R$ is C^1 and satisfying that $M_3(\tilde{x}, \tilde{y}, 0) = 0$,

And for all (x, y, z) then $M_3(x, y, z) > 0$,

$$\frac{dM_3}{dt} < -n_3q_1E_1\, (x - \tilde{x})^2 - n_4q_2E_2(y - \tilde{y})^2 - \{(rf + \alpha_1)\tilde{x} + (s + \alpha_2)\tilde{y} + d\}\, z$$

Consequently, conditions (50-51), implies that $\frac{dM_2}{dt} < 0$, in R_+^3. Also, Q_3 is asymptotically globally stable.

Theorem 6.5: As stated earlier by Theorem 5.5, the point Q_4 is asymptotically locally. Then, for the guaranteed condition below, Q_4 is asymptotically globally stable in R_+^3:

$$\frac{q_2E_2}{n_3E_2 + n_4y} < s \tag{52}$$

Proof: let $M_4 = \left(x - \check{x}\, ln\frac{x}{\check{x}}\right) + y + \left(z - \check{z}\, ln\frac{z}{\check{z}}\right)$**,** is a positive real value Lyapunov function

Where $M_4(t): R_+^3 \to R$ is C^1 and satisfying that $M_4(\tilde{x}, \tilde{y}, 0) = 0$,

And for all (x, y, z) , $M_4(x, y, z) > 0$,

$$\frac{dM_4}{dt} < -n_2 q_1 (x - \check{x})^2 - d(z - \check{z})^2 - rf(x - \check{x})(z - \check{z}) - \left\{\frac{q_2 E_2}{n_3 E_2 + n_4 y} - s\right\} y$$

Consequently, condition (52), implies that $\frac{dM_4}{dt} < 0$, in R_+^3. Also, Q_4 is asymptotically globally stable.

Theorem 6.6: As stated earlier by Theorem 5.6, the point Q_5 is asymptotically locally. Then, for the guaranteed condition below, Q_5 is asymptotically globally stable in R_+^3:

$$\left\{\frac{s}{k} + \frac{q_2 n_4 E_2}{(n_3 E_2 + n_4 y)(n_3 E_2 + n_4 \hat{y})}\right\} (y - \hat{y})^2 + \frac{c_2 \alpha_2 \{1 - \gamma(1-m) y \hat{y}\}}{\{1 + \gamma(1-m) y^2\}\{1 + \gamma(1-m)\hat{y}^2\}} (y - \hat{y})(z - \check{z}) +$$
$$d(z - \check{z})^2 > \frac{rx}{1 + fz} + \left\{\frac{(c_1 \alpha_1 x + c_2 \alpha_2 y)}{1 + \gamma(1-m)\hat{y}^2}\right\} z \tag{53}$$

$$y > \hat{y} \tag{54}$$

$$z > \check{z} \tag{55}$$

Proof: let $M_5 = x + \left(y - \hat{y}\, ln\frac{y}{\hat{y}}\right) + \left(z - \hat{z}\, ln\frac{z}{\hat{z}}\right)$**,** is a positive real value Lyapunov function

Where $M_5(t): R_+^3 \to R$ is C^1 and satisfying that $M_5(\tilde{x}, \tilde{y}, 0) = 0$,

And for all (x, y, z) , $M_5(x, y, z) > 0$,

$$\frac{dM_5}{dt} < -\left\{\frac{s}{k} + \frac{q_2 n_4 E_2}{(n_3 E_2 + n_4 y)(n_3 E_2 + n_4 \hat{y})}\right\} (y - \hat{y})^2 - \frac{e_2 \alpha_2 \{1 - \gamma(1-m) y \hat{y}\}}{\{1 + \gamma(1-m) y^2\}\{1 + \gamma(1-m)\hat{y}^2\}} (y - \hat{y})(z -$$
$$\check{z}) - d(z - \check{z})^2 + \frac{rx}{1 + fz} + \left\{\frac{(e_1 \alpha_1 x + c_2 \alpha_2 y)}{1 + \gamma(1-m)\hat{y}^2}\right\} z$$

Consequently, conditions (53-55), implies that $\frac{dM_5}{dt} < 0$, in R_+^3. Also, Q_5 is asymptotically globally stable.

Theorem 6.7: Whenever satisfying conditions (41-44) in Theorem 5.7, tend the basin of attraction of $Q_6 = (\tilde{x}, \tilde{y}, \tilde{z})$, ensuring that it is globally asymptotically stable, as follows: the conditions hold.

$$\frac{q_1n_2E_1}{(n_1E_1+n_2x)(n_1E_1+n_2\tilde{x})}(x-\tilde{x})^2+\left\{\frac{\alpha_2\big(1-c_2(1-m)\big)}{\{1+\gamma(1-m)y^2\}\{1+\gamma(1-m)\hat{y}^2\}}\right\}(y-\tilde{y})(z-\tilde{z})<\left(\frac{s}{k}+\frac{q_2n_4E_2}{(n_3E_2+n_4y)(n_3E_2+n_4\tilde{y})}\right)(y-\tilde{y})^2+m(z-\tilde{z})^2+\left\{\frac{rf+\alpha_1g(1-c_1)}{(\delta x+\beta y+g)(\delta\tilde{x}+\beta\tilde{y}+g)}\right\}(x-\tilde{x})(z-\tilde{z}), \tag{56}$$

$$x>\tilde{x} \tag{57}$$

$$y>\tilde{y} \tag{58}$$

$$z>\tilde{z} \tag{59}$$

$$1>\max\{m,c_2\} \tag{60}$$

Proof: let $M_6=\left(x-\tilde{x}\,ln\frac{x}{\tilde{x}}\right)+\left(y-\tilde{y}\,ln\frac{y}{\tilde{y}}\right)+\left(z-\tilde{z}\,ln\frac{z}{\tilde{z}}\right)$**,** is a positive real value Lyapunov function

Where $M_6(t): R_+^3\to R$ is C^1 and satisfying that $Q_6=(\tilde{x},\tilde{y},\tilde{z})=0$,

