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THE INFLUENCE OF HUNTING COOPERATION, FEAR AND ANTI-PREDATOR ON THE DYNAMIC OF A PREY-PREDATOR MODEL WITH DISEASE IN PREY

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Abstract: In this study, we examine an eco-epidemiological prey-predator model that incorporates hunting cooperation, fear, and anti-predator behavior including linear harvest. There are two subclasses of prey: susceptible and diseased. We investigate thorough mathematical analysis, including the presence and stability of equilibria, the boundedness of the model, and the existence and uniqueness of solutions. The conditions under which local bifurcation could occur near the equilibrium points were discovered. Numerical simulations were run to validate the model's long-term behavior and comprehend the impact of the model's main parameters. The purpose is to demonstrate the analytical findings numerically and study the impact of changing the parameters on the dynamical behavior of the system, and control settings are determined by numerical simulations using MATLAB, R2021a.

Keywords: eco-epidemiological; fear; anti-predator; hunting cooperation; harvest.
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1. INTRODUCTION

Models that include diseases in ecological communities are referred to as eco-epidemiological models [1]. Anderson and May presented the first eco-epidemiological model that included an infectious illness in prey [2]. Subsequently, eco-epidemiological models involving several biological components were created and studied by several researchers [3–10].

The interaction between prey and predators cannot be adequately described by direct predation

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alone, according to certain evolutionary scientists and theoretical ecologists; the cost of fear must also be considered. The first mathematical model incorporating the fear effect in a two-species predator-prey model was introduced by Wang et al. in 2016. Prey populations may shift their grazing zones to safer locations out of fear of predation, giving up places where they could get the maximum intake rates. They might become more vigilant and change how they reproduce [11].

Many researchers have incorporated cooperation items into the modeling of functional responses for predation rates since cooperation between species of the same species is widespread in nature. These include cooperative hunting, where wolves hunt together against larger creatures [14], lions pursue quicker animals [12–13], and numerous academics have examined the importance of hunting collaboration [15–17]. Pal et al. [18] recently examined how predator-prey dynamics can be impacted by cooperation and fear effects in a predator-prey model with hunting cooperation among predators and the fear put on the prey population. Numerous scholars have examined the eco-epidemiological relationship between hunting collaboration and fear [17, 19–21].

However, a research of prey-predator interactions to prevent prey extinctions was presented [22], using anti-predator behavior as a means of protecting prey from predation pressure. When the prey feels threatened, it naturally responds with anti-predator behavior at the expense of specific bodily parts. To protect themselves from predators, lizards, for instance, may let go of their tails. The spines on fish and insects keep birds and predators from eating them. Different types of anti-predator behaviors in different prey species have been researched by numerous behavioral ecologists [23–25]. Prey animals display inducible defense, which is characterized as protective actions acquired from previous attacks, when a predator is present. Chemicals in various parts of the prey's body are triggered by inducible defense to create new structures or cleverly fend off predators.

Furthermore, [26] investigated the relationship between predator hunting and prey anti-predator behavior in the environment and used a stochastic predator-prey model that includes hunting cooperation and fear effects.

Conversely, harvesting is an important and regular event. Fishermen frequently use harvesting because ecosystems are essentially regenerative. In a capitalized hunting system with two interacting species, scientists are examining the capture of either prey or predator species, or both. Numerous different techniques for harvesting have been used. Some employ nonlinear harvesting [27–29], while others use continuous threshold harvesting, proportional harvesting, and constant

harvesting [30–32]. The impact of fear and harvesting in a prey-predator paradigm with sickness on the prey was proposed by Ibrahim and Naji [33]. They discovered that while fear causes the system to stabilize, sickness and harvesting lead to the extinction of one or more species.
A general prey-predator model that included fear, harvest, cooperative hunting, anti-predator, and approach was developed based on the previous studies. Both susceptible and diseased prey made up a significant portion of the prey population. Predators are said to consume both healthy and sick prey because they can't tell the difference.

2. MODEL FORMULATION

The model includes three main species indicated as $S(t), I(t),$ and $Y(t)$ which represented the densities at time t for the susceptible prey, Infected prey and predators, respectively. The mathematical model can be formulated according to the following assumptions

- The predators feed on their prey on the Lotka-Volterra functional response; without the predator, prey numbers increase logistically.
- Fear effect from the predation causes decrease in the growth rate with constant fear rate.
- The disease is meant for dissemination within the prey species, the infected prey competing for resources and being genetically inherited, the predator makes no distinction between infected and susceptible prey; it consumes both.
- As the predator has a hunting cooperation capability, it will successfully acquire prey. therefore, the predator population's attack rate, $a_1 > 0$, can be increased by the cooperation term to become $(c_1 + a_1 Y)$, where $a_2 \geq 0$, denotes the predator cooperation in hunting.
- Harvesting is imposed on susceptible prey and predator populations by an external force.
- Prey has an anti-predator ability that decreases predation.

Consequently, the subsequent system of nonlinear first-order differential equations may characterize the dynamics of the specified eco-epidemiological system

$$\begin{aligned} \frac{dS}{dT} &= \frac{r}{1+\alpha Y}\left(1 - \frac{S+I}{k}\right) S - (c_1 + a_1 Y) SY - \beta SI - q_1 E_1 S = g_1(S, I, Y), \\ \frac{dI}{dT} &= \beta SI - c_2 IY - d_1 I = g_2(S, I, Y), \\ \frac{dY}{dT} &= e_1 (c_1 + a_1 Y) SY + e_2 c_2 IY - d_2 Y - a_2 SY - q_2 E_2 Y = g_3(S, I, Y) \end{aligned} \quad , \qquad (1)$$

where $S(0) = S_0 \geq 0$, $I(0) = I_0 \geq 0$, and $Y(0) = Y_0 \geq 0$ indicates the initial point of the system (1), with every parameter positive and described in Table 1.

Table 1: The description of the model parameters

Parameters	**Description**
r	The intrinsic growth rate of the prey.
k	Environmental carrying capacity
α	Fear level
c_1	the consumption rate by the predator
a_1	The rate of hunting cooperation
e_1	The conversion rate of devouring susceptible prey by predator
e_2	The conversion rate of devouring infected prey by predator.
a_2	The rate of anti-predator
β	The rate of infection
d_1	The mortality rate of infected prey
d_2	The mortality rate of predator
q_1E_1	The harvesting catchability constant and the effort rate of susceptible prey
q_2E_2	The harvesting catchability constant and the effort rate of predator

Therefore, system (2) has the following domain

$$\Omega = \{(S, I, Y) \in R^3, S \geq 0, I \geq 0, Y \geq 0\}.|$$

System (1) has a continuous interaction function with a continuous partial derivatives, and hence the solution exists and is unique. Moreover, in order to guarantees the convergent of the solution to an attractor, the solution of system (1) is proved to be uniformly bounded as shown in the following theorem.

Theorem 1. Solutions of system (1) starting in $\mathbb{R}^3_+$, are uniformly bounded under the prey's survival condition

$$r > q_1E_1 \tag{2}$$

Proof. From the susceptible prey equation in system (1) yields that

$$\frac{dS}{dt} \leq r\left(1 - \frac{S}{k}\right)S - q_1E_1S,$$

Then direct computation leads to $S \leq \frac{K(r-q_1E_1)}{r}$

Now, define the function $W_2\,(t) = S\,(t) + I\,(t) + Y(t)$, Differentiating the function $W_2\,(t)$, yields

$$\frac{dW_2}{dT} \le 2(r-q_1E_1)S-\rho_1W_2.$$

Therefore,

$$\frac{dW_2}{dT}+\rho_1W_2 \le \rho_2,$$

where $\rho_1 = min\{r-q_1E_1, d_1, d_2+q_2E_2\}$

Then, according to the above differential inequality, direct computation shows that for $t \to \infty$, it is obtained

$$W_2(t) \le \frac{\rho_2}{\rho_1} = \mu,$$

where $\mu = \frac{2K(r-q_1E_1)^2}{r}$.

Thus, the solutions of system (1) in the region Ω are uniformly bounded.

3. Equilibrium Points and Their Local Stability Analysis

In this section, the existence of non-negative equilibria is examined, and the stability of these critical points is established. The non-negative equilibrium points are determined as follows

- The trivial equilibrium point (TEP), $\hat{P}_0 = (0,0,0)$, always exists
- The axial equilibrium point (AEP), $\check{P}_1 = (\check{S},0,0)$, where $\check{S} = \frac{K(r-q_1E_1)}{r}$, which exists under condition (2).
- The free predator equilibrium point $(FPEP)$, $\bar{P}_2 = (\bar{S},\bar{I},0)$, where

  $\bar{S} = \frac{d_1}{\beta}$, $\bar{I} = \frac{K\beta(r-E_1q_1)-rd_1}{\beta(r+K\beta)}$, exists provided that

$$\frac{rd_1}{\beta} < (r-E_1q_1) \tag{3}$$

- The free infected prey equilibrium point $(FIPEP)$,$\tilde{P}_3 = (\tilde{S},0.\tilde{Y})$

$$\tilde{S} = \frac{d_2+q_2E_2}{e_1(c_1+a_1\tilde{Y})-a_2},$$

while $\tilde{Y}$ represents a positive root for the equation

$$\delta_4\tilde{Y}^4+\delta_3\tilde{Y}^3+\delta_2\tilde{Y}^2+\delta_1\tilde{Y}+\delta_0=0, \tag{4}$$

with

$$\delta_4 = ka_1^2e_1\alpha > 0,$$
$$\delta_3 = Ke_1a_1^2+Ka_1\alpha(e_1c_1-a_2)+Ke_1a_1c_1\alpha,$$
$$\delta_2 = Kc_1(e_1a_1+\alpha)+Ka_1(e_1c_1-a_2),$$

$$\delta_1 = Kq_1E_1\alpha + Kc_1(e_1c_1 - a_2) - Kre_1a_1,$$

$$\delta_0 = Kq_1E_1 + r(q_2E_2 + d_2) - Kr(e_1c_1 - a_2).$$

By “Descartes rule of signs”, equation (4) has one positive root under the conditions

$$\delta_0 < 0. \tag{5a}$$

$$e_1c_1 > a_2. \tag{5b}$$

- The interior equilibrium point (IEP), $P_4^* = (S^*, I^*, Y^*)$, where

$$S^* = \frac{c_2Y^* + d_1}{\beta},$$

$$I^* = \frac{\beta(Kr - KY^{*2}a_1 - KY^{*3}\alpha a_1 - KY^*c_1 - KY^{*2}\alpha c_1 - Ke_1q_1 - KY^*\alpha e_1q_1) - r(Y^*c_2 + d_1)}{(r + K\beta(1+\alpha Y^*))},$$

while Y^* is a positive root of the following

$$D_3Y^{*3} + D_2Y^{*2} + D_1Y^* + D_0 = 0, \tag{6a}$$

where

$$D_3 = K\alpha\beta a_1c_2(e_2 - e_1).$$

$$D_2 = K\beta a_1c_2(e_2 - e_1) - ra_1e_1c_2 + K\alpha\beta c_1c_2(e_2 - e_1) - K\alpha\beta a_1e_1d_1$$

$$D_1 = KS^*\alpha\beta^2a_2 - re_1c_1c_2 - K\beta e_1c_1c_2 + K\beta e_2c_1c_2 + re_2c_2^2 - ra_1e_1d_1 - K\beta a_1e_1d_1 - K\alpha\beta e_1c_1d_1 + K\alpha\beta^2d_2 + K\alpha\beta e_2c_2E_1q_1 + K\alpha\beta^2E_2q_2 \; .$$

$$D_0 = r\beta d_2 + K\beta^2d_2 + K\beta e_2c_2E_1q_1 + r\beta E_2q_2 + K\beta^2E_2q_2 + rS^*\beta a_2 + KS^*\beta^2a_2 - Kr\beta e_2c_2 - re_1c_1d_1 - K\beta e_1c_1d_1 + re_2c_2d_1$$

