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TIME-DELAY-INDUCED HOPF BIFURCATION IN A FRACTIONAL-ORDER ECOLOGICAL SYSTEM

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Abstract. In this paper, we investigate a fractional-order delayed dynamical system. Taking the time delay τ as the bifurcation parameter, we establish that the equilibrium is asymptotically stable for all $\tau < \tau_0$, and a Hopf bifurcation occurs as τ passes through the critical value τ_0. We derive the characteristic equation and verify the transversality condition, which identifies the onset of oscillations and characterizes the stability of the emerging periodic solutions. Numerical experiments carried out in MATLAB with a predictor–corrector scheme support the analysis and illustrate the resulting oscillatory behavior.

Keywords: fractional calculus; stability; bifurcation analysis; numerical simulation; dynamical system.
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1. Introduction

Delayed differential equation systems have an important place in many fields of science, since many processes, both natural and man-made, such as biology, medicine, chemistry, physics, mathematics, engineering and economics, involve time delays. The best example of time delay from nature is the afforestation of forest areas. After a tree is cut, it takes twenty years for the planted tree to reach maturity. A mathematical model that studies this process must include

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a time delay. By using ordinary differential equations, existing delays are always ignored in systems where modeling is desired, but even very small amounts of delay in the system, may cause major changes in the current state of the system. For this reason, it is more realistic to use delayed differential equations when modeling many of the problems encountered. In the last few decades, the study of dynamical systems of population models has received much attention by theoreticians and experimentalists [1, 18, 19, 20, 21, 28, 31]. To study past-to-present impacts, researchers augment population models with delay terms—most notably in delayed predator–prey systems.

Fractional calculus extends differentiation and integration to non-integer orders. In recent decades, the area has advanced rapidly and found broad applications. Because fractional differential equations naturally encode memory and hereditary effects, they have been used in contexts such as earthquake dynamics [34] and biological modeling [2, 13, 32, 33]. They also provide a flexible *methodology* for a range of mathematical models, including epidemic processes [8, 29, 35], neural networks [30, 33], control systems [4], and chaotic dynamics [36, 37]. Compared with classical integer-order formulations, fractional-order models often provide greater fidelity and adaptability for nonclassical phenomena in the natural and engineering sciences—spanning economics, biology, and electroanalytical chemistry—with particularly notable gains in biological systems, where long-memory and history-dependent molecular dynamics are captured more faithfully [2, 13, 32, 33].

Because fractional operators accumulate past information with weighted influence, they provide a global, memory-aware description of functions. Evidence across many fields indicates that this formalism surpasses the expressive scope of integer-order calculus. Accordingly, recent progress and applications—especially in dynamical systems—have intensified attention on stability properties and qualitative behaviors.

Bifurcation occurs when a small change in the bifurcation parameter chosen in a system around the positive equilibrium point of the system causes a topological change in the behavior of the system. The change in the behavior of the solution occurs when the parameter changes the stability of the equilibrium point. The type of bifurcation that occurs in systems containing two or more first-order differential equations is called “Hopf bifurcation”. At the same time, French mathematician Jules Henri Poincaré (1854-1912), Russian mathematician Alexander A. Andronov (1901-1952), and German mathematician Heinz Hopf (1894-1971) made contributions to the development of this theory. For this reason, it is also referred to as the Poincaré–Andronov–Hopf bifurcation.

Çelik C. and Çekiç G. previously performed the Hopf bifurcation and stability analysis of the following model [1], which is of integer order

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)[r_1 - a_{11}x(t) - a_{12}y(t-\tau)] \\ \frac{dy(t)}{dt} &= y(t)[-r_2 + a_{21}x(t) - a_{22}y(t-\tau)] \end{aligned} \tag{1.1}$$

where $x(t)$ is the population density of the prey and $y(t)$ is the population density of the predator at time t, $\tau > 0$ is the feedback time delay of the predator species to the growth of the species itself, $r_1 > 0$ symbolizes the intrinsic growth rate of the prey and $r_2 > 0$ symbolizes the death rate of the predator. This paper investigates the predator–prey model (1.1) in a fractional-order setting:

$$\begin{aligned} {}_{t_0}^{c}D_t^q x(t) &= x(t)[r_1 - a_{11}x(t) - a_{12}y(t-\tau)] \\ {}_{t_0}^{c}D_t^q y(t) &= y(t)[-r_2 + a_{21}x(t) - a_{22}y(t-\tau)] \end{aligned} \tag{1.2}$$