And for all (x,y,z) , $M_6(x,y,z)>0$,

$$\frac{dM_6}{dt}\leq(x-\tilde{x})\left\{\frac{q_1n_2E_1(x-\tilde{x})}{(n_1E_1+n_2x)(n_1E_1+n_2\tilde{x})}-\frac{\{rf+\alpha_1g\}(z-\tilde{z})}{(\delta x+\beta y+g)(\delta\tilde{x}+\beta\tilde{y}+g)}\right\}-(y-\tilde{y})\left\{\left(\frac{s}{k}+\frac{q_2n_4E_2}{(n_3E_2+n_4y)(n_3E_2+n_4\tilde{y})}\right)(y-\tilde{y})+\frac{\alpha_2(z-\tilde{z})}{\{1+\gamma(1-m)y^2\}\{1+\gamma(1-m)\hat{y}^2\}}\right\}-(z-\tilde{z})\left\{m(z-\tilde{z})-\frac{e_1\alpha_1g(x-\tilde{x})}{(\delta x+\beta y+g)(\delta\tilde{x}+\beta\tilde{y}+g)}-\frac{e_2\alpha_2(1-m)(y-\tilde{y})}{\{1+\gamma(1-m)y^2\}\{1+\gamma(1-m)\hat{y}^2\}}\right\}$$

$$\frac{dM_6}{dt}\leq\frac{q_1n_2E_1}{(n_1E_1+n_2x)(n_1E_1+n_2\tilde{x})}(x-\tilde{x})^2-\left(\frac{s}{k}+\frac{q_2n_4E_2}{(n_3E_2+n_4y)(n_3E_2+n_4\tilde{y})}\right)(y-\tilde{y})^2-m(z-\tilde{z})^2-\left\{\frac{rf+\alpha_1g(1-c_1)}{(\delta x+\beta y+g)(\delta\tilde{x}+\beta\tilde{y}+g)}\right\}(x-\tilde{x})(z-\tilde{z})+\left\{\frac{\alpha_2\big(1-c_2(1-m)\big)}{\{1+\gamma(1-m)y^2\}\{1+\gamma(1-m)\hat{y}^2\}}\right\}(y-\tilde{y})(z-\tilde{z}),$$

Consequently, conditions (56-60), implies that $\frac{dM_6}{dt}<0$, in R_+^3. Also, Q_6 is asymptotically globally stable.

7. NUMERICAL ANALYSIS

The conception of the global dynamics of the system and to detect the influence of varying all the parameter values on its dynamic attitude. Employed time series and phase portrait, to analyze the solutions of the proposed biological system (1), which are presented via the MATLAB Program. All the results are based on the following initial points (4,5,2), (6,7,4), (5,6,3).

Table 1: hypothetical collection of proposed parameters:

parameter	r	f	α_1	δ	β	g	s	k	m	α_2	γ	e_1
value	0.8	0.5	0.6	0.3	0.2	0.5	0.6	8	0.5	0.15	0.1	0.4
	e_2	c	d	q_1	q_2	E_1	E_2	n_1	n_2	n_3	n_4	
	0.4	0.1	0.25	0.1	0.1	0.01	0.01	1	1	1	1	

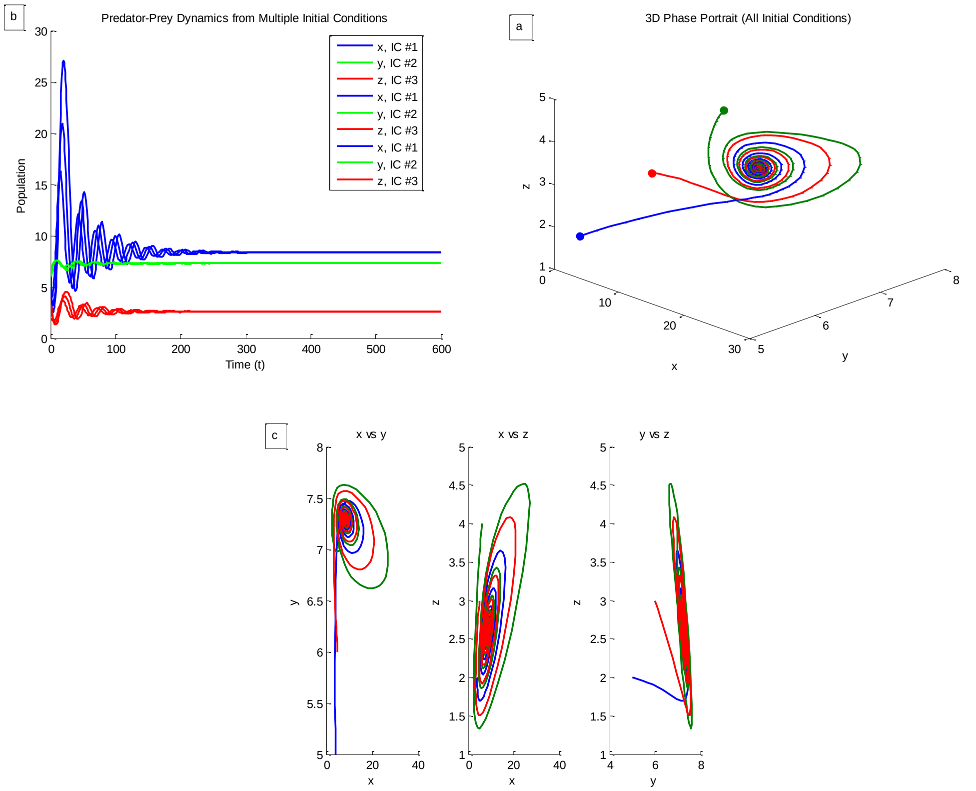

Figure.2. Employing both Table 1 and distinct initial points (4,5,2), (6,7,4), (5,6,3)., getting an asymptotically globally stable positive point of equilibrium Q_6 (a) $3D$-Phase sketch of the ecological model (1) (b) Time series of the first prey x, second prey y and the predator z trajectories respectively. (c) Clarify the projection on the y-z plane. Clarify the projection on the x-z plane. Clarify the projection on the x-y plane.

Figure2, illustrates that fear and refuge concerning the first prey and the second prey, respectively, stabilize the system; however, the influence of nonlinear predator harvesting does not destabilize it, but all species of the ecological system tend to the long-term coexistence.

The following scenario changes every parameter within the biological range that simulates the reality described for the proposed model, analyzes and discusses the results mathematically and biologically to provide a clearer understanding of the model's nature and its results.

Starting with varying $(0.001 \leq r < 0.01)$, the head asymptotically approaches Q_2 is illustrated in Figure 3a and Figure 3b, typically at $r = 0.001$. Biologically, the first prey's growth rate (r) is very low, so it cannot heal from the united effort of predation and harvesting. It dies out first. As a result, the predator population goes extinct because it is already under pressure of intra competition $(-cz^2)$ and natural death $(-dz)$, but the second prey (y) using refuge (m), which the predator was less dependent upon, is now safe from predation. This leads to a stabilizing population of the second prey at its own carrying capacity $(k), (0.01 \leq r < 0.015)$, asymptotically to Q_3 illustrated in Figure 3c and Figure 3d, typically at $r = 0.013$. Biologically, here (r): the first prey's growth rate is now nearly sufficient to remain alive under the harvesting pressure. However, the predator cannot maintain a growth rate and goes extinct because the energy it supplies from the first prey (x_1) (turns into e_1) is still too low, joint with the energy coming from predation of the second prey, to keep the predator population against the pressure of its natural death (d) and intra competition $(-cz^2)$. As a result, no common enemy of the two prey species populations exists, leading to survival of these two prey populations determined solely by their respective growth rates and harvesting parameters. $(0.015 \leq r < 0.92)$ head to asymptotically to Q_6 illustrated at Figure3e and Figure3f typically at $r = 0.25$.Biologically, this is an excellent zone for r. It is high enough to keep the first prey alive under the pressure of predation and harvesting, besides providing sufficient energy (via e_1) to support the existence of the predator population. While the second prey's refuge (m) plays a vital role in preventing being wiped out.Therefore, ecological results in a stable, complex food web. Which is the most ecologically diverse and flexible consequence. Finally $(0.92 \leq r < 1)$, the head asymptotically approaches Q_1, illustrated in Figure 3g and Figure 3h, typically at $r = 0.93$. Biologically (very high r), this Result asserts competitive insularity. The system collapses to only one creature with the fastest-growing; in other words, very high (r) can destroy biodiversity.

