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BIFURCATION AND PERSISTENCE CONDITIONS OF SOKOL-HOWELL PREY-PREDATOR MODEL WITH TOXINS AND ADDITIONAL FOOD FOR PREDATORS

SAFAA HASAN RASOOL*, AZHAR ABBAS MAJEED

Department of Mathematics, College of Science, University of Baghdad, Iraq

Abstract: This work investigates the conditions for local bifurcation in a three-species food chain model, comprising prey, predator, and top predator, incorporating factors such as fear, a weak Allee effect, toxins, and supplementary food in the predator population, utilizing the Sokol-Howell functional response. This system possesses thirteen equilibrium points. Transcritical and pitchfork bifurcations occur near the extinction equilibrium point (Q_0), the predator-top predator-free equilibrium point (Q_1), the prey-top predator-free equilibrium point (Q_3), while near and the top predator-free equilibrium point (Q_4), while saddle node bifurcations are found near the prey-free equilibrium point (Q_8), and the interior equilibrium point (Q_{10}). Moreover, the persistence conditions have been derived to ensure the coexistence and stability of all species. Numerical simulation performed using matlab support the analytic finding and illustrate the complex dynamic behaviors near the equilibrium points. This study provides deeper insight into how Allee effect, toxin effects and additional food sources influence the bifurcation structure and persistence of the three-species ecological system.

Keywords: prey-predator, Sokol-Howell, fear, Allee-effect, additional food, toxin, bifurcation, persistence.

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*Corresponding author

E-mail address: safaa.hasan2303@sc.uobaghdad.edu.iq

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1. INTRODUCTION

In the last few decades, the study of how prey-predator models behave over time has become one of the most active areas in theoretical ecology and population dynamic. Ecosystem in the real word are always changing because of changes in biological and environmental factors. These changes can have a big effect on how species interact and how they are stable [1],[2]. In these kinds of biological systems, things like fear, toxicity and food availability can change the basic relationships between predators and prey. For example, the fear effect makes prey change how they act and hunt when predators are around, which in turn affects population growth and stability [3],[4].

To encapsulate these intricate interactions, various mathematical models have been formulated, from the Lotka-Volterra model to more advance frameworks that integrate functional responses and behavioral adaptations [5],[6]. The Sokol-Howell functional response has been employed to characterize predator saturation and adaptive predation rates, rendering it appropriate for the analysis has also become a very useful way to find parameter thresholds that cause the system's dynamic to change in important ways, like when stable equilibria turn into oscillations or chaos [8-10].

Along with the fear effect, the Allee effect is receiving a lot of attention for how it slows population growth when prey density drops below a certain level. This effect can change the boundaries of persistence and extinction in predator-prey interaction, which can lead to interesting dynamic results [11-13]. Research indicates that the Allee effect can stabilize the system or produce oscillatory dynamics, depending on its intensity and interaction with other ecological variables [14],[15]. Furthermore, the introduction of toxic substances (toxins) into the environment can destabilize ecological balance by directly influencing prey or predator populations. Toxins can either slow down the growth of predator or help prey populations by making it less likely that they will be eaten [16-18]. For example, toxins released by certain marine organisms can affected the growth of predatory fish or help product smaller creatures, by reducing the likelihood of them being preyed upon.

Supplementary food availability for predators is a significant ecological factor that enhances predator survival and alters the pressure on prey populations [19]. Numerous studies have examined the impact of supplementary food, revealing its significant influence on persistence conditions and bifurcation structure within predator-prey systems [20],[21]. For example, adding food source can increase the range of coexistence or create new bifurcation phenomena, like Hopf or saddle-node bifurcations [22].

A. A. Majeed and his colleagues have recently conducted comprehensive investigations into the effects of fear and toxins within diverse predator-prey framework, delineating bifurcation conditions that characterize transitions between stable and oscillatory states [23]. The impacts of stage structure, migration, and eco-toxicants on population persistence have been examined to improve understanding of complex ecological interactions [24]. Furthermore, the interplay of fear, Allee effect, toxins, and supplementary food has demonstrated complex dynamical behaviors, such as multistability and coexistence regions [25-28].

This paper examines the Sokol-Howell prey-predator model, which involves fear, the Allee effect, toxins, and supplementary food sources for predators. Our primary objective is to identify the bifurcation and persistence conditions that enable the coexistence and continued existence of both populations. Through the use of bifurcation theory, we delineate parameter areas that indicate transitions between several dynamic regimes and develop persistence criteria that ensure positive population densities. The findings reported here improve existing ecological models and provide a deep understanding of the complicated interactions that arise when various ecological influences coexist in natural systems. This study can assist in biological conservation, managing pollutants, and maintaining ecological stability in frequently polluted and altered environments.

2. Mathematical Model

This article presents a study model comprising Prey, Predator, and Top Predator, utilizing the Sokol-Howel functional response that given in [19]. Which of the following denotes the overall population density at time T, $S_1(T), S_2(T)$ and $S_3(T)$ in that sequence.

$$\left.\begin{aligned}
\frac{dS_1}{dT} &= r_1 S_1\left(1-\frac{S_1}{k_1}\right)\left(\frac{1}{1+f_1S_2}\right)\left(\frac{S_1}{S_1+A_1}\right)-\frac{c_1S_1S_2}{a_1+\gamma_1S_1^2}-b_1S_1^2S_2-d_1S_1,\\
\frac{dS_2}{dT} &= r_2 S_2\left(1-\frac{S_2}{k_2}\right)\left(\frac{1}{1+f_2S_3}\right)\left(\frac{S_2}{S_2+A_2}\right)+\frac{\ell_1S_1S_2}{a_1+\gamma_1S_1^2}-\frac{c_2S_2S_3}{a_2+\gamma_2S_2^2}+(1-\eta)\alpha S_2-d_2S_2-\delta_1S_2^2,\\
\frac{dS_3}{dT} &= \frac{\ell_2S_2S_3}{a_2+\gamma_2S_2^2}+\eta\alpha S_3-d_3S_3-\delta_2S_3.
\end{aligned}\right\} \quad (1)$$

With the following initial condition $S_1(T) \geq 0, S_2(T) \geq 0$ and $S_3(T) \geq 0$. In spite of the fact that the biological significance of the system (1) attributes is shown in Table 1:

Table 1 Biological meaning of the parameters of system 1

Parameters	**Biological meaning**
$r_i > 0\,, i = 1,2$	The prey and predator respective inherent growth rates
$k_i > 0\,, i = 1,2$	The carrying capacity of the prey and predator in that order
$f_i > 0\,, i = 1,2$	The fear rate of the prey species from predator and predator species from top predator in that order
$A_i > 0\,, i = 1,2$	The Allee effect parameter**s** of the prey and predator in that order
$c_i > 0\,, i = 1,2$	The rates at which the prey and predator attack in that order
$\ell_i > 0, \ell_i \leq c_i,$ $i = 1,2$	Both the rates of food conversion to the predator and the apex predator are shown here
$a_i > 0\,, i = 1,2$	The interference constant of Sokol-Howell function response of the prey and predator in that order
$\gamma_i > 0\,, i = 1,2$	The saturation coefficient of Sokol-Howell function response of the prey-predator in that order
$b_1 > 0$	The toxin rate of the prey
$\alpha > 0$	The additional food for predator
$(1-\eta)\alpha$, $0<\eta<1$	The benefited of top predator by some portion of the additional food of predator
$d_m > 0, m = 1,2,3$	Rates of natural mortality among prey, predators, and top predators
$\delta_i > 0\,, i = 1,2,3$	The rates of toxins in the prey and top predators

3. DIMENSIONLESS MATHEMATICAL MODEL [19]

Non-dimensionalization is applied to system (1) to simplify the model and reduce the number of parameters for improved interpretability. In system (1), the parameter count is diminished from 24 to 19 through the integration of dimensionless parameters in the prey equation, yielding the subsequent dimensionless representation:

$$\left.\begin{aligned}
\frac{dx}{dt} &= x\left[x(1-x)\left(\frac{1}{1+u_1 y}\right)\left(\frac{1}{x+u_2}\right)-\frac{u_3 y}{u_4+x^2}-u_5 xy-u_6\right]=f_1(x,y,z),\\
\frac{dy}{dt} &= y\left[u_7 y(1-y)\left(\frac{1}{1+u_8 z}\right)\left(\frac{1}{y+u_9}\right)+\frac{u_{10}\,x}{u_4+x^2}-\frac{u_{11}\,z}{u_{12}+y^2}+u_{13}-u_{14}-u_{15}y\right]=f_2(x,y,z),\\
\frac{dz}{dt} &= z\left[\frac{u_{16}\,y}{u_{12}+y^2}+u_{17}-u_{18}-u_{19}\right]=f_3(x,y,z).
\end{aligned}\right\}(2)$$

Where the parameters of dimensionless are given by:

$x=\frac{s_1}{k_1},\ y=\frac{s_2}{k_2},\ z=\frac{s_3}{k_3},\ t=r_1 T,\ u_1=k_1 f_1,\ u_2=\frac{A_1}{k_1},\ u_3=\frac{c_1 k_2}{r_1\gamma_1 k_1^2}, u_4=\frac{a_1}{\gamma_1 k_1^2},\ u_5=\frac{b_1 k_1 k_2}{r_1},$

$u_6=\frac{d_1}{r_1}, u_7=\frac{r_2}{r_1},\ u_8=k_2 f_2,\ u_9=\frac{A_2}{k_2},\ u_{10}=\frac{\ell_1}{r_1\gamma_1 k_1},\ u_{11}=\frac{c_2}{r_1\gamma_2 k_2},\ u_{12}=\frac{a_2}{\gamma_2 k_2^2},\ u_{13}=\frac{(1-\eta)\alpha}{r_2},$

$u_{14}=\frac{d_2}{r_2}, u_{15}=\frac{\delta_1 k_2}{r_2},$ where $\eta\in(0,1),\ u_{16}=\frac{\ell_2}{r_1\gamma_2 k_2},\ u_{17}=\frac{\eta\alpha}{r_2},\ u_{18}=\frac{d_3}{r_1},\ u_{19}=\frac{\delta_2}{r_2}.$

As far as the dependent variables x, y, and z are concerned, it's easy to make sure that all of system (2) interaction functions are continuous and have continuous partial derivatives on R_+^3. System (2) can provide only one solution for any non-negative initial condition since the functions in problem are Lipschitz functions.