2. PRELIMINARIES

Definition 2.1 (Podlubny, 1999). *Let $f(t) \in C_{-1}^n$, with $n \in \mathbb{N} \cup \{0\}$. The Caputo fractional derivative is defined by*

$$ {}_0^{C}D_t^q f(t) = \frac{1}{\Gamma(p-q)} \int_0^t \frac{f^{(p)}(\tau)}{(t-\tau)^{q-p+1}} d\tau, \tag{2.1}$$

where $p-1 < q \leq p$.

Definition 2.2 (Podlubny, 1999). *The Laplace transform of the Caputo fractional derivative function $f(t)$ is expressed as follows.*

$$L\{{}_0^{c}D_t^q f(t); s\} = s^q F(s) - \sum_{p=0}^{n-1} s^{q-p-1} f^p(0) \quad n-1 < q \leq n, \tag{2.2}$$

$f^p(0)$, $p = 0, 1, 2, ..., n-1$ are the initial conditions.

3. Analysis of Stability and Hopf Bifurcation for Fractional Dynamics

System (1.1) possesses a unique positive equilibrium $E^* = (x^*, y^*)$, where $x^* = \frac{r_1 a_{22}+r_2 a_{12}}{a_{11}a_{22}+a_{12}a_{21}}$ and $y^* = \frac{r_1 a_{21}-r_2 a_{11}}{a_{11}a_{22}+a_{12}a_{21}}$. To assess the local stability of E^*, we apply the linear shift $u_1(t) = x(t) - x^*$ and $u_2(t) = y(t) - y^*$. For fractional orders $n, p \in (0,1)$, this change of variables recenters the dynamics at the origin and enables the linearized analysis of system (1.1) about E^*. This translation moves the equilibrium to the origin and rewrites system (1.1) in the variables $\mathbf{u} = (u_1, u_2)$; linearizing at $\mathbf{u} = \mathbf{0}$ yields the Jacobian $J(E^*)$ governing the local dynamics (higher-order terms neglected).

$$
\begin{aligned}
\frac{du_1}{dt} &= (u_1(t)+x^*)[r_1 - a_{11}(u_1(t)+x^*) - a_{22}(u_2(t-\tau)+y^*)] \\
\frac{du_2}{dt} &= (u_2(t)+y^*)[-r_2 + a_{21}(u_1(t)+x^*) - a_{22}(u_2(t-\tau)+y^*)]
\end{aligned}
\tag{3.1}
$$

and using relations $r_1 - a_{11}x^* - a_{12}y^* = 0$ and $-r_2 + a_{21}x^* - a_{22}y^* = 0$, we obtain the following linear system

$$
\begin{aligned}
\frac{du_1}{dt} &= \quad -a_{11}x^*u_1(t) - a_{12}x^*u_2(t-\tau) \\
\frac{du_2}{dt} &= \quad a_{21}y^*u_1(t) - a_{22}y^*u_2(t-\tau).
\end{aligned}
\tag{3.2}
$$

By replacing the left-hand side of equation (3.2) with the fractional derivative of order $q \in (0,1]$, the following system is obtained.

$$
\begin{aligned}
{}_{t_0}^{c}D_t^q u_1(t) &= \quad -a_{11}x^*u_1(t) - a_{12}x^*u_2(t-\tau) \\
{}_{t_0}^{c}D_t^q u_2(t) &= \quad a_{21}y^*u_1(t) - a_{22}y^*u_2(t-\tau).
\end{aligned}
\tag{3.3}
$$

When the Laplace transform of the system (3.3) is applied, the following equations are obtained.

$$
L\{{}_{t_0}^{c}D_t^q u_1(t)\} \quad = L\{-a_{11}x^*u_1(t) - a_{12}x^*u_2(t-\tau)\}
$$

and

$$
s^q U_1(s) - s^{q-1}U_1(0) = \quad -a_{11}x^*U_1(s) - a_{12}x^*e^{-s\tau}\left(\int_{-\tau}^{0}\theta(t)e^{-st}dt + U_2(s)\right)
$$