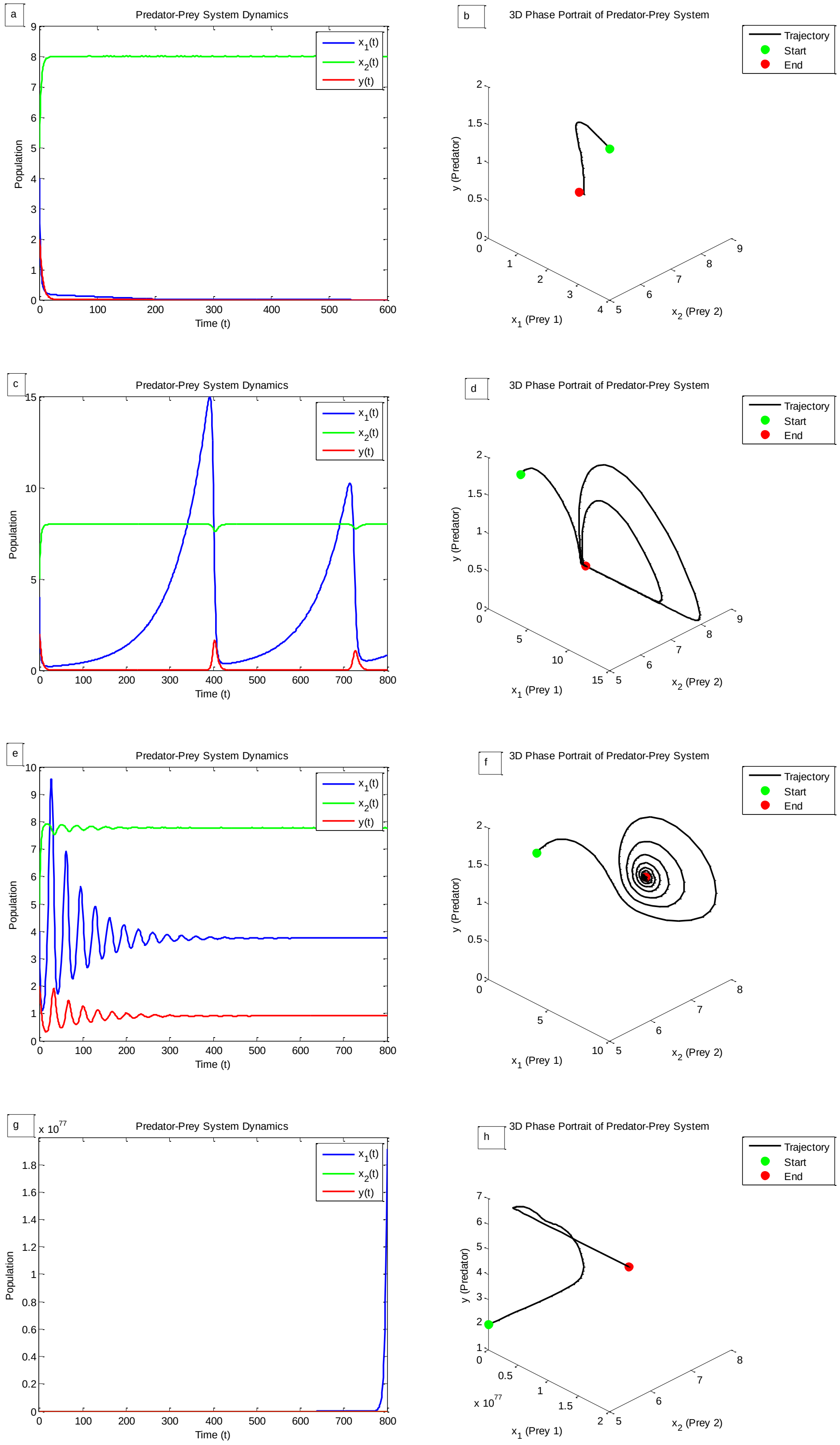

Figure 3. Time series and 3D plot employ Table 1of Model (1) with different values of r. (a) for

$r = 0.001$, approach stable point Q_2 (b) phase portrait in 3D of approach Q_2 (c) for r = 0.013, approach stable point Q_3 (b) phase portriat in 3D of approach Q_3 (e) for r = 0.25, approach stable point Q_6 (f) phase portrait in $3D$ of approach Q_6 (g) for $r = 0.93$, approach stable point Q_1; (h) phase portrait in $3D$ of approach Q_1.

- For $0.001 \leq \mathrm{f} < 0.41$, it heads asymptotically to Q_1 as visible in Figure 3g and Figure 3h, while at $0.41 \leq \mathrm{f}$, it heads asymptotically to Q_6 as visible in Figure 3e and Figure 3f. Biologically, this result's most outstanding feature is that the influence of fear from predation is a powerful force concerning the prey population, which makes the difference between a thriving ecosystem and a collapsed one. The fear response is an adaptive sign that serves at the group level of the prey population by averting overpopulation and its tragic consequences.

Now, about the above results of the growth rate of the first prey the next figure below illustrates varying of the parameters (r) and (f) concerning it.

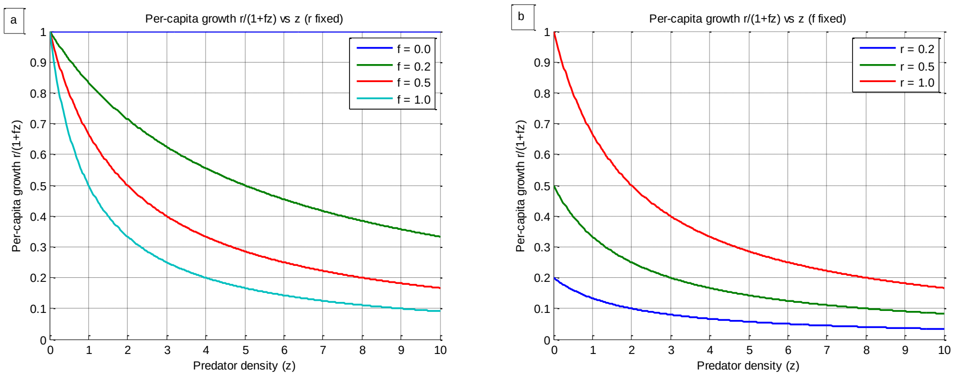

Figure 4: (a) At different values of the fear (f), plot of pre capita growth and effect on the predator (z), (b) At different values of the intrinsic growth rate (r), plot of pre capita growth and effect on the predator (z)

Fear parameter (f) illustrated in Figure 4a. Rising fear drops the influential growth of the first prey, curtailing the food available to the predator. Therefore, the predator population declines or becomes extinct because it cannot obtain enough energy.