4. LOCAL BIFURCATION ANALYSIS (LBA):

This section investigated (LBA) of model (2), concentrating on the alterations occurring around each equilibrium point as the parameters of dynamic behavior change. Our objective is to establish higher order conditions that guarantee the emergence of the predominant local bifurcations, as proven by Sotomayor's theorem. According to the Jacobean matrix J(x, y, z) of system (2) presented in [19], the following is accurate:

$$J=\left[m_{ij}\right]_{3\times 3}, \tag{3}$$

$$m_{11}=\frac{x(1-x)}{(1+u_1 y)(x+u_2)}-\frac{u_3 y}{u_4+x^2}-u_5 xy-u_6+x\left(\frac{u_2(1-2x)-x^2}{(1+u_1 y)(x+u_2)^2}+\frac{2u_3 xy}{(u_4+x^2)^2}-u_5 y\right),$$

$$m_{12} = x\left(\frac{u_1 x(x-1)}{(1+u_1 y)^2 (x+u_2)} - \frac{u_3}{u_4+x^2} - u_5 x\right), \quad m_{13} = 0,$$

$$m_{21} = u_{10}\frac{(u_4 - x^2)}{(u_4+x^2)^2} y,$$

$$m_{22} = u_7 \frac{(1-y)}{(1+u_8 z)} \frac{y}{(y+u_9)} + u_{10}\frac{x}{u_4+x^2} - u_{11}\frac{z}{u_{12}+y^2} + u_{13} - (u_{14} + u_{15} y) + y\left(\frac{u_7\left(u_9(1-2y)-y^2\right)}{(1+u_8 z)(y+u_9)^2} + \frac{2u_{11} y\, z}{(u_{12}+y^2)^2} - u_{15}\right),$$

$$m_{23} = y\left(\frac{u_7 u_8 (y-1)}{(1+u_8 z)^2}\frac{y}{(y+u_9)} - \frac{u_{11}}{u_{12}+y^2}\right),$$

$$m_{31} = 0, \quad m_{32} = z\left(\frac{u_{16}(u_{12} - y^2)}{(u_{12}+y^2)^2}\right),$$

$$m_{33} = \frac{u_{16}\, y}{u_{12}+y^2} + u_{17} - u_{18} - u_{19}.$$

For any non-zero vector, it is self-evident that $\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_3)^{\mathrm{T}}$:

$$D^2F(Q,\mu)(\Upsilon,\Upsilon) = [W_{i1}]_{3\times1}, \tag{4}$$

$$W_{11} = 2\Bigg[\left[\frac{u_2(1-2x)-x^2}{(1+u_1 y)(x+u_2)^2} + \frac{2u_3 xy}{(u_4+x^2)^2} - u_5 y + x\left(-u_2\frac{(u_2+1)}{(1+u_1 y)(x+u_2)^3} + u_3\frac{(u_4-3x^2)}{(u_4+x^2)^3} y\right)\right]{\Upsilon_1}^2 + \left[\frac{u_1 x(x-1)}{(1+u_1 y)^2(x+u_2)} - \frac{u_3}{u_4+x^2} - 2u_5 x + x\left(\frac{u_1\left(u_2(2x-1)+x^2\right)}{(1+u_1 y)^2(x+u_2)^2} + \frac{2u_3 x}{(u_4+x^2)^2}\right)\right]\Upsilon_1\Upsilon_2 + {u_1}^2\frac{(1-x)}{(1+u_1 y)^3}\frac{x}{(x+u_2)}{\Upsilon_2}^2\Bigg],$$

$$W_{21} = 2\Bigg[\frac{u_{10} xy\left(x^2-3u_4\right)}{(u_4+x^2)^3}{\Upsilon_1}^2 + u_{10}\frac{(u_4-x^2)}{(u_4+x^2)^2}\Upsilon_1\,\Upsilon_2 + \Bigg[\frac{u_7\left(u_9(1-2y)-y^2\right)}{(1+u_8 z)(y+u_9)^2} + \frac{2u_{11} y\, z}{(u_{12}+y^2)^2} - u_{15} + y\left(\frac{-u_7}{(1+u_8 z)}\frac{u_9(u_9+1)}{(y+u_9)^3} + u_{11}\frac{z\left(u_{12}-3y^2\right)}{(u_{12}+y^2)^3}\right)\Bigg]{\Upsilon_2}^2 + \Bigg[u_7 u_8\frac{(y-1)}{(1+u_8 z)^2}\frac{y}{(y+u_9)} - \frac{u_{11}}{u_{12}+y^2} + y\left(u_7 u_8\frac{\left(u_9(2y-1)+y^2\right)}{(1+u_8 z)^2(y+u_9)^2} + 2u_{11}\frac{y}{(u_{12}+y^2)^2}\right)\Bigg]\Upsilon_2\Upsilon_3 + u_7 {u_8}^2\frac{(1-y)}{(1+u_8 z)^3}\frac{y}{(y+u_9)}{\Upsilon_3}^2\Bigg],$$

$$W_{31} = 2\left[\frac{u_{16} yz\left(y^2-3u_{12}\right)}{(u_{12}+y^2)^3}{\Upsilon_2}^2 + u_{16}\frac{(u_{12}-y^2)}{(u_{12}+y^2)^2}\,\Upsilon_2\Upsilon_3\right],$$

and

$$D^3F(Q,\mu)(\Upsilon,\Upsilon,\Upsilon) = [L_{i1}]_{3\times1}, \tag{5}$$

$$L_{11} = 6\Bigg[\left[\frac{-u_2(u_2+1)}{(1+u_1 y)(x+u_2)^3} + \frac{u_3 y(u_4-3x^2)}{(u_4+x^2)^3} + x\left(\frac{u_2(u_2+1)}{(1+u_1 y)(x+u_2)^4} - \frac{4u_3 xy(u_4-x^2)}{(u_4+x^2)^4}\right)\right]{\Upsilon_1}^3 + \left[u_1\frac{\left(u_2(2x-1)+x^2\right)}{(1+u_1 y)^2(x+u_2)^2} + 2u_3\frac{x}{(u_4+x^2)^2} - u_5 + x\left(\frac{u_1 u_2(u_2+1)}{(1+u_1 y)^2(x+u_2)^3} + \frac{u_3(u_4-3x^2)}{(u_4+x^2)^3}\right)\right]{\Upsilon_1}^2\Upsilon_2 + u_1^2\frac{x\left(x(1-2x)+u_2(2-3x)\right)}{(1+u_1 y)^3(x+u_2)^2}\Upsilon_1{\Upsilon_2}^2 + {u_1}^3\frac{x^2(x-1)}{(1+u_1 y)^4(x+u_2)}{\Upsilon_2}^3\Bigg],$$

$L_{21} = 6\Big[\frac{u_{10}y(u_4(6x^2-u_4)-x^4)}{(u_4+x^2)^4}\Upsilon_1{}^3 + u_{10}\frac{x(x^2-3u_4)}{(u_4+x^2)^3}\Upsilon_1{}^2\,\Upsilon_2 + \Big[\frac{-u_7}{(1+u_8z)}\frac{u_9(u_9+1)}{(y+u_9)^3} + u_{11}\frac{(u_{12}-3y^2)}{(u_{12}+y^2)^3}z +$

$y\left(\frac{u_7}{(1+u_8z)}\frac{u_9(u_9+1)}{(y+u_9)^4} + 4\frac{u_{11}yz(y^2-u_{12})}{(u_{12}+y^2)^4}\right)\Big]\Upsilon_2{}^3 + \Big[\frac{u_7u_8(u_9(2y-1)+y^2)}{(1+u_8z)^2(y+u_9)^2} + 2u_{11}\frac{y}{(u_{12}+y^2)^2} +$

$y\left(\frac{u_7u_8u_9(u_9+1)}{(1+u_8z)^2(y+u_9)^3} + u_{11}\frac{(u_{12}-3y^2)}{(u_{12}+y^2)^3}\right)\Big]\Upsilon_2{}^2\,\Upsilon_3 + u_7u_8{}^2\frac{y(y(1-2y)+u_9(2-3y))}{(1+u_8z)^3(y+u_9)^2}\Upsilon_2\,\Upsilon_3{}^2 +$

$u_7u_8{}^3\frac{y(y-1)}{(1+u_8z)^4(y+u_9)}\Upsilon_3{}^3\Big],$

$L_{31} = 6\Big[\frac{u_{16}z(u_{12}(6y^2-u_{12})-y^4)}{(u_{12}+y^2)^4}\Upsilon_2{}^3 + u_{16}\frac{y(y^2-3u_{12})}{(u_{12}+y^2)^3}\,\Upsilon_2{}^2\,\Upsilon_3\Big],$

where $Q = (x, \mathrm{y}, \mathrm{z})^T$ and μ is any parameter.