Similarly,

$$\begin{aligned}
L\{{}^{c}_{t_0}D^q_t u_2(t)\} &= L\{a_{21}y^*u_1(t) - a_{22}y^*u_2(t-\tau)\} \\
L\{{}^{c}_{t_0}D^q_t u_2(t)\} &= a_{21}y^*U_1(s) - a_{22}y^*e^{-s\tau}(\int_{-\tau}^{0}\theta(t)e^{-st}dt + U_2(s)) \\
s^qU_2(s) - s^{q-1}U_2(0) &= a_{21}y^*U_1(s) - a_{22}y^*e^{-s\tau}(\int_{-\tau}^{0}\theta(t)e^{-st}dt + U_2(s))
\end{aligned}$$

The system can be reformulated as follows:

$$\Delta(s)\begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} = \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}$$

where

$$\Delta(s) = \begin{pmatrix} s^q + a_{11}x^* & a_{12}x^*e^{-s\tau} \\ -a_{21}y^* & s^q + a_{22}y^*e^{-s\tau} \end{pmatrix},$$

$$\begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix} = \begin{pmatrix} s^{q-1}U_1(0) - a_{12}x^*e^{-s\tau}(\int_{-\tau}^{0}\theta(t)e^{-st}dt) \\ s^{q-1}U_2(0) - a_{22}y^*e^{-s\tau}(\int_{-\tau}^{0}\theta(t)e^{-st}dt) \end{pmatrix}.$$

$$\begin{aligned}
det(\Delta(s)) =&(s^q + a_{11}x^*)(s^q + a_{22}y^*e^{-s\tau}) - (-a_{21}y^*a_{12}x^*e^{-s\tau}) \\
=&s^{2q} + a_{11}x^*s^q + a_{22}y^*e^{-s\tau}s^q + (a_{11}a_{22} + a_{12}a_{21})x^*y^*e^{-s\tau}.
\end{aligned}$$

$k_i (i = 1,2,3)$ are determined by

$$k_1 = a_{11}x^*, \quad k_2 = a_{22}y^*, \quad k_3 = (a_{11}a_{22} + a_{12}a_{21})x^*y^*.$$

Thus,

$$det(\Delta(s)) = \quad s^{2q} + k_1s^q + k_2e^{-s\tau}s^q + k_3e^{-s\tau}. \tag{3.4}$$

For $s = i\omega$ $(\omega > 0)$,

$$det(\Delta(s)) =(i\omega)^{2q} + k_1(i\omega)^q + k_2e^{-i\omega\tau}(i\omega)^q + k_3e^{-i\omega\tau} = 0. \tag{3.5}$$

Using appropriate trigonometric identities for i^{2q}, i^q, and $e^{-i\omega\tau}$, we obtain

$$\begin{aligned}
det(\Delta(s)) =&\omega^{2q}(\cos(q\pi) + i\sin(q\pi)) + k_1\omega^q(\cos(\frac{q\pi}{2}) + i\sin(\frac{q\pi}{2})) \\
&+k_2(\cos(\omega\tau) - i\sin(\omega\tau))w^q(\cos(\frac{q\pi}{2}) + i\sin(\frac{q\pi}{2}))
\end{aligned}$$

$$+k_3(cos(\omega\tau)-isin(\omega\tau))$$

$$=0. \tag{3.6}$$

Real part of Eq. (3.6):

$$\cos(q\pi)\omega^{2q}+k_1\cos(\frac{q\pi}{2})\omega^q \quad = -k_2\omega^q\cos(\tfrac{q\pi}{2}-\omega\tau)-k_3cos(\omega\tau)$$

Imaginary part of Eq. (3.6);

$$\sin(q\pi)\omega^{2q}+k_1\sin(\frac{q\pi}{2})\omega^q \quad = -k_2\omega^q\sin(\tfrac{q\pi}{2}-\omega\tau)+k_3cos(\omega\tau).$$

If we square both sides of the real and imaginary parts separately and add them, we obtain the following equation.

$$\omega^{4q} \quad +2k_1cos(\tfrac{q\pi}{2})\omega^{3q}+(k_1^2-k_2^2)\omega^{2q}-2k_2k_3cos(\tfrac{q\pi}{2})\omega^q-k_3^2=0. \tag{3.7}$$

