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# THE STABILITY AND RANDOM ATTRACTORS OF THE STOCHASTIC MODEL DESCRIBING THE ONCOLYTIC VIROTHERAPY WITH TUMOR-VIRUS INTERACTIONS

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**Abstract:** The treatment of cancer tumors varies based on cancer type, stage, location, and patient health. This research focuses on viral therapy, which employs viruses that target and eliminate malignant cells while sparing healthy ones. A stochastic model investigates suicidal viral tumor therapy, emphasizing the interaction between tumorous and non-tumorous cells. The study analyzes the stochastic dynamics and stability of the system's equilibrium points through stochastic differential equations. Key model factors affecting viral treatment outcomes are identified through bifurcation parameters and sensitivity analysis, validated by numerical simulation.

**Keywords:** cancer, oncolytic virotherapy; stability; stochastic differential equations; random dynamical systems; stochastic stability; stochastic Lyapunov function.

**2020 AMS Subject Classification**: 37H10, 37N25.

## 1. INTRODUCTION

There has been a lot of interest in viruses as potential tumor-destroying agents since the late 1880s. Oncolytic virus history reveals that physicians have noted that some patients with cancer do experience remission following viral infection [1]. Therefore, the viral therapy method has received significant attention from researchers interested in studying cancer tumor treatments. The chief problem that the researchers sought to answer is how to undermine the ability of those viruses that cause the disease so that they convert appropriate as medicines. It turns out that

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viruses can kill cancer patients' tumor tissues in the right circumstances. It has been discovered that tumor tissues sustain far more damage than normal host tissues.

The majority of these viruses were deemed unsafe for use in cancer treatment due to their pathogenicity. However, most viruses can have their pathogenicity removed without losing their oncolytic effectiveness thanks to adaptability and genetic engineering approaches [2].

A type of immunotherapy called oncolytic virotherapy uses certain viruses to target and infect tumor cells, causing them to proliferate and die. Viruses multiply inside infected tumor cells during this process, releasing more virion particles that can infect more tumor cells. Because the viruses utilized in this treatment can only replicate in cancer cells, healthy cells are largely unaffected.By infecting neighboring or distant tumor cells, the freshly released viruses from the lysed cells may initiate many cycles of infection.

Considerably work has recently been done to comprehend the molecular mechanics and dynamics of oncolytic virus cytotoxicity. These initiatives offered an intriguing potential substitute treatment strategy to aid in the recovery of cancer patients.

However, the outcomes of virotherapy are complicatedly influenced by both the immune response and the virus-cancer interaction [3-7]. The majority of cancer treatments currently in use were created empirically [8].

However, several mathematical models have recently been developed to describe the outcome of such interactions. [9–15]. Other models and techniques are being developed to study the dynamics of virotherapy [16–22].

This may not be the case, though, as it is obvious that local connections as well as the spatial component play a significant role in population expansion (see, for example, [12]). Several computational models of virotherapy were created with particular standard mathematical frameworks, like diffusion-reaction models and Lotka-Volterra models, which typically assume that populations are closely mixed. In general, there hasn't been any experimental evidence to support many of the modeling techniques used today.

This problem has been addressed by developing an in vitro computational model which may be explicit represent the relationships between viral and tumor populations [3,10]. They explored several important characteristics of oncolytic viruses using the computational model after informing the model parameters using in vitro 2D and 3D data. They demonstrated how adding a third dimension drastically changes the dynamics, which has substantial effects on how well

therapy works. Nevertheless, their research did not examine the model's mathematical and qualitative characteristics. The qualitative characteristics of the model from [3,10] are examined in [23], in addition to the behavior of the solution and the stability of rest points. Significant features of the model make it intriguing from a mathematical and clinical standpoint. In [24], Under particular circumstances, the study's stochastic model for the growing of malignant cancers through targeted chemotherapy reveals a stable tumor-free equilibrium state. The paper provides a thorough mathematical analysis of the cancer-virus interaction by examining equilibrium points stability, sensitivity analysis, parameter impacts, and numerical simulations. The results are compared with previous research.

This is how the paper is structured. A comprehensive overview of random dynamical systems and stochastic differential equations is given in Section 2. Section 3 introduces the system of stochastic differential equations that describes our situation and establishes its parameters. Section 4 shows the boundedness and positive invariance of the stochastic model. In Section 5, the stochastic model's long behavior is examined. Section 6 looks at the random attractors and stability analysis of the random dynamical system produced by the stochastic model. Section 7 displays the model's numerical simulations and sensitivity analysis. Finally, conclusions and remarks are offered in Sections 8 and 9.

## 2. PRELIMINARIES

This section focuses on presenting some basic concepts related to stochastic differential equations and stochastic dynamic systems relevant to the research topic.

Let $\{\mathfrak{F}_t\}_{t\geq 0}$ be a filtration of a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and $\mathfrak{B}(t) = (\mathfrak{B}_1(t), \dots, \mathfrak{B}_m(t))^T$, $t \geq 0$ be an $m-$dimensional Brownian motion defined on $(\Omega, \mathfrak{F}, \mathbb{P})$. Let $\mathfrak{f}: \mathfrak{R}^d \times [t_0, T] \longrightarrow \mathfrak{R}^d$ and $\mathfrak{g}: \mathfrak{R}^d \times [t_0, T] \longrightarrow \mathfrak{R}^{d\times m}$ be two Boral measurable functions. Let

$$dx(t) = \mathfrak{f}(x(t), t)dt + \mathfrak{g}(x(t), t)d\mathfrak{B}(t) \text{ on } t_0 \leq t \leq T \tag{1}$$

be the Ito stochastic differential equation of $d-$ dimensional with initial value $x(t_0) = x_0$, where $x_0: \Omega \longrightarrow \mathfrak{R}^d$ be an $\mathfrak{F}_{t_0}$_measurable with $\mathbb{E}|x_0|^2 < \infty$. Equation (1) is equivalent to the equation:

$$x(t) = x_0 + \int_{t_0}^{t} \mathfrak{f}(x(s), s)ds + \int_{t_0}^{t} \mathfrak{g}(x(s), s)d\mathfrak{B}(s) \quad \text{on } t_0 \leq t \leq T \tag{2}$$

For more detail see [25].

**Definition 2.1 [25]**: The solution of the equation (1) is the stochastic process $\{x(t)\}_{t_0\le t\le T}$, where $x(t)\in\Re^d$ such that

(a) $\{x(t)\}$ is $\mathfrak{F}_t$- adapted and continuous;

(b) $\{\mathfrak{f}(x(t),t)\}\in\mathcal{L}^1([t_0,T];\Re^d)$ and $\{\mathfrak{g}(x(t),t)\}\in\mathcal{L}^2([t_0,T];\Re^{d\times m})$;

(c) equation (2) holds for every $t\in[t_0,T]$ with full measure.

**Remark 2.2 [25]:** From (2), we have

$$x(t)=x(s)+\int_s^t \mathfrak{f}(x(r),r)dr+\int_s^t \mathfrak{g}(x(r),r)d\mathfrak{B}(r) \text{ on } s\le t\le T,\ s\in[t_0,T] \tag{3}$$

**Definition 2.3 [26]:** A collection of measurable actions $\{\theta_t:\Omega\longrightarrow\Omega,t\in\Re\}$ is measurable dynamical system (MDS) if every member in $\mathfrak{F}$ is $\mathbb{P}-$invariant.

**Definition 2.4 [26]:** Consider the function

$$\varphi:\Re\times\Omega\times\mathcal{X}\longrightarrow\mathcal{X},\ (t,\omega,x)\mapsto\varphi(t,\omega,x)\ ,$$

with the following axioms:

(i) for every $t\in\Re$ and $\omega\in\Omega$ the function $x\mapsto\varphi(t,\omega,x)\equiv\varphi(t,\omega)x$ is continuous,

(ii) the function $\varphi(t,\omega):=\varphi(t,\omega,\cdot)$ satisfy:

$$\varphi(0,\omega)x=x,\ \varphi(t+s,\omega)x=\varphi(t,\theta_s\omega)\circ\varphi(s,\omega)x.$$

Then the pair $(\theta,\varphi)$ is called random dynamical system (RDS).

**Definition 2.5 [26]:** The RDS $(\theta,\varphi)$ is said to be affine if $\mathcal{X}$ is a linear Polish space and

$$\varphi(t,\omega)x=\Phi(t,\omega)x+\psi(t,\omega) \tag{4}$$

where $\Phi(t,\omega)$ is a co-cycle over $\theta$ and the function $\psi:\Re\times\Omega\longrightarrow\mathcal{X}$ is a measurable. The RDS $(\theta,\varphi)$ is called linear whenever $\psi(t,\omega)\equiv 0$ and it is written by LRDS. If $(\theta,\Phi)$ is a LRDS, then $\varphi$ in (4) is corresponding to

$$\varphi(t+s,\omega)=\Phi(t,\theta_s\omega)\psi(s,\omega)+\psi(t,\theta_s\omega),\ t,s\ge 0 \tag{5}$$

**Definition 2.6** [26]: Let $\mathcal{X}\equiv(\mathcal{X},d)$ be a metric space. If the function $\psi:\Omega\longrightarrow\Re$, given by $\psi(\omega):=\mathrm{dist}_{\mathcal{X}}(x,\mathcal{D}(\omega))$, is measurable for each $x$, then a multifunction $\mathcal{D}:\Omega\longrightarrow 2^{\mathcal{X}}/\{\emptyset\}$ is a random set. $\mathcal{D}$ is considered closed (rep. compact, bounded) if $D(\omega)$ is closed (compact, bounded) in $\mathcal{X}$ for $\omega$.

**Definition 2.7 [26]**: The random set $\mathcal{D}$ is called tempered if there exist $r:\Omega\longrightarrow\Re$ and $y\in\mathcal{X}$ with $\mathcal{D}(\omega)\subset\{x:d(x,y)\le r(\omega)\}$ and

$$\sup_{\tau\in\Re}\{e^{-\alpha|\tau|}|r(\theta_\tau\omega)|\}<\infty \quad \text{for every } \alpha>0 \text{ and } \omega\in\Omega. \tag{6}$$

**Definition 2.8 [26]:** A collection of closed random sets that are closed under inclusions called the universe of sets.

**Definition 2.9 [26]:** A member $\mathcal{B}$ in the universe $\mathfrak{D}$ is said to be absorbing for $(\theta, \varphi)$, if there is a $t_0(\omega)$ with $\varphi(t, \theta_{-t}\omega)\mathcal{D}(\theta_{-t}\omega) \subset \mathcal{B}(\omega)$ for all $t \geq t_0(\omega)$, for all $\mathcal{D} \in \mathfrak{D}$ and $\omega \in \Omega$.

**Definition 2.10 [26]:** An RDS $(\theta, \varphi)$ is dissipative in $\mathfrak{D}$ if there exist a random variable $r(\omega)$, $x_0 \in \mathcal{X}$ and absorbing set $\mathcal{B}$ in $\mathfrak{D}$ such that $\mathcal{B}(\omega)$ containing in the random closed ball $\mathcal{B}_{r(\omega)}(x_0)$.

**Definition 2.11 [26]:** A random equilibrium of $(\theta, \varphi)$ is a measurable function $\mathrm{u}: \Omega \mapsto \mathcal{X}$ with

$$\varphi(t, \omega)u(\omega) = u(\theta_t\omega), \ t \geq 0, \ \omega \in \Omega.$$

**Definition 2.12 [26]:** The Lyapunov exponent for $(\theta, \varphi)$ is the smallest number $\lambda$ satisfy :

$$\lambda(\omega, x) := \lim_{t \to +\infty} \frac{1}{t} \log \|\varphi(t, \omega)x\|, \ \omega \in \Omega^*, t > 0 \tag{7}$$

where $\mathbb{P}(\Omega^*) = 1$.

**Proposition 2.13 [26]:** Suppose that $\mathfrak{D}$ is a universe with the following properties:

(i) $\{0\} \in \mathfrak{D}$,

(ii) $\lambda\mathcal{D}(\omega) := \{x: x\lambda^{-1} \in \mathcal{D}(\omega)\} \in \mathcal{D}$ for every $\mathcal{D}(\omega) \in \mathfrak{D}$ and $\lambda > 0$,

(iii) $\mathfrak{D}$ containing an attracting compact random set $\mathcal{B}_0(\omega)$ of asymptotically compact affine RDS $(\theta, \varphi)$.