Theorem (1): Assume that condition (15) stated in reference [19] and a further condition are met:

$$u_7u_9 \neq u_{15} \tag{6}$$

then system (2) at the (EP) $Q_0 = (0\,,0\,,0\,)$ with $u_{14}^{\mathrm{o}} = u_{14} = u_{13}$ a transcritical bifurcation or pitchfork bifurcation can occur, but a saddle-node bifurcation is not possible.

Proof: using the Jacobian matrix J_0, as stated in Eq. (25) in reference [19], If the eigenvalue ($\lambda_{0y} = 0$) of system (2) at $u_{14} = u_{13}$, then J_0 with $u_{14}^{\mathrm{o}} = u_{14} = u_{13}$ becom

$$J_0^{\mathrm{o}} = J_0(Q_0\,, u_{14}^{\mathrm{o}}) = \begin{bmatrix} -u_6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_{17}-u_{18}-u_{19} \end{bmatrix}.$$

Now, the eigenvector that corresponds to the eigenvalue $\lambda_{0y} = 0$ is written in $\Upsilon^{\mathrm{o}[0]} = \left(\Upsilon_1^{\mathrm{o}[0]}, \Upsilon_2^{\mathrm{o}[0]}, \Upsilon_3^{\mathrm{o}[0]}\right)^T$.

Hence $\left(J_0^{\mathrm{o}} - \lambda_{0\mathrm{y}}I\right)\Upsilon^{\mathrm{o}[0]} = 0$, by condition (15) which is given in [19] that gives $\Upsilon^{\mathrm{o}[0]} = \left(0\,, \Upsilon_2^{\mathrm{o}[0]}, 0\right)^T$ where the real constant $\Upsilon_2^{\mathrm{o}[0]} \neq 0$.

For $\lambda_{0y} = 0$, the eigenvector of $J_0^{\mathrm{o}T}$ is $\psi^{\mathrm{o}[0]} = \left(\psi_1^{\mathrm{o}[0]}, \psi_2^{\mathrm{o}[0]}, \psi_3^{\mathrm{o}[0]}\right)^T$.

We find $\left(J_0^{\mathrm{o}T} - \lambda_{0\mathrm{y}}I\right)\psi^{\mathrm{o}[0]} = 0$, we solving this equation for $\psi^{\mathrm{o}[0]}$ we obtain $\psi^{\mathrm{o}[0]} = \left(0\,, \psi_2^{\mathrm{o}[0]}, 0\right)^T$, where the real constant $\psi_2^{\mathrm{o}[0]} \neq 0$.

Right now, assume $\frac{\partial f}{\partial u_{14}} = f_{u_{14}}(Q\,, u_{14}) = \left(\frac{\partial f_1}{\partial u_{14}}, \frac{\partial f_2}{\partial u_{14}}, \frac{\partial f_3}{\partial u_{14}}\right)^T = (0\,, -\mathrm{y}\,, 0)^T$.

So, $f_{u_{14}}(Q_0\,, u_{14}^{\mathrm{o}}) = (0\,,0\,,0)^T$ then $\left(\psi^{\mathrm{o}[0]}\right)^T f_{u_{14}}(Q_0\,, u_{14}^{\mathrm{o}}) = 0$.

Hence, according "Sotomayor's theorem," we conclude the conditions for a saddle-node bifurcation are not be fulfilled. The first condition for the transcritical bifurcation is fulfilled.

Currently, as a result, $Df_{u_{14}}(Q\ ,u_{14}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

where $Df_{u_{14}}(Q\ ,u_{14})$ is the derivative of $f_{u_{14}}(Q\ ,u_{14})$ with respect to $Q = (x, \mathrm{y}, \mathrm{z})^T$.

Further, it is observed that $Df_{u_{14}}(Q_0\ ,u_{14}^{\mathrm{o}})\Upsilon^{\mathrm{o}[0]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \Upsilon_2^{\mathrm{o}[0]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\Upsilon_2^{\mathrm{o}[0]} \\ 0 \end{bmatrix}$, so

$$\left(\psi^{\mathrm{o}[0]}\right)^T \left[Df_{u_{14}}(Q_0\ ,u_{14}^{\mathrm{o}})\Upsilon^{\mathrm{o}[0]}\right] = \left(\ 0\ , \psi_2^{\mathrm{o}[0]}\ ,0\right)\left(0\ ,-\Upsilon_2^{\mathrm{o}[0]}\ ,0\ \right)^T = -\Upsilon_2^{\mathrm{o}[0]}\psi_2^{\mathrm{o}[0]} \neq 0.$$

Moreover, by substituting $\Upsilon^{\mathrm{o}[0]}$ in (4) we get:

$$D^2 f(Q_0\ ,u_{14}^{\mathrm{o}})\left(\Upsilon^{\mathrm{o}[0]}, \Upsilon^{\mathrm{o}[0]}\right) = \begin{bmatrix} 0 \\ 2(u_7 u_9 - u_{15})\left(\Upsilon_2^{\mathrm{o}[0]}\right)^2 \\ 0 \end{bmatrix}.$$

Hence, it obtains that

$$\left(\psi^{\mathrm{o}[0]}\right)^T \left[D^2 f(Q_0\ ,u_{14}^{\mathrm{o}})\left(\Upsilon^{\mathrm{o}[0]}, \Upsilon^{\mathrm{o}[0]}\right)\right] = 2(u_7 u_9 - u_{15})\psi_2^{\mathrm{o}[0]}\left(\Upsilon_2^{\mathrm{o}[0]}\right)^2 \neq 0.$$

According to "Sotomayor's theorem," if condition (6) is met, Q_0 exhibits a transcritical bifurcation at $u_{14}^{\mathrm{o}} = u_{14}$.

However, there is no transcirtical bifurcation if condition (6) is not hold and by using $\Upsilon^{\mathrm{o}[0]}$ in Eq.(5) we get:

$$D^3 F(Q_0\ ,u_{14}^{\mathrm{o}})\left(\Upsilon^{\mathrm{o}[0]}, \Upsilon^{\mathrm{o}[0]}\ ,\Upsilon^{\mathrm{o}[0]}\right) = \begin{bmatrix} 0 \\ \frac{-\ 6u_7(u_9+1)}{{u_9}^2}\left(\Upsilon_2^{\mathrm{o}[0]}\right)^3 \\ 0 \end{bmatrix},$$

$$\left(\psi^{o[0]}\right)^{T}\left[D^{3}F(Q_0\,,u_{14}^{o})\left(\Upsilon^{o[0]},\Upsilon^{o[0]}\,,\Upsilon^{o[0]}\right)\right]=\frac{-6u_7(u_9+1)}{{u_9}^2}\psi_2^{o[0]}\left(\Upsilon_2^{o[0]}\right)^3\neq 0.$$

Hence again, according to "Sotomayor's theorem" Q_0 exhibits a pitchfork bifurcation at $u_{14}^{o}=u_{14}$, but saddle-node bifurcation cannot be occurs.

Theorem (2): Assume that conditions (15) and (26.2) stated in reference $[19]$ and a further condition are met:

$$\frac{u_1\bar{x}(\bar{x}-1)}{(\bar{x}+u_2)}>\frac{u_3}{u_4+\bar{x}^2}+u_5\bar{x}, \quad (7)$$

$$u_4<\frac{\bar{x}^2}{3}, \quad (8)$$

$$\frac{u_{10}(\bar{x}^2-u_4)}{(u_4+\bar{x}^2)^2}\bar{v}+u_{15}\neq\frac{u_7}{u_9}, \quad (9)$$

$$\frac{u_{10}\bar{x}(\bar{x}^2-3u_4)}{(u_4+\bar{x}^2)^3}\bar{v}^2\neq\frac{u_7(u_9+1)}{{u_9}^2}, \quad (10)$$

Where $\bar{v}=\frac{-\bar{J}_{12}}{\bar{J}_{11}}\,,\bar{J}_{12}=\frac{\bar{x}(1-\bar{x})}{(\bar{x}+u_2)}-u_6+\bar{x}\left(\frac{u_2(1-2\bar{x})-\bar{x}^2}{(\bar{x}+u_2)^2}\right)$ and $\bar{J}_{11}=\bar{x}\left(\frac{u_1\bar{x}(\bar{x}-1)}{(\bar{x}+u_2)}-\frac{u_3}{u_4+\bar{x}^2}-u_5\bar{x}\right)$.

then system (2) at the (EP) $Q_1=(\,\bar{x},0\,,0\,)$ with $\bar{u}_{14}=u_{14}=\frac{u_{10}\,\bar{x}}{u_4+\bar{x}^2}+u_{13}$, a transcritical bifurcation or pitchfork bifurcation can occur, but a saddle-node bifurcation is not possible.