Since $\cos(\frac{q\pi}{2})>0$, $\omega^q>0$ and $0<q<1$. Let $v=\omega^q$, then we get $h(v)=v^4+2k_1cos(\frac{q\pi}{2})v^3+(k_1^2-k_2^2)v^2-2k_2k_3cos(\frac{q\pi}{2})v-k_3^2=0$. Since $h(0)=-(k_3)^2<0$ and $\lim_{v\to\infty}h(v)=\infty$, there exists a $v_0>0$ such that $h(v_0)=0$. Next, we determine the critical delay τ.
Since,

$$det(\Delta(s)) = \quad s^{2q}+k_1s^q+k_2e^{-s\tau}s^q+k_3e^{-s\tau}.$$

Then, we get

$$\begin{aligned} det(\Delta(s)) &= \quad s^{2q}+k_1s^q+k_2e^{-s\tau}s^q+k_3e^{-s\tau}=0,\\ A &= \quad s^{2q}+k_1s^q,\\ E &= \quad k_2s^q+k_3,\\ A+Ee^{-s\tau} &= \quad 0.\end{aligned}$$

Let $s=i\omega(\omega>0)$, and let A_1 represent the real part of A and A_2 the imaginary part of A. Similarly, let E_1 represent the real part of E and E_2 the imaginary part of E. Thus,

$$(A_1+iA_2)+(E_1+iE_2)(\cos(\omega\tau)-i\sin(\omega\tau))=0$$

where

$$\begin{aligned} A_1 = & \quad \omega^{2q}\cos(q\pi) + k_1\omega^q\cos(\tfrac{q\pi}{2}), \\ A_2 = & \quad \omega^{2q}\sin(q\pi) + k_1\omega^q\sin(\tfrac{q\pi}{2}), \\ E_1 = & \quad k_2\omega^q cos(\tfrac{q\pi}{2}) + k_3, \\ E_2 = & \quad k_2\omega^q sin(\tfrac{q\pi}{2}). \end{aligned}$$

We can write

$$\begin{aligned} A_1 + E_1\cos(\omega\tau) + E_2\sin(\omega\tau) = & \quad 0, \\ A_2 + E_2\cos(\omega\tau) - E_1\sin(\omega\tau) = & \quad 0. \end{aligned}$$

$$\begin{aligned} \frac{\sin(\omega\tau)}{\cos(\omega\tau)} = & \quad \tfrac{A_1E_2 - A_2E_1}{A_1E_1 + A_2E_2}, \\ \tan(\omega\tau) = & \quad \tfrac{A_1E_2 - A_2E_1}{A_1E_1 + A_2E_2}, \\ \omega\tau = & \quad \arctan(\tfrac{A_1E_2 - A_2E_1}{A_1E_1 + A_2E_2}), \end{aligned}$$

which leads to

$$\tau_k = \frac{1}{\omega}\arctan(\frac{A_1E_2 - A_2E_1}{A_1E_1 + A_2E_2}) + \frac{k\pi}{\omega} \quad \text{for} \quad k = 0, 1, 2, ...$$

Let $s(\tau) = \alpha(\tau) + i\,\omega(\tau)$ be a root of (3.4) in a neighborhood of $\tau = \tau_k$ such that $\alpha(\tau_k) = 0$ and $\omega(\tau_k) = \omega_1$ for $k = 0, 1, 2, \ldots$. Then the following result holds.

Lemma 3.1. *Assume $g'(z_1) \neq 0$, then the following transversality condition is satisfied:*

$$\frac{d(Res(\tau_k))}{d\tau} \neq 0, \quad k = 0, 1, 2, 3, ...$$

and $g'(z_1)$ and $\frac{d(Res(\tau_k))}{d\tau}$ have the same sign.