So by “Descartes’ rule of sign”, equation (6a) has a unique positive root and hence, system (1) has a unique IEP if one of the following sets of conditions

$$\left.\begin{array}{r} e_2 > e_1 \\ D_0 < 0 \\ D_2 > 0 \;\; OR \;\; D_1 < 0 \end{array}\right\} \tag{6b}$$

Or else:

$$\left.\begin{array}{r} e_2 < e_1 \\ D_0 > 0 \\ D_2 < 0 \;\; OR \;\; D_1 > 0 \end{array}\right\} \tag{6c}$$

Now, to establish the local stability, the Jacobain matrix (JM) of system (1) about (S, I, Y)

$$J = (u_{ij})_{3\times3}, \tag{7}$$

where

$$u_{11} = \frac{-rS}{K(1+\alpha Y)} + \frac{r}{1+\alpha Y} - \frac{rS}{K(1+\alpha Y)} - \frac{rI}{K(1+\alpha Y)} - (c_1 + a_1Y)Y - \beta I - q_1E_1,$$

$$u_{12} = -(\frac{r}{K(1+\alpha Y)} + \beta)S, \ \ u_{13} = \left(\frac{-r\alpha}{(1+\alpha Y)^2} + \frac{rK\alpha S}{(K(1+\alpha Y))^2} + \frac{rK\alpha I}{(K(1+\alpha Y))^2} - c_1 - 2a_1Y\right)S,$$

$$u_{21} = \beta I;\ u_{22} = \beta S - c_2Y - d_1;\ u_{23} = -c_2I,$$

$$u_{31} = (e_1(c_1 + a_1Y) - a_2)Y;\ u_{32} = e_2c_2Y,$$

$$u_{33} = e_1a_1SY + e_1(c_1 + a_1Y)S + e_2c_2I - d_2 - a_2S - q_2E_2.$$

It is clear that the system (1) has JM at EEP, $\hat{P}_0 = (0,0,0)$ specified by

$$J(\hat{P}_0) = \begin{bmatrix} r - q_1E_1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 - q_2E_2 \end{bmatrix}. \tag{8a}$$

The eigenvalues of $J(\hat{P}_0)$ are $\lambda_{01} = r - q_1E_1,\ \lambda_{02} = -d_1 < 0, \lambda_{03} = -d_2 - q_2E_2 < 0$. Therefore, $\hat{P}_0$ is locally asymptotically stable (LAS) under the condition

$$r < q_1E_1 \tag{8b}$$

The JM at $AEP, \check{P}_1 = (\check{S}, 0,0)$, is determent by

$$J(\check{P}_1) = \begin{bmatrix} -(r - q_1E_1) & -(\frac{r}{K} + \beta)\check{S} & (-r\alpha + \frac{r\alpha\check{S}}{K} - c_1)\check{S} \\ 0 & \beta\check{S} - d_1 & 0 \\ 0 & 0 & e_1c_1\check{S} - d_2 - a_2\check{S} - q_2E_2 \end{bmatrix} \tag{9a}$$

Therefore, the eigenvalues of $J(\check{P}_1)$ are $\lambda_{11} = -(r - q_1E_1)$; $\lambda_{12} = \beta\check{S} - d_1, \lambda_{13} = e_1c_1\check{S} - d_2 - a_2\check{S} - q_2E_2$.

Hence, all the eigenvalues are negative, and $\check{P}_1$is LAS under the condition (2) and the following conditions

$$e_1c_1\tilde{S} < d_2 + a_2\tilde{S} + q_2E_2 \tag{9b}$$

$$\beta\tilde{S} < d_1. \tag{9c}$$

The JM at $FPEP$, $\bar{P}_2 = (\bar{S}, \bar{I}, 0)$, is determined

$$J(\bar{P}_2) = \begin{bmatrix} -\frac{r\bar{S}}{k} & -\left(\frac{r}{K} + \beta\right)\bar{S} & (-r\alpha + \frac{r\alpha\bar{S}}{K} + \frac{r\alpha\bar{\imath}}{K} - c_1)\bar{S} \\ \beta\bar{I} & 0 & -c_2\bar{I} \\ 0 & 0 & e_1c_1\bar{S} - a_2\bar{S} - e_2c_2\bar{I} - d_2 - q_2E_2 \end{bmatrix}. \tag{10}$$

The characteristic equation of $J(P_2)$ is

$$(\lambda_2^2 - T_1\lambda_2 + D_1)(e_1c_1\bar{S} - a_2\bar{S} - e_2c_2\bar{I} - d_2 - q_2E_2) = 0 \tag{10a}$$

where

$$D_1 = \left(\frac{r\bar{S}}{K} + \beta\right)\beta\bar{I}\bar{S} > 0$$

$$T_1 = -\frac{r\bar{S}}{K} < 0$$

Obviously, $T_1 < 0$ and $D_1 > 0$. Therefore, the eigenvalues are written as

$$\lambda_{21} = \frac{T_1}{2} + \frac{1}{2}\sqrt{{T_1}^2 - 4D_1};\ \lambda_{22} = \frac{T_1}{2} - \frac{1}{2}\sqrt{{T_1}^2 - 4D_1};$$

$$\lambda_{23} = e_1 c_1 \bar{S} + e_2 c_2 \bar{I} - d_2 - a_2 \bar{S} - q_2 E_2.$$

Therefore, the eigenvalues λ_{21} and λ_{22} have negative real parts, and then $\bar{P}_2$ is LAS under the condition

$$e_1 c_1 \bar{S} + e_2 c_2 \bar{I} < d_2 + a_2 \bar{S} + q_2 E_2 \tag{10b}$$

The JM at IEP, $\tilde{P}_3 = (\tilde{S}, 0, \tilde{Y})$ is determine

$$J(\tilde{P}_3) = \begin{bmatrix} -\frac{r\tilde{S}}{K(1+\alpha\tilde{Y})} & -\left(\frac{r\tilde{S}}{K(1+\alpha\tilde{Y})} + \beta\right)\tilde{S} & \left(-\frac{r\alpha}{(1+\alpha\tilde{Y})^2} + \frac{r\alpha\tilde{S}}{K(1+\alpha\tilde{Y})^2} - c_1 - 2a_1\tilde{Y}\right)\tilde{S} \\ 0 & \beta\tilde{S} - d_1 & 0 \\ (e_1c_1 + e_1a_1\tilde{Y} - a_2)\tilde{Y} & e_2c_2\tilde{Y} & e_1a_1\tilde{S}\tilde{Y} \end{bmatrix}.$$

Hence, the characteristic equation of $J(\tilde{P}_3)$ is given by

$$(\lambda_3^2 - T_2\lambda_3 + D_2)(\beta\tilde{S} - d_1) = 0 \tag{11a}$$

where,

$$T_2 = \frac{-r\tilde{S}}{K(1+\alpha\tilde{Y})} + e_1 a_1 \tilde{S}\tilde{Y}$$

$$D_2 = \left(-\frac{r\alpha}{(1+\alpha\tilde{Y})^2} + \frac{r\alpha\hat{S}}{K(1+\alpha\tilde{Y})^2} - c_1 - 2a_1\tilde{Y}\right)(e_1c_1 + e_1a_1\tilde{Y} - a_2)\tilde{S}\tilde{Y}.$$

Obviously, $T_2 < 0$ and $D_2 > 0$. Therefore, the eigenvalues are written as

$$\lambda_{31} = \frac{T_2}{2} + \frac{1}{2}\sqrt{{T_2}^2 - 4D_2};\ \lambda_{32} = \frac{T_2}{2} - \frac{1}{2}\sqrt{{T_2}^2 - 4D_2};\ \text{and ,}\ \lambda_{33} = \beta\tilde{S} - d_1$$

Direct computation shows that the eigenvalues λ_{31} and λ_{32} have negative real parts if

$$e_1 a_1 \tilde{S}\tilde{Y} < \frac{r\tilde{S}}{K(1+\alpha\tilde{Y})} \tag{11b}$$

$$\left(-\frac{r\alpha}{(1+\alpha\tilde{Y})^2} + \frac{r\alpha\tilde{S}}{K(1+\alpha\tilde{Y})^2} - c_1 - 2a_1\tilde{Y}\right)(e_1c_1 + e_1a_1\tilde{Y} - a_2)\tilde{S}\tilde{Y} > 0 \tag{11c}$$

while the third eigenvalue λ_{33} is negative if

$$\beta\tilde{S} < d_1 \tag{11d}$$

Finally, the JM at the IEP, $P_4^* = (S^*, I^*, Y^*)$ is

$$J(P_4^*) = [a_{ij}^*]_{3\times 3} \tag{12a}$$

where

$$a_{11}^* = \frac{-rS^*}{K(1+\alpha Y^*)};\ a_{12}^* = -\left(\frac{rS^*}{K(1+\alpha Y^*)} + \beta\right)S^*$$

$$a_{13}^* = \left(\frac{-r\alpha}{(1+\alpha Y^*)^2} + \frac{r\alpha S^*}{K(1+\alpha Y^*)^2} + \frac{r\alpha I^*}{K(1+\alpha Y^*)^2} - c_1 - 2a_1 Y^*\right)S^*;$$