Then $u(\omega) := \lim_{t \to +\infty} \psi(t, \theta_{-t}\omega)$ exists (it is an equilibrium) and $u(\omega)$ is globally asymptotically stable (GAS), that is,

$$\lim_{t \to +\infty} \sup_{v \in \mathcal{D}(\theta_{-t}\omega)} \|\varphi(t, \theta_{-t}\omega)v - u(\omega)\| = 0, \ \mathcal{D} \in \mathfrak{D}.$$

**Proposition 2.14.[26]** If the LRDS $(\theta, \Phi)$ admits a negative top Lyapunov exponent $\lambda$ as well as for every $\omega \in \Omega$ there is a tempered compact set $\mathcal{B}_0(\omega)$ with $\lim_{t \to \infty} \sup_{b \in \mathcal{B}_0(\omega)} \|\psi(t, \theta_{-t}\omega) - b\| = 0$. Then

$$u(\omega) := \lim_{t \to +\infty} \psi(t, \theta_{-t}\omega) \in \mathcal{B}_0(\omega), \ \omega \in \Omega^*.$$

Additionally, $u(\omega)$ is the unique almost surely equilibrium on $\Omega^*$ and

$$\lim_{t \to +\infty} \{e^{\gamma t} \sup_{v \in \mathcal{D}(\theta_{-t}\omega)} \|\varphi(t, \theta_{-t}\omega)v - u(\omega)\|\} = 0, \quad \omega \in \Omega^*, \tag{8}$$

where $\mathcal{D} \subset \mathcal{X}$ is a tempered closed random set and $\gamma < -\lambda$.

**Proposition 2.15 [25]:** Consider the SDE

$$dx = (\alpha x + \gamma)dt + (\beta x + \delta)d\mathfrak{B},\ x(0) = x_0 \tag{9}$$

where $\mathfrak{B}$ is a standard Brownian motion and $\alpha, \beta, \gamma,$ and $\delta$ are real . The function

$$x(t,\omega) = \psi(t)(x_0 + (\gamma - \beta\delta)\int_0^1 \frac{1}{\psi(s)}ds + \delta\int_0^1 \frac{1}{\psi(s)}d\mathfrak{B}(s)) \tag{10}$$

form a solution of (9), where

$$\psi(t) = \exp\left(\left(\alpha - \tfrac{1}{2}\beta^2\right)t + \beta\mathfrak{B}(t)\right) \tag{11}$$

Scalar linear SDEs with multiplicative noise are common in financial applications because they may be used to simulate strictly positive processes.

**Theorem 2.16 [25]** The solution $x(t)$ of the nonhomogeneous linear SDE

$$dx(t) = [\mathfrak{f}_1(t) + \mathfrak{f}_2(t)x(t)]dt + [\mathfrak{g}_1(t) + \mathfrak{g}_2(t)x(t)]d\mathfrak{B}(t) \tag{12}$$

can be written

$$x(t) = x_0(t)\left\{x(t) + \int_0^t x_0^{-1}(s)[\mathfrak{f}_1(s) - \mathfrak{g}_1(s)\mathfrak{g}_2(s)]\,ds + \int_0^t x_0^{-1}(s)\mathfrak{g}_1(s)\,d\mathfrak{B}(s)\right\}$$

where $x_0(t) = \exp\left\{\int_0^t \left[\mathfrak{f}_2(s) - \frac{1}{2}\mathfrak{g}_2^2(s)\right]ds + \int_0^t \mathfrak{g}_2(s)\,d\mathfrak{B}(s)\right\}$.

**Proposition 2.17 [25]** The function

$$x_t = \frac{\exp\left\{\left(rK - \frac{1}{2}\sigma^2\right)t + \sigma\mathfrak{B}_t\right\}}{x^{-1} + r\int_0^t \exp\left\{\left(rK - \frac{1}{2}\sigma^2\right)s + \sigma\mathfrak{B}_s\right\}ds};\ t \geq 0.$$

form a solution of the SDE

$$dx_t = rx_t(K - x_t)dt + \sigma x_t d\mathfrak{B}_t\ .$$

Let $\mathcal{C}^{2,1}(\mathfrak{R}^d \times \mathfrak{R}^+;\ \mathfrak{R})$ be the class of all functions $\mathcal{V}: \mathfrak{R}^d \times \mathfrak{R}^+ \longrightarrow \mathfrak{R}$ with the property that they are continuously twice differentiable in $x$ and once in $t$. For $\mathcal{V} \in \mathcal{C}^{2,1}(\mathfrak{R}^d \times \mathfrak{R}^+;\ \mathfrak{R})$, define

$$\mathcal{V}_t = \frac{\partial V}{\partial t},\quad \mathcal{V}_t = \left(\frac{\partial \mathcal{V}}{\partial x_1},\quad \ldots,\quad \frac{\partial \mathcal{V}}{\partial x_d}\right),\ \mathcal{V}_{xx} = \left(\frac{\partial^2 \mathcal{V}}{\partial x_i \partial x_j}\right)_{d\times d} = \begin{pmatrix} \frac{\partial^2 \mathcal{V}}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 \mathcal{V}}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{V}}{\partial x_d \partial x_1} & \cdots & \frac{\partial^2 \mathcal{V}}{\partial x_d \partial x_d} \end{pmatrix}.$$

Let $\mathcal{V} \in \mathcal{C}^{2,1}(S_h \times \mathfrak{R}^+;\ \mathfrak{R}^+)$, where $0 < h \leq \infty$. Define

$$\mathcal{L} \coloneqq \frac{\partial}{\partial t} + \sum_{i=1}^d \mathfrak{f}_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d [\mathfrak{g}(x,t)\mathfrak{g}^T(x,t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j},$$

as the differential operator related with (2). If $\mathcal{L}$ acts on a function $\mathcal{V} \in \mathcal{C}^{2,1}(S_h \times \mathfrak{R}^+; \mathfrak{R})$, then

$$\mathcal{L}\mathcal{V} = \mathcal{V}_t(x,t) + \mathcal{V}_x(x,t)f(x,t) + \frac{1}{2}trace[\mathfrak{g}^T(x,t)V_{xx}(x,t)\mathfrak{g}(x,t)].$$

**Definition 2.18[25](i)** The trivial solution of equation (2) is said to be stochastically stable if for every $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta \equiv \delta(\varepsilon, r, t_0) > 0$ with

$$\mathbb{P}\{|x(t; t_0, x_0)| < r : t > t_0\} > 1 - \varepsilon$$

when $|x_0| < \delta$. Else, called a stochastically unstable.

(ii) When the trivial solution is stochastically stable as well as for any $\varepsilon \in (0,1)$, there is a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ with

$$\mathbb{P}\left\{\omega: \lim_{t\to\infty} x(t; t_0, x_0) = 0\right\} \geq 1 - \varepsilon$$

when $|x_0| < \delta_0$, then it called stochastically asymptotically stable (SAS).

**Theorem 2.19** [**25**] If there is $\mathcal{V}(x,t) \in \mathcal{C}(\mathcal{S}_h \times [t,\infty); \Re^+)$, with $\mathcal{V}(x,t) > 0$ and $\mathcal{L}\mathcal{V}(x,t) \leq 0$

for all $(x,t) \in \mathcal{S}_h \times [t,\infty)$, then the trivial solution of (2) is stochastic stable.

**Lemma 2.20** [27, 28]. If $x: [0,\infty) \times \Omega \longrightarrow \Re^n$ is a stochastic process fulfill

$$\mathbb{E}|x(t) - x(s)|^{\alpha_1} \leq c|t-s|^{1+\alpha_2},\ 0 \leq s,\ t < \infty, \tag{13}$$

where $\alpha_1,\ \alpha_2, \mathrm{c} > 0$ and there is a continuous modification $\tilde{x}(t)$ of $x(t)$ so that, $\forall v \in (0, \alpha 2/\alpha 1)$, $\exists\ \psi: \Omega \longrightarrow \Re^+/\{0\}$ which is measurable with

$$\mathbb{P}\left\{\omega: \sup_{0<|t-s|<\psi(\omega), 0\leq s,t<\infty} \frac{|\tilde{z}(t,\omega) - z(t,\omega)|}{|t-s|^v} \leq \frac{2}{1-2^{-v}}\right\} = 1. \tag{14}$$

**Definition 2.21 [29].** The solution $x(t)$ of model (1) is called stochastically ultimately bounded (SUB), if, for every $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon) > 0$, so that for every $x_0 \in \Re^3_+$, we have

$$\lim_{t\to\infty} \sup \mathbb{P}\{|x(t)| > \delta\} < \varepsilon. \tag{15}$$

**Definition 2.22 [29].** The solution $x(t)$ of (1) has stochastic permanent property, if there is a couple of real numbers $\varphi = \varphi(v) > 0$ and $\chi = \chi(v) > 0$, $v \in (0,1)$ so that for any $x_0 \in \Re^3_+$, we have

$$\lim_{t\to\infty} \inf \mathbb{P}\{|x(t)| \geq \varphi\} \geq 1 - v,\ \lim_{t\to\infty} \inf \mathbb{P}\{|x(t)| \leq \chi\} \geq 1 - v. \tag{16}$$

## 3. MODEL FORMULATION

This section focus to study the deterministic and stochastic model for our main problem.

### 3.1 DETERMINISTIC MODEL

A classic three-species Lotka-Volterra system is the model under investigation; these systems have been essential for modeling interspecies competition, this has a considerable impact on

research of different competition models in biology, ecology, and medicine. Take [2,8,30] as an example. Our model depicts three distinct cell types: normal cells ($x$), infected cancer cells ($z$), and cancer cells ($y$). Based on predator-prey interactions, this mean-field model explains how viral infection of tumor cells and tumor growth are related.

**Table1:** Parameters Description, ref. [10]

| Parameter | Description | Value | Unit |
|---|---|---|---|
| $r$ | Proliferation of normal cells | 0.5 | $1/h$ cell |
| $a$ | Death rate of normal population | 0.2 | $1/h$ cell |
| $s$ | Proliferation of the uninfected cells | 1.0 | $mm^3\ h/$ cell |
| $b$ | Death rate of uninfected population | 0.1 | $1/h$ cell |
| $c$ | Proliferation of the infected cells | 1.2 | $mm^3\ h/$ cell |
| $d$ | Death rate of the infected cells | 0.1 | $1/h$ cell |
| $\sigma_1$ | | 0.1 | Estimate |
| $\sigma_2$ | | 0.7 | Estimate |
| $\sigma_3$ | | 0.2 | Estimate |

Although the important contact is mostly between infected and uninfected cells, the three model parts can describe the fundamental dynamics of such a communication. The following situation is used to formulate the model: One method for simulating virotherapy under discussion is to use a network with nodes that are both empty and populated by the three different types of cells. Since the virus is only meant to attack cancer cells and spreads from cell to cell, infected cells can only attack and take over a node that is occupied by a cancer cell. In contrast, A neighboring empty node must be occupied by the newly created cell when a normal or cancerous cell multiplies. ([31,32]). Since the virus primarily targets cancer cells and travels from cell to cell, Only a node that is occupied by a cancer cell can be attacked and taken over by infected cells. . On the other hand, whether a cancerous or normal cell proliferates, it needs to take up a neighboring empty node ([31,32]). The viruses arrive at different times, but they all follow the Poisson process, which has exponential distributions for the time to the next event. The three different cell kinds' growth and death rates can be changed, according to the model. It is also possible to specify the parameters of virus infection [3,10].

All of the parameters in the model mentioned above are nonnegative, as well as it is controlled by the differential equation system below:

$$\begin{cases} \frac{dx}{dt} = & rx(1 - x - - z) - ax \\ \frac{dy}{dt} = & sy(1 - x - y - z) - by - cyz \\ \frac{dz}{dt} = & cyz - \delta z \end{cases} \quad (17)$$

and the initial conditions are: $x(0) = x_0 > 0$ ، $y(0) = y_0 > 0$ ،and $z(0) = z_0 > 0$:

where $a, b,$ and $d$ stand for the corresponding population's death rates, and $r$ for proliferation. The model is based on mass action kinetics and was fitted to data from in vitro studies [3,10].

Now, using the boundedness of the model equation (17), we offer specific constraints to bound the system's solutions.

**Theorem 3.1[23]** With nonnegative initial conditions in the invariance region,

$$\Delta := \{(x, y, z) \in \Re_+^3 : x \leq 1, y + z \leq 1\}.$$

every solution to system (17) is bounded and nonnegative.

**Theorem 3.2** With positive initial conditions, every solution of system (17) in $\mathbb{R}_+^3$ is uniformly bounded.