Proof. Assume that at $\tau = \tau_k$ the characteristic equation (3.4) admits a purely imaginary root $s = i\omega$ with $\omega \in \mathbb{R}$, $\omega > 0$. Differentiating (3.4) with respect to τ yields

$$2qs^{2q-1}\frac{ds}{d\tau}+k_1qs^{q-1}\frac{ds}{d\tau}+(k_2qs^{q-1}\frac{ds}{d\tau})e^{-s\tau}-e^{-s\tau}(\tau\frac{ds}{d\tau}+s)(k_2s^q+k_3)=0.$$

So,

$$\frac{d\tau}{ds}=\frac{2qs^{2q-1}+k_1qs^{q-1}}{s(k_2s^q+k_3)}e^{s\tau}+\frac{k_2qs^{q-1}}{s(k_2s^q+k_3)}-\frac{\tau}{s}.$$

If we take $s=i\omega$, we can get

$$\begin{aligned} Re(\frac{d\tau}{ds})|_{s=i\omega} &= Re[\tfrac{2q(i\omega)^{2q-1}+k_1q(i\omega)^{q-1}}{(i\omega)(k_2(i\omega)^q+k_3)}e^{i\omega\tau}+\tfrac{k_2q(i\omega)^{q-1}}{(i\omega)(k_2(i\omega)^q+k_3)}-\tfrac{\tau}{i\omega}], \\ Re(\frac{d\tau}{ds})|_{s=i\omega} &= Re[\tfrac{(2q(i\omega)^{2q-1}+k_1q(i\omega)^{q-1})(\cos(\omega\tau)+i\sin(\omega\tau))+k_2q(i\omega)^{q-1}}{(i\omega)(k_2(i\omega)^q+k_3)}], \end{aligned}$$

$$\begin{aligned} & Re(\frac{d\tau}{ds})|_{s=i\omega} \\ = & \frac{(k_1k_2\omega^{2q-2}q\cos(\frac{q-2}{2}\pi)\cos(\frac{q-1}{2}\pi)\cos(\omega\tau-\frac{q\pi}{2}))}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})}+\frac{k_2^2q\omega^{2q-2}\cos(\frac{q\pi}{2})\cos(\frac{q-2}{2}\pi)}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})} \\ + & \frac{(k_1k_3q\omega^{q-2}(\sin(\omega\tau)\cos(\frac{q-1}{2}\pi)+\cos(\frac{q-2}{2}\pi))}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})}-\frac{q\omega^{q-2}(k_1\sin(\omega\tau+\frac{q\pi}{2})+k_2)(\sin(\frac{q\pi}{2})\cos(\frac{(q-1)\pi}{2}))}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})} \\ + & \frac{2q\omega^{2q-2}\cos((q-1)\pi)(\omega^qk_2\cos(\frac{q\pi}{2})+k_3)}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})}-\frac{2q\omega^{2q-2}\sin(\frac{q\pi}{2})\cos(\frac{(2q-1)\pi}{2})}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})}. \end{aligned}$$

Since the denominators are positive, analysis of the numerators is sufficient to determine the value of the sign.

$$\begin{aligned} & sgn(Re(\frac{d\tau}{ds})|_{s=i\omega}) \\ = & (k_1k_2\omega^{2q-2}q\cos(\frac{q-2}{2}\pi)\cos(\frac{q-1}{2}\pi)\cos(\omega\tau-\frac{q\pi}{2}))+k_2^2q\omega^{2q-2}\cos(\frac{q\pi}{2})\cos(\frac{q-2}{2}\pi) \\ + & (k_1k_3q\omega^{q-2}(\sin(\omega\tau)\cos(\frac{q-1}{2}\pi)+\cos(\frac{q-2}{2}\pi)) \\ - & q\omega^{q-2}(k_1\sin(\omega\tau+\frac{q\pi}{2})+k_2)(\sin(\frac{q\pi}{2})\cos(\frac{(q-1)\pi}{2})) \\ + & 2q\omega^{2q-2}\cos((q-1)\pi)(\omega^qk_2\cos(\frac{q\pi}{2})+k_3)-2q\omega^{2q-2}\sin(\frac{q\pi}{2})\cos(\frac{(2q-1)\pi}{2}) \end{aligned}$$

$0<q<1$, $\omega^q>0$, $0<\cos\frac{q\pi}{2},\sin\frac{q\pi}{2}<1$, $0<\cos\frac{(q-1)\pi}{2}<1$ and ,$-1<\cos\frac{(q-2)\pi}{2}<0$,

So,

$$Re(\frac{d\tau}{ds})|_{s=i\omega}\neq 0.$$

Thus, lemma follows. □

Theorem 3.1. *For system (3.3), the following statements are valid:*

i) If $\tau \in [0, \tau_0)$, then the equilibrium at the origin $(0,0)$ is asymptotically stable.

ii) If $g'(z_1) \neq 0$, then a Hopf bifurcation occurs at the origin when $\tau = \tau_k$ with $k = 0, 1, 2, \ldots$.