$a_{21}^{*} = \beta I^{*};\ a_{22}^{*} = 0; a_{23}^{*} = -c_2 I^{*};$

$a_{31}^{*} = (e_1 c_1 + e_1 a_1 Y^{*} - a_2)Y^{*};\ a_{32}^{*} = e_2 c_2 Y^{*};\ a_{33}^{*} = e_1 a_1 S^{*} Y^{*}$

The corresponding characteristic equation

$$\lambda_4^3 + C_1 \lambda_4^2 + C_2 \lambda_4 + C_3 = 0, \tag{12b}$$

where

$$C_1 = -(a_{11}^{*} - a_{33}^{*})$$
$$C_2 = [-a_{12}^{*} a_{21}^{*} + (a_{11}^{*} a_{33}^{*} - a_{13}^{*} a_{31}^{*}) - a_{23}^{*} a_{32}^{*}]$$
$$C_3 = [a_{12}^{*}(a_{23}^{*} a_{31}^{*} - a_{21}^{*} a_{33}^{*}) + a_{32}^{*}(a_{13}^{*} a_{21}^{*} - a_{11}^{*} a_{23}^{*})]$$

with

$$\begin{aligned}\Delta = {} & C_1 C_2 - C_3 \\ = {} & -(a_{11}^{*} + a_{33}^{*})(a_{11}^{*} a_{33}^{*} - a_{13}^{*} a_{31}^{*}) + a_{12}^{*}(a_{11}^{*} a_{21}^{*} + a_{23}^{*} a_{31}^{*}) \\ & + a_{32}^{*}(a_{23}^{*} a_{33}^{*} + a_{13}^{*} a_{21}^{*})\end{aligned}$$

The characteristic equation (12b), according to the "Routh-Hurwitz criterion", has three eigenvalues with negative real portions if the following conditions are met parts if $C_1 > 0$, $C_3 > 0$, and $\Delta = C_1 C_2 - C_3$Moreover, the "Routh-Hurwitz requirements" are satisfied if the conditions given in the following theorem hold.

Theorem 3. The IEP of system (1) is LAS if the following conditions are met.

$$\frac{-rS^{*}}{K(1+\alpha Y^{*})} + e_1 a_1 S^{*} Y^{*} < 0 \tag{13a}$$

$$a_{11}^{*} a_{33}^{*} - a_{13}^{*} a_{31}^{*} > 0 \tag{13b}$$

$$\frac{-r\alpha}{(1+\alpha Y^{*})^2} + \frac{r\alpha S^{*}}{K(1+\alpha Y^{*})^2} + \frac{r\alpha I^{*}}{K(1+\alpha Y^{*})^2} - c_1 - 2a_1 Y^{*} < 0 \tag{13c}$$

$$e_1 c_1 + e_1 a_1 Y^{*} > a_2 \tag{13d}$$

$$a_{12}^{*}(a_{23}^{*} a_{31}^{*} - a_{21}^{*} a_{33}^{*}) + a_{32}^{*}(a_{13}^{*} a_{21}^{*} - a_{11}^{*} a_{23}^{*}) > 0 \tag{13e}$$

Proof. Assuming $C_1 > 0$, $C_3 > 0$, and $\Delta > 0$, the roots of the Jacobian matrix $J(a_{ij}^{*})$ are considered to comprise negative real parts according to the "Routh-Hurwitz criterion". The satisfaction of the "Routh–Hurwitz criterion" requirements is guaranteed by conditions (13a)-(13e), as demonstrated by direct computation.

4. GLOBAL STABILITY ANALYSIS

In this part, the global stability of system (1) is studied as shown in the next theorems, through applying suitable Lyapunov functions. The basin of attraction of a trajectory to the dynamical system can be described as the state space or a particular region in it, depending on the state variables of t.

Theorem 4. The TEP, $\hat{P}_0 = (0,0,0)$, is a LAS, then it is globally asymptotically stable (G.AS).

Proof. We choose a suitable function about $\hat{P}_0$ as

$\mu_0 = S + I + Y$, where μ_0 is a C^1 function, which is a positive definite real-valued function, then we have

$$\frac{d\mu_0}{dT} = \frac{rS}{1+\alpha Y} - (c_1 + a_1 Y)SY - q_1 E_1 S - c_2 IY - d_1 I + e_1(c_1 + a_1 Y)SY + e_2 c_2 IY - d_2 Y - q_2 E_2 Y,$$

Further simplification leads to the following

$$\frac{d\mu_0}{dT} \leq -(q_1 E_1 - r)S - d_1 I - (d_2 - q_2 E_2)Y.$$

So, the function $\frac{d\mu_0}{dt}$ is negative definite due to the above given condition (8b). Thus $\hat{P}_0$ is G.AS.

Theorem 5. The AEP, $\check{P}_1 = \left(\check{S}, 0,0\right)$ is a LAS, then it is G.AS if the following conditions are met

$$(c_1 + a_1 \mu)\check{S} + r\alpha\check{S} < \frac{r\alpha\check{S}^2}{K(1+\alpha\mu)} + d_2 + q_2 E_2 \tag{14a}$$

$$\beta\check{S} + \frac{r\check{S}}{k} < d_1 \tag{14b}$$

$$\frac{r\alpha}{1+\alpha\mu} + a_2 < \frac{r\alpha\check{S}}{K} \tag{14c}$$

Proof. We choose a suitable function about $\check{P}_1$ as

$\mu_1 = \left(S - \check{S} - \check{S}\ln\frac{S}{\check{S}}\right) + I + Y$, where μ_1 is C^1 function, which is a positive definite real-valued function, then we have

$$\frac{d\mu_1}{dT} = -\frac{r\left(S-\check{S}\right)^2}{K(1+\alpha Y)} - \frac{r\alpha SI}{1+\alpha Y} + \frac{r\alpha\check{S}I}{1+\alpha Y} + \frac{r\alpha\check{S}\,SY}{K(1+\alpha Y)} - \frac{r\alpha\check{S}^2 Y}{K(1+\alpha Y)} - \frac{rSI}{K(1+\alpha Y)} + \frac{r\check{S}\,I}{K(1+\alpha Y)} - (c_1 + a_1 Y)SY + (c_1 + a_1 Y)\check{S}Y + \beta\check{S}I - c_2 IY - d_1 I + e_1(c_1 + a_1 Y)SY + e_2 c_2 IY - a_2 SY - d_2 Y - q_2 E_2 Y$$

Further simplification leads to the following

$$\frac{d\mu_1}{dT} \leq -\frac{r}{K(1+\alpha Y)}\left(S-\check{S}\right)^2 - \left[d_1 - \beta\check{S} - \frac{r\check{S}}{k}\right]I - \left[d_2 + q_2 E_2 - (c_1 + a_1\mu)\check{S} - r\alpha\check{S} + \frac{r\alpha\check{S}^2}{K(1+\alpha\mu)}\right]Y - \left[\frac{r\alpha}{1+\alpha\mu} - \frac{r\alpha\check{S}}{K} + a_2\right]SY.$$

So, the function $\frac{d\mu_1}{dT}$ is negative definite under the conditions (14a)-(14c). Thus $\check{P}_1$ is G.AS.

Theorem 6. The $FPEP$, $\bar{P}_2 = (\bar{S}, \bar{I}, 0)$ is a LAS, then it is G.AS if the following conditions are met

$$c_2\bar{I} + (c_1 + a_1\mu)\bar{S} < d_2 + q_2E_2 + \frac{r\alpha\bar{S}}{K(1+\alpha\mu)}(\bar{S} + \bar{I}) \tag{15a}$$

$$\frac{r\alpha}{k}(\bar{S} + \bar{I}) < a_2 + \frac{r}{(1+\alpha\mu)} \tag{15b}$$

Proof. We choose a suitable function about $\bar{P}_2$ as

$$\mu_2 = \left(S - \bar{S} - \bar{S}\ln\frac{S}{\bar{S}}\right) + \left(I - \bar{I} - \bar{I}\ln\frac{I}{\bar{I}}\right) + Y,$$

where μ_2 is C^1 function, which is a positive definite real-valued function, then

$$\frac{d\mu_2}{dT} = -\frac{r\alpha SY}{1+\alpha Y} - \frac{r}{K(1+\alpha Y)}(S-\bar{S})^2 + \frac{r\alpha\bar{S}SY}{K(1+\alpha Y)} - \frac{r\alpha\bar{S}^2Y}{K(1+\alpha Y)}$$
$$- \frac{r}{K(1+\alpha Y)}(S-\bar{S})(I-\bar{I}) + \frac{r\alpha\bar{I}SY}{K(1+\alpha Y)} - \frac{r\alpha\bar{S}\bar{I}Y}{K(1+\alpha Y)} - (c_1 + a_1Y)SY$$
$$+ (c_1 + a_1Y)\bar{S}Y - \beta(S-\bar{S})(I-\bar{I}) - c_2IY + c_2\bar{I}Y + e_1(c_1 + a_1Y)SY + e_2c_2IY$$
$$- d_2Y - a_2SY - q_2E_2Y$$

Further simplification leads to the following

$$\frac{d\mu_2}{dT} \le -\frac{2r}{K(1+\alpha\mu)}(S-\bar{S})^2 - \left[\frac{r}{(1+\alpha\mu)} - \frac{r\alpha\bar{S}}{K} - \frac{r\alpha\bar{I}}{K} + a_2\right]SY - \left[d_2 + q_2E_2 - c_2\bar{I} - (c_1 + a_1\mu)\bar{S} + \frac{r\alpha\bar{S}^2}{K(1+\alpha\mu)} + \frac{r\alpha\bar{S}\bar{I}}{K(1+\alpha\mu)}\right]Y - \frac{r}{K(1+\alpha\mu)}(I-\bar{I})^2.$$

So, the function $\frac{d\mu_2}{dT}$ is negative definite under the conditions (15a)-(15b). Thus $\bar{P}_2$ is G.AS.