Intrinsic growth rate (r) of the first prey is illustrated in Figure 4b. At a small value of (r), the first prey face a strong impact of both predation and harvesting, which implies predator extinction.

While at an intermediate value of (r), the first prey supplies enough energy for predator survival, which implies stable coexistence in all species populations. Finally, at a very high value of (r), the first prey heavily competes to tend the system, destabilizing biodiversity.

- For $0.001 \leq \alpha_1 < 0.58$, it heads asymptotically to Q_1 as Figure3g, while at $0.58 \leq \alpha_1 < 1$, it heads asymptotically to Q_6 as Figure3e. In essence, the biological result emphasizes an essential principle in ecology, often to the detriment of the predator itself, when a predator is sufficiently influential and effective at hunting to vindicate its place in the ecosystem. Conversely, an inefficient predator fails to perform its regulatory role, leading to an unstable system that falls into a less assorted state.
- For $0.001 \leq \delta < 0.32$, it heads asymptotically to Q_6 as Figure 3e, while at $0.32 \leq \delta < 3$, it heads asymptotically to Q_1 as Figure 3g. The parameter δ represents a double role as a measure of a prey's edibility, firstly, and the ecosystem is confirmed to be complex and stable, secondly, which suggests that the predator is an effective hunter. When (high δ) indicates a prey species that is hard or time-consuming to eat, it not only acquires a powerful defensive feature but also eventually starves the predators that rely on it, leading to a fall in the entire predatory relationship.
- For $0.001 \leq \beta < 0.11$, it heads asymptotically to Q_1 as Figure 3g, while at $0.11 \leq \beta < 0.25$, it heads asymptotically to Q_6 as Figure 3e, but again returns to Q_1 at $0.25 \leq \beta < 3$, as Figure 3g.
- For $0.001 \leq g < 0.81$, it heads asymptotically to Q_6 as Figure 3e, while at $0.81 \leq g < 5$, it heads asymptotically to Q_1 as Figure 3g. This parameter (g) emphasizes the importance of foraging attitude and acclimation in preserving ecological balance. So (high g) illustrated that a predator must be an expert searcher to vindicate its role in the ecosystem. Conversely, bad searching ability leads to predator starvation and, therefore, collapses of the entire food web.

Consequently, figure.5 illustrated the effect of varying the parameters $(\alpha_1, \beta, \delta, g)$ with the consumption function with respect to the first prey.

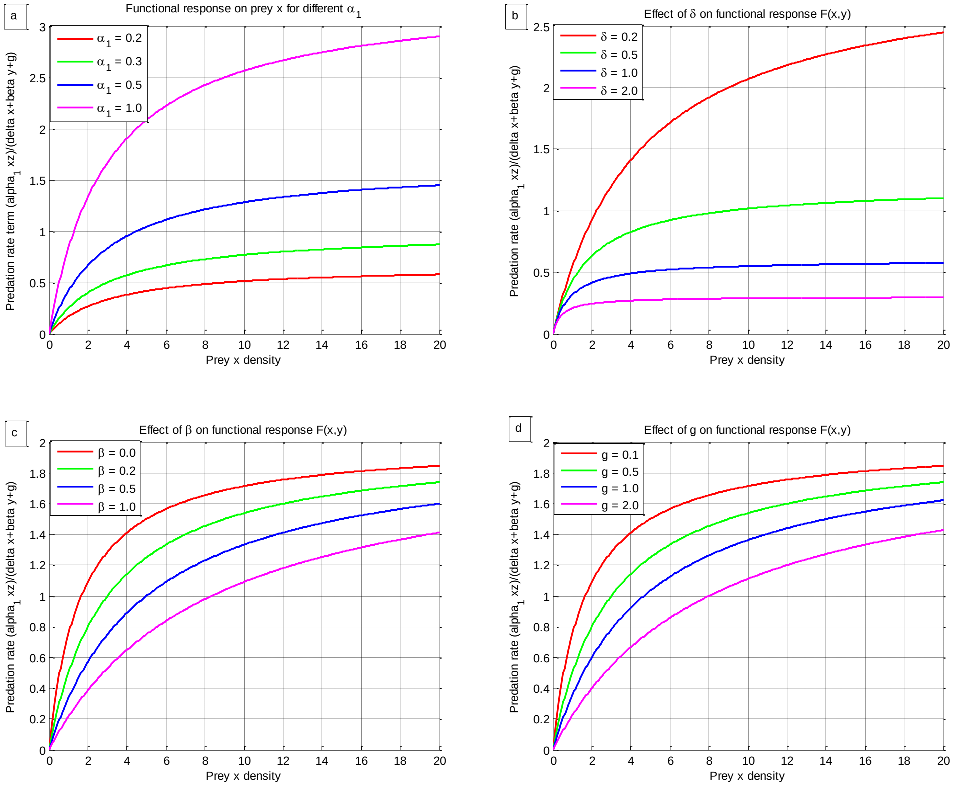

Figure 5: (a) At different values of (α_1), plot of predation term ($\frac{\alpha_1 xz}{\delta x+\beta y+g}$) and effect on the prey (x). (b) At different values of (δ), plot of predation term ($\frac{\alpha_1 xz}{\delta x+\beta y+g}$) and effect on the prey (x). (c) At different values of (β), plot of predation term ($\frac{\alpha_1 xz}{\delta x+\beta y+g}$) and effect on the prey (x). (d) At different values of (g), plot of predation term ($\frac{\alpha_1 xz}{\delta x+\beta y+g}$) and effect on the prey (x).

Biological meaning of Predation strength and efficiency in Figure 5. When predation parameters $(\alpha_1, \beta,\ \delta, g)$ are low, the first prey reaches near its natural carrying capacity, and the predator cannot preserve a stable population. While intermediate parameter values, predation adjusts the first prey, and energy moves to the predator, allowing coexistence. Finally, with high parameter values, the first prey becomes heavily exploited, which can cause the predator's hunger and destabilize the system.

While, Figure 6 illustrated effect of varying (q_1, E_1, n_1, n_2) with respect of the non-linear harvesting function $(\frac{q_1\, E_1}{n_1 E_1 + n_2 x})$ of the first prey.