Proof: using the Jacobian matrix J_1, as stated in Eq.(26.1) in reference [19], If the eigenvalue $(\lambda_{1y}=0)$ of system (2) at $\bar{u}_{14}=u_{14}=\frac{u_{10}\,\bar{x}}{u_4+\bar{x}^2}+u_{13}$, J_1 with $\bar{u}_{14}=u_{14}$ becom

$$\bar{J}_1=J_1(Q_1\,,\bar{u}_{14})=\begin{bmatrix}\frac{\bar{x}(1-\bar{x})}{(\bar{x}+u_2)}-u_6+\bar{x}\left(\frac{u_2(1-2\bar{x})-\bar{x}^2}{(\bar{x}+u_2)^2}\right) & \bar{x}\left(\frac{u_1\bar{x}(\bar{x}-1)}{(\bar{x}+u_2)}-\frac{u_3}{u_4+\bar{x}^2}-u_5\bar{x}\right) & 0\\ 0 & 0 & 0\\ 0 & 0 & u_{17}-u_{18}-u_{19}\end{bmatrix}.$$

Now, the eigenvector that corresponds to the eigenvalue $\lambda_{1y}=0$ is written in $\bar{\Upsilon}^{[1]}=\left(\bar{\Upsilon}_1^{[1]}\,,\bar{\Upsilon}_2^{[1]}\,,\bar{\Upsilon}_3^{[1]}\right)^{T}$.

Hence $\left(\bar{J}_1-\lambda_{1y}I\right)\bar{\Upsilon}^{[1]}=0$, by conditions (15),(26.2) stated in reference $[19]$ and (7) that gives $\bar{\Upsilon}^{[1]}=\left(\bar{v}\bar{\Upsilon}_2^{[1]}\,,\bar{\Upsilon}_2^{[1]}\,,0\right)^{T}$ where the real constant $\bar{\Upsilon}_2^{[1]}\neq 0$.

Let $\bar{\psi}^{[1]}=\left(\bar{\psi}_1^{[1]}\,,\bar{\psi}_2^{[1]}\,,\bar{\psi}_3^{[1]}\right)^{T}$ be the eigenvector of $\bar{J}_1^{T}$ for $\lambda_{1y}=0$.

We find $\left(\bar{J}_1^{T}-\lambda_{1y}I\right)\bar{\psi}^{[1]}=0$, then by conditions (15) and (26.2) stated in reference [19], we solving this equation for $\overline{\Psi}^{[1]}$ we obtain

$\overline{\Psi}^{[1]} = \left(0\,, \overline{\Psi}_2^{[1]}\,, 0\right)^T$, where the real constant $\overline{\Psi}_2^{[1]} \neq 0$.

Now, assume $\frac{\partial f}{\partial u_{14}} = f_{u_{14}}(Q\,, u_{14}) = \left(\frac{\partial f_1}{\partial u_{14}}, \frac{\partial f_2}{\partial u_{14}}, \frac{\partial f_3}{\partial u_{14}}\right)^T = (0\,, -y\,, 0)^T$.

So, $f_{u_{14}}(Q_1\,, \bar{u}_{14}) = (0\,, 0\,, 0)^T$ and hence $\left(\overline{\Psi}^{[1]}\right)^T f_{u_{14}}(Q_1\,, \bar{u}_{14}) = 0$.

Hence, according to "Sotomayor's theorem," we conclude the conditions for a saddle-node bifurcation are not be fulfilled. The first condition for the transcritical bifurcation is fulfilled.

Currently, as a result, $Df_{u_{14}}(Q\,, u_{14}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

where $Df_{u_{14}}(Q\,, u_{14})$ is the derivative of $f_{u_{14}}(Q\,, u_{14})$ with regarding to $Q = (x, \mathrm{y}, \mathrm{z})^T$.

Further, it is observed that $Df_{u_{14}}(Q_1\,, \bar{u}_{14})\overline{\Upsilon}^{[1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{v}\overline{\Upsilon}_2^{[1]} \\ \overline{\Upsilon}_2^{[1]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\overline{\Upsilon}_2^{[1]} \\ 0 \end{bmatrix}$, so

$$\left(\overline{\Psi}^{[1]}\right)^T \left[Df_{u_{14}}(Q_1\,, \bar{u}_{14})\overline{\Upsilon}^{[1]}\right] = \left(0\,, \overline{\Psi}_2^{[1]}\,, 0\right)\left(0\,, -\overline{\Upsilon}_2^{[1]}\,, 0\right)^T = -\,\overline{\Psi}_2^{[1]}\overline{\Upsilon}_2^{[1]} \neq 0.$$

Moreover, by substituting $\overline{\Upsilon}^{[1]}$ in (4) we get

$$D^2 f(Q_1\,, \bar{u}_{14})\left(\overline{\Upsilon}^{[1]}\,, \overline{\Upsilon}^{[1]}\right) = \begin{bmatrix} \overline{w} \\ 2\left[\frac{u_{10}(u_4 - \bar{x}^2)}{(u_4 + \bar{x}^2)^2}\,\bar{v} + \frac{u_7}{u_9} - u_{15}\right]\left(\overline{\Upsilon}_2^{[1]}\right)^2 \\ 0 \end{bmatrix}, \text{ where}$$

$$\overline{w} = 2\left[\left[\frac{u_2(1-2\bar{x}) - \bar{x}^2}{(\bar{x}+u_2)^2} - \frac{u_2(u_2+1)\bar{x}}{(\bar{x}+u_2)^3}\right]\left(\bar{v}\overline{\Upsilon}_2^{[1]}\right)^2 + \left[\frac{u_1\bar{x}(\bar{x}-1)}{(\bar{x}+u_2)} - \frac{u_3}{u_4+\bar{x}^2} - 2u_5\bar{x} + \bar{x}\left(\frac{u_1\left(u_2(2\bar{x}-1)+\bar{x}^2\right)}{(\bar{x}+u_2)^2} + \frac{2u_3\bar{x}}{(u_4+\bar{x}^2)^2}\right)\right]\bar{v}\left(\overline{\Upsilon}_2^{[1]}\right)^2 + \frac{{u_1}^2\bar{x}(1-\bar{x})\,{\overline{\Upsilon}_2}^2}{(\bar{x}+u_2)}\right].$$

Hence, it obtains that:

$$\left(\overline{\Psi}^{[1]}\right)^T\left[D^2 f(Q_1\,, \bar{u}_{14})\left(\overline{\Upsilon}^{[1]}\,, \overline{\Upsilon}^{[1]}\right)\right] = 2\left[\frac{u_{10}(u_4 - \bar{x}^2)}{(u_4+\bar{x}^2)^2}\,\bar{v} + \frac{u_7}{u_9} - u_{15}\right]\overline{\Psi}^{[1]}\left(\overline{\Upsilon}_2^{[1]}\right)^2 \neq 0.$$

According to "Sotomayor's theorem," if conditions (8) and (9) are met, it exhibits a transcritical bifurcation at $\bar{u}_{14} = u_{14}$.

However, there is no transcritical bifurcation if conditions (8) and (9) do not hold, and by using $\bar{\Upsilon}^{[1]}$ in Eq. (5), we get:

$$D^3F(Q_1, \bar{u}_{14})(\bar{\Upsilon}^{[1]}, \bar{\Upsilon}^{[1]}, \bar{\Upsilon}^{[1]}) = \begin{bmatrix} \bar{w}' \\ 6\left[\frac{u_{10}\bar{x}(\bar{x}^2-3u_4)}{(u_4+\bar{x}^2)^3}\bar{v}^2 - \frac{u_7(u_9+1)}{u_9{}^2}\right]\left(\bar{\Upsilon}_2^{[1]}\right)^3 \\ 0 \end{bmatrix}, \text{ where}$$

$$\bar{w}' = 6\Bigg[\left[\frac{-u_2(u_2+1)}{(\bar{x}+u_2)^3} + \frac{u_2(u_2+1)\bar{x}}{(\bar{x}+u_2)^4}\right]\bar{v}^3 + \left[\frac{u_1\left(u_2(2\bar{x}-1)+\bar{x}^2\right)}{(\bar{x}+u_2)^2} + \frac{2u_3\bar{x}}{(u_4+\bar{x}^2)^2} - u_5 + \bar{x}\left(\frac{u_1u_2(u_2+1)}{(\bar{x}+u_2)^3} + \frac{u_3(u_4-3\bar{x}^2)}{(u_4+\bar{x}^2)^3}\right)\right]\bar{v}^2 + \frac{u_1^2\bar{x}\left(\bar{x}(1-2\bar{x})+u_2(2-3\bar{x})\right)}{(\bar{x}+u_2)^2}\bar{v} + \frac{u_1{}^3\bar{x}^2(\bar{x}-1)}{(\bar{x}+u_2)}\Bigg]\left(\bar{\Upsilon}_2^{[1]}\right)^3,$$

$$\left(\bar{\Psi}^{[1]}\right)^{\mathrm{T}}\left[D^3F(Q_1, \bar{u}_{14})(\bar{\Upsilon}^{[1]}, \bar{\Upsilon}^{[1]}, \bar{\Upsilon}^{[1]})\right] = 6\left[\frac{u_{10}\bar{x}(\bar{x}^2-3u_4)}{(u_4+\bar{x}^2)^3}\bar{v}^2 - \frac{u_7(u_9+1)}{u_9{}^2}\right]\bar{\Psi}^{[1]}\left(\bar{\Upsilon}_2^{[1]}\right)^3 \neq 0.$$

Hence again, according to “Sotomayor’s theorem” Q_1 possesses a pitchfork bifurcation at $\bar{u}_{14} = u_{14}$, confirming condition (10) hold, but saddle $-$ node bifurcation cannot be occurs. Similarly, for the (EP) Q_2.