Theorem 7. The basin of attraction of $FIPEP, \tilde{P}_3 = (\tilde{S}, 0.\tilde{Y})$ satisfies the following conditions, when $\tilde{P}_3$ is a LAS

$$\frac{r}{KA} > \frac{B_1}{2} \tag{16a}$$

$$B_2 > \frac{B_1}{2} \tag{16b}$$

$$d_1 > \beta\tilde{S} + \frac{r\tilde{S}}{KA} \tag{16c}$$

$$e_2c_2\tilde{Y} > e_2c_2\mu + c_2 \tag{16d}$$

Proof. We choose a suitable function about $\tilde{P}_3$ as

$$\mu_3 = \left(S - \tilde{S} - \tilde{S}\ln\frac{S}{\tilde{S}}\right) + I + \frac{(Y-\tilde{Y})^2}{2}$$

where μ_3 is C^1 function, which is a positive definite real-valued function, then we have

$$\frac{d\mu_3}{dT} = (S-\tilde{S})[\frac{-r\alpha(Y-\tilde{Y})}{A\tilde{A}} - \frac{r(S-\tilde{S})}{KA} + \frac{r\alpha\tilde{S}(Y-\tilde{Y})}{KA\tilde{A}} - \frac{rI}{KA} - (c_1 + a_1(Y+\tilde{Y})(Y-\tilde{Y}) - \beta I]$$
$$+ [\beta IS - c_2IY - d_1I] + (Y-\tilde{Y})[e_1S(c_1 + a_1(Y+\tilde{Y})(Y-\tilde{Y})$$
$$+ e_1\tilde{Y}(c_1 + a_1\tilde{Y})(S-\tilde{S}) + e_2c_2I - d_2(Y-\tilde{Y}) - a_2S(Y-\tilde{Y}) - a_2\tilde{Y}(S-\tilde{S})$$
$$- q_2E_2(Y-\tilde{Y})]$$

Therefore, we obtain

$$\frac{d\mu_3}{dT} = \frac{-r}{KA}(S-\tilde{S})^2$$
$$-\left[\frac{r}{KA} - \frac{r\alpha\tilde{S}}{KA\tilde{A}} + \left(c_1 + a_1(Y+\tilde{Y})\right) - e_1\tilde{Y}(c_1 + a\tilde{Y}_1) + a_2\tilde{Y}\right](S-\tilde{S})(Y-\tilde{Y})$$
$$-\left[q_2E_2 + d_2 - e_1S\left(c_1 + a_1(Y+\tilde{Y})\right) + a_2S\right](Y-\tilde{Y})^2 - \left[d_1 - \beta\tilde{S} - \frac{r\tilde{S}}{KA}\right]I$$
$$- [e_2c_2\tilde{Y} - e_2c_2Y - c_2]IY$$

Therefore, we obtain

$$\frac{d\mu_3}{dt} < -\left[\frac{r}{KA} - \frac{B_1}{2}\right](S-\tilde{S})^2 - \left[B_2 - \frac{B_1}{2}\right](Y-\tilde{Y})^2 - \left[d_1 - \beta\tilde{S} - \frac{r\tilde{S}}{KA}\right]I - [e_2c_2\tilde{Y} - e_2c_2Y - c_2]IY.$$

where $A = 1 + \alpha Y, \tilde{S} = 1 + \alpha\tilde{Y}, B_1 = [\frac{r}{KA} - \frac{r\alpha\tilde{S}}{KA\tilde{A}} + \left(c_1 + a_1(Y+\tilde{Y})\right) - e_1\tilde{Y}(c_1 + a\tilde{Y}_1) + a_2\tilde{Y}\,]$ and $B_2 = [q_2E_2 + d_2 - e_1S\left(c_1 + a_1(Y+\tilde{Y})\right) + a_2S]$

So, in the region that meets condition (16a-16d), $\frac{d\mu_3}{dT}$ is negative definite. Hence $\tilde{P}_3$ is G.AS.

Theorem 8. The basin of attraction of $IEP, P_4^* = (S^*, I^*, Y^*)$ satisfies the following conditions, when P_4^* is LAS

$$\frac{r}{2KA} > \frac{B_3}{2} \quad (17a)$$

$$(d_1 - c_2Y^* - \beta\mu) > \frac{r}{2KA} + \frac{B_4}{2} \quad (17b)$$

$$B_5 > \frac{B_4}{2} + \frac{B_3}{2} \quad (17c)$$

Proof. We choose a suitable function about P_4^* as

$$\mu_4 = \left(S - S^* - S^*\, ln\frac{S}{S^*}\right) + \frac{(I-I^*)^2}{2} + \frac{(Y-Y^*)^2}{2}$$

where μ_3 is C^1 function, which is a positive definite real-valued function, then we have

$$\frac{d\mu_4}{dT} = (S-S^*)[\frac{-r}{AA^*}(Y-Y^*) - \frac{r}{KA}(S-S^*) - \frac{r\alpha S^*}{KAA^*}(Y-Y^*) - \frac{r}{KA}(I-I^*)$$
$$+\frac{r\alpha I^*}{KAA^*}(Y-Y^*) - (c_1 + a_1(Y+Y^*))(Y-Y^*) - \beta(I-I^*)]$$
$$+ (I-I^*)[\beta(I-I^*) - \beta I^*(S-S^*) + c_2 I(Y-Y^*) + c_2 Y^*(I-I^*) - d_1(I-I^*)]$$
$$+ (Y-Y^*)[e_1 Y^*(c_1 + a_1 Y^*)(S-S^*) + e_1 S(c_1 + a_1(Y+Y^*))(Y-Y^*)$$
$$+ e_2 c_2 I(Y-Y^*) + e_2 c_2 Y^*(I-I^*) - d_2(Y-Y^*) - a_2 S(Y-Y^*)$$
$$- a_2 Y^*(S-S^*) - q_2 E_2(Y-Y^*)].$$

Then further simplification leads to the following.

$$\frac{d\mu_4}{dT} \le \frac{-r}{KA}(S-S^*)^2$$
$$- \left[\frac{r}{AA^*} + \frac{r\alpha S^*}{KAA^*} + (c_1 + a_1(Y+Y^*) - \frac{r\alpha I^*}{KAA^*} - e_1 Y^*(c_1 + a_1 Y^*) + a_2 Y^*\right](S$$
$$- S^*)(Y-Y^*) - \frac{-r}{KA}(S-S^*)(I-I^*) + c_2(I + e_2 Y^*)(I-I^*)(Y-Y^*)$$
$$- [d_1 - c_2 Y^* - \beta S](I-I^*)^2 - [q_2 E_2 + a_2 + d_2 - e_2 c_2 \mathrm{I} - e_2 e_1 S(c_1$$
$$+ a_1(Y+Y^*)](Y-Y^*)^2$$

Therefore, we obtain

$$\frac{d\mu_4}{dT} \le \frac{-r}{KA}(S-S^*)^2$$
$$- \left[\frac{r}{AA^*} + \frac{r\alpha S^*}{KAA^*} + (c_1 + a_1(Y+Y^*) - \frac{r\alpha I^*}{KAA^*} - e_1 Y^*(c_1 + a_1 Y^*) + a_2 Y^*\right](S$$
$$- S^*)(Y-Y^*) - \frac{-r}{KA}(S-S^*)(I-I^*) + c_2(I + e_2 Y^*)(I-I^*)(Y-Y^*)$$
$$- [d_1 - c_2 Y^* - \beta S](I-I^*)^2 - [q_2 E_2 + a_2 + d_2 - e_2 c_2 \mathrm{I} - e_2 e_1 S(c_1$$
$$+ a_1(Y+Y^*)](Y-Y^*)^2$$

Hence

$$\frac{d\mu_4}{dT} \le -\left[\frac{r}{2KA} - \frac{B_3}{2}\right](S-S^*)^2 - \left[(d_1 - c_2 Y^* - \beta S) - \frac{r}{2KA} - \frac{B_4}{2}\right](I-I^*)^2 - [B_5 - \frac{B_4}{2} - \frac{B_3}{2}](Y-Y^*)^2$$

where $A^* = 1 + \alpha Y^*, B_3 = \frac{r}{AA^*} + \frac{r\alpha S^*}{KAA^*} + (c_1 + a_1(Y+Y^*) - \frac{r\alpha I^*}{KAA^*} - e_1 Y^*(c_1 + a_1 Y^*) + a_2 Y^*, B_4 = c_2(I + e_2 Y^*), B_5 = q_2 E_2 + a_2 + d_2 - e_2 c_2 \mathrm{I} - e_2 e_1 S(c_1 + a_1(Y+Y^*)]$

Therefore, in the region that meets the conditions (17a)-(17c), then $\frac{d\mu_4}{dt}$ is negative- definite. Hence, P_4^* is a G.A.S.

5. BIFURCATION ANALYSIS

The analysis of changing the parameter values on the dynamic of the system (1) is investigated in this section. Now, to compute the second derivative of the JM system (1), rewrite it in the vector form as follows

$$\frac{dX}{dt} = G(X), \text{with } X = (S, I, Y)^T \text{and } G = (Sg_1, Ig_2, Yf_3)^T$$

Let $V = (v_1, v_2, v_3)^T$ be any nonzero vector. Thus, system (1)'s second directional derivatives can be expressed as

$$D^2G(V,V) = [\pi_{ij}]_{3\times1} \tag{18}$$

where

$$\pi_{11} = \frac{-2}{K(1+\alpha Y)^3}\big[r(1+\alpha Y)^2 v_1^2 + Sv_3[-r\alpha(1+\alpha Y)v_2 + \big(r(I-K+S)\big)\alpha^2 + K(1+\alpha Y)^3 a_1)v_3] + (1+\alpha Y)v_1[(1+\alpha Y)(r+K\beta+K\beta\alpha Y)v_2 + (r(-I+k-2S)\alpha + k(1+\alpha Y)^2(2a_1 Y + c_1))v_3\big].$$

$$\pi_{21} = 2(\beta v_1 - c_1 v_3)v_2.$$

$$\pi_{31} = 2(-a_2 v_1 + e_2 c_2 v_2 + e_1\big((2a_1 Y + c_1)v_1 + a_1 S v_3\big))v_3.$$

So, the third directional derivative for system (1) is given by

$$D^3G(V,V,V) = \left[\sigma_{ij}\right]_{3\times1}, \tag{19}$$

where

$$\sigma_{11} = \frac{6v_3}{K(1+Y\alpha)^4}\big[r\alpha(1+Y\alpha)^2 v_1^2 + rS\alpha^2 v_3[-(1+\alpha Y)v_2 + (I-K+S)\alpha v_3] + (1+\alpha Y)v_1[r\alpha(1+Y\alpha)v_2 - (r(I-K+2S)\alpha^2 + K(1+\alpha Y)^3 a_1)v_3]\big].$$

$$\sigma_{21} = 0.$$

$$\sigma_{31} = 6a_1 e_1 v_1 v_3^2.$$

Theorem 9. The system (1) at $\hat{P}_0$ undergoes a transcritical bifurcation (TB) when $r = q_1 E_1 = r^*$.

Proof. It is easy to verify the JM of the model (1) at $\hat{P}_0$ with $r = r^*$, we get

$$J_0^* = J^*(P_0, r^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 - q_2 E_2 \end{bmatrix}$$

Thus $\lambda_{01}^* = 0$, $\lambda_{02}^* = -d_1$ and $\lambda_{03}^* = -d_2 - q_2 E_2$, represent the eigenvalues for J_0^*.