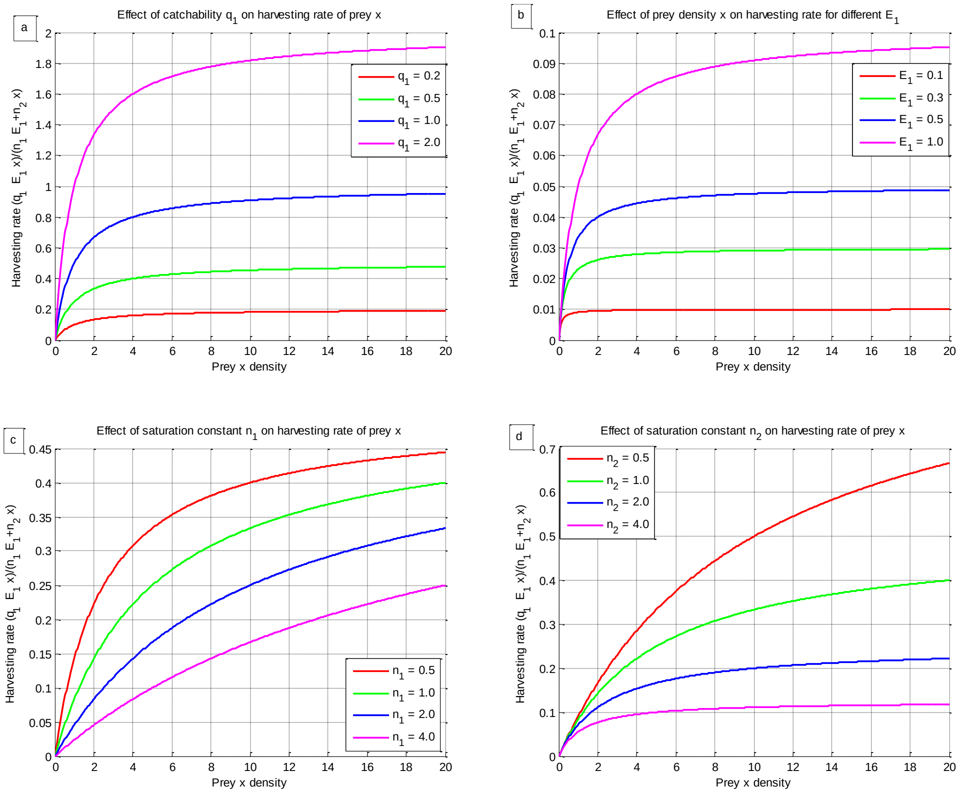

Figure 6: Plot of the non-linear harvesting $(\frac{q_1\, E_1}{n_1 E_1 + n_2 x})$ with respect to the first prey (x) (a) At different values of (q_1), plot of non-linear harvesting $(\frac{q_1\, E_1}{n_1 E_1 + n_2 x})$ and effect on the prey (x). (b) At different values of (E_1), plot of non-linear harvesting term $(\frac{q_1\, E_1}{n_1 E_1 + n_2 x})$ and effect on the prey (x). (c) At different values of (n_1), plot of non-linear harvesting term $(\frac{q_1\, E_1}{n_1 E_1 + n_2 x})$ and effect on the prey (x), d) At different values of (n_2), plot of non-linear harvesting term $(\frac{q_1\, E_1}{n_1 E_1 + n_2 x})$ and effect on the prey (x).

Biological meaning of Figure 6: Investigate the effect of the nonlinear harvesting act on the first prey. With low nonlinear harvesting pressure, the first prey survives and can conserve both the predator and the second prey in the ecological system. When a moderate value of the non-linear harvesting pressure is used, the first Prey decreases in amplitude without going extinct. Finally, the high pressure of nonlinear harvesting on the first Prey prevents it from conserving itself and breaks down, which makes the predator lose its main food source, and also goes extinct.

- For $0.001 \leq s < 0.34$, it heads asymptotically to Q_1 as in Figure 3g, while at $0.34 \leq s < 3$, it heads asymptotically to Q_6 as in Figure 3e. Biologically, in essence, this result explains that the health of a full predator population can rely on the vitality of the second prey, which is not excessive; it supports the apex of the proposed model. Its ability to grow betimes is sufficient as a fundamental necessity for a complex, stable ecosystem.
- For $1 \leq \mathrm{k} < 6$, it heads asymptotically to Q_1 as Figure 3g, while at $6 \leq \mathrm{k} < 10$, it heads asymptotically to Q_6 as Figure 3e, but again return to Q_1 at $10 \leq \mathrm{k}$

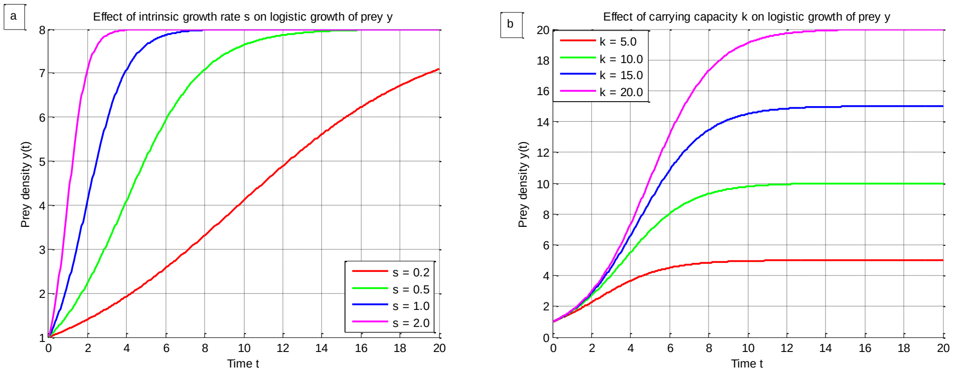

Figure 7: Plot of growth rate of the second prey (y). (a) At different values of (s), plot of effect (s) on logistic growth of the prey (y). (b) At different values of (k), plot of effect (k) on logistic growth of the prey (y).

Biological Interpretation of Figure 7a illustrated the impact of varying the (growth rate s) of the second prey on its population dynamics. With a low growth rate of the second prey, despite refuge protection, it is unable to recover from predation. By providing a steady food source for the predator, the Intermediate growth rate of the second prey preserves a sustainable population. But the high growth rate of the second prey can grow rapidly, even under predation. Figure 7b, (k) regulates the scope to which the second prey can influence the system. Predator extinction occurs at a low value of k, while moderate values (k) encourage the coexistence of all species of the system, but risks of competitive exclusion come from a high value (k)

Figure 8 illustrated effect of varying (q_1, E_1, n_1, n_2) with respect of the non-linear harvesting function $(\frac{q_1 E_1}{n_1 E_1 + n_2 x})$ of the first prey.