Theorem (3): Assume that conditions (27.2) and (27.3) stated in reference [19] and a further condition are met:

$$\frac{u_{16}\,\hat{y}}{u_{12}+\hat{y}^2} + u_{17} > u_{19}, \tag{11}$$

$$\frac{u_7u_8\hat{y}(\hat{y}-1)}{(\hat{y}+u_9)} > \frac{u_{11}}{u_{12}+\hat{y}^2}, \tag{12}$$

$$u_{12} \neq \hat{y}^2, \tag{13}$$

$$3u_{12} \neq \hat{y}^2, \tag{14}$$

then system (2) at the (EP) $Q_3 = (0, \hat{y}, 0)$ with $\hat{u}_{18} = u_{18} = \frac{u_{16}\,\hat{y}}{u_{12}+\hat{y}^2} + u_{17} - u_{19}$ a transcritical bifurcation or pitchfork bifurcation can occur, but a saddle-node bifurcation is not possible.

Proof: using the Jacobian matrix J_3, as stated in Eq. (27.1) in reference [19], If the eigenvalue ($\lambda_{3z} = 0$) of system (2) at $\hat{u}_{18} = u_{18} = \frac{u_{16}\,\hat{y}}{u_{12}+\hat{y}^2} + u_{17} - u_{19}$, with condition (11) so the Jacobian matrix J_3 with $\hat{u}_{18} = u_{18}$ becom

$$\hat{J}_3 = J_3(Q_3\,, \hat{u}_{18}) =$$

$$\begin{bmatrix} \frac{-u_3\hat{y}}{u_4} - u_6 & 0 & 0 \\ \frac{u_{10}\hat{y}}{u_4} & \frac{u_7\hat{y}(1-\hat{y})}{(\hat{y}+u_9)} + u_{13} - u_{14} - 2u_{15}\hat{y} + u_7\frac{(u_9(1-2\hat{y})-\hat{y}^2)}{(\hat{y}+u_9)^2}\hat{y} & \hat{y}\left(\frac{u_7u_8\hat{y}(\hat{y}-1)}{(\hat{y}+u_9)} - \frac{u_{11}}{u_{12}+\hat{y}^2}\right) \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, the eigenvector that corresponds to the eigenvalue $\lambda_{3z} = 0$ is written in $\widehat{\Upsilon}^{[3]} = \left(\widehat{\Upsilon}_1^{[3]}, \widehat{\Upsilon}_2^{[3]}, \widehat{\Upsilon}_3^{[3]}\right)^{\mathrm{T}}$.

Hence $(\hat{J}_3 - \lambda_{3z}I)\widehat{\Upsilon}^{[3]} = 0$, by condition (27.2), (27.3) which is given in [19] and (12) that gives $\widehat{\Upsilon}^{[3]} = \left(0\,, \hat{v}\widehat{\Upsilon}_3^{[3]}, \widehat{\Upsilon}_3^{[3]}\right)^T$ where $\hat{v} = -\frac{\hat{j}_{23}}{\hat{j}_{22}}$, $\hat{j}_{23} = \hat{y}\left(\frac{u_7u_8\hat{y}(\hat{y}-1)}{(\hat{y}+u_9)} - \frac{u_{11}}{u_{12}+\hat{y}^2}\right)$, $\hat{j}_{22} = \frac{u_7\hat{y}(1-\hat{y})}{(\hat{y}+u_9)} + u_{13} - u_{14} - 2u_{15}\hat{y} + \frac{u_7(u_9(1-2\hat{y})-\hat{y}^2)\hat{y}}{(\hat{y}+u_9)^2}$ and $\widehat{\Upsilon}_3^{[3]} \neq 0$ is any real number.

Let $\widehat{\Psi}^{[3]} = \left(\widehat{\Psi}_1^{[3]}, \widehat{\Psi}_2^{[3]}, \widehat{\Psi}_3^{[3]}\right)^T$ be the eigenvector of $\hat{J}_3^T$ for $\lambda_{3z} = 0$.

We find $(\hat{J}_3 - \lambda_{3z}I)\widehat{\Psi}^{[3]} = 0$, then by conditions (27.2) and (27.3) stated in reference [19], we solving this equation for $\widehat{\Psi}^{[3]}$ we obtain

$\widehat{\Psi}^{[3]} = \left(0, 0, \widehat{\Psi}_3^{[3]}\right)^T$, where $\widehat{\Psi}_3^{[3]} \neq 0$ is any real number.

Now, assume $\frac{\partial f}{\partial u_{18}} = f_{u_{18}}(Q\,, u_{18}) = \left(\frac{\partial f_1}{\partial u_{18}}, \frac{\partial f_2}{\partial u_{18}}, \frac{\partial f_3}{\partial u_{18}}\right)^T = (0\,, 0\,, -z)^T$.

So, $f_{u_{18}}(Q_3\,, \hat{u}_{18}) = (0\,, 0\,, 0)^T$ and hence $\left(\widehat{\Psi}^{[3]}\right)^T f_{u_{18}}(Q_3\,, \hat{u}_{18}) = 0$.

Hence, using "Sotomayor's theorem," we conclude the conditions for a saddle-node bifurcation are not be fulfilled. The first condition for the transcritical bifurcation is fulfilled. Currently, as a result

$$Df_{u_{18}}(Q\,, u_{18}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $Df_{u_{18}}(Q\,, u_{18})$ is the derivative of $f_{u_{18}}(Q\,, u_{18})$ with respect to $Q = (x, \mathrm{y}, \mathrm{z})^T$.

Further, it is observed that $Df_{u_{18}}(Q_3\,, \hat{u}_{18})\widehat{\Upsilon}^{[3]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ \hat{v}\widehat{\Upsilon}_3^{[3]} \\ \widehat{\Upsilon}_3^{[3]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\widehat{\Upsilon}_3^{[3]} \end{bmatrix}$, so

$$\left(\widehat{\Psi}^{[3]}\right)^T\left[Df_{u_{18}}(Q_3\,,\hat{u}_{18})\widehat{\Upsilon}^{[3]}\right]=\left(\,0\,,0,\widehat{\Psi}_3^{[3]}\right)\,\left(0,0,-\widehat{\Upsilon}_3^{[3]}\right)^T=-\,\widehat{\Psi}_3^{[3]}\widehat{\Upsilon}_3^{[3]}\neq 0.$$

Moreover, by substituting $\widehat{\Upsilon}^{[3]}$ in (4) we get

$$D^2f(Q_3\,,\hat{u}_{18})\left(\widehat{\Upsilon}^{[3]}\,,\widehat{\Upsilon}^{[3]}\right)=\begin{bmatrix}0\\ \\ \widehat{w}\\ \\ \frac{2u_{16}\left(u_{12}-\hat{y}^2\right)}{(u_{12}+\hat{y}^2)^2}\left(\widehat{\Upsilon}_3^{[3]}\right)^2\end{bmatrix}\text{, were}$$

$$\widehat{w}=2\left[\left[\frac{u_7\left(u_9(1-2\hat{y})-y^{\hat{}2}\right)}{(\hat{y}+u_9)^2}-u_{15}-\frac{u_7u_9(u_9+1)\hat{y}}{(\hat{y}+u_9)^3}\right]\hat{v}^2+\left[\frac{u_7u_8\hat{y}(\hat{y}-1)}{(\hat{y}+u_9)}-\frac{u_{11}}{u_{12}+\hat{y}^2}+\right.\right.$$

$$\left.\left.\hat{y}\left(\frac{u_7u_8\left(u_9(2\hat{y}-1)+\hat{y}^2\right)}{(\hat{y}+u_9)^2}+\frac{2u_{11}\hat{y}}{(u_{12}+\hat{y}^2)^2}\right)\right]\hat{v}+\frac{u_7{u_8}^2\hat{y}(1-\hat{y})}{(\hat{y}+u_9)}\right]\left(\widehat{\Upsilon}_3^{[3]}\right)^2,$$

Hence, it obtains that:

$$\left(\widehat{\Psi}^{[3]}\right)^T\left[D^2f(Q_3\,,\hat{u}_{18})\left(\widehat{\Upsilon}^{[3]}\,,\widehat{\Upsilon}^{[3]}\right)\right]=2\left[\frac{u_{16}\left(u_{12}-\hat{y}^2\right)}{(u_{12}+\hat{y}^2)^2}\right]\widehat{\Psi}_3^{[3]}\left(\widehat{\Upsilon}_3^{[3]}\right)^2\neq 0.$$

According to "Sotomayor's theorem" if condition (13) is met, it exhibits a transcritical bifurcation at $\hat{u}_{18}=u_{18}$.