4. THE NUMERICAL EXAMPLE

In this section, to support our theoretical results, we illustrate an example numerically using MATLAB programming. To see the behavior of the solution for stability and Hopf bifurcation, we consider q = 0.72 in the underlying system. To obtain the graphs, we modified the Adams - Bashforth - Moulton predictor - corrector scheme and used MATLAB programming as in [9].

For $r_1 = 1.5$, $r_2 = 0.6$, $a_{11} = 0.7$, $a_{12} = 0.8$, $a_{21} = 0.45$ and $a_{22} = 0.006$ in system (1.2). We obtain the following system

$$\begin{aligned} {}_{t_0}^{c}D_t^{0.82}x(t) &= \quad x(t)[1.5 - 0.7x(t) - 0.8y(t-\tau)] \\ {}_{t_0}^{c}D_t^{0.82}y(t) &= \; y(t)[-0.6 + 0.45x(t) - 0.006y(t-\tau)] \end{aligned} \tag{4.1}$$

$$\begin{aligned} x^* &= \tfrac{r_1 a_{22} + r_2 a_{12}}{a_{11}a_{22} + a_{12}a_{21}} = \tfrac{1.50.006 + 0.60.8}{0.70.006 + 0.80.45} = 1.3427 \\ y^* &= \tfrac{r_1 a_{21} - r_2 a_{11}}{a_{11}a_{22} + a_{12}a_{21}} = \tfrac{1.50.45 - 0.60.7}{0.70.006 + 0.80.45} = 0.7001. \end{aligned}$$

So, we get $E^* = (x^*, y^*) = (1.3427, 0.7001)$.

$$k_1 = a_{11}x^* = (0.7)(1.3427) = 0.9399, \quad k_2 = a_{22}y^* = (0.006)(0.7001) = 0.0042,$$

$$k_3 = (a_{11}a_{22} + a_{12}a_{21})x^*y^* = ((0.7)(0.006) + (0.8)(0.45))(1.3427)(0.7001) = 0.3424.$$

Taking $x^* = 1.3427$, $y^* = 0.7001$, $q = 0.82$, $k_1 = 0.9399$, $k_2 = 0.0042$, and $k_3 = 0.3424$ in equation (3.7), for $v = \omega^{0.82}$ the following equation is obtained

$$v^4 + 0.5244v^3 + 0.8834v^2 - (8.024e - 04)v - 0.1172 = 0. \tag{4.2}$$

For which the solutions are $v = 0.3192$ and $\omega = 0.2484$.

$$\begin{aligned} A_1 &= \omega^{2q}\cos(q\pi) + k_1\omega^q\cos(\frac{q\pi}{2}) = -0.0023, \\ A_2 &= \omega^{2q}\sin(q\pi) + k_1\omega^q\sin(\frac{q\pi}{2}) = 0.3427, \\ E_1 &= k_2\omega^q cos(\frac{q\pi}{2}) + k_3 = 0.3428, \end{aligned}$$

$$
\begin{aligned}
E_2 &= k_2\omega^q sin(\frac{q\pi}{2}) = 0.0013, \\
\tau_k &= \frac{1}{\omega}\arctan(\frac{A_1E_2 - A_2E_1}{A_1E_1 + A_2E_2}) + \frac{k\pi}{\omega} \quad \text{for} \quad k = 0,1,2,... \\
\tau_0 &= 6.3115.
\end{aligned}
$$

We know the following equation from Lemma 3.1.