Let $V_0 = (v_{01}, v_{02}, v_{03})^T$ represents the eigenvector associated with $\lambda_{01}^* = 0$.

Then direct computation gives that $V_0 = (v_{01}, 0, 0)^T$, where $(v_{01} \neq 0)$ any real number.

Let $\Psi_0=(\psi_{01},\psi_{02},\psi_{03})^T$ represent the eigenvalues for $\lambda_{01}^*=0$ for the J_0^{*T}.
Then, the direct computation gives that $\Psi_0=(\psi_{01},0{,}0)^T$, where $(\psi_{01}\neq 0)$ any real number.
Accordingly, the following is obtained

$$\frac{\partial G}{\partial r}=G_r=(\frac{S}{1+\alpha Y}(1-\frac{S+I}{k}),0{,}0)^T\Rightarrow\Psi_0^T[G_r(\hat{P}_0,r^*)]=0.$$

Hence, the system (1) at $\hat{P}_0$ has no saddle-node bifurcation (S.NB).

Now, $\Psi_0^T[DG_r(\hat{P}_0,r^*)V_0]=v_{01}\psi_{01}\neq 0$

Moreover,$\Psi_0^T[D^2G(\hat{P}_0,r^*)(V_0,V_0)]=\frac{-2r^*}{k}v_{01}^2\psi_{01}\neq 0$. by "Sotomayor theorem"[27], system (1) undergoes a Transcritical Bifurcation (T.B) at the $\hat{P}_0$.

Theorem 10. The system (1) at $\breve{P}_1$ undergoes a T.B when $\beta=\frac{d_1}{\breve{S}}=\beta^*$.

Proof. The JM at at $\breve{P}_1$,we get

$$J_1^*=J^*(\breve{P}_1,\beta^*)=\begin{bmatrix}-(r-q_1E_1) & -(\frac{r}{K}+\beta^*)\breve{S} & (-r\alpha+\frac{r\alpha\breve{S}}{K}-c_1)\breve{S}\\ 0 & 0 & 0\\ 0 & 0 & -d_2-q_2E_2\end{bmatrix}.$$

Thus, $\lambda_{11}^*=-(r-q_1E_1)$, $\lambda_{12}^*=0$ and $\lambda_{13}^*=-d_2-q_2E_2$, represent the eigenvalues for J_1^*.
Let $V_1=(v_{11},v_{12},v_{13})^T$ represents the eigenvector associated with $\lambda_{12}^*=0$, Then direct computation gives that $V_1=(\tau_1v_{12},v_{12},0)^T$, where $(v_{12}\neq 0)$ with $\tau_1=\frac{-\breve{a}_{12}}{\breve{a}_{11}}<0$.

Let $\Psi_1=(\psi_{11},\psi_{12},\psi_{13})^T$ represent the eigenvalues for $\lambda_{01}^*=0$ for the J_0^{*T}, Then, the direct computation gives that $\Psi_1=(0,\psi_{12},0)^T$, where $(\psi_{12}\neq 0)$.
Accordingly, the following is obtained

$$\frac{\partial G}{\partial\beta}=G_\beta=(-SI,SI,0)^T\Rightarrow\Psi_1^T[G_\beta(\breve{P}_1,\beta^*)]=0.$$

Hence, the system (1) at $\breve{P}_1$ has no saddle-node bifurcation (S.NB).

Now, we have $\Psi_1^T[DG_r(\breve{P}_1,\beta^*)V_0]=2\breve{S}\tau_1v_{12}\psi_{12}\neq 0$

Moreover,$\Psi_1^T[D^2G(\breve{P}_1,\beta^*)(V_1,V_1)]=2\beta^*\tau_1{v_1}^2\neq 0$. system (1) undergoes a T.B at the $\breve{P}_1$.

Theorem 11. The system (1) at $\bar{P}_2$ undergoes a T.B when $a_2=\frac{e_1c_1\bar{S}-e_2c_2\bar{I}-d_2-q_2E_2}{\bar{S}}={a_2}^*$ provided that the following condition is hold

$$e_2c_2\tau_3+e_1c_1\tau_2+a_1\bar{S}\neq a_2\tau_2 \tag{20}$$

Otherwise it has a P.B.

Proof. The JM at at $\bar{P}_2$, we get

$$J_2^* = J^*(\bar{P}_2, {a_2}^*) = \begin{bmatrix} -\frac{r\bar{S}}{k} & -(\frac{r\bar{S}}{k}+\beta) & (-r\alpha+\frac{r\alpha\bar{S}}{k}+\frac{r\alpha\bar{I}}{k}+c_1)\bar{S} \\ \beta\bar{I} & 0 & -c_2\bar{S} \\ 0 & 0 & 0 \end{bmatrix} = [\bar{a}_{ij}]_{3\times 3}$$

Therefore, the eigenvalues of $J^*(\bar{P}_2, {a_2}^*)$ are determined as

$\lambda_{21} = \frac{T_1+\sqrt{{T_1}^2-4D_1}}{2}$, $\lambda_{22} = \frac{T_1-\sqrt{{T_1}^2-4D_1}}{2}$ and $\lambda_{23}^* = 0$ where $T_1$and $D_1$are given in (10a).

Let $V_2 = (v_{21}, v_{22}, v_{23})^T$ represents the eigenvector associated with $\lambda_{23}^* = 0$, Then direct computation gives that $V_2 = (\tau_2 v_{23}, \tau_3 v_{23}, v_{23})^T$, where $(v_{23} \neq 0)$ any real number with $\tau_2 = \frac{-\bar{a}_{23}}{\bar{a}_{21}} > 0$ and $\tau_2 = \frac{\bar{a}_{23}\bar{a}_{11}-\bar{a}_{21}\bar{a}_{13}}{\bar{a}_{12}\bar{a}_{21}}$

Let $\Psi_2 = (\psi_{21}, \psi_{22}, \psi_{23})^T$ represent the eigenvalues for $\lambda_{23}^* = 0$ for the J_2^{*T}. Then, the direct computation gives that $\Psi_2 = (0, 0, \psi_{23})^T$, where $(\psi_{23} \neq 0)$.

Accordingly, the following is obtained

$$\frac{\partial G}{\partial a_2} = G_{a_2} = (0,0,-SY)^T \Rightarrow \quad \Psi_2^T\left[G_{a_2}(\bar{P}_2, {a_2}^*)\right] = 0.$$

Hence, the system (1) at $\bar{P}_2$ has no saddle-node bifurcation (S.NB).

Now, we have $\Psi_2^T\left[DG_{a_2}(\bar{P}_2, {a_2}^*)V_2\right] = -\bar{S}v_{23}\psi_{23} \neq 0$

Moreover, $\Psi_2^T[D^2G(\bar{P}_2, {a_2}^*)(V_2, V_2)] = 2v_{23}\psi_{23}(-a_2\tau_2 + e_2c_2\tau_3 + e_1c_1\tau_2 + a_1\bar{S})$. Then the system (1) undergoes a T.B at the $\bar{P}_2$ under the condition (20).

However, violating condition (20) leads to $\Psi_2^T[D^3G(\bar{P}_2, {a_2}^*)(V_2, V_2, V_2)] = 6a_1e_1\tau_2 v_{23}^3 \neq 0$. Hence system (1) undergoes a P.B.

Theorem 12. The system (1) at $\tilde{P}_3$ undergoes a T.B when the parameter $d_1 = \beta\tilde{S} = {d_1}^*$ provided that the following condition hold

$$\beta\tau_4 \neq c_1\tau_5 \tag{21}$$

Proof. The JM at at $\tilde{P}_3$, we get

$J_3^* = J^*(\tilde{P}_3, {d_1}^*)$

$$= \begin{bmatrix} -\frac{r\hat{S}}{k(1+\alpha\hat{Y})} & -(\frac{r\hat{S}}{k(1+\alpha\hat{Y})}+\frac{d_1}{\hat{S}})\hat{S} & (\frac{-r\alpha}{k(1+\alpha\hat{Y})^2}+\frac{r\alpha k\hat{S}}{\left(k(1+\alpha\hat{Y})\right)^2}+c_1-2a_1\hat{Y})\hat{S} \\ 0 & 0 & 0 \\ (e_1c_1+e_1a_1\hat{Y}-a_2)\hat{Y} & e_2c_2\hat{Y} & e_1a_1\hat{S}\hat{Y} \end{bmatrix}$$

$= [\tilde{a}_{ij}]_{3\times 3}$

Therefore, the eigenvalues of $J^*(\tilde{P}_3, {a_2}^*)$ are determined as

$\lambda_{31} = \frac{T_2+\sqrt{{T_2}^2-4D_2}}{2}$, $\lambda_{33} = \frac{T_2-\sqrt{{T_2}^2-4D_2}}{2}$ and $\lambda_{32}^{*} = 0$ where $T_2$and $D_2$are given in (11a).

Let $V_3 = (v_{31}, v_{32}, v_{33})^T$ represents the eigenvector associated with $\lambda_{32}^{*} = 0$, Then direct computation gives that $V_3 = (\tau_4 v_{32}, v_{32}, \tau_5 v_{32})^T$, where $(v_{32} \neq 0)$ any real number with $\tau_4 = \frac{\tilde{a}_{13}\tilde{a}_{32} - \tilde{a}_{12}\tilde{a}_{33}}{\tilde{a}_{11}\tilde{a}_{33} - \tilde{a}_{13}\tilde{a}_{31}}$ and $\tau_5 = \frac{\tilde{a}_{12}\tilde{a}_{31} - \tilde{a}_{11}\tilde{a}_{32}}{\tilde{a}_{11}\tilde{a}_{33} - \tilde{a}_{13}\tilde{a}_{31}}$

Let $\Psi_3 = (\psi_{31}, \psi_{32}, \psi_{33})^T$ represent the eigenvalues for $\lambda_{32}^{*} = 0$ for the J_3^{*T}. Then, the direct computation gives that $\Psi_3 = (0, \psi_{32}, 0)^T$, where $(\psi_{32} \neq 0)$ any real number.

Accordingly, the following is obtained

$$\frac{\partial G}{\partial d_1} = G_{d_1} = (0, -I, 0)^T \Rightarrow \quad \Psi_3^T\left[G_\beta(\tilde{P}_3, {d_1}^*)\right] = 0.$$

Hence, the system (1) at$\tilde{P}_3$ has no S.NB. Now, we have

$$\Psi_3^T\left[DG_{d_1}(\tilde{P}_3, {d_1}^*)V_3\right] = v_{32}\psi_{32} \neq 0,$$

Moreover, $\Psi_3^T\left[D^2G\left(\tilde{P}_3, {d_1}^*\right)(V_3, V_3)\right] = 2v_{32}\psi_{32}(\beta\tau_4 - c_1\tau_5)$. Then the system (1) undergoes a T.B at the $\tilde{P}_3$ under the condition (21).