**Proof.** Set

$$w = x + y + z. \quad (18)$$

So,

$\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$

$= [rx(1 - x) - rx(y + z) - ax] + [s y (1 - y) - s y(x + z) - (by + c yz)] +$

$(c y - \delta)z$

$\leq rx(1 - x) - rx(y + z) + s y (1 - y) - s y(x + z) - \eta(x + y + z),$

where $\eta = \min\{a, b, \delta\}$. So, $\frac{dw}{dt} + \eta w \leq rx(1 - x) + s y (1 - y) \leq r + s$. Assume that $\lambda = r + s > 0$, this implies $\frac{dw}{dt} + \eta w \leq \lambda$. Hence

$$0 \leq w(x, y, z) \leq \frac{\lambda}{\eta}(1 - e^{-\eta t}) + w(x(0), y(0), z(0))e^{-\eta t}, \quad (19)$$

and letting $t \longrightarrow +\infty$, from (15) yield

$$0 \leq w(x, y, z) < \frac{\lambda}{\eta}. \quad (20)$$

Consequently, the solution space of model (11) belongs to

$$\mathcal{D} = \left\{(x, y, z) \in \Re_+^3 : w = \frac{\lambda}{\eta} + \varepsilon, \text{ for every } \varepsilon > 0\right\}. \quad (21)$$

Henceforth, the proof is completed.

### 3.2 STOCHASTIC MODEL

As in [33], the stochastic model that agrees to (17) can be expressed as follows: It may happen that the effector cells' natural death rate ($d_1$), intrinsic growth rate of tumor cells ($r_1$) , maximum carrying capacity of tumor cells ($1/b_1$), normal cells' growth rate ($r_2$) and decay rate of targeted chemo-drug ($d_2$) are not totally identified nonetheless conditional on some random ecological effects, so that

$$r(t) \mapsto r(t) + \sigma_1 \dot{\mathfrak{B}}_1, \quad s(t) \mapsto s(t) + \sigma_2 \dot{\mathfrak{B}}_2 \text{ , and } \delta(t) \mapsto \delta(t) - \sigma_3 \dot{\mathfrak{B}}_3,$$

where the probability distribution of the noise terms $\sigma_i \dot{\mathfrak{B}}_i$ is unknown, but $\mathfrak{B}_i(t)$ denotes the typical independent Brownian motions and $\sigma_i > 0,\ i = 1,2,3$. It is supposed that the functions $r(t), s(t),$ and $\delta(t)$ are constants and nonrandom. Consequently, system (17) becomes into

$$\begin{cases} dx = & [rx\,(1 - x - y - z) - ax]dt + \sigma_1 x d\mathfrak{B}_1 \\ dy = & [s\,y\,(1 - x - y - z) - b\,y - c\,yz]dt + \sigma_2\,y\,d\mathfrak{B}_2 \\ dz = & [c\,yz - \delta z]dt + \sigma_3 z d\mathfrak{B}_3 \end{cases} \tag{22}$$

or in the matrix form

$$dX = \mathfrak{f}(X,t)dt + \mathfrak{g}(X,t)d\mathfrak{B}, \tag{23}$$

where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \mathfrak{f}(X,t) = \begin{pmatrix} rx\,(1 - x - y - z) - ax \\ s\,y\,(1 - x - y - z) - b\,y - c\,yz \\ c\,yz - \delta z \end{pmatrix},\ g(X,t) = \begin{pmatrix} \sigma_1 x & 0 & 0 \\ 0 & \sigma_2\,y & 0 \\ 0 & 0 & \sigma_3 z \end{pmatrix},$$

and $\mathfrak{B} = \begin{pmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \\ \mathfrak{B}_3 \end{pmatrix}$. Also, all $x(0) = x_0$ ، $y(0) = y_0$ ،and $z(0) = z_0$ are positive.

### 3.3 MEDICAL INTERPRETATION OF THE MATHETICAL MODEL FOR NORMAL, CANCEROUS, AND INFECTED CELLS

This mathematical model describes the temporal dynamics of three kinds of cells in the body: normal cells ($x$), cancerous cells ( $y$), and infected cancerous cells ($z$). Random effects have been included to represent environmental, therapeutic, or genetic mutation factors.

The first equation: natural cells ($x$)

- $rx(1 - x)$: Logistic growth of natural cells.
- $-rx(\,y + z)$: Decrease in normal cells due to competition with cancerous and infected cells.
- $-ax$: Natural loss of cells.
- $\sigma_1 x d\mathfrak{B}_1$: The effect of random environmental or therapeutic factors on normal cells.

The second equation: cancer cells ( $y$)

- $s\, y(1-\ y)$: Autologous growth of cancer cells.

- $-s\, y(x+\ z)$: Negative effect of interaction with normal and infected cells.

- $-b\, y$: Natural loss or due to the immune system.

- $-c\, yz$: The transfer of some cancer cells to the affected type.

- $\sigma_2\, y d\mathfrak{B}_2$: The influence of randomness on the growth or deterioration of cancer cells.

The third equation: infected cancer cells ($z$)

- $c\, yz$: The increase in the number of infected cells due to the interaction of cancer with the infection.

- $-\delta z$: Loss of infected cells due to treatment or natural deterioration.

- $\sigma_3 z\ d\mathfrak{B}_3$: A direct random effect on the affected cells.

**THE GENERAL INTERPRETATION OF THE MODEL**

The model represents a dynamic communication between different types of cells in the body, taking into account competition, interaction, and transformation from one type to another, in addition to unexpected fluctuations that affect the system. This model helps in understanding tumor development and treatment response, and can be used to simulate future therapeutic strategies.

## 4. POSITIVE INVARIANCE AND BOUNDEDNESS OF THE STOCHASTIC

In this section we will show whether the solutions of the model (22) are biologically acceptable or not for all the values of the parameters adopted in the model. So, we use the principle of stochastic comparison to ensure that the solutions are positive and constrained [11].

**Theorem 4.1.** The solutions of (22) are bounded and nonnegative in $\Delta = \{(x, y, z) \in \Re^3 : x \leq \lambda_1, y \leq \lambda_2, v\}$,

where $\lambda_1 := \frac{\left(r-\frac{1}{2}\sigma_1^2\right)}{r}$, $\lambda_2 := \frac{\left(s-\frac{1}{2}\sigma_2^2\right)}{s}$, and $\lambda_3 := \frac{c\left(s-\frac{1}{2}\sigma_2^2\right)}{s}$.

**Proof.** First, we have

$$dx = [rx(1-x) - rx(\ y+z) - ax]dt + [\sigma_1 x\ (1-x\ ) - \sigma_1 x(\ y+z)]d\mathfrak{B}_1$$
$$\leq rx(1-x)dt + \sigma_1 x\, dW_1.$$

Now, consider $d\bar{x} = r\bar{x}\ (1-\bar{x}\ )dt + \sigma_1 \bar{x}\, dW_1$.

Then $\bar{x}(t) = \frac{\exp\left\{\left(r-\frac{1}{2}\sigma_1^2\right)t+\sigma_1\mathfrak{B}_1(t)\right\}}{(\bar{x}(0))^{-1}+r\int_0^t \exp\left\{\left(r-\frac{1}{2}\sigma_1^2\right)\tau+\sigma_1\mathfrak{B}_1(\tau)\right\}d\tau}$.

Since $dx \le d\bar{x}$. Then $\lim\limits_{t\to\infty}\sup x(t) \le \lim\limits_{t\to\infty}\sup \bar{x}(t)$.But $\lim\limits_{t\to\infty}\sup \bar{x}(t) \le \frac{\left(r-\frac{1}{2}\sigma_1^2\right)}{r}$ , so

$$\lim_{t\to\infty}\sup x(t) \le \frac{\left(r-\frac{1}{2}\sigma_1^2\right)}{r}.$$

Now,

$d\,y = [s\,y(1-\,y\,) - s\,y(\,x+z) - (\,b\,y + \,c\,yz)]dt + \sigma_2\,y\,d\mathfrak{B}_2 \le s\,y(1-y)dt + \sigma_2\,y\,d\mathfrak{B}_2$

Consider the SDE $d\bar{y} = s\bar{y}\,(1-\bar{y}\,)dt + \sigma_2\bar{y}\,d\mathfrak{B}_2$. Then

$$\bar{y}(t) = \frac{\exp\left\{\left(s-\frac{1}{2}\sigma_2^2\right)t+\sigma_2\mathfrak{B}_2(t)\right\}}{(\bar{y}(0))^{-1}+s\int_0^t \exp\left\{\left(s-\frac{1}{2}\sigma_2^2\right)\tau+\sigma_2\mathfrak{B}_2(\tau)\right\}d\tau}\ .$$

Since $dy \le d\bar{y}$. Then $\lim\limits_{t\to\infty}\sup\ y(t) \le \lim\limits_{t\to\infty}\sup \bar{y}(t)$. But $\lim\limits_{t\to\infty}\sup \bar{y}(t) \le \frac{\left(s-\frac{1}{2}\sigma_2^2\right)}{s}$ , so

$$\lim_{t\to\infty}\sup\ y(t) \le \frac{\left(s-\frac{1}{2}\sigma_2^2\right)}{s}.$$

Finally, the solution of the SDE $dz = (\,c\,y\, - \delta)dt + \sigma_3 z d\mathfrak{B}_3$ is given by

$$z(t) = z(0)\exp\left[\int_0^t c\,y(\tau)\,d\tau - \delta t + \sigma_3\mathfrak{B}_3(t)\right].$$

Thus

$$\lim_{t\to\infty}\sup z(t,\omega) \le z(0)\exp\left[\frac{c\left(s-\frac{1}{2}\sigma_2^2\right)}{s}t + \sigma_3\mathfrak{B}_3(t)\right].$$

So, we have $\Delta = \{(x,y,z)\in\mathfrak{R}^3 : x\le\lambda_1, y\le\lambda_2, v\le\lambda_3\}$, where $\lambda_1 := \frac{\left(r-\frac{1}{2}\sigma_1^2\right)}{r}$, $\lambda_2 := \frac{\left(s-\frac{1}{2}\sigma_2^2\right)}{s}$, $\lambda_3 := \frac{c\left(s-\frac{1}{2}\sigma_2^2\right)}{s}$.

The domain region $\Delta$ is positively invariant, which verifies that the model system (22) is biologically feasible.

**Theorem 4.2** The system (22) admits a unique positive local solution $(x(t), y(t), z(t))$ for $(x_0,\ y_0, z_0)\in Int(\mathfrak{R}^3_+)$ and $t\in[0,\tau_e)$ almost surely, where $\tau_e$ is the explosion time.

**Proof.** Take the transformation of variables $N = \ln x$, $T = \ln\ y$, and $I = \ln z$. Using the Itô formula,

$$\mathcal{L}\mathcal{V} = \mathcal{V}_t(t,x) + \mathcal{V}_x(t,x)f(t,x) + \frac{1}{2}trace\ (\mathrm{g}^T(t,x)\mathcal{V}_{xx}(t,x)\mathrm{g}(t,x)),$$

we get

$$e^N dN = \left[re^N\,(1-e^N) - re^N(e^T + e^I) - ae^N - \frac{1}{2}\sigma_1^2 e^N\right]dt + \sigma_1 e^N d\mathfrak{B}_1.$$

Then

$$\mathcal{L}N = \left[r\,(1-e^N) - r(e^T + e^I) - (a + \frac{1}{2}\sigma_1^2)\right] \text{ and } \quad dN = \mathcal{L}N + \sigma_1 d\mathfrak{B}_1.$$

Likewise, we get, from (22)

$$\begin{cases} dN & = \left[r\,(1-e^N) - r(e^T + e^I) - (a + \frac{1}{2}\sigma_1^2)\right]dt + \sigma_1 d\mathfrak{B}_1, \\ dT & = \left[s(1 - e^T) - s(e^N + e^I) - ce^I - (b + \frac{1}{2}\sigma_2^2)\right]dt + \sigma_2 e^T\, d\mathfrak{B}_2, \\ dI & = \left[ce^T - \left(\delta + \frac{1}{2}\sigma_3^2\right)\right]dt + \sigma_3 d\mathfrak{B}_3. \end{cases} \tag{24}$$

with $N(t) = \ln x(t)$, $T(t) = \ln y(t)$, and $I(t) = \ln z(t)$. At the present, the functions conforming to (24) admit initial growth and they fulfill the local Lipchitz condition. Thus, there is a unique local solution $(N, T, I)$ defined in $[0, \tau_e)$.