$$
\begin{aligned}
Re(\frac{d\tau}{ds})|_{s=i\omega} = & \frac{(k_1k_2\omega^{2q-2}q\cos(\frac{q-2}{2}\pi)\cos(\frac{q-1}{2}\pi)\cos(\omega\tau-\frac{q\pi}{2}))}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})} + \frac{k_2^2q\omega^{2q-2}\cos(\frac{q\pi}{2})\cos(\frac{q-2}{2}\pi)}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})} \\
+ & \frac{(k_1k_3q\omega^{q-2}(\sin(\omega\tau)\cos(\frac{q-1}{2}\pi)+\cos(\frac{q-2}{2}\pi))}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})} - \frac{q\omega^{q-2}(k_1\sin(\omega\tau+\frac{q\pi}{2})+k_2)(\sin(\frac{q\pi}{2})\cos(\frac{(q-1)\pi}{2}))}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})} \\
+ & \frac{2q\omega^{2q-2}\cos((q-1)\pi)(\omega^qk_2\cos(\frac{q\pi}{2})+k_3)}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})} - \frac{2q\omega^{2q-2}\sin(\frac{q\pi}{2})\cos(\frac{(2q-1)\pi}{2})}{(k_2\omega^q\cos(\frac{q\pi}{2})+k_3)^2+\sin^2(\frac{q\pi}{2})}
\end{aligned}
$$

Using all values, $Re(\frac{d\tau}{ds})|_{s=i\omega}$ is obtained as follows:

$$Re(\frac{d\tau}{ds})|_{s=i\omega} = -0.8459 \neq 0.$$

Using the Adams- Bashforth - Moulton predictor-corrector method that we mentioned in the first part of this section, we also plot our predator and prey functions to illustrate the dynamic behavior of our fractional model (3.3), modified with parameters τ_0 and ω .

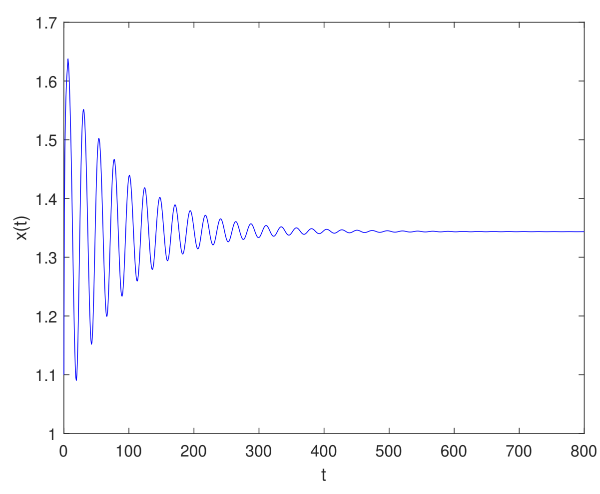

FIGURE 1. State paths for the fractional system with the initial conditions $x_0 = 1.1, y_0 = 0.4$, $q = 0.82$, and $\tau = 5.9 < \tau_0 = 6.3115$.

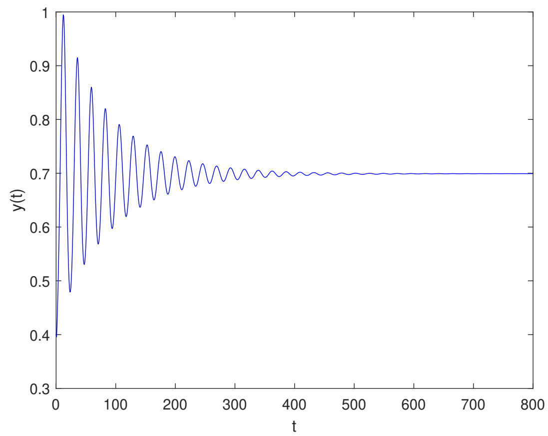

FIGURE 2. State paths for the fractional system with the initial conditions $x_0 = 1.1, y_0 = 0.4$, $q = 0.82$, and $\tau = 5.9 < \tau_0 = 6.3115$.

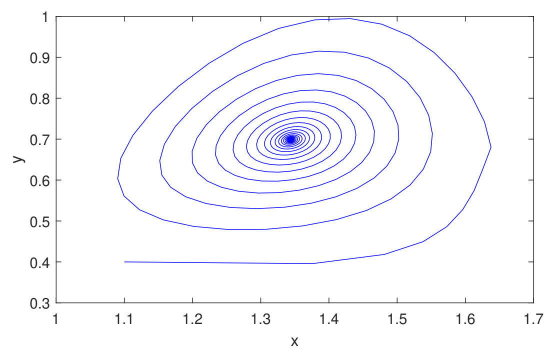

FIGURE 3. State paths of x versus y when $\tau = 5.9 < \tau_0$.