However, violating condition (21) leads to $\Psi_3^T\left[D^3G\left(\tilde{P}_3, {d_1}^*\right)(V_3, V_3, V_3)\right] = 0$. Hence system (1) at $\tilde{P}_3$has no P.B.

Theorem 13. The system (1) at P_4^* undergoes a S.NB when $e_2 = \frac{a_{12}^*(a_{21}^* a_{33}^* - a_{23}^* a_{31}^*)}{(a_{13}^* a_{21}^* - a_{11}^* a_{23}^*)\, c_2 Y^*} = e_2^*$, provided that

$$\pi_{11}\tau_8 + \pi_{21}\tau_9 + \pi_{31} \neq 0 \tag{22}$$

Proof. The JM of the system (2) at P_4^* , we get

$$J_4^* = J^*(P_4^*, e_2^*) = [a_{ij}^*]_{3\times 3}.$$

Therefore, if put $C_3 = 0$ at $e_2 = e_2^*$ in equation (12b). Hence the characteristic equation has a zero root.

Let $V_4 = (v_{41}, v_{42}, v_{43})^T$ represents the eigenvector associated with $\lambda_{42}^{*} = 0$, Then direct computation gives that $V_4 = (\tau_6 v_{43}, \tau_7 v_{43}, v_{43})^T$, where $(v_{43} \neq 0)$ any real number with $\tau_6 = \frac{-a_{23}^*}{a_{21}^*} > 0$ and $\tau_7 = \frac{a_{11}^* a_{23}^* - a_{21}^* a_{13}^*}{a_{12}^* a_{21}^*}$

Let $\Psi_4 = (\psi_{41}, \psi_{42}, \psi_{43})^T$ represent the eigenvalues for $\lambda_{42}^{*} = 0$ for the J_4^{*T}. Then, the direct computation gives that $\Psi_4 = (\tau_8\psi_{43}, \tau_9\psi_{43}, \psi_{43})^T$, where $(\psi_{43} \neq 0)$ any real number with $\tau_8 = \frac{-a_{32}^*}{a_{12}^*} > 0$ and $\tau_9 = \frac{a_{11}^* a_{32}^* - a_{12}^* a_{31}^*}{a_{12}^* a_{21}^*}$

Accordingly, the following is obtained

$$\frac{\partial G}{\partial e_2} = G_{e_2} = (0,0,c_2IY)^T, \ \Rightarrow \Psi_4^T\left[G_{e_2}(P_4^*, e_2^*)\right] = c_2I^*Y^*\psi_{43} \neq 0$$

Hence, S.NB takes place near P_4^*.

Clearly, straightforward computation shows that

$\Psi_4{}^T[D^2G(P_4^*, e_2^*)(V_4, V_4)] = (\pi_{11}\tau_8 + \pi_{21}\tau_9 + \pi_{31})v_{43}\psi_{43} \neq 0$, under the condition (22), and hence the system (1) undergoes a S.NB near P_4^* but neither T.B nor P.B can occur.

6. NUMERICAL SIMULATIONS

It is well known that the natural environment's interaction between prey and predator is one of mutual constraint and control. To further understand the dynamic connection between prey and predator, numerical simulations of the model (1) will be run to demonstrate some complicated dynamic behaviors. For simplicity, we set the parameter values as follows

$$r = 2;\ a_0 = 0.1;\ K = 20;\ c_1 = 0.75;\ a_1 = 0.05;\ b = 0.25;\ q_1 = 0.5;\ E_1 = 2;$$
$$c_2 = 0.75; d_1 = 0.1;\ e_1 = 0.5;\ e_2 = 0.6; d_2 = 0.1;\ a_2 = 0.05; q_2 = 0.5;\ E_2 = 2. \quad (23)$$

It is obtained that the system (1) asymptotically approaches the IEP under set (23), as shown in Figure (1).

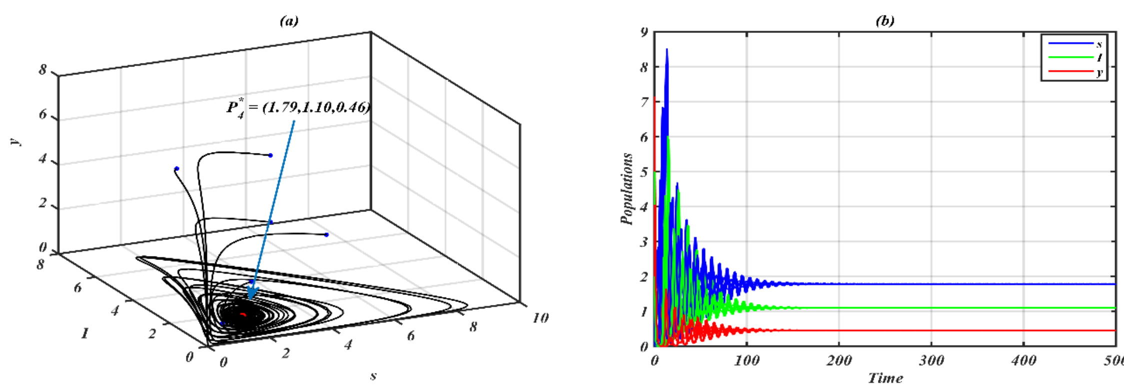

Figure 1. Using the set (23) with different initial points, the system(1) solutions converges to P_4^*= (1.79, 1.10, 0.46) (a) 3D Phaseplot. (b) Solutions as a function of time.

Moreover, the impact of the varying parameter r is studied numerically on the dynamic of the system (1), and it is noted that for $r \geq 2.3$, the system approaches to $FIPEP$, for $r \leq 1.7$ the system (1) approaches to $FPEP$, while $r < 1$, the solution approaches to TEP, as illustrated Fig. (2). It is noted that system (1) approaches to IEP of the system (1) for $r \in [1.8,2.2]$, as showed in Fig. (1).

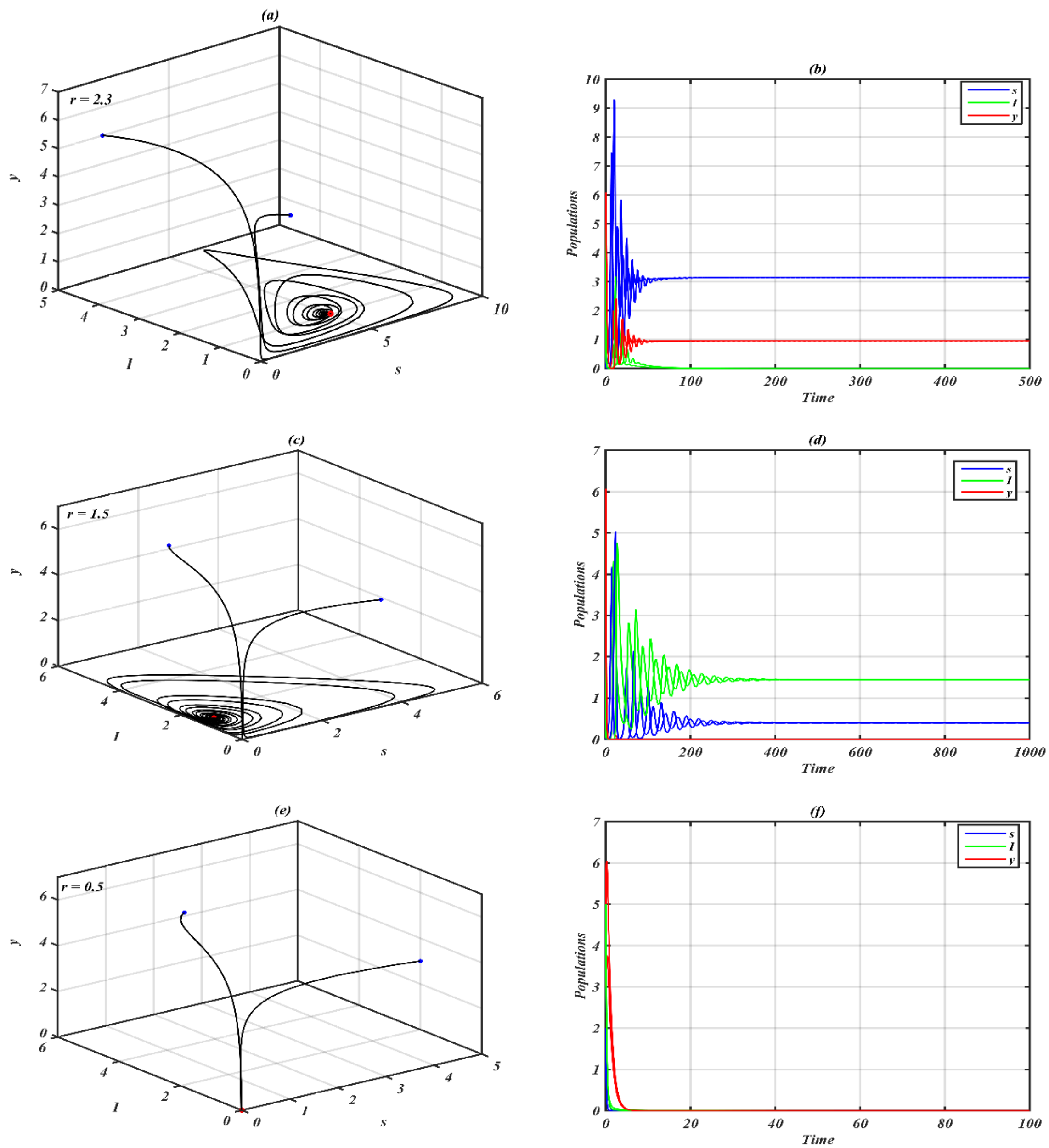

Figure 2. The trajectories of system (1) using parameter values (23) with different value r (a) Trajectories approach $\tilde{P}_3 = (3.14,0,0.96)$ for $r = 2.3$.(b) Time series for $r = 2.3$. (c)Trajectories approach $\bar{P}_2 = 0.49,1.44,0)$ for $r = 1.5$ (d) b) Time series for $r = 1.5$. (e) Trajectories approach to $\hat{P}_0 = (0,0,0)$ for $r = 0.5$.(f) time series for $r = 0.5$.