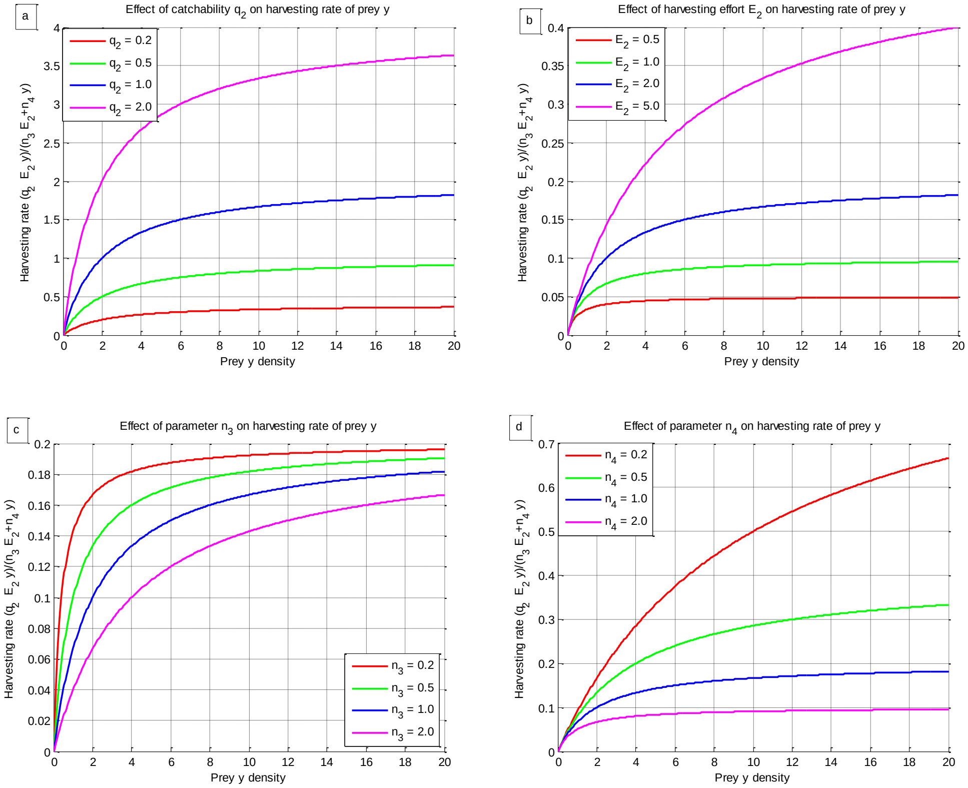

Figure 8: Plot of the non-linear harvesting with respect to the second prey, (a) At different values of (q_2), plot of non-linear harvesting term $(\frac{q_2 E_2 y}{n_3 E_2 + n_4 y})$ and effect on the prey (y). (b) At different values of (E_2), plot of non-linear harvesting term $(\frac{q_2 E_2 y}{n_3 E_2 + n_4 y})$ and effect on the prey (y). (c) At different values of (n_3), plot of non-linear harvesting term $(\frac{q_2 E_2 y}{n_3 E_2 + n_4 y})$ and effect on the prey (y). (d) At different values of (n_4), plot of non-linear harvesting term $(\frac{q_2 E_2 y}{n_3 E_2 + n_4 y})$ and effect on the prey (y).

Figure 8 illustrates that the nonlinear harvesting of the second prey seriously controls the ecosystem consequences. Low harvesting permit coexistence, but the collapse of the second prey with predator extinction, a high harvesting level leads to the persistence of the first prey.

- For $0.001 \leq \alpha_2 < 0.11$, it heads asymptotically to Q_1 as Figure 3g, while at $0.11 \leq \alpha_2 < 0.27$, it heads asymptotically to Q_6 as Figure 3e. $0.27 \leq \alpha_2 < 0.69$, It heads asymptotically to Q_1 as Figure 3g, while at $0.69 \leq \alpha_2 < 0.83$, it heads asymptotically to Q_6 as Figure 3e. Furthermore, $0.83 \leq \alpha_2 < 1$, it heads asymptotically to Q_5 illustrate in Figure 9a and Figure 9b.
- For $0.0001 \leq m < 0.83$, it heads asymptotically to Q_6 as Figure 3e while at $0.83 \leq m < 1$, it heads asymptotically to Q_1 as Figure3g.
- For $0.001 \leq \gamma < 0.17$, it heads asymptotically to Q_6 as Figure 3e while at $0.17 \leq \gamma < 2$, it heads asymptotically to Q_1 as Figure 3g. (γ) . Represent the level of defense concerning refuge of the second prey. At a higher value of (γ), it means a stronger anti-predator (refuge) gain for higher densities in a refuge space, which indicates for predators (z) the energy gain from consuming prey (y), is reduced, possibly lowering predator population growth, but conversely, for the prey (y), it increases the chance of persistence in the system.

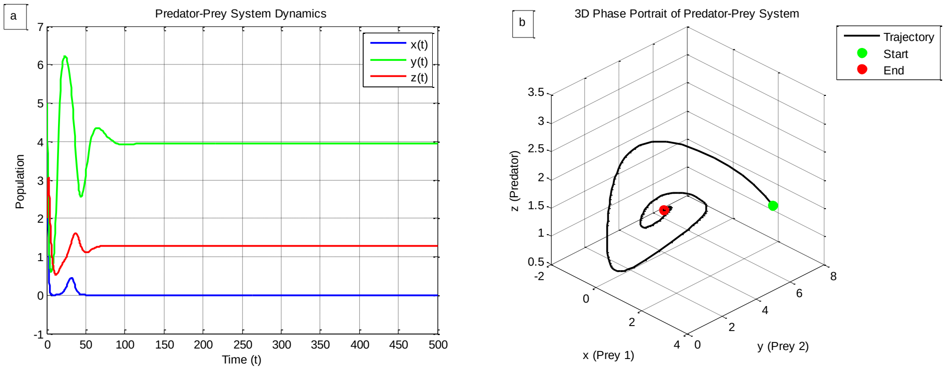

Figure 9: Time series and 3D plot employ Table 1 of Model 1 with varying value of α_2. (a) At $\alpha_2 = 0.85$, approach stable point Q_5, (b) Phase portrait in 3D of approach Q_5.

Figure 10 illustrated effect of varying the parameters (α_2, m, γ) of the predation function of the second prey by the predator (z).