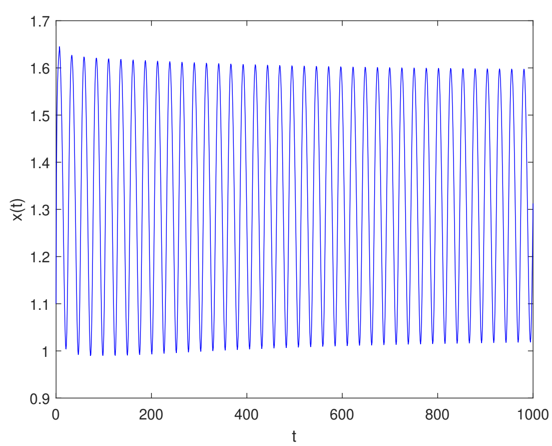

FIGURE 4. State paths for the fractional system with the initial conditions $x_0 = 1.1, y_0 = 0.4$, and $\tau = 6.32$

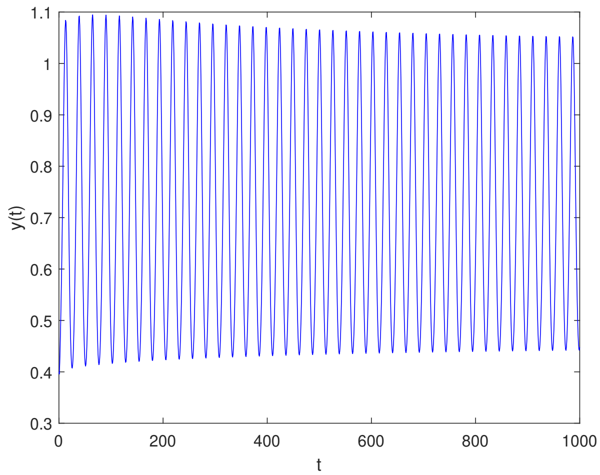

FIGURE 5. State paths for the fractional system with the initial conditions $x_0 = 1.1, y_0 = 0.4$, and $\tau = 6.32$

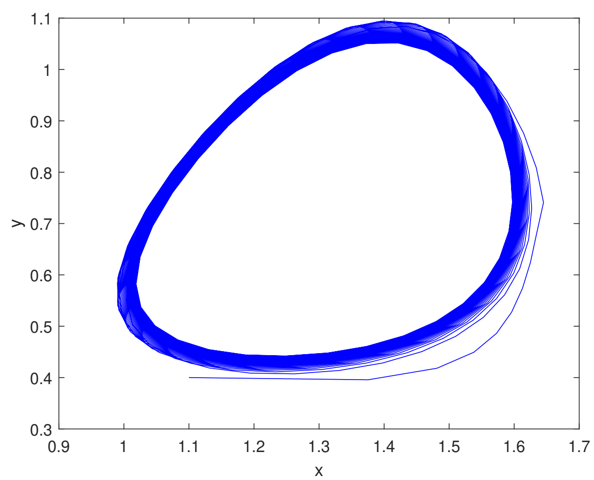

FIGURE 6. State paths of x versus y when $\tau = 6.32$

5. CONCLUSION

In this paper, a fractional-order delayed ecology model was examined. The existence and stability conditions for the equilibrium point were determined using the Laplace transform and linearization theory. By selecting the time delay τ as the bifurcation parameter, the critical value of τ was calculated to determine the point of Hopf bifurcation. Moreover, a numerical example is given to verify the theoretical results. In the example, the equilibrium point is obtained as $E^* = (1.3427, 0.7001)$, $\omega = 0.2484$ and $\tau_0 = 6.3115$. For $q = 0.82$, first $\tau = 5.9 < \tau_0$ is taken and the graphs of Figure-1,2 are obtained, respectively. These figures show that the positive equilibrium point is asymptotically stable when $\tau < \tau_0$. However, in Figure-4,5, while $q = 0.82$ and $\tau = 6.32$ are close enough to τ_0, the existence of periodic solutions bifurcating from the equilibrium point E^* is shown. Figure 3,6 also show the relative positions of the x and y for the values of $\tau = 5.9$ and $\tau = 6.32$, respectively.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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