However, there is no transcritical bifurcation if condition (13) does not hold, and by using $\widehat{\Upsilon}^{[3]}$ in Eq. (5), we get:

$$D^3F(Q_3\,,\hat{u}_{18})\left(\widehat{\Upsilon}^{[3]}\,,\widehat{\Upsilon}^{[3]}\,,\widehat{\Upsilon}^{[3]}\right)=\begin{bmatrix}0\\ \\ \widehat{w}'\\ \\ \frac{6u_{16}\hat{y}\left(\hat{y}^2-3u_{12}\right)}{(u_{12}+\hat{y}^2)^3}\,\hat{v}^2\left(\widehat{\Upsilon}_3^{[3]}\right)^3\end{bmatrix}\text{, where}$$

$$\widehat{w}'=6\left[\left[\frac{-u_7u_9(u_9+1)}{(\hat{y}+u_9)^3}+\frac{u_7u_9(u_9+1)\hat{y}}{(\hat{y}+u_9)^4}\right]\hat{v}^3+\left[\frac{u_7u_8\left(u_9(2\hat{y}-1)+\hat{y}^2\right)}{(\hat{y}+u_9)^2}+\frac{2u_{11}\hat{y}}{(u_{12}+\hat{y}^2)^2}+\hat{y}\left(\frac{u_7u_8u_9(u_9+1)}{(\hat{y}+u_9)^3}+\right.\right.\right.$$

$$\left.\left.\left.\frac{u_{11}\left(u_{12}-3\hat{y}^2\right)}{(u_{12}+\hat{y}^2)^3}\right)\right]\hat{v}^2+\frac{u_7{u_8}^2\hat{y}\left(\hat{y}(1-2\hat{y})+u_9(2-3\hat{y})\right)}{(\hat{y}+u_9)^2}\hat{v}+\frac{u_7{u_8}^3\hat{y}(\hat{y}-1)}{(\hat{y}+u_9)}\right]\left(\widehat{\Upsilon}_3^{[3]}\right)^3,$$

$$\left(\widehat{\Psi}^{[3]}\right)^{\mathrm{T}}\left[D^3F(Q_3\,,\hat{u}_{18})\left(\widehat{\Upsilon}^{[3]}\,,\widehat{\Upsilon}^{[3]}\,,\widehat{\Upsilon}^{[3]}\right)\right]=\frac{6u_{16}\hat{y}\left(\hat{y}^2-3u_{12}\right)}{(u_{12}+\hat{y}^2)^3}\,\hat{v}^2\left(\widehat{\Upsilon}_3^{[3]}\right)^3\widehat{\Psi}_3^{[3]}\neq 0.$$

Hence again, according to “Sotomayor’s theorem” Q_3 possesses a pitchfork bifurcation at $\hat{u}_{18}=$

u_{18}, confirming condition (14) hold, but $\text{saddle} - \text{node}$ bifurcation cannot be occurs.

Theorem (4): Assume that conditions $(28.2 - 28.7)$ stated in reference $[19]$ and a further condition are met:

$$\frac{u_{16}\,\bar{\bar{y}}}{u_{12}+\bar{\bar{y}}^2} + u_{17} > u_{19}, \tag{15}$$

$$\frac{u_7 u_8 \bar{\bar{y}}(\bar{\bar{y}}-1)}{(\bar{\bar{y}}+u_9)} > \frac{u_{11}}{u_{12}+\bar{\bar{y}}^2}, \tag{16}$$

$$u_{12} \neq \bar{\bar{y}}^2, \tag{17}$$

$$3u_{12} \neq \bar{\bar{y}}^2, \tag{18}$$

then system (2) at the (EP) $Q_4 = (\,\bar{\bar{x}}, \bar{\bar{y}}\,, 0\,)$ with $\bar{\bar{u}}_{18} = u_{18} = \frac{u_{16}\,\bar{\bar{y}}}{u_{12}+\bar{\bar{y}}^2} + u_{17} - u_{19}$, a transcritical bifurcation or pitchfork bifurcation can occur, but a saddle-node bifurcation is not possible.

Proof: using the Jacobian matrix J_4, as stated in Eq. (28.1) in reference [19], if the eigenvalue $(\lambda_{4z} = 0)$ of system (2) at $\bar{\bar{u}}_{18} = u_{18} = \frac{u_{16}\,\bar{\bar{y}}}{u_{12}+\bar{\bar{y}}^2} + u_{17} - u_{19}$, with condition (15) so the Jacobian matrix J_4 with $\bar{\bar{u}}_{18} = u_{18}$ becom

$\bar{\bar{J}}_4 = J_1(Q_4\,, \bar{\bar{u}}_{18}) = \left[\bar{\bar{e}}_{ij}\right]_{3\times 3}$,

where $\bar{\bar{e}}_{ij} = e_{ij}$, $i, j = 1,2,3$ as shown in Eq. (28.1) in reference [19] accept $\bar{\bar{e}}_{33} = 0$.

Now, the eigenvector that corresponds to the eigenvalue $\lambda_{4z} = 0$ is written in $\bar{\bar{\Upsilon}}^{[4]} = \left(\bar{\bar{\Upsilon}}_1^{[4]}\,, \bar{\bar{\Upsilon}}_2^{[4]}\,, \bar{\bar{\Upsilon}}_3^{[4]}\right)^{\mathrm{T}}$.

Henc $(\bar{\bar{J}}_4 - \lambda_{4z} I)\bar{\bar{\Upsilon}}^{[4]} = 0$, by conditions $(28.2 - 28.7)$ which is given in $[19]$, (16) that gives

$\bar{\bar{\Upsilon}}^{[4]} = \left(\bar{\bar{v}}_1 \bar{\bar{\Upsilon}}_3^{[4]}, \bar{\bar{v}}_2 \bar{\bar{\Upsilon}}_3^{[4]}\,, \bar{\bar{\Upsilon}}_3^{[4]}\right)^T$,

where $\bar{\bar{v}}_1 = \frac{e_{12}e_{23}}{e_{11}e_{22}-e_{12}e_{21}}, \bar{\bar{v}}_2 = \frac{e_{11}e_{23}}{e_{12}e_{21}-e_{11}e_{22}}$ and $\bar{\bar{\Upsilon}}_3^{[4]} \neq 0$ is any real number.

Let $\bar{\bar{\Psi}}^{[4]} = \left(\bar{\bar{\Psi}}_1^{[4]}\,, \bar{\bar{\Psi}}_2^{[4]}\,, \bar{\bar{\Psi}}_3^{[4]}\right)^T$ be the eigenvector of $\bar{\bar{J}}_4^T$ for $\lambda_{4z} = 0$.

We find $(\bar{\bar{J}}_4^T - \lambda_{4z} I)\bar{\bar{\Psi}}^{[4]} = 0$, then by conditions (28.2), (28.4) stated in reference [19] and (16) we solving this equation for $\bar{\bar{\Psi}}^{[4]}$ we obtain

$\bar{\bar{\Psi}}^{[4]} = \left(\,0, 0, \bar{\bar{\Psi}}_3^{[4]}\right)^T$, where the real constant $\bar{\bar{\Psi}}_3^{[4]} \neq 0$.

Now, assume $\frac{\partial f}{\partial u_{18}} = f_{u_{18}}(Q\,, u_{18}) = \left(\frac{\partial f_1}{\partial u_{18}}, \frac{\partial f_2}{\partial u_{18}}, \frac{\partial f_3}{\partial u_{18}}\right)^T = (0\,,0\,,-z)^T$.

So, $f_{u_{18}}(Q_4\,, \bar{\bar{u}}_{18}) = (0\,,0\,,0)^T$ and hence $\left(\bar{\bar{\Psi}}^{[4]}\right)^T f_{u_{18}}(Q_4\,, \bar{\bar{u}}_{18}) = 0$.

Hence, using "Sotomayor's theorem," we conclude the conditions for a saddle-node bifurcation are not fulfilled. The first condition for the transcritical bifurcation is fulfilled. Currently, as a result

$$Df_{u_{18}}(Q\,, u_{18}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $Df_{u_{18}}(Q\,, u_{18})$ is the derivative of $f_{u_{18}}(Q\,, u_{18})$ according to $Q = (x, \mathrm{y}, \mathrm{z})^T$.

Further, it is observed that $Df_{u_{18}}(Q_4\,, \bar{\bar{u}}_{18})\bar{\bar{\Upsilon}}^{[4]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{\bar{v}}_1 \bar{\bar{\Upsilon}}_3^{[4]} \\ \bar{\bar{v}}_2 \bar{\bar{\Upsilon}}_3^{[4]} \\ \bar{\bar{\Upsilon}}_3^{[4]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\bar{\bar{\Upsilon}}_3^{[4]} \end{bmatrix}$, so

$$\left(\bar{\bar{\Psi}}^{[4]}\right)^T \left[Df_{u_{18}}(Q_4\,, \bar{\bar{u}}_{18})\bar{\bar{\Upsilon}}^{[4]}\right] = \left(0\,, 0, \bar{\bar{\Psi}}_3^{[4]}\right) \left(0, 0, -\bar{\bar{\Upsilon}}_3^{[4]}\right)^T = -\bar{\bar{\Psi}}_3^{[4]} \bar{\bar{\Upsilon}}_3^{[4]} \neq 0.$$