**Theorem 4.3.** The model (22) has a unique solution $(x(t), y(t), z(t))$ for $t \in [0, \tau_e)$ and for every $(x_0, y_0, z_0) \in Int(\mathfrak{R}_+^3)$ and $\mathbb{P}\{(x(t), y(t), z(t)) \in Int(\mathfrak{R}_+^3): t \geq 0\} = 1$.

**Proof.** In order to prove the existence of the global solutions it is sufficient to demonstration that the global solution exists, it is sufficient to verify that $\mathbb{P}\{\tau_\infty = \infty\} = 1$. If $\kappa_0 \in \mathbb{Z}_+$ is a sufficient large such that $(x_0, y_0, z_0)$ lies in the closed ball $\mathcal{B}(\kappa_0) \subset \mathfrak{R}_+^3$. For $\kappa \geq \kappa_0$, we take and describe the stop-time as

$$\tau_\kappa = \inf\left\{t \in [0, \tau e): x \notin \left(\frac{1}{\kappa}, \kappa\right) \text{ or } y \notin \left(\frac{1}{\kappa}, \kappa\right) \text{ or } z \notin \left(\frac{1}{\kappa}, \kappa\right)\right\}. \tag{25}$$

Now, $\inf \emptyset = \infty$ ($\emptyset$ is the empty set). So, $\tau_\kappa$ is growing as $\kappa \longrightarrow \infty$. Let $\tau_\infty = \lim_{\kappa \to \infty} \tau_\kappa$; then, $\tau_\infty \leq \tau_e$ almost surely. If $\mathbb{P}\{\tau_\infty = \infty\} = 1$, then $\mathbb{P}\{\tau_e = \infty\} = 1$. If this declaration is false, i.e., if $\tau_\infty \neq \infty$, then $T > 0$ and $\varepsilon \in (0, 1)$ exist with

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon. \tag{26}$$

Hence, put $\Omega_\kappa = \{\tau_\kappa \leq T\}$, then $\kappa_1 \geq \kappa_0$ is an integer such that, for all $\kappa \geq \kappa_1$,

$$\mathbb{P}\{\tau_\kappa \leq T\} \geq \varepsilon \tag{27}$$

Define a function $\mathcal{V}: \text{Int}(\mathfrak{R}_+^3) \longrightarrow \text{Int}(\mathfrak{R}_+)$ as follows

$$\mathcal{V}(x, y, z) = (x - 1 - \ln x) + (y - 1 - \ln y) + (z - 1 - \ln z), \tag{28}$$

where $\mathcal{V}(x, y, z) > 0$ for all $(x, y, z) \in \text{Int}(\mathfrak{R}_+^3)$. By Itô's formula, yield

$$\mathcal{L}\mathcal{V} = (x-1)[r(1 - x - y - z) - a] + (y-1)[s\,(1 - x - y - z) - b + cz)]$$
$$+(z-1)(cy - \delta) + \left(\frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2}\right).$$

Taking the differential of $\mathcal{V}(x, y, z)$, one gets

$$d\mathcal{V}(x, y, z) = \mathfrak{f}(x, y, z)dt + \mathfrak{g}(x, y, z)d\mathfrak{B}, \tag{29}$$

where $\mathfrak{g}(x, y, z) = \sigma_1(x-1) + \sigma_2(y-1) + \sigma_3(z-1)$ , and

$\mathfrak{f}(x, y, z) = \left(\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{2}\right) - (x-1)[a - r(1-x-y-z)] - (y-1)[b - s\,(1-x-\,y-z) - cz)]\; -(z-1)(\delta - cy\,)$.

Then , there is $M > 0$ so, $|\mathfrak{f}(x, y, z)| \leq M$, for all $(x, y, z) \in \mathfrak{R}_+^3$. It follows (29) that

$$\int_0^{\tau_m \wedge T} d\mathcal{V}(x, y, z) \leq \int_0^{\tau_m \wedge T} M\, dt + \int_0^{\tau_m \wedge T} \mathfrak{g}(x, y, z)\, d\mathfrak{B}(t), \tag{30}$$

where $\tau_k \wedge T = \min\{\tau_k, T\}$. Considering the aforementioned inequality's expectations, yield

$$\mathbb{E}\mathcal{V}\big(x(\tau_k \wedge T), y(\tau_k \wedge T), z(\tau_k \wedge T)\big) \leq \mathcal{V}\big(x(0), y(0), z(0)\big) + MT \tag{31}$$

Note that no less than one of $x(\tau_k, \omega)$, $y(\tau_k, \omega)$, and $z(\tau_k, \omega)$ lies in $\{\kappa, 1/\kappa\}$, for every $\omega \in \Omega_\kappa$; consequently,

$$\mathcal{V}\big(x(\tau_k \wedge T), y(\tau_k \wedge T), z(\tau_k \wedge T)\big) \geq (\kappa - 1 - \ln\kappa) \wedge \left(\frac{1}{\kappa} - 1 - \ln\frac{1}{\kappa}\right) \tag{32}$$

Hence, from (27)

$$\begin{aligned}\mathbb{E}\mathcal{V}\big(x(\tau_k \wedge T), y(\tau_k \wedge T), z(\tau_k \wedge T)\big) &\geq \mathbb{E}\big[\mathbb{I}_{\Omega_\kappa(\omega)} V\big(x(\tau_k \wedge T), y(\tau_k \wedge T), z(\tau_k \wedge T)\big)\big] \\ &\geq \varepsilon(\kappa - 1 - \ln\kappa) \wedge \left(\frac{1}{\kappa} - 1 - \ln\frac{1}{\kappa}\right)\end{aligned} \tag{33}$$

Here $\mathbb{I}_{\Omega_\kappa(\omega)}$ represents the characteristic function of $\Omega_\kappa$. Using (32), we get

$$\mathcal{V}\big(x(0), y(0), z(0)\big) + MT \geq \varepsilon(\kappa - 1 - \ln\kappa) \wedge \left(\frac{1}{\kappa} - 1 - \ln\frac{1}{\kappa}\right) \tag{34}$$

$\kappa \longrightarrow \infty$ implies to a contradiction: $\infty > \mathcal{V}\big(x(0), y(0), z(0)\big) + MT = \infty$ : Therefore, $\mathbb{P}\{\tau_\infty = \infty\} = 1$. Thus, $\mathbb{P}\big\{\big(x(t), y(t), z(t)\big) \in Int(\mathfrak{R}_+^3) : t \geq 0\big\} = 1$.

**Theorem 4.4.** The solutions of system (22) are stochastically ultimately bounded, for every initial value $w_0 = (x_0, y_0, z_0) \in \mathfrak{R}_+^3$.

**Proof.** From Theorem 4.3, we have $\mathbb{P}\{w(t) \in Int(\mathfrak{R}_+^3) : t \geq 0\} = 1$. Suppose that $\mathcal{V}_1(t, x) = e^t x^\theta$ for $\theta > 0$. The Itô formula implies that

$$\begin{aligned}\mathcal{L}\mathcal{V}_1(t, x) &= \frac{\partial \mathcal{V}_1(t,x)}{\partial t} + \frac{\partial \mathcal{V}_1(t,x)}{\partial x}\mathfrak{f}_1(t, x, y, z) + \frac{1}{2}\frac{\partial^2 \mathcal{V}_1(t,x)}{\partial x^2}\mathfrak{g}_1^2(t, x, y, z) \\ &= e^t x^\theta \left\{1 + \theta[r\,(1-x-y-z) - a] + \frac{\sigma_1^2}{2}\theta(\theta-1)\right\} \\ &\leq e^t \left\{\left[1 + r\theta + \frac{\sigma_1^2}{2}\theta(\theta-1)\right]x^\theta - r\theta x^{\theta+1}\right\} \leq M_1(\theta)e^t.\end{aligned} \tag{35}$$

Taking the integral and expectation on two sides of the above inequality, yield $e^t\mathbb{E}(x^\theta(t)) - \mathbb{E}(x_0^\theta) \le M_1(\theta)e^t$. Thus, $\limsup_{t\to\infty}\mathbb{E}x^\theta(t) \le M_1(\theta) < +\infty$. Suppose that $\mathcal{V}_2(t,y) = e^t y^\theta$ for $\theta > 0$; the Itô formula, implies that

$$\mathcal{L}\mathcal{V}_2(t,y) = \frac{\partial \mathcal{V}_2(t,y)}{\partial t} + \frac{\partial \mathcal{V}_2(t,y)}{\partial y}\mathfrak{f}_2(t,x,y,z) + \frac{1}{2}\frac{\partial^2 \mathcal{V}_2(t,y)}{\partial y^2}\mathfrak{g}_2^2(t,x,y,z)$$

$$= e^t y^\theta\left\{1 + \theta[s\,(1 - y - x - z - cz) - b] + \frac{\sigma_1^2}{2}\theta(\theta-1)\right\}$$

$$\le e^t\left\{\left[1 + \theta s + \frac{\sigma_1^2}{2}\theta(\theta-1)\right]y^\theta - \theta s y^{\theta+1}\right\} \le M_2(\theta)e^t \tag{36}$$

Taking the integral and expectation on two sides of the above inequality , yield

$$e^t\mathbb{E}(y^\theta(t)) - \mathbb{E}(y_0^\theta) \le M_2(\theta)e^t.$$

Thus,

$$\limsup_{t\to\infty}\mathbb{E}y^\theta(t) \le M_2(\theta) < +\infty.$$

Similarly, suppose that $\mathcal{V}_3(t,z) = e^t z^\theta$ for $\theta > 0$; it follows from Itô formula that

$$\mathcal{L}\mathcal{V}_3(t,z) = \frac{\partial \mathcal{V}_3(t,z)}{\partial t} + \frac{\partial \mathcal{V}_3(t,z)}{\partial z}\mathfrak{f}_3(t,x,y,z) + \frac{1}{2}\frac{\partial^2 \mathcal{V}_2(t,z)}{\partial z^2}\mathfrak{g}_3^2(t,x,y,z)$$

$$= e^t z^\theta + \theta e^t z^{\theta-1}(\,cz - \delta)z + \frac{1}{2}\theta(\theta-1)e^t z^{\theta-2}\sigma_3^2 z^2$$

$$\le e^t\left\{1 + \theta c y + \frac{\sigma_3^2}{2}\theta(\theta-1)\right\}z^\theta \le M_3(\theta)e^t \tag{37}$$

Taking the integral and expectation on two sides of the above inequality , yield

$$e^t\mathbb{E}(z^\theta(t)) - \mathbb{E}(z_0^\theta) \le M_3(\theta)e^t.$$

Thus

$$\limsup_{t\to\infty}\mathbb{E}z^\theta(t) \le M_3(\theta) < +\infty.$$

For $w(t) = (x(t), y(t), z(t)) \in \Re_+^3$, we may get

$$|w(t)|^\theta \le (3\max\{x^3(t), y^3(t), z^3(t)\})^{\theta/3} \le 3^{\theta/3}(x^\theta(t) + y^\theta(t) + z^\theta(t)). \tag{38}$$

$$\limsup_{t\to\infty}\mathbb{E}|w(t)|^\theta \le M_4(\theta) < +\infty, \tag{39}$$

where $M_4(\theta) = 3^{\frac{\theta}{3}}(M_1(\theta) + M_2(\theta) + M_3(\theta))$. The Chebyshev inequality leads us to the conclusion that every solution is stochastically bounded.

It is remain to demonstrate that the positive solution using basic features and appropriate Lyapunov functions

$w(t) = (x(t), y(t), z(t))$ is uniformly Hölder continuous.

**Theorem 4.5.** Every sample path of $(x(t), y(t), z(t))$ is uniformly continuous, where $(x(t), y(t), z(t))$ is a solution of system (22) on $t \geq 0$ with $(x_0, y_0, z_0) \in \Re_+^3$.