Using data (23), the effect of varying the parameters K on the dynamic of the system (1) is numerically studied., it is observed that for $K \in [0.9,11]$, the system approaches to $FPEP$, for $K \leq 0.8$, the system (1) approaches to $FIPEP$, while $K \geq 52$, the system (1) approaches to periodic dynamics in 3D, as illustrated Fig. (2). It is noted that system (1) approaches to IEP of the system (1) for $K \in [12,51]$, as showed in Fig.(1).

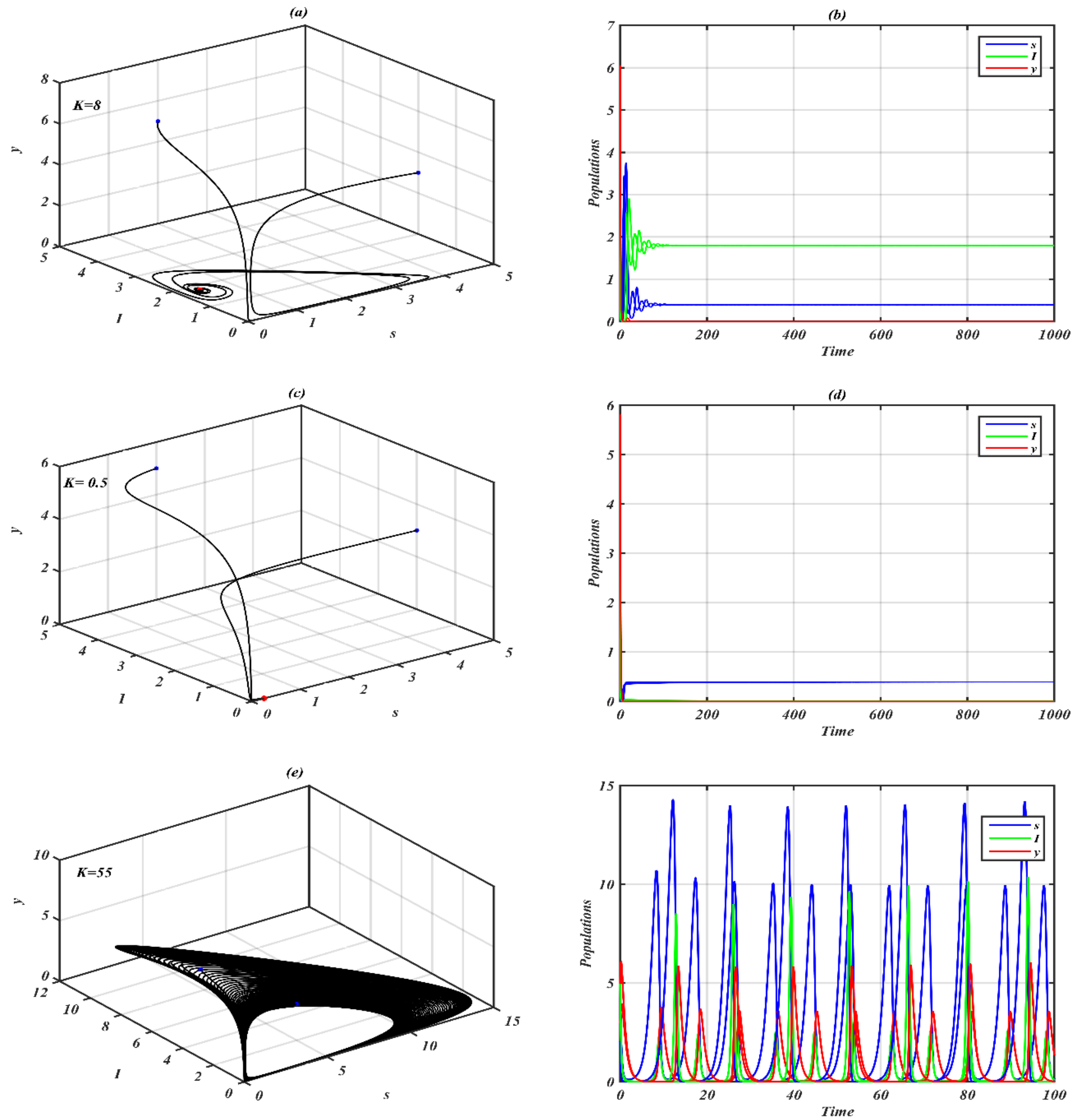

Figure 3. The trajectories of system (1) using parameter values (23) with different values for K. (a) Trajectories approach $\bar{P}_2 = (0.40,1.79,0)$ for $K = 8$. (b) Time series for $K = 8$. (c)Trajectories approach $\check{P}_1 = (0.39,0,0)$ for $K = 0.5$ (d) Time series for $K = 0.5$. (e) Periodic dynamics in $\mathcal{R}_+^3$ for $K = 55$.. (f) Time series for $K = 55$.

Moreover, it is observed that for α, it observed that when $\alpha \geq 0.3$, that system (1) approaches to periodic dynamics in 3D and when $\alpha < 0.3$, as illustrated in Fig. (3). Otherwise, the system approaches to IEP as showed in Fig. (1).

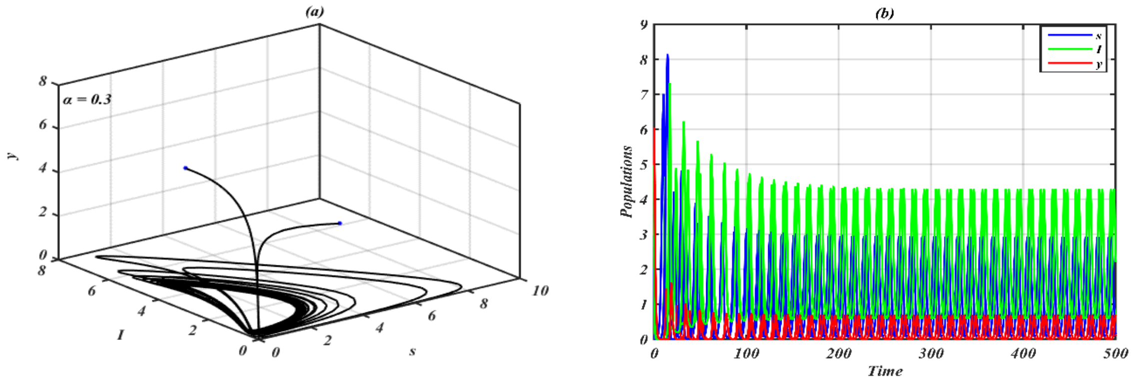

Figure 4. The trajectories of system (1) using parameter values (23) with different value α. (a) Periodic dynamics in $\mathcal{R}_{+}^{3}$ for $\alpha = 0.3$.(b) Time series for $\alpha = 0.3$.

It is observed that varying the parameters $a_1$and a_2 has a similar effect as that shown with varying α.

The analysis of the impact of varying the parameter β on the system's (1) dynamics reveals that it approaches $FPEP$ when $\beta \geq 0.35$. Also, it approaches $FIPEP$, when $\beta \leq 0.19$, as illustrated in Fig.(4). While the system (1) approaches to IEP for $\beta \in [0.2,0.34]$, as showed in Fig. (1).

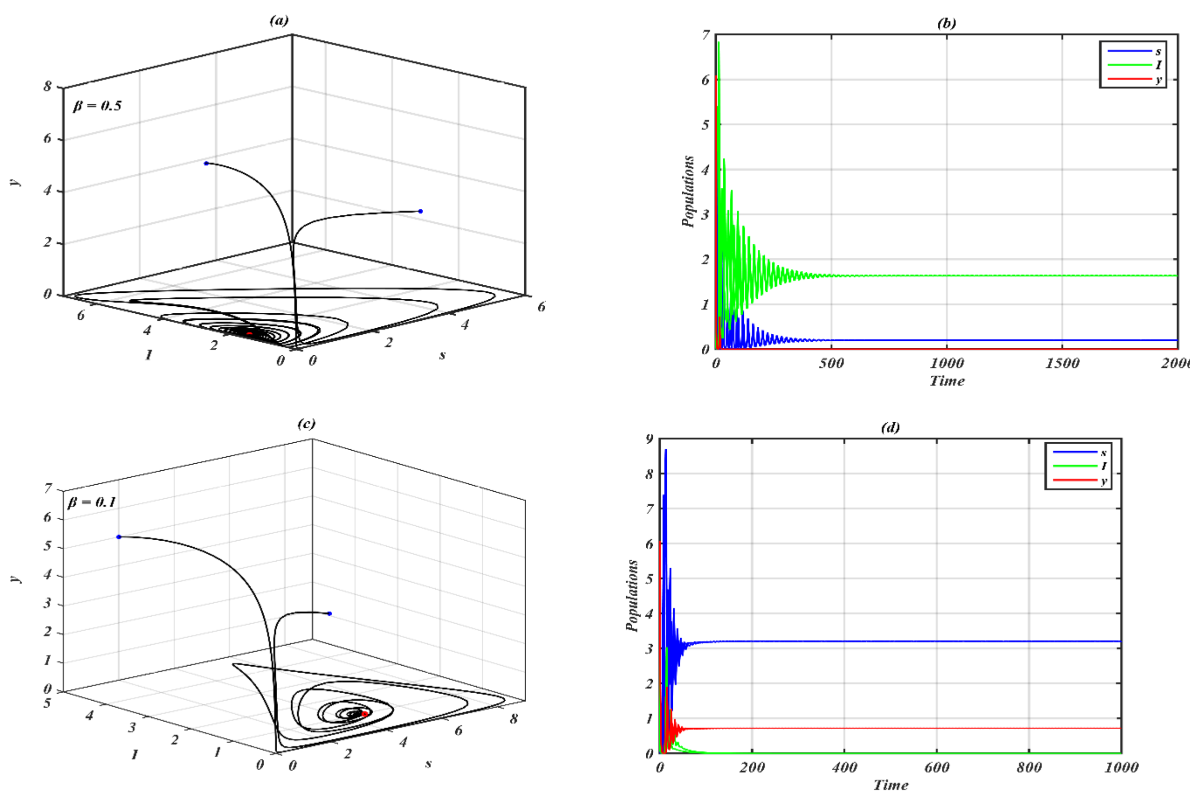

Figure 5. The trajectories of system (1) using parameter values (23) with different values for β.

(a) Trajectories approach $\bar{P}_2 = (0.19,1.63,0)$ for $\beta = 0.5$. (b) Time series for $\beta = 0.5$. (c)Trajectories approach $\tilde{P}_3 = (3.20,0,0.72)$ for $\beta = 0.1$ (d) Time series for $\beta = 0.1$.

It noted that changing the parameters $q_1, q_2, E_1,$ and E_2 has a similar impact as that shown with varying β.