Moreover, by substituting $\bar{\bar{\Upsilon}}^{[4]}$ in (4) we get

$$D^2 f(Q_4\,, \bar{\bar{u}}_{18})\left(\bar{\bar{\Upsilon}}^{[4]}\,, \bar{\bar{\Upsilon}}^{[4]}\right) = \begin{bmatrix} \bar{\bar{\mathrm{w}}}_{11} \\ \bar{\bar{\mathrm{w}}}_{21} \\ \frac{2u_{16}\left(u_{12} - \bar{\bar{y}}^2\right)}{(u_{12} + \bar{\bar{y}}^2)^2}\ \bar{\bar{v}}_2 \left(\bar{\bar{\Upsilon}}_3^{[4]}\right)^2 \end{bmatrix},$$ were

$$\bar{\bar{\mathrm{w}}}_{11} = 2\Bigg[\left[\frac{u_2(1-2\bar{\bar{x}}) - \bar{\bar{x}}^2}{(1+u_1\bar{\bar{y}})(\bar{\bar{x}}+u_2)^2} + \frac{2u_3\bar{\bar{x}}\bar{\bar{y}}}{(u_4+\bar{\bar{x}}^2)^2} - u_5\bar{\bar{y}} + \bar{\bar{x}}\left(\frac{-u_2(u_2+1)}{(1+u_1\bar{\bar{y}})(\bar{\bar{x}}+u_2)^3} + \frac{u_3\bar{\bar{y}}(u_4-3\bar{\bar{x}}^2)}{(u_4+\bar{\bar{x}}^2)^3}\right)\right]\bar{\bar{v}}_1{}^2 +$$

$$\left[\frac{u_1\bar{\bar{x}}(\bar{\bar{x}}-1)}{(1+u_1\bar{\bar{y}})^2(\bar{\bar{x}}+u_2)} - \frac{u_3}{u_4+\bar{\bar{x}}^2} - 2u_5\bar{\bar{x}} + \bar{\bar{x}}\left(\frac{u_1\left(u_2(2\bar{\bar{x}}-1)+\bar{\bar{x}}^2\right)}{(1+u_1\bar{\bar{y}})^2(\bar{\bar{x}}+u_2)^2} + \frac{2u_3\bar{\bar{x}}}{(u_4+\bar{\bar{x}}^2)^2}\right)\right]\bar{\bar{v}}_1\bar{\bar{v}}_2 + \frac{u_1{}^2\bar{\bar{x}}(1-\bar{\bar{x}})\,\bar{\bar{v}}_2{}^2}{(1+u_1\bar{\bar{y}})^3(\bar{\bar{x}}+u_2)}\Bigg]\left(\bar{\bar{\Upsilon}}_3^{[4]}\right)^2$$

and

$$\bar{\bar{\mathrm{w}}}_{21} = 2\Bigg[\frac{u_{10}\bar{\bar{x}}\bar{\bar{y}}\left(\bar{\bar{x}}^2-3u_4\right)}{(u_4+\bar{\bar{x}}^2)^3}\bar{\bar{v}}_1{}^2 + \frac{u_{10}(u_4-\bar{\bar{x}}^2)}{(u_4+\bar{\bar{x}}^2)^2}\bar{\bar{v}}_1\,\bar{\bar{v}}_2 + \left[\frac{u_7\left(u_9(1-2\bar{\bar{y}})-\bar{\bar{y}}^2\right)}{(\bar{\bar{y}}+u_9)^2} - u_{15} + \frac{-u_7u_9(u_9+1)\bar{\bar{y}}}{(\bar{\bar{y}}+u_9)^3}\right]\bar{\bar{v}}_2{}^2 +$$

$$\left[\frac{u_7u_8\bar{\bar{y}}(\bar{\bar{y}}-1)}{(\bar{\bar{y}}+u_9)} - \frac{u_{11}}{u_{12}+\bar{\bar{y}}^2} + y\left(\frac{u_7u_8\left(u_9(2\bar{\bar{y}}-1)+\bar{\bar{y}}^2\right)}{(\bar{\bar{y}}+u_9)^2} + \frac{2u_{11}\bar{\bar{y}}}{(u_{12}+\bar{\bar{y}}^2)^2}\right)\right]\bar{\bar{v}}_2 + \frac{u_7u_8{}^2\bar{\bar{y}}(1-\bar{\bar{y}})}{(\bar{\bar{y}}+u_9)}\Bigg]\left(\bar{\bar{\Upsilon}}_3^{[4]}\right)^2.$$

Hence, it obtains that:

$$\left(\bar{\bar{\Psi}}^{[4]}\right)^{T}\left[D^2 f(Q_4\,,\bar{\bar{u}}_{18})\left(\bar{\bar{\Upsilon}}^{[4]}\,,\bar{\bar{\Upsilon}}^{[4]}\right)\right] = 2\,\frac{u_{16}(u_{12}-\bar{\bar{y}}^2)}{(u_{12}+\bar{\bar{y}}^2)^2}\;\bar{\bar{v}}_2\bar{\bar{\Psi}}_3^{[4]}\left(\bar{\bar{\Upsilon}}_3^{[4]}\right)^2 \neq 0.$$

According to "Sotomayor's theorem" if condition (17) is met, Q_4 exhibits a transcritical bifurcation at $\bar{\bar{u}}_{18} = u_{18}$.

However, there is no transcritical bifurcation if condition (17) does not hold, and by using $\bar{\bar{\Upsilon}}^{[4]}$ in Eq. (5), we get:

$$D^3F(Q_4\,,\bar{\bar{u}}_{18})\left(\bar{\bar{\Upsilon}}^{[4]}\,,\bar{\bar{\Upsilon}}^{[4]}\,,\bar{\bar{\Upsilon}}^{[4]}\right) = \begin{bmatrix} \bar{\bar{w}}'_{11} \\ \bar{\bar{w}}'_{21} \\ 6u_{16}\bar{\bar{y}}\,\frac{(\bar{\bar{y}}^2-3u_{12})}{(u_{12}+\bar{\bar{y}}^2)^3}\;\bar{\bar{v}}_2^{\,2}\left(\bar{\bar{\Upsilon}}_3^{[4]}\right)^3 \end{bmatrix}, \text{ where}$$

$$\bar{\bar{w}}'_{11} = 6\Bigg[\left[\frac{-u_2(u_2+1)}{(1+u_1\bar{\bar{y}})(\bar{\bar{x}}+u_2)^3} + \frac{u_3\bar{\bar{y}}(u_4-3\bar{\bar{x}}^2)}{(u_4+\bar{\bar{x}}^2)^3} + \bar{\bar{x}}\left(\frac{u_2(u_2+1)}{(1+u_1\bar{\bar{y}})(\bar{\bar{x}}+u_2)^4} - \frac{4u_3\bar{\bar{x}}\bar{\bar{y}}(u_4-\bar{\bar{x}}^2)}{(u_4+\bar{\bar{x}}^2)^4}\right)\right]\bar{\bar{v}}_1^{\,3} +$$
$$\left[\frac{u_1\left(u_2(2\bar{\bar{x}}-1)+\bar{\bar{x}}^2\right)}{(1+u_1\bar{\bar{y}})^2(\bar{\bar{x}}+u_2)^2} + \frac{2u_3\bar{\bar{x}}}{(u_4+\bar{\bar{x}}^2)^2} - u_5 + \bar{\bar{x}}\left(\frac{u_1u_2(u_2+1)}{(1+u_1\bar{\bar{y}})^2(\bar{\bar{x}}+u_2)^3} + \frac{u_3(u_4-3\bar{\bar{x}}^2)}{(u_4+\bar{\bar{x}}^2)^3}\right)\right]\bar{\bar{v}}_1^{\,2}\,\bar{\bar{v}}_2 +$$
$$\frac{u_1^2\bar{\bar{x}}\left(\bar{\bar{x}}(1-2\bar{\bar{x}})+u_2(2-3\bar{\bar{x}})\right)}{(1+u_1\bar{\bar{y}})^3(\bar{\bar{x}}+u_2)^2}\,\bar{\bar{v}}_1\,\bar{\bar{v}}_2^{\,2} + \frac{u_1^{\,3}\bar{\bar{x}}^2(\bar{\bar{x}}-1)\,\bar{\bar{v}}_2^{\,3}}{(1+u_1\bar{\bar{y}})^4(\bar{\bar{x}}+u_2)}\Bigg]\left(\bar{\bar{\Upsilon}}_3^{[4]}\right)^3 \text{ and}$$

$$\bar{\bar{w}}'_{21} = 6\Bigg[\frac{u_{10}\bar{\bar{y}}\left(u_4(6\bar{\bar{x}}^2-u_4)-\bar{\bar{x}}^4\right)}{(u_4+\bar{\bar{x}}^2)^4}\bar{\bar{v}}_1^{\,3} + \frac{u_{10}\bar{\bar{x}}(\bar{\bar{x}}^2-3u_4)}{(u_4+\bar{\bar{x}}^2)^3}\bar{\bar{v}}_1^{\,2}\bar{\bar{v}}_2 + \left[\frac{-u_7u_9(u_9+1)}{(\bar{\bar{y}}+u_9)^3} + \frac{u_7u_9(u_9+1)\bar{\bar{y}}}{(\bar{\bar{y}}+u_9)^4}\right]\bar{\bar{v}}_2^{\,3} +$$
$$\left[\frac{u_7u_8\left(u_9(2\bar{\bar{y}}-1)+\bar{\bar{y}}^2\right)}{(\bar{\bar{y}}+u_9)^2} + \frac{2u_{11}\bar{\bar{y}}}{(u_{12}+\bar{\bar{y}}^2)^2} + \bar{\bar{y}}\left(\frac{u_7u_8u_9(u_9+1)}{(\bar{\bar{y}}+u_9)^3} + \frac{u_{11}\left(u_{12}-3\bar{\bar{y}}^2\right)}{(u_{12}+\bar{\bar{y}}^2)^3}\right)\right]\bar{\bar{v}}_2^{\,2} +$$
$$\frac{u_7u_8^{\,2}\bar{\bar{y}}\left(\bar{\bar{y}}(1-2\bar{\bar{y}})+u_9(2-3\bar{\bar{y}})\right)\bar{\bar{v}}_2}{(\bar{\bar{y}}+u_9)^2} + \frac{u_7u_8^{\,3}\bar{\bar{y}}(\bar{\bar{y}}-1)}{(\bar{\bar{y}}+u_9)}\Bigg]\left(\bar{\bar{\Upsilon}}_3^{[4]}\right)^3,$$

$$\left(\bar{\bar{\Psi}}^{[4]}\right)^{\mathrm{T}}\left[D^3F(Q_4\,,\bar{\bar{u}}_{18})\left(\bar{\bar{\Upsilon}}^{[4]}\,,\bar{\bar{\Upsilon}}^{[4]}\,,\bar{\bar{\Upsilon}}^{[4]}\right)\right] = 6\,\frac{u_{16}\bar{\bar{y}}(\bar{\bar{y}}^2-3u_{12})}{(u_{12}+\bar{\bar{y}}^2)^3}\;\bar{\bar{v}}_2^{\,2}\left(\bar{\bar{\Upsilon}}_3^{[4]}\right)^3\bar{\bar{\Psi}}_3^{[4]} \neq 0.$$

Hence again, according to “Sotomayor’s theorem” Q_4 possesses a pitchfork bifurcation at $\bar{\bar{u}}_{18} = u_{18}$ confirming condition (18) hold but saddle $-$ node bifurcation cannot be occurs. Similarly, for the (EP) Q_5, Q_6 and Q_7.