**Proof.** It follows from the first equation of (22) that

$$x(t) = x_0 + \int_0^t \left[rx(u)\left(1 - x(u) - y(u) - z(u)\right) - ax(u)\right] du + \int_0^t \sigma_1 x(u)\, d\mathfrak{B}_1(u).$$

Assume that $\mathfrak{f}_1(u) = \left[rx(u)\left(1 - x(u) - y(u) - z(u)\right) - az(u)\right]$ and $\mathfrak{f}_2(u) = \sigma_1 x(u)$. We infer from Theorem (4.4) that

$$\begin{aligned}
\mathbb{E}|\mathfrak{f}_1(t)|^\theta &= \mathbb{E}|rx\,(1 - x - y - z) - az|^\theta \\
&\leq \tfrac{1}{2}\mathbb{E}|x|^{2\theta} + \tfrac{1}{2}\mathbb{E}|r + rx + ry + rz)|^{2\theta} \\
&\leq \tfrac{1}{2}\mathbb{E}|x|^{2\theta} + 4^{4\theta - 3/2}\left[r^{2\theta} + r^{2\theta}\mathbb{E}|x|^{2\theta} + r^{2\theta}\mathbb{E}|y|^{2\theta} + r^{2\theta}\mathbb{E}|z|^{2\theta}\right] \\
&\leq \tfrac{1}{2}M_1(2\theta) + 4^{4\theta - \frac{3}{2}}\left[r^{2\theta} + r^{2\theta}M_1(2\theta) + r^{2\theta}M_2(2\theta) + r^{2\theta}M_3(2\theta)\right] = F_1(\theta), \\
\mathbb{E}|\mathfrak{f}_2(t)|^\theta &= \mathbb{E}|\sigma_1 x(t)|^\theta = \sigma_1^\theta \mathbb{E}|x|^\theta \leq \sigma_1^\theta M_1(\theta) \leq F_2(\theta). \qquad (40)
\end{aligned}$$

For stochastic integrals, we use the moment inequality for $0 \leq t_1 \leq t_2$ and $\theta > 2$, to obtain The moment inequality for$0 \leq t_1 \leq t_2$ as well as $\theta > 2$is used for stochastic integrals to derive

$$\begin{aligned}
\mathbb{E}\left|\int_{t_1}^{t_2} \mathfrak{f}_2(u)\, dW_1(u)\right|^\theta &\leq \left(\tfrac{\theta(\theta-1)}{2}\right)^{\theta/2} (t_2 - t_1)^{(\theta-2)/2} \int_{t_1}^{t_2} \mathbb{E}|\mathfrak{f}_2(u)|^\theta\, du \\
&\leq \left(\tfrac{\theta(\theta-1)}{2}\right)^{\theta/2} (t_2 - t_1)^{\theta/2} F_2(\theta). \qquad (41)
\end{aligned}$$

Thus, for $0 < t_1 < t_2 < \infty$, $t_2 - t_1 \leq 1$, $\left(\frac{1}{\theta}\right) + \left(\frac{1}{\kappa}\right) = 1$ (or $\frac{\theta}{\kappa} + 1 = \theta$), we get

$$\begin{aligned}
&\mathbb{E}|x(t_2) - x(t_1)|^\theta = \mathbb{E}\left|\int_{t_1}^{t_2} \mathfrak{f}_1(u)\, du + \int_{t_1}^{t_2} \mathfrak{f}_2(u)\, d\mathfrak{B}_1(u)\right|^\theta \\
&\leq 2^{\theta-1}\mathbb{E}\left|\int_{t_1}^{t_2} \mathfrak{f}_1(u)\, du\right|^\theta + 2^{\theta-1}\mathbb{E}\left|\int_{t_1}^{t_2} \mathfrak{f}_2(u)\, d\mathfrak{B}_1(u)\right|^\theta \\
&\leq 2^{\theta-1}(t_2 - t_1)^{\theta/2}\left\{(t_2 - t_1)^{\theta/2} F_1(\theta) + \left(\tfrac{\theta(\theta-1)}{2}\right)^{\theta/2} F_2(\theta)\right\} \\
&\leq 2^{\theta-1}(t_2 - t_1)^{\frac{\theta}{2}}\left\{1 + \left(\tfrac{\theta(\theta-1)}{2}\right)^{\frac{\theta}{2}}\right\} F(\theta) \qquad (42)
\end{aligned}$$

where $F(\theta) = \max\{F_1(\theta), F_2(\theta)\}$. By Lemma 2.20, for each exponent $\upsilon \in (0, (\theta - 2)/2\theta)$, the uniform continuity of each sample path of $x(t)$ on $\Re_+^3$ is demonstrated by the fact that each sample path of $x(t)$ is uniformly and locallylder continuous. Each sample path of $x(t)$ on $\Re_+^3$ is uniformly and locally continuous, proving the uniform continuity of each sample path. Similarly,

on $\Re_+^3$, the uniform continuity of $y(t)$ and $z(t)$ is demonstrated. As a result, on $t > 0$, we obtain the uniform continuity of each sample path of $(x(t), y(t), z(t))$ to system (22).

## 5. LONG TIME BEHAVIOR OF SYSTEM

We focus our attention in this section on the long-term behavior of the system. For this purpose, we will define two hypotheses that will be useful in the stability analysis later.

(H1): $\frac{L}{\eta}\max\{r,a\} + \frac{1}{2}\max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\} < \min\{r-s, r-\delta\}$

(H2): $r - a - \frac{\sigma_1^2}{2} < 0, \quad s - b - \frac{\sigma_2^2}{2} < 0, \; c - \delta - \frac{\sigma_3^2}{2} < 0$ (43)

We will first demonstrate stochastic persistence, which is crucial to population dynamics. Here is how we talk about this property:

**Theorem 5.1.** If $(H_1)$ fulfills , then model (22) is stochastically permanent.

**Proof.** For $w(0) = (x(0), y(0), z(0)) \in \Re_+^3$, we show that a solution $w(t) = (x(t), y(t), z(t))$ exists such that

$$\lim_{t\to\infty} \sup \mathbb{E}\left(\frac{1}{|w(t)|^\gamma}\right) \le M, \tag{44}$$

where $\gamma \in \Re_+$ fulfills

$$\frac{L}{\eta}\max\{r,a\} + \frac{(\gamma+1)}{2}\max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\} < \min\{r-s, r-\delta\} \tag{45}$$

By (45), there is a $\rho > 0$ fulfills

$$\min\{r-s, r-\delta\} - \rho - \frac{L}{\eta}\max\{r,a\} - \frac{1}{2}\max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\} > 0. \tag{46}$$

Define $V(x,y,z) = x + y + z$ for $(x,y,z) \in \Re_+^3$ and $\bar{V}(x,y,z) = 1/V(x,y,z)$; so

$$\begin{aligned} dV(x,y,z) = \{&[rx(1-x-y-z) - ax] + [sy(1-y-x-z) - (by + cyz)] + (cy \\ &- \delta)z\}dt \\ &+(\sigma_1 x d\mathfrak{B}_1 + \sigma_2 y\, d\mathfrak{B}_2 + \sigma_3 z d\mathfrak{B}_3) \end{aligned}$$

$$\begin{aligned} d\bar{V}(x,y,z) &= -\bar{V}^2(x,y,z)\{[rx(1-x-y-z) - ax] + [sy(1-y-x-z) - (by + \\ &cyz)] + (cy - \delta)z\}dt \\ &+\bar{V}^2(x,y,z)[(\sigma_1 x)^2 + (\sigma_2 y)^2 + (\sigma_3 z)^2]dt - \bar{V}^2(x,y,z)(\sigma_1 x d\mathfrak{B}_1 + \sigma_2 y\, d\mathfrak{B}_2 + \sigma_3 z d\mathfrak{B}_3) \\ &= \mathcal{L}\bar{V}(x,y,z)dt - \bar{V}^2(x,y,z)(\sigma_1 x d\mathfrak{B}_1 + \sigma_2 y\, d\mathfrak{B}_2 + \sigma_3 z d\mathfrak{B}_3) \end{aligned} \tag{47}$$

We choose $\gamma > 0$ under (H1) so that condition (45) is satisfied. Therefore

$$\begin{aligned} \mathcal{L}(1+\bar{V}(x,y,z))^\gamma &= \gamma(1+\bar{V}(x,y,z))^{\gamma-1}\mathcal{L}\bar{V}(x,y,z) \\ &+\frac{1}{2}\gamma(\gamma-1)(1+\bar{V}(x,y,z))^{\gamma-2}\bar{V}^4(x,y,z) \times [(\sigma_1 x)^2 + (\sigma_2 y)^2 + (\sigma_3 z)^2]. \end{aligned} \tag{48}$$

Then, we select $\rho > 0$ so that (46) is satisfied. So,

$$\mathcal{L}e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma} = \rho e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma} + e^{\rho t}\mathcal{L}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma}$$
$$= e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma-2}[\rho(1+\bar{\mathcal{V}}(x,y,z))^2 + \mathcal{A}] \quad (49)$$

where

$$\mathcal{A} = -\gamma\bar{\mathcal{V}}^2(x,y,z)\{[rx\,(1-x-y-z) - ax] + [sy\,(1-y-x-z) - (by + cyz)] + (cy-\delta)z\} - \gamma\bar{\mathcal{V}}^3(x,y,z)\{[rx\,(1-x-y-z) - ax] + [sy\,(1-y-x-z) - (by + cyz)] + (cy-\delta)z\} + \gamma\bar{\mathcal{V}}^3(x,y,z)[(\sigma_1 x)^2 + (\sigma_2 y)^2 + (\sigma_3 z)^2] + \frac{\gamma(\gamma+1)}{2}\bar{\mathcal{V}}^4(x,y,z)[(\sigma_1 x)^2 + (\sigma_2 y)^2 + (\sigma_3 z)^2]$$

The upper bound of the function $(1+\bar{\mathcal{V}}(x,y,z))^{\gamma-2}[\rho(1+\bar{\mathcal{V}}^2(x,y,z)) + \mathcal{A}]$ is described as:

$$\gamma\bar{\mathcal{V}}^3(x,y,z)[(\sigma_1 x)^2 + (\sigma_2 y)^2 + (\sigma_3 z)^2] \le \gamma\bar{\mathcal{V}}(x,y,z)\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\},$$
$$\frac{\gamma(\gamma+1)}{2}\bar{\mathcal{V}}^4(x,y,z)[(\sigma_1 x)^2 + (\sigma_2 y)^2 + (\sigma_3 z)^2] \le \frac{\gamma(\gamma+1)}{2}\bar{\mathcal{V}}^2(x,y,z)\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}, \quad (50)$$

$$\mathcal{L}e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma} = e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma-2}[e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^2 + \mathcal{A}]$$
$$\le e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma-2}\Big\{\rho + \Big[2\rho - \gamma\min\{r-s,r-\delta\} + \frac{\gamma L}{\eta}\max\{r,a\} + \gamma\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}\Big]\bar{\mathcal{V}}(x,y,z)\Big[\rho - \gamma\min\{r-s,r-\delta\} + \frac{\gamma L}{\eta}\max\{r,a\} + \frac{\gamma(\gamma+1)}{2}\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}\Big]\bar{\mathcal{V}}^2(x,y,z)\Big\}. \quad (51)$$

As of (45) and (46), we get a nonnegative constant $Q$ satisfying $\mathcal{L}e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma} \le Qe^{\rho t}$. This implies that

$$\mathbb{E}[e^{\rho t}(1+\bar{\mathcal{V}}(x,y,z))^{\gamma}] \le (1+\bar{\mathcal{V}}(x,y,z))^{\gamma} + \frac{Q(e^{\rho t}-1)}{\rho} \quad (52)$$

Therefore,

$$\lim_{t\to\infty}\sup\mathbb{E}[\bar{\mathcal{V}}(x(t),y(t),z(t))^{\gamma}] \le \lim_{t\to\infty}\sup\mathbb{E}\big[(1+\bar{\mathcal{V}}\big(x(t),y(t),z(t)\big)^{\gamma})\big] \le \frac{Q}{\rho}. \quad (53)$$

Note that,

$$(x+y+z)^{\gamma} \le 3^{\gamma}(x^3+y^3+z^3)^{\gamma/3} = 3^{\gamma}|w|^{\gamma}, \quad (54)$$

where $w = (x,y,z) \in \Re_+^3$. Accordingly

$$\lim_{t\to\infty}\sup\mathbb{E}\left[\frac{1}{|w(t)|^{\gamma}}\right] \le 3^{\gamma}\lim_{t\to\infty}\sup\mathbb{E}[\bar{\mathcal{V}}(x(t),y(t),z(t))^{\gamma}] \le 3^{\gamma}\frac{Q}{\rho} := \mathcal{R}. \quad (55)$$

Assume that $\mathcal{Q} = (v/\mathcal{R})^{1/\gamma}$ for any $v > 0$; then, it follows from Chebyshev's inequality that

$$\mathbb{P}\{|w(t)| < \mathcal{Q}\} = \mathbb{P}\{|w(t)|^{-\gamma} < \mathcal{Q}^{-\gamma}\} \le \mathbb{E}[|w(t)|^{-\gamma}]/\mathcal{Q}^{-\gamma} \le \mathbb{E}[|w(t)|^{-\gamma}]/\mathcal{Q}^{-\gamma}, \quad (56)$$

i.e.,

$$\lim_{t\to\infty}\inf \mathbb{P}\{|\mathcal{w}(t)| \geq \mathcal{Q}\} \geq 1-\nu. \tag{57}$$

Likewise, we may obtain $\chi > 0$ for every $\varepsilon > 0$ so that

$$\lim_{t\to\infty}\inf \mathbb{P}\{|\mathcal{w}(t)| \leq \chi\} \geq 1-\nu.$$

System (22) is therefore stochastically persistent according to Definition 2.22.