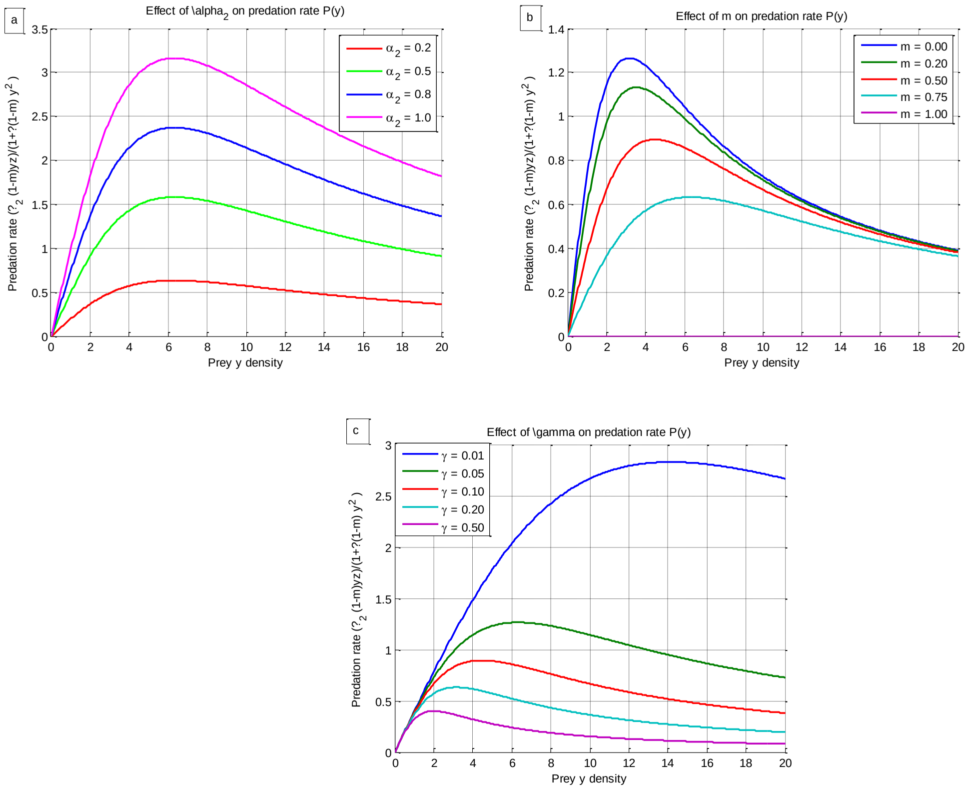

Figure 10: Plot of the predation function of the second prey by the predator (z). (a) For different values of (α_2). (b) For different values of (m). (c) For different values of (γ).

Even with copious amounts of the second prey in the system, the lower value of the predation efficiency (〖α〗 _2) in Figure 10a means the predator cannot gain sufficient energy. This implies that the Predator declines or goes extinct, and the second prey species dominates. With an intermediate value, the second prey supplies sufficient energy to the predator, which implies that this supports the coexistence of all species in the system. Finally, a high value of the predation efficiency heavily exploits the second prey. Which implies that the second Prey grows smaller, ultimately causing the food base to disappear for the predator.

The refuge (m) in Figure 10b acts as an anti-predator defense. A moderate value of the refuge strategy stabilizes ecosystems by avoiding over-predation, but a higher refuge value threatens predator survival.

(γ): Is the refuge defense level of the (second prey y) against the (predator z). It represents the effectiveness of the refuge in keeping the second prey away from predation. While m acts as the basic amount of prey use refuge, γ strongly regulates the effect of this refuge protection, which scales with prey density. A low (γ) in Figure 10c indicates that it favors predators, a high (γ) level favors prey, and finally, a moderate (γ) level balances the system for coexistence.

- For $0.001 \leq e_1 < 0.39$, it heads asymptotically to Q_1 as Figure 3g, while at $0.39 \leq e_1 < 1$, it heads asymptotically to Q_6, as Figure 3e. Biologically, this result shows that it is not adequate for prey to be plentiful; they must also be a sustaining and applicable food source for the predator. The matching of biochemical and physiological compatibility between a predator and its prey, represented by (e_1), is a basic determinant of whether a complex ecosystem will collapse into a simpler state.
- For $0.001 \leq \mathrm{e}_2 < 0.24$, it heads asymptotically to Q_1 as Figure 3g, while at $0.24 \leq \mathrm{e}_1 < 1$, it heads asymptotically to Q_6 as Figure 3e. Biological interpretation same for e_1.
- For $0.0001 \leq d < 0.28$, it heads asymptotically to Q_6 as Figure 3e, while at $0.28 \leq d < 1$, it heads asymptotically to Q_1 as Figure 3g. Normally, at a low rate of death, the system tends to a coexistence point, while the converse is true. At low densities of the predator population, only the (d) effect in population declines, but not too fast. Conversely, at higher densities of the predator population, natural death plus intraspecific competition cause declines to crash steadily faster due to competition.
- For $0.001 \leq c < 0.08$, it heads asymptotically to Q_1 as Figure 3g, while at 0.08≤c<0.11, it heads asymptotically to Q_6as Figure 3e, but again returns to Q_1 at 0.11≤c<1. Biologically, Density effects strongly limit biological growth at very low c, predators, so the system tends to Q_1(absence of the predator). At intermediate (c), crowding becomes important, pushing the system into coexistence ecological Q_6. At higher (c), intraspecific competition is so strong that predator persistence is heavily limited, causing the system back return to Q_1 .

Figure 11 illustrates the effect of varying the conversion rate of food consumption by the predator that relies on both of the prey populations and the effect of the natural death of (z) and intraspecific competition among the predator individuals of the population. An extra plot illustrated the effect of the natural death of (z), and merged with the intraspecific competition of the predator population.