On the other hand, for the parameter c_2, the system (1) approaches $FPEP$ when $c_2 \leq 0.58$, as illustrated in Fig.(5). Otherwise, the system approaches IEP as showed in Fig.(1).

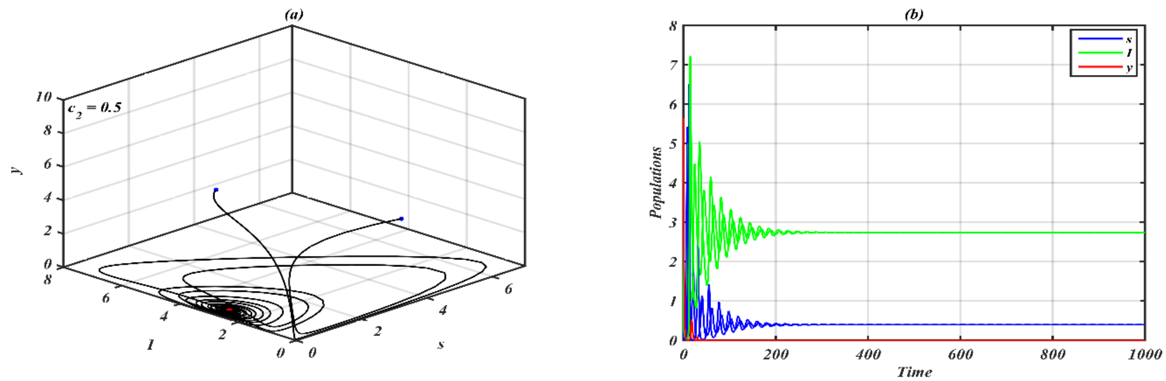

Figure 6. The trajectories of system (1) using parameter values (23) with different value c_2. (a) Trajectories approach $\bar{P}_2 = (0.39,2.27,0)$ for $c_2 = 0.5$. (b) Time series for $c_2 = 0.5$.

Now, the impact of parameter d_1, the system (1) approaches $FIPEP$ when $d_1 \geq 0.27$, as illustrated in Fig. (6). Otherwise, the system approaches IEP as showed in Fig. (1).

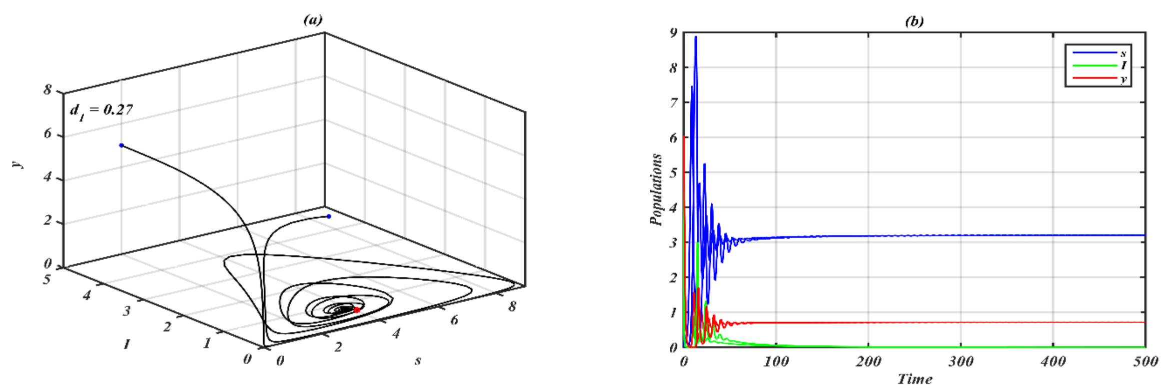

Figure 7. The trajectories of system (1) using parameter values (23) with different values for d_1. (a) Trajectories approach $\tilde{P}_3 = (3.20,0,0.72)$ for $d_1 = 0.27$. (b) Time series for $d_1 = 0.27$.

In Fig. (7), the influence of varying the parameter e_1 is shown at a selected value. It is noted that for $e_1 \geq 0.58$, the system (1) approaches $FIPEP$, and when $e_1 \leq 0.35$, the system (1) approaches periodic dynamics in 3D, as illustrated in Fig. (7). While when $e_1 \in [0.36,0.57]$, the system approaches IEP as showed in Fig. (1).

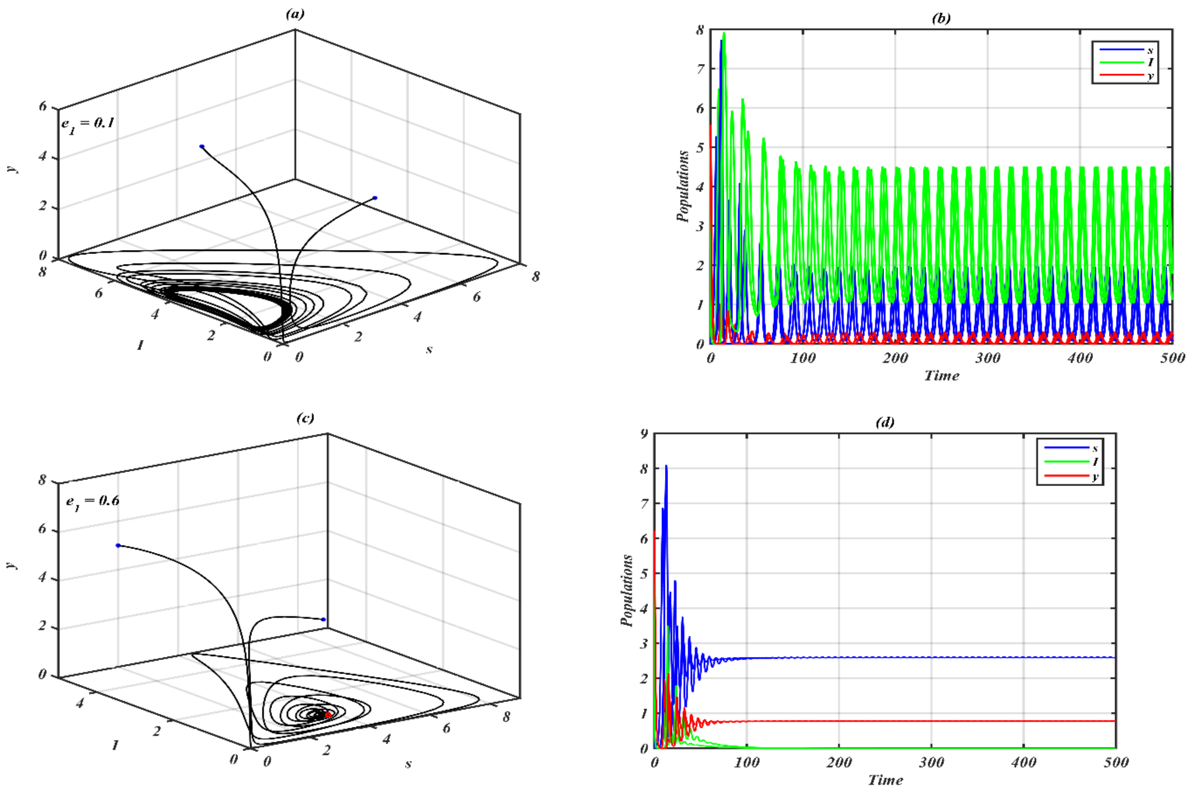

Figure 8. The trajectories of system (1) using parameter values (23) with different values for e_1. (a)Trajectories approach $\tilde{P}_3 = (2.59,0,0.78)$ for $e_1 = 0.6$. (b) Time series for $e_1 = 0.6$. (c) Periodic dynamics in $\mathcal{R}_+^3$ for $e_1 = 0.1$. (d) Time series for $e_1 = 0.1$.

Finally, the effect of varying the parameter d_2 on the system's dynamic shows when $d_2 \geq 0.4$, the system (1) approaches $FPEP$ when $d_2 \geq 0.4$, as illustrated in Fig. (9). Otherwise, the system approaches to IEP as showed in Fig. (1).

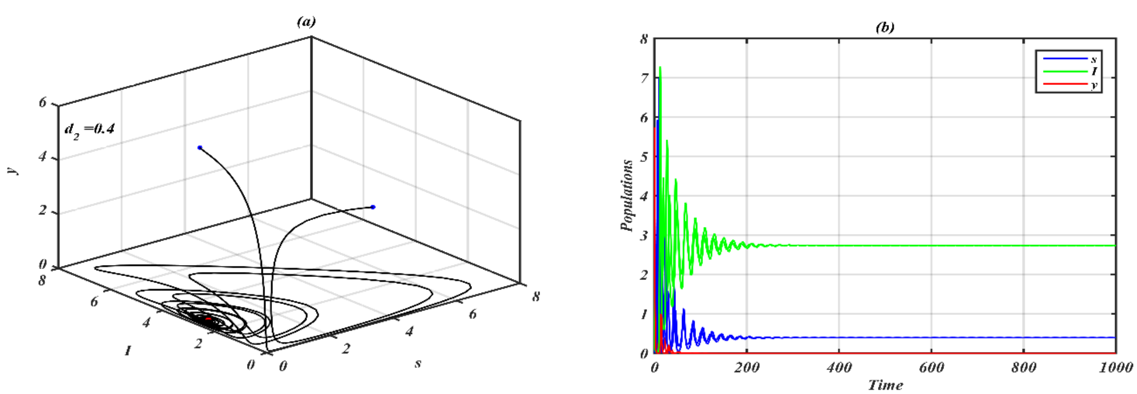

Figure 9. The trajectories of system (1) using parameter values (23) with different value d_2. (a) Trajectories approach $\bar{P}_2 = (0.40,2.74,0)$ for $d_2 = 0.4$.(b) Time series for $d_2 = 0.4$.

7. CONCLUSIONS

Predicting and controlling environmental dynamics requires an understanding of the intricate interaction between hunting cooperation, anti-predator behavior, Fear and harvest in eco-epidemiological models include disease in prey.

This paper proposes an eco-epidemiological model consisting of prey-predator system that includes the role of fear from predation, hunting cooperation, fear and anti-predator on the dynamic of a prey-predator model with disease in prey. The proposed model's solution characteristics are studied. All of the biologically feasible EPs have been found. the local and global stability requirements for each equilibrium point are determined. It was established what was needed for local bifurcation to happen. Lastly, using an approximated data set, the system's overall dynamical behavior is numerically analyzed to comprehend and validate the theoretical findings and investigate the impact of changing a parameter on the system's dynamics. Then the obtained numerical solutions are drawn numerically to obtain different phase portraits with the help of the MATLAB R2021a program.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest.

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