The possibility of a species' population completely disappearing exists in population dynamics. Therefore, the study of species extinction is crucial to the ecosystem.

**Theorem 5.2.** The solution $\mathcal{w}(t) = (x(t), \mathcal{y}(t), z(t))$ of system (22) will be extinct with probability one for every given initial value $\mathcal{w}(0) = (x(0), \mathcal{y}(0), z(0)) \in \Re_+^3$ when (H2) is fulfills.

**Proof.** Suppose that $\mathcal{V}_4(x, \mathcal{y}, z) = \ln x$. Therefore

$$d(\ln x) = \left[r\,(1 - x - \mathcal{y} - z\,) - a - \tfrac{1}{2}\sigma_1^2\right]dt + \sigma_1 d\mathfrak{B}_1 \tag{58}$$

Then

$$\int_0^t d(\ln x(\tau)) = \int_0^t \left[r\,(1 - x(\tau) - \mathcal{y}(\tau) - z(\tau)\,) - a - \tfrac{1}{2}\sigma_1^2\right]d\tau + \int_0^t \sigma_1\, d\mathfrak{B}_1(\tau).$$

Consequently,

$$\ln x(t) = \ln x(0) + \left(r - a - \tfrac{1}{2}\sigma_1^2\right)t - r\int_0^t [x(u) + \mathcal{y}(u) + z(u)]\,dt + \sigma_1 \int_0^t d\mathfrak{B}_1\,(u). \tag{59}$$

Then

$$\ln x(t) \leq \ln x(0) + \left(r - a - \tfrac{1}{2}\sigma_1^2\right)t + \sigma_1\mathfrak{B}_1(t). \tag{60}$$

Hence

$$\lim_{t\to\infty}\frac{\ln x(t)}{t} \leq \lim_{t\to\infty}\frac{\ln x(0)}{t} + \lim_{t\to\infty}\frac{\left(r-a-\frac{1}{2}\sigma_1^2\right)t}{t} + \lim_{t\to\infty}\frac{\sigma_1\mathfrak{B}_1(t)}{t}.$$

Therefore,

$$\lim_{t\to\infty}\sup\frac{\ln x(t)}{t} \leq r - a - \tfrac{1}{2}\sigma_1^2 < 0, \text{ almost surely.} \tag{61}$$

Also, define the Lyapunov function $\mathcal{V}_5(x, \mathcal{y}, z) = \ln \mathcal{y}$; use Itô's formula yield

$$d(\ln \mathcal{y}) = \left[s - b - (\,\mathcal{y} + x + (1+c)z)) - \tfrac{1}{2}\sigma_2^2\right]dt + \sigma_2\, d\mathfrak{B}_2. \tag{62}$$

Then

$$\int_0^t d(\ln \mathcal{y}) = \int_0^t \left[s - b - (\,\mathcal{y} + x + (1+c)z)) - \tfrac{1}{2}\sigma_2^2\right]dt + \int_0^t \sigma_2\, d\mathfrak{B}_2.$$

Thus

$$\ln y(t) = \ln y(0) + \left(s - b - \frac{1}{2}\sigma_2^2\right)t - \int_0^t (\, y(u) + x(u) + (1+c)z(u))\, du + \sigma_2 \int_0^t d\mathfrak{B}_2(u) \tag{63}$$

Consequently,

$$\ln y(t) \leq \ln y(0) + \left(s - b - \frac{1}{2}\sigma_2^2\right)t + \sigma_2 \mathfrak{B}_2(t).$$

Then

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} \leq \lim_{t\to\infty} \frac{\ln y(0)}{t} + \lim_{t\to\infty} \frac{\left(s-b-\frac{1}{2}\sigma_2^2\right)t}{t} + \lim_{t\to\infty} \frac{\sigma_2 \mathfrak{B}_2(t)}{t}.$$

Hence

$$\lim_{t\to\infty} \sup \frac{\ln y(t)}{t} \leq s - b - \frac{1}{2}\sigma_2^2 < 0, \text{ almost surely.} \tag{64}$$

Likewise, define $\mathcal{V}_6(x, y, z) = \ln z$ as the Lyapunov function, and then

$$d(\ln z) = (cy\,(u) - \delta - \frac{1}{2}\sigma_3^2)dt + \sigma_3 d\mathfrak{B}_3 \tag{65}$$

Therefore

$$\ln z(t) = \ln z(0) + \left(-\delta - \frac{1}{2}\sigma_3^2\right)t + \int_0^t cy\,(u)\; du + \sigma_3 \int_0^t\; d\mathfrak{B}_3(u). \tag{66}$$

Consequently

$$\ln z(t) \leq \ln z(0) + \left(c - \delta - \frac{1}{2}\sigma_3^2\right)t + \sigma_3 \mathfrak{B}_3(t). \tag{67}$$

Then

$$\lim_{t\to\infty} \frac{\ln z(t)}{t} \leq \lim_{t\to\infty} \frac{\ln z(0)}{t} + \lim_{t\to\infty} \frac{\left(c-\delta-\frac{1}{2}\sigma_3^2\right)t}{t} + \lim_{t\to\infty} \frac{\sigma_3 \mathfrak{B}_3(t)}{t}.$$

Therefore,

$$\lim_{t\to\infty} \sup \frac{\ln z(t)}{t} \leq c - \delta - \frac{1}{2}\sigma_3^2 < 0, \text{ almost surely.} \tag{68}$$

Thus, the required claim is validated.

## 6. Stochastic Stability and Random Attractors

Our focus in this section is on studying the stability of model (22), where we will show under certain conditions that the coexistence equilibrium points and the level equilibrium points are stochastically asymptotically stable. In addition, we will study the exponential stability, asymptotic stability, and random attractors of the random dynamical system generated by the SDEs system described in system (22).

**Theorem 6.1.** The GAS for $E_0 = (0, 0, 0)$ of (22) occur when $\mu_1 := \frac{a}{r} > 1$ and $\mu_2 := \frac{b}{s} > 1$.

**Proof.** Suppose that a Lyapunov function $\mathcal{V}: \Re^3 \longrightarrow \Re$, defined by $\mathcal{V}(x, y, z) = x + y + z$. Since $x, y, z > 0$, then $\mathcal{V}(x, y, z) > 0$. The derivative of $\mathcal{V}$ along the solutions of (22), can be computed by Itô formula ,

$$\mathcal{L}\mathcal{V}(X, t) = (r - a)x + (s - b)y - (r + s)xy - rxz - syz - rx^2 - sy^2.$$

Now, $\mathcal{L}\mathcal{V}(O, t) = 0$, $O = (0,0,0)$.Hence $\mathcal{L}\mathcal{V}(X, t) < 0$ , whene $r < a$ and $s < b$. Thus $E_0 = (0, 0, 0)$ is GAS [34,35] and hence by Theorem 2.2 in [25] it is stochastically stable.

Note that the equilibrium points of (22) are the steady-state solutions. Model (22) has one trivial equilibrium point which is $E_0 = (0, 0, 0)$: signifies a free equilibrium wherever totally populations become extinct.

**Theorem 6.2.** The equilibrium $E_1 = (\mu_1, 0, 0)$, where $\mu_1 = 1 - \frac{a}{r}$, of (22) is SAS on $\Delta$, whenever

$$\mu_1 \geq 1, \quad \mu_1 > \frac{2+s-b-\delta}{2} \text{ and } \frac{1}{2}\frac{\sigma_1^2 x_1}{x^2} \leq \eta_1(x, y, z), \tag{69}$$

Where $\eta_1(x, y, z) = [x(x_1 - \mu_1) + y(b - x_1 - s) + z(\delta - x_1) + \mu_1 x_1]$.

**Proof.** We create the appropriate Lyapunov function in the manner described below:

$$\mathcal{V}(x, y, z) = \left(x - x_1 - x_1 \ln \frac{x}{x_1}\right) + y + z. \tag{70}$$

We define $\mathcal{L}$ on $\mathcal{V}$ to obtain

$$\begin{aligned}
\mathcal{L}\mathcal{V}(x, y, z) &= x\left(1 + x_1 - \frac{a}{r}\right) + y(x_1 + s - b) + z(x_1 - \delta) - xy - xz \\
&\quad + x_1\left(\frac{a}{r} - 1\right) - syx - sy^2 - syz - x^2 + \frac{1}{2}\frac{\sigma_1^2 x_1}{x^2} \\
&\leq -[x(x_1 - \mu_1) + y(b - x_1 - s) + z(\delta - x_1) + \mu_1 x_1] + \frac{1}{2}\frac{\sigma_1^2 x_1}{x^2} \\
&= -\eta_1(x, y, z) + \frac{1}{2}\frac{\sigma_1^2 x_1}{x^2}
\end{aligned} \tag{71}$$

If $\mu_1 \geq 1$, $\mu_1 > \frac{2+s-b-\delta}{2}$, it follows that $\eta_1(x, y, z) \geq 0$. By assumptions, $\mathcal{L}\mathcal{V}(x, y, z) < 0$ on $\Delta = \{(x, y, z) \in \Re_+^3 : \eta_1(x, y, z) = 0\}$. Therefore, $E_1$ is SAS.

Biological interpolation of Theorem 6.2: The cancer extinction equilibrium $E_1 = (1 - \mu_1, 0, 0)$, $\mu_1 = \frac{a}{r}$: if the death rate of the infected cells is positive, they will follow the extinction of cancer.

**Theorem 6.3** The equilibrium $E_2 = (0, 1 - \mu_2, 0)$ of (22) is SAS on $\mathcal{D}$, whenever

$$\mu_2 = 1, \ \frac{s}{c(s-1)} > 1, \ \frac{s}{r-a} > 1 \text{ and } \frac{1}{2}\frac{\sigma_2^2 y_2}{y^2} \leq \eta_2(x, y, z) \tag{72}$$

where

$$\eta_2(x,y,z) = [(y_2 - r + a + s)x + \left(\frac{b}{s} - 1 - y_2\right)y + \left(\delta - y_2 - \frac{c}{s}y_2\right)z + \left(1 + \frac{c}{s} - c\right)yz + \left(1 - \frac{b}{s}\right)y_2].$$

**Proof.** For $E_2$, define a Lyapunov function as follows

$$\mathcal{V}(x,y,z) = x + \frac{1}{s}\left(y - y_2 - y_2 \ln\frac{y}{y_2}\right) + z.$$

So,

$$\mathcal{L}\mathcal{V}(x,y,z) = (r - a - s - y_2)x + \left(1 + y_2 - \frac{b}{s}\right)y + \left(y_2 - \delta + \frac{c}{s}y_2\right)z + \left(c - 1 - \frac{c}{s}\right)yz + \left(\frac{b}{s} - 1\right)y_2 - rx^2 - rxy - rxz - y^2 + \frac{1}{2}\frac{\sigma_2^2 y_2}{y^2},$$

where $y_2 = 1 - \frac{b}{s}$. Thus

$$\mathcal{L}\mathcal{V}(x,y,z) \le (r - a - s - y_2)x + \left(1 + y_2 - \frac{b}{s}\right)y + \left(y_2 - \delta + \frac{c}{s}y_2\right)z + \left(c - 1 - \frac{c}{s}\right)yz + \left(\frac{b}{s} - 1\right)y_2 + \frac{1}{2}\frac{\sigma_2^2 y_2}{y^2}.$$

$$= -\eta_2(x,y,z) + + \frac{1}{2}\frac{\sigma_2^2 y_2}{y^2}$$

Thus $\eta_2(x,y,z) \ge 0$ if $\mu_2 = 1$, $\frac{s}{c(s-1)} \ge 1$, and $\frac{s}{r-a} \ge 1$. By assumptions, $\mathcal{L}\mathcal{V}(x,y,z) < 0$ on $\{(x,y,z) \in \Re_+^3 : \eta_2(x,y,z) = 0\}$. Therefore, $E_2$ is SAS.