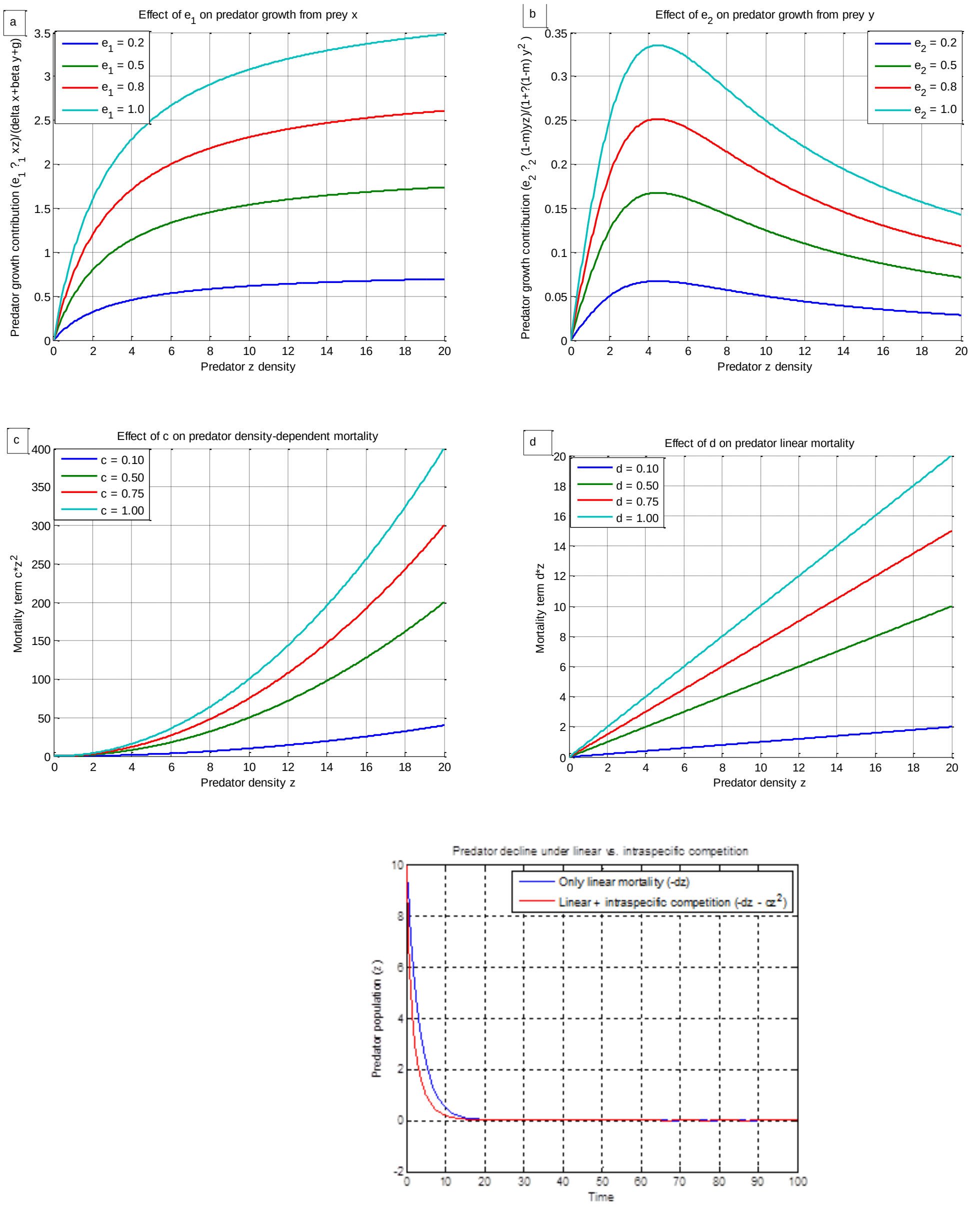

Figure 11: Plot of the conversion of food to the predator (z) via the two prey populations with the parameters of decay. (a) For different values of (e_1). (b) For different values of (e_2). (c) For different values of (c). (d) For different values of (d). (e) Comparison plot between natural death and intraspecific competition of the predator (z).

Conversion efficiency of food $(e_1$ and $e_2)$ in Figure (11a-11b), to the predator (z). At low values, the translation of food to the predator's growth cannot be beneficial. Implies predator population declines. Predators can grow while still allowing prey to persist at intermediate efficiency levels, which implies that this produces a balanced coexistence among all the species. Finally, overexploited prey occurs at very high efficiency levels, which implies that Prey collapse follows, and predator starvation eventually destabilizes the system. Even if prey are abundant, at higher natural death (d) in Figure 11d, of the predator (z), predator persistence is reduced; conversely, at lower levels (d), it allows predator survival, and the supply of energy from prey is sufficient. Intraspecific competition among predator populations is (c) in Figure 11c. So, at a low level of (c), meaning minimal crowding pressure, predator growth relies mainly on prey supply. But the excellent level occurs at moderate(c), with double benefit; competition balances predator numbers and prevents overexploitation of prey. Even when prey is available. At a high level of (c), competition limits predator survival .

The combined effect of $(d$ and $c)$ in Figure 11e acts together as self-control mechanisms for the predator population. Their balance plays a crucial role in whether the predator persists with both prey and collapses.

The following parameters, no matter how their values change, remain close to the positive point Q_6 as Figure3e in the system without any difference in the closeness, i.e no effect on the system: $(q_1 , q_2, E_1, E_2, n_i \; for \; i = 1,2,3,4)$.

Merge the plot of both effect of (α_1 , α_2) against $F_x = \frac{\alpha_1 xz}{\delta x+\beta y+g}$ and $F_y = \frac{\alpha_2(1-m)yz}{1+\gamma(1-m)y^2}$ respectively as illustrated in figure 12.

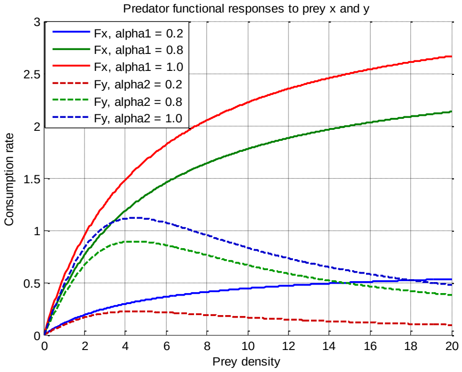

Figure 12: At different values of (α_1 , α_2) the effect of these parameters on the consumptions functions F_x (solid lines) and F_y (dash lines)

Biological interpretation of Figure 12 explains the joint balance of predation on both prey species and the ecosystem stability. Stable consumption implies the coexistence of all species in the system, while relying on a single prey increases sensitivity and limits biodiversity.

8. CONCLUSION

In this work, we proposed and analyzed a mixed model of the merge impact of prey fear concerning the first prey, prey refuge concerning the second prey, in the presence of nonlinear predator harvesting on both of the prey populations, and two different functional responses Beddington–DeAngelis and Holling type IV for the first and second prey, respectively on an ecological system contain predator–two prey. Via strict mathematical analysis, we determined the existence of boundedness, positivity, and equilibria, accompanied by studying stability conditions under various parameter regimes of both local and global stability, with the assistance of the Lyapunov functions and Jacobian analysis, which confirmed under biologically feasible conditions that the system possesses globally asymptotically stable equilibria.

After all these analyses, further support of the numerical simulations asserts the analytical results, focusing attention on the serious roles of prey defense strategies, predator predation merit, and harvesting consistency in foundry system dynamics. Specifically, a fear effect on the first prey and refuge effect on the second prey influence to stabilize the system and permit long-term coexistence. While the influence of nonlinear predator harvesting did not destabilize population persistence. Revealed a spectrum of ecological outcomes ranging from parameter variations from extinction of one or more species to coexistence equilibria and, at extreme values, collapse of biodiversity.

Biologically, in preserving ecosystem stability, these findings confirm that indirect effects such as fear and refuge are as pivotal as direct predation. Moderate levels of predation, harvesting, and refuge reinforce resilience and keep biodiversity; on the other hand, extravagant growth destabilizes the .Ecosystems, balance complexity with persistence between the interplay of functional responses, and nonlinear harvesting illustrates.Overall, this work is particularly relevant to theoretical ecology by merging various ecological mechanisms within a proposed system, presenting insights into how activities (harvesting) and natural animal attitude (prey defense strategies) jointly determine ecosystem outcomes.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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