Biological interpolation of Theorem 6.3: The virus population reaches zero at the virus extinction equilibrium $E_2 = (0, 1 - \mu_2, 0)$, indicating that normal cells are also extinct.

**Theorem 6.4.** The equilibrium $E_3 = (0, \frac{d}{c}, \frac{sc - bc - ds}{s + c})$ of (22) is SAS on $\mathcal{D}$, whenever

$$\frac{a - r + 1}{2(s - b - 1)} > 1 \text{ and } \frac{1}{2}\frac{\sigma_2^2 y_3}{y^2} + \frac{1}{2}\frac{\sigma_3^2 y_3}{y^2} \le \eta_3(x,y,z) \tag{73}$$

where

$$\eta_3(x,y,z) = [(sy_3 - r + a)x + (b + c - s - sy_3)y + \left(\frac{\delta}{z_3} - sy_3 - cy_3\right)z + \left(c + s - \frac{c}{z_3}\right)yz + (sy_3 - by_3 - \delta)]$$

**Proof.** Define

$$\mathcal{V}(x,y,z) = x + \left(y - y_3 - y_3 \ln\frac{y}{y_3}\right) + \frac{1}{z_3}\left(z - z_3 - z_3 \ln\frac{z}{z_3}\right),$$

where $y_3 = \frac{d}{c}$ and $z_3 = \frac{sc - bc - ds}{s + c}$. Therefore,

$$\mathcal{LV}(x,y,z) = (r-a-sy_3)x + (s+sy_3-b-c)y + \left(sy_3+cy_3-\frac{\delta}{z_3}\right)z + \left(\frac{c}{z_3}-s-c\right)yz + (by_3-sy_3+\delta) - rx^2 - rxy - rxz - syx - sy^2 + \frac{1}{2}\frac{\sigma_2^2 y_3}{y^2} + \frac{1}{2}\frac{\sigma_3^2 z_3}{z^2}$$

$$\leq (r-a-sy_3)x + (s+sy_3-b-c)y + \left(sy_3+cy_3-\frac{\delta}{z_3}\right)z + \left(\frac{c}{z_3}-s-c\right)yz + (by_3-sy_3+\delta) + \frac{1}{2}\frac{\sigma_2^2 y_3}{y^2} + \frac{1}{2}\frac{\sigma_3^2 z_3}{z^2}$$

$$= -\eta_3(x,y,z) + \frac{1}{2}\frac{\sigma_2^2 y_3}{y^2} + \frac{1}{2}\frac{\sigma_3^2 z_3}{z^2}$$

Clearly that $\eta_3(x,y,z) \geq 0$ whenever $\frac{s\delta}{c(a-r)} \leq 1$, $\frac{s\delta}{c(b-s+c)} \leq 1$, $\frac{sc-bc-s\delta}{c} \leq 1$, $\frac{s}{c+b} \geq 1$, and $\frac{sc-bc-s\delta}{c} \geq 1$. Combining these conditions, we get the following condition $\frac{a-r+1}{2(s-b-1)} \geq 1$. By assumptions, $\mathcal{LV}(x,y,z) < 0$ on $\mathcal{D}$. So, $E_3$ is SAS.

Biological interpolation of Theorem 6.4: According to the cancer-virus equilibrium $E_3 = (0, \frac{d}{c}, \frac{sc-bc-ds}{s+c})$, cancer cells and virus-infected cells with constant sizes are sent off while normal cells go extinct.

**Theorem 6.5.** The equilibrium $E_4 = (\frac{c(r-a)-r(\delta+b)-as}{rc}, \frac{\delta}{c}, \frac{as}{rc}-\frac{b}{c})$ of (22) is SAS on $\mathcal{D}$, whenever $\frac{2c+b}{c+s} < \mu_1 < \frac{sa\delta-2br\delta}{r(c-\delta+b)-a(c+s)}$ and $\frac{1}{2}\frac{\sigma_1^2 x_4}{x^2} + \frac{1}{2}\frac{\sigma_2^2 y_4}{y^2} + \frac{1}{2}\frac{\sigma_3^2 z_4}{z^2} \leq \eta_4(x,y,z)$, where

$$\eta_4(x,y,z) = -(1+x_4+y_4) - (1+x_4+y_4)y - \left(x_4+y_4+\frac{c}{s}y_4\right)z - \left(\frac{a}{r}x_4+\frac{b}{s}y_4+\frac{\delta}{s}z_4\right)$$

**Proof.** For the coexistence equilibrium, create the appropriate Lyapunov function as follows.

$$\mathcal{V}(x,y,z) = \frac{1}{r}\left(x-x_2-x_2\ln\frac{x}{x_2}\right) + \frac{1}{s}\left(y-y_2-y_2\ln\frac{y}{y_2}\right) + \frac{1}{s}\left(z-z_2-z_2\ln\frac{z}{z_2}\right),$$

Define $\mathcal{L}$ on $\mathcal{V}(x,y,z)$ and grouping coefficients of the positive terms we get

$$\mathcal{LV}(x,y,z) = (1+x_4+y_4) + (1+x_4+y_4)y + \left(x_4+y_4+\frac{c}{s}y_4\right)z + \left(\frac{a}{r}x_4+\frac{b}{s}y_4+\frac{\delta}{s}z_4\right) - x^2 - xy - xz - \frac{a}{r}x - x_4 - yx - y^2 - \frac{b}{s}y - y_4 - \frac{\delta}{s}z - \frac{c}{s}z_4 y + \frac{1}{2}\frac{\sigma_1^2 x_4}{x^2} + \frac{1}{2}\frac{\sigma_2^2 y_4}{y^2} + \frac{1}{2}\frac{\sigma_3^2 z_4}{z^2}$$

$$\leq (1+x_4+y_4)+(1+x_4+y_4)y+\left(x_4+y_4+\frac{c}{s}y_4\right)z+\left(\frac{a}{r}x_4+\frac{b}{s}y_4+\frac{\delta}{s}z_4\right)+\frac{1}{2}\frac{\sigma_1^2x_4}{x^2}$$
$$+\frac{1}{2}\frac{\sigma_2^2y_4}{y^2}+\frac{1}{2}\frac{\sigma_3^2z_4}{z^2}$$
$$=-\eta_4(x,y,z)+\tfrac{1}{2}\tfrac{\sigma_1^2x_4}{x^2}+\tfrac{1}{2}\tfrac{\sigma_2^2y_4}{y^2}+\tfrac{1}{2}\tfrac{\sigma_3^2z_4}{z^2}$$

We find that $\eta_4(x,y,z)\geq 0$ if $\frac{2c+b}{c+s}<\mu_1<\frac{sa\delta-2br\delta}{r(c-\delta+b)-a(c+s)}$. By assumptions, $\mathcal{LV}(x,y,z)<0$ on $\mathcal{D}=\{(x,y,z)\in\Re_+^3:\eta_4(x,y,z)=0\}$. Therefore, the coexistence equilibrium $E_4$ of (22) is SAS.

Biological interpolation of Theorem 6.5: Three population coexistence equilibrium $E_4=(\frac{c(r-a)-r(\delta+b)-as}{rc},\frac{\delta}{c},\frac{as}{rc}-\frac{b}{c})$: There are all three cell types, and their populations are steady.

The remaining part of this section is dedicated to studying the asymptotic stability, exponential stability, Lyapunov exponent, and random attractors of the stochastic dynamic system generated by the stochastic differential equations system (22).

**Theorem 6.6.** Consider the system

$$\begin{cases} dx = & rx\,(1-x\,)dt+\sigma_1 x d\mathfrak{B}_1 \\ dy = & sy\,(1-y\,)dt+\sigma_2 y d\mathfrak{B}_2 \\ dz = & -\delta z dt+\sigma_3 z d\mathfrak{B}_3 \end{cases} \tag{74}$$

corresponding to the original (non-linear) system (22). The random equilibriums of this linear system are GAS and exponentially stable.

**Proof** From the first equation we have

$$x(t)=\frac{\exp\left\{\left(r-\frac{1}{2}\sigma_1^2\right)t+\sigma_1\mathfrak{B}_1(t)\right\}}{x^{-1}+r\int_0^t\exp\left\{\left(r-\frac{1}{2}\sigma_1^2\right)\tau+\sigma_1\mathfrak{B}_1(\tau)\right\}d\tau} \tag{75}$$

The equation (75) induce the RDS $(\theta,\varphi_x)$ where

$$\varphi_x(t,\omega)x=\begin{cases}\dfrac{\exp\left\{\left(r-\frac{1}{2}\sigma_1^2\right)t+\sigma_1\mathfrak{B}_1(t)\right\}}{x^{-1}+r\int_0^t\exp\left\{\left(r-\frac{1}{2}\sigma_1^2\right)\tau+\sigma_1\mathfrak{B}_1(\tau)\right\}d\tau}, & x>0\\ 0 & x=0\end{cases}$$

This RDS is strictly order-preserving in $\mathbb{R}^+$. the random set $\mathcal{A}(\omega)=[0,u(\omega)]$ is a random attractor for $(\theta,\varphi_x)$ in $\Re^+$, where $u(\omega)\geq 0$ is a random equilibrium.

Consequently $\mathcal{A}(\omega)=\begin{cases}\{0\}, & r<0\\ [0,u_{\alpha,\beta,N}(\omega)], & r>0\end{cases}$, where $u_{\alpha,\beta,N}(\omega):=r\int_{-\infty}^0\exp\{r\tau+\sigma_1\mathfrak{B}_1(\tau,\omega)\}\,d\tau$

In addition, it can be shown that there exists a $\gamma > 0$ with

$$\lim_{t\to\infty} e^{\gamma}\left|\varphi_x(t,\theta_{-t}\omega)x - u_{\alpha,\beta,N}(\omega)\right| = 0 \ , \ \forall \ x > 0 \ ,\omega \in \Omega. \tag{76}$$

If $n = 2m+1$ is odd, $m \geq 1$, then the first equation of the system (74) is invariant with respect to the transformation $x \mapsto -x$.

The random set $\mathcal{A}_x(\omega) = \begin{cases} \{0\}, & \alpha < 0 \\ \left[-u_{\alpha,\beta,n}(\omega), u_{\alpha,\beta,n}(\omega)\right], & \alpha > 0 \end{cases}$, is the random attractor of $(\theta, \varphi_x)$ .In the latter case $u_{\alpha,\beta,n}(\omega)$ ( resp. $-u_{\alpha,\beta,n}(\omega)$ ) is globally stable random equilibrium and $u_0 \equiv 0$ is an unstable random equilibrium. So, as $\alpha$ increases through $0$.

From the second equation we have

$$\mathcal{y}(t) = \frac{\exp\left\{\left(s-\frac{1}{2}\sigma_2^2\right)t+\sigma_2\mathfrak{B}_2(t)\right\}}{\mathcal{y}^{-1}+s\int_0^t \exp\left\{\left(S-\frac{1}{2}\sigma_2^2\right)\tau+\sigma_2\mathfrak{B}_2(\tau)\right\}d\tau} \tag{77}$$

The equation (77) induce the RDS $(\theta, \varphi_{\mathcal{y}})$ where

$$\varphi_{\mathcal{y}}(t,\omega)\mathcal{y} = \begin{cases} \dfrac{\exp\left\{\left(s-\frac{1}{2}\sigma_2^2\right)t+\sigma_2\mathfrak{B}_2(t)\right\}}{\mathcal{y}^{-1}+s\int_0^t \exp\left\{\left(S-\frac{1}{2}\sigma_2^2\right)\tau+\sigma_2\mathfrak{B}_2(\tau)\right\}d\tau}, & \mathcal{y} > 0 \\ 0 & \mathcal{y} = 0 \end{cases}.$$

This RDS is strictly order-preserving in $\Re^+$. The random set $\mathcal{B}_{\mathcal{y}}(\omega) = [0, v(\omega)]$ is a random attractor for $(\theta, \varphi_{\mathcal{y}})$ in $\Re^+$, where $v(\omega) \geq 0$ is a random equilibrium. Consequently $\mathcal{B}_{\mathcal{y}}(\omega) = \begin{cases} \{0\}, & s < 0 \\ \left[0, v_{\alpha,\beta,n}(\omega)\right], & s > 0 \end{cases}$, where

$$v_{\alpha,\beta,n}(\omega) \coloneqq s\int_{-\infty}^{0} \exp\left\{\left(s-\frac{1}{2}\sigma_2^2\right)\tau + \sigma_2\mathfrak{B}_2(\tau,\omega)\right\}d\tau.$$

In addition, it can be shown that (see Proposition 1.14) there exists a $\gamma > 0$ with

$$\lim_{t\to\infty} e^{\gamma}\left|\varphi_{\mathcal{y}}(t,\theta_{-t}\omega)\mathcal{y} - v_{\alpha,\beta,n}(\omega)\right| = 0 \ , \ \forall \ x > 0 \ ,\omega \in \Omega. \tag{78}$$

If $n = 2m+1$ is odd, $m \geq 1$, then the second equation of the system (74) is invariant with respect to the transformation $\mathcal{y} \mapsto -\mathcal{y}$.

The random set $A_{\mathcal{y}}(\omega) = \begin{cases} \{0\}, & \alpha < 0 \\ \left[-v_{\alpha,\beta,n}(\omega), v_{\alpha,\beta,n}(\omega)\right], & \alpha > 0 \end{cases}$ is an attractor of $(\theta, \varphi_{\mathcal{y}})$ .In the latter case $v_{\alpha,\beta,n}(\omega)$ ( resp. $-v_{\alpha,\beta,N}(\omega)$) is globally stable random equilibrium and $u_0 \equiv 0$ is an unstable random equilibrium. So, as $\alpha$ increases through $0$.

From the third equation we have $dz \geq -\delta z dt + \sigma_3 z dW_3$. The equation $dz = -\delta z dt + \sigma_3 z d\mathfrak{B}_3$ creates an affine RDS $(\theta, \varphi_z)$ in $\mathbb{R}$. Then $\varphi_z$ admits the form (4), where

$$\varphi_z(t,\omega)z = \Phi_z(t,\omega)z = z\exp\left\{-\left(\delta + \frac{\sigma_3^2}{2}\right)t + \sigma_3\mathfrak{B}_3(t,\omega)\right\}.$$

Then $\lambda_3 := -\left(\delta + \frac{\sigma_3^2}{2}\right)$ is the (top) Lyapunov exponent for $(\theta, \varphi_z)$. Since $\lambda_3 := -\left(\delta + \frac{\sigma_3^2}{2}\right) < 0$ ( from the fact that $\left(\delta + \frac{\sigma_3^2}{2}\right) > 0$) , consequently $(\theta, \varphi)$ is dissipative[36] in the universe of all tempered subsets of $\mathfrak{R}$. It follows from Proposition 1.13 that , $(\theta, \varphi_z)$ admits a unique equilibrium $u(\omega)$. Since

$$u(\omega) := \lim_{t \to +\infty} \psi(t, \theta_{-t}\omega) \text{ and } \psi(t, \omega) = \beta \int_0^t \exp\{\lambda(t-\tau) + \sigma(\mathfrak{B}_t(\omega) - \mathfrak{B}_\tau(\omega))\}\, d\tau$$

then $z^*(\omega) = \{0\}$. This equilibrium is measurable with respect to the past $\sigma$ −algebra $\mathcal{F}_-$ (see [26] Sec.1.10) and exponentially stable[18,19] .This equilibrium is GAS [18,19]. From the above discussion we have two random equilibriums of the linear system (74) Corresponding to the original system (22). They are $X_1(\omega) = (x^*(\omega), y^*(\omega), 0)$, and $X_2(\omega) = (u_{\alpha,\beta,n}(\omega), v_{\alpha,\beta,n}(\omega), 0)$.The last one occur when $r, s < 0$. The equilibrium $X_1(\omega) = (x^*(\omega), y^*(\omega), 0)$ is the random super-equilibrium of the original system (22) which is measurable with respect to the future $\sigma$ −algebra $\mathcal{F}_+$. This equilibrium is GAS and exponentially stable Proposition 1.13.

**Theorem 6.7** The random equilibrium $X_0(\omega) = (0,0,0)$ of the linear stochastic system

$$\begin{cases} dx = & rx\, dt + \sigma_1 x d\mathfrak{B}_1 \\ dy = & sy\, dt + \sigma_2 y d\mathfrak{B}_2 \\ dz = & -\delta z dt + \sigma_3 z d\mathfrak{B}_3 \end{cases} \tag{79}$$

is a super-equilibrium of the (non-linear) system (22) which is GAS and exponentially stable.

**Proof**. The equation $dx = rxdt + \sigma_1 x dW_1$ creates an affine RDS $(\theta, \varphi_x)$ in $\mathfrak{R}$. Then $\varphi_x$ admits the form (5), wherever

$$\varphi_x(t, \omega)x = \Phi_x(t, \omega)x = x \exp\{rt + \sigma_1 \mathfrak{B}_1(t, \omega)\}.$$

Then $\lambda_1 := r$ is the (top) Lyapunov exponent for $(\theta, \varphi_x)$.

If $\lambda_1 := r > 0$, then $(\theta, \varphi_x)$ possesses the equilibrium $x^*(\omega) = 0$ which is measurable with respect to the future $\sigma$ −algebra $\mathcal{F}_+$. This equilibrium is GAS (see Proposition 1.13). The equation $dy = sydt + \sigma_2 y d\mathfrak{B}_2$ creates an affine RDS $(\theta, \varphi_y)$ in $\mathfrak{R}$. So, $\varphi_y$ admits the form (5), wherever $\varphi_y(t, \omega)y = \Phi_y(t, \omega)y = y \exp\{st + \sigma_2 \mathfrak{B}_2(t, \omega)\}$. Then $\lambda_2 := s$ is the (top) Lyapunov exponent for $(\theta, \varphi_y)$. Since $\lambda_2 := s > 0$, then the RDS $(\theta, \varphi_y)$ possesses the equilibrium $y^*(\omega) = 0$ which is measurable with respect to the future $\sigma$ −algebra $\mathcal{F}_+$.This equilibrium is GAS (see Proposition 1.13.).

The equation $dz = -\delta z dt + \sigma_3 z dW_3$ creates an affine RDS $(\theta, \varphi_z)$ in $\mathfrak{R}$. Then $\varphi_z$

admits the form (5), wherever $\varphi_z(t,\omega)z = \Phi_z(t,\omega)z = z\exp\{-\delta t + \sigma_3\mathfrak{B}_3(t,\omega)\}$.Then $\lambda_3 := -\delta$ is the (top) Lyapunov exponent for $(\theta,\varphi_z)$. Since $\lambda_3 := -\delta < 0$( from the fact that $\delta > 0$) , consequently $(\theta,\varphi_z)$ is dissipative in the universe of all tempered subsets of $\Re$. It follows from Proposition 1.13 hat $(\theta,\varphi_z)$ admits a unique equilibrium $u(\omega)$. As in Theorem 6.6, we get $z^*(\omega) = \{0\}$. This equilibrium is measurable with respect to the past $\sigma-$algebra $\mathcal{F}_-$ (see [37]) and exponentially stable. This equilibrium is GAS. From the above discussion we get the point $X_0(\omega) = (0,0,0)$ is the random equilibrium of the system (79) corresponding to the system (22). Thus $X_0(\omega) = (0,0,0)$ is the random super-equilibrium of the (non-linear) system (22) which is measurable with respect to the future $\sigma-$algebra $\mathcal{F}_+$. This equilibrium is GAS and exponentially stable(see [26,38]).

## 7. NUMERICAL SIMULATION

Now, we do a numerical simulation to confirm the results and see how well they match reality, as well as to make our conclusions more realistic. The comparable estimation formulas are as follows:

$$\begin{aligned} x_{k+1} &= x_k + [rx_k(1-x_k) - rx_k(y_k+z_k) - ax_k]\Delta t + x_k\left[\sigma_1\sqrt{\Delta t}\xi_{k,1} + \frac{\sigma_1^2}{2}(\xi_{k,1}^2 - 1)\Delta t\right] \\ y_{k+1} &= y_k + [sy_k(1-y_k) - sy_k(x_k+z_k) - (by_k + cy_kz_k)]\Delta t \\ &\quad + y_k\left[\sigma_2\sqrt{\Delta t}\xi_{k,2} + \frac{\sigma_2^2}{2}(\xi_{k,2}^2 - 1)\Delta t\right] \\ z_{k+1} &= z_k + (cy_k - \delta)z_k\Delta t + z_k\left[\sigma_3\sqrt{\Delta t}\xi_{k,3} + \frac{\sigma_3^2}{2}(\xi_{k,3}^2 - 1)\Delta t\right]. \end{aligned} \tag{80}$$

All of the numerical simulations were performed using the parameter values listed in Table 1 as in [9]. The choice of parameter units was made at random.

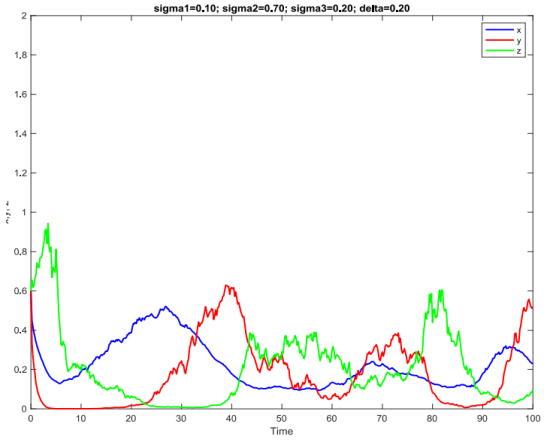


**Figure 1:** Model dynamics using starting values of $x_0 = 0.6$, $y_0 = 0.6$, and $z_0 = 0.6$.

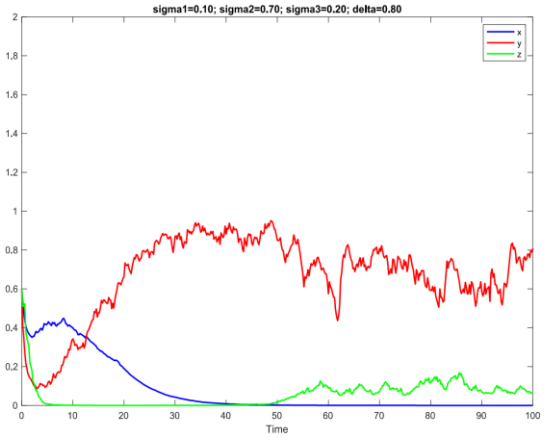


**Figure 2:** Model dynamics using starting values of $x_0 = 0.5$, $y_0 = 0.5$, and $z_0 = 0.6$.

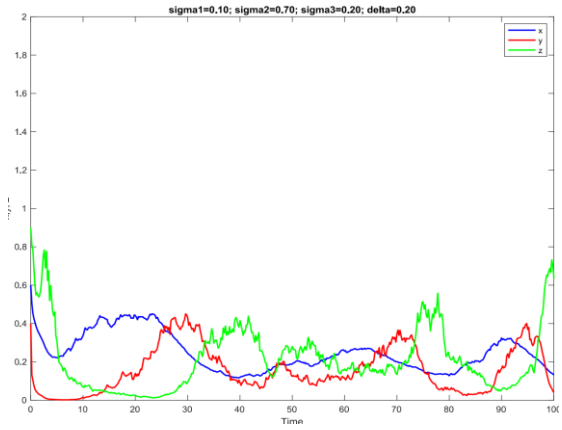


**Figure 3:** Model dynamics using starting values of $x_0 = 0.6$, $y_0 = 0.4$, and $z_0 = 0.09$.

## 8. DISCUSSION

In this work, we examined an oncolytic virotherapy model that was created in the research [5,6] to explain the relationship between viruses and cancer cells. We investigated the qualitative characteristics of the solutions in our study, and we utilized the findings to enhance and broaden the previous research. We anticipate that our research will help us better understand how virus cells interact with cancer cells. It is demonstrated in the aforementioned texts that a high viral clearance rate may lead to treatment failure. Nonetheless, treatment success is achieved by marginally lowering the viral clearance rate while maintaining the other parameter values. This is consistent with the outcome found in [6].

## 9.CONCLUSIONS

According to the research, the effectiveness of the action is independent of the virus's propagation and death values when certain conditions on the death rates of normal and cancer cells are met. The

study emphasizes the significance of understanding the dynamics of virotherapy as both stochastic models and deterministic approaches reveal different behavioral insights. Key findings indicate that specific death and replication rate parameters significantly impact the stochastic model's dynamics, with sensitivity analysis showing that certain factors are crucial for the fundamental reproduction number. The paper concludes that further research is needed to explore additional stochastic effects to better explain virotherapy dynamics.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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