Theorem (5): Assume that conditions $(29.2, 29.3 \text{ and } 29.5)$ stated in reference $[19]$ and a further condition are met:

$$\tilde{v} + \frac{u_{16}\,\tilde{y}}{u_{12}+\tilde{y}^2} + u_{17} > u_{19}\,, \tag{19}$$

where $\tilde{v} = \frac{-\tilde{g}_{23}\tilde{g}_{32}}{\tilde{g}_{22}}$, $\tilde{g}_{23} = y\left(\frac{u_7 u_8 \tilde{y}(\tilde{y}-1)}{(1+u_8\tilde{z})^2(\tilde{y}+u_9)} - \frac{u_{11}}{u_{12}+\tilde{y}^2}\right)$, $\tilde{g}_{32} = \frac{u_{16}(u_{12}-\tilde{y}^2)\tilde{z}}{(u_{12}+\tilde{y}^2)^2}$ and

$\tilde{g}_{22} = \frac{u_7\tilde{y}(1-\tilde{y})}{(1+u_8\tilde{z})(\tilde{y}+u_9)} - \frac{u_{11}\,\tilde{z}}{u_{12}+\tilde{y}^2} + u_{13} - u_{14} - u_{15}\tilde{y} + \tilde{y}\left(\frac{u_7(u_9(1-2\tilde{y})-\tilde{y}^2)}{(1+u_8\tilde{z})(\tilde{y}+u_9)^2} + \frac{2u_{11}\tilde{y}}{(u_{12}+\tilde{y}^2)^2} - u_{15}\right)$.

$$u_{12} < \tilde{y}^2, \tag{20}$$

$$\tilde{y}^2 < 3u_{12}, \tag{21}$$

$$\tilde{\tau}_1 \neq \tilde{\tau}_2, \tag{22}$$

where

$$\tilde{\tau}_1 = \left[\left[\frac{u_7(u_9(1-2\tilde{y})-\tilde{y}^2)}{(1+u_8\tilde{z})(\tilde{y}+u_9)^2} - u_{15} + \tilde{y}\left(\frac{-u_7u_9(u_9+1)}{(1+u_8\tilde{z})(\tilde{y}+u_9)^3} + \frac{u_{11}\tilde{z}(u_{12}-3\tilde{y}^2)}{(u_{12}+\tilde{y}^2)^3}\right)\right]\left(\frac{\tilde{v}}{\tilde{g}_{32}}\right)^2\frac{\tilde{v}}{\tilde{g}_{23}} + \left[\frac{u_7u_8\tilde{y}(\tilde{y}-1)}{(1+u_8\tilde{z})^2(\tilde{y}+u_9)} + \tilde{y}\left(\frac{u_7u_8(u_9(2\tilde{y}-1)+\tilde{y}^2)}{(1+u_8\tilde{z})^2(\tilde{y}+u_9)^2} + \frac{2u_{11}\tilde{y}}{(u_{12}+\tilde{y}^2)^2}\right)\right]\frac{\tilde{v}}{\tilde{g}_{32}}\frac{\tilde{v}}{\tilde{g}_{23}} + \frac{u_7{u_8}^2\tilde{y}(1-\tilde{y})}{(1+u_8\tilde{z})^3(\tilde{y}+u_9)}\frac{\tilde{v}}{\tilde{g}_{23}} + \frac{u_{16}(u_{12}-\tilde{y}^2)}{(u_{12}+\tilde{y}^2)^2}\frac{\tilde{v}}{\tilde{g}_{32}}\right],$$

$$\tilde{\tau}_2 = -\frac{2u_{11}\tilde{y}\tilde{z}}{(u_{12}+\tilde{y}^2)^2}\left(\frac{\tilde{v}}{\tilde{g}_{32}}\right)^2\frac{\tilde{v}}{\tilde{g}_{23}} + \frac{u_{11}}{u_{12}+\tilde{y}^2}\frac{\tilde{v}}{\tilde{g}_{32}}\frac{\tilde{v}}{\tilde{g}_{23}} + \frac{u_{16}\tilde{y}\tilde{z}(3u_{12}-\tilde{y}^2)}{(u_{12}+\tilde{y}^2)^3}\left(\frac{\tilde{v}}{\tilde{g}_{32}}\right)^2,$$

then system (2) at the (EP) $Q_8 = (0, \tilde{y}, \tilde{z})$ with $\tilde{u}_{18} = u_{18} = \tilde{v} + \frac{u_{16}\,\tilde{y}}{u_{12}+\tilde{y}^2} + u_{17} - u_{19}$, a saddle-node bifurcation can occur, but transcritical bifurcation or pitchfork bifurcation is not possible.

Proof: using the Jacobian matrix J_8 as stated in Eq. (29.1) in references [19], If the eigenvalue ($\lambda_{8z} = 0$) of (EP) Q_8 in system (2) when $\tilde{u}_{18} = u_{18} = \tilde{v} + \frac{u_{16}\,\tilde{y}}{u_{12}+\tilde{y}^2} + u_{17} - u_{19}$, with conditions (29.2, 29.3 and 29.5) stated in reference [19] and also conditions (19) and (20) so the Jacobian matrix J_8 with $\tilde{u}_{18} = u_{18}$ becom

$\tilde{J}_8 = J_8(Q_8, \tilde{u}_{18}) = [\tilde{g}_{ij}]_{3\times3}$,

where $\tilde{g}_{ij} = g_{ij}$, $i,j = 1,2,3$ as shown in Eq.(29.1) stated in reference [19] accept $\tilde{g}_{33} = -\tilde{v}$. Now, the eigenvector that corresponds to the eigenvalue $\lambda_{8z} = 0$ is written in $\tilde{\Upsilon}^{[8]} = \left(\tilde{\Upsilon}_1^{[8]}, \tilde{\Upsilon}_2^{[8]}, \tilde{\Upsilon}_3^{[8]}\right)^T$.

Hence $(\tilde{J}_8 - \lambda_{8z}I)\tilde{\Upsilon}^{[8]} = 0$, with the same conditions (29.2, 29.3 and 29.5) and (20) that gives: $\tilde{\Upsilon}^{[8]} = \left(0, \frac{\tilde{v}}{\tilde{g}_{32}}\tilde{\Upsilon}_3^{[8]}, \tilde{\Upsilon}_3^{[8]}\right)^T$, where the real constant $\tilde{\Upsilon}_3^{[8]} \neq 0$.

Let $\tilde{\Psi}^{[8]} = \left(\tilde{\Psi}_1^{[8]}, \tilde{\Psi}_2^{[8]}, \tilde{\Psi}_3^{[8]}\right)^T$ be the eigenvector of $\tilde{J}_8^T$ for $\lambda_{8z} = 0$.

We find $(\tilde{J}_8^T - \lambda_{8z} I)\widetilde{\Psi}^{[8]} = 0$, then by conditions $(29.2, 29.3 \text{ and } 29.5)$ in reference $[19]$ and (20) we solving this equation for $\widetilde{\Psi}^{[8]}$ we obtain

$\widetilde{\Psi}^{[8]} = \left(\frac{-\tilde{v}\tilde{g}_{21}}{\tilde{g}_{11}\tilde{g}_{23}}\widetilde{\Psi}_3^{[8]}, \frac{\tilde{v}}{\tilde{g}_{23}}\widetilde{\Psi}_3^{[8]}, \widetilde{\Psi}_3^{[8]}\right)^T$, where $\widetilde{\Psi}_3^{[8]} \neq 0$ is any real number.

Now, consider: $\frac{\partial f}{\partial u_{18}} = f_{u_{18}}(Q, u_{18}) = \left(\frac{\partial f_1}{\partial u_{18}}, \frac{\partial f_2}{\partial u_{18}}, \frac{\partial f_3}{\partial u_{18}}\right)^T = (0, 0, -z)^T$.

So, $f_{u_{18}}(Q_8, \tilde{u}_{18}) = (0, 0, -\tilde{z})^T$ and hence $\left(\widetilde{\Psi}^{[8]}\right)^T f_{u_{18}}(Q_8, \tilde{u}_{18}) = -\tilde{z}\widetilde{\Psi}_3^{[8]} \neq 0$.

Moreover, by substituting $\widetilde{\Upsilon}^{[8]}$ in (4) we get $D^2 f(Q_8, \tilde{u}_{18})\left(\widetilde{\Upsilon}^{[8]}, \widetilde{\Upsilon}^{[8]}\right) = [\widetilde{w}_{i1}]_{3\times 1}$, where

$\widetilde{w}_{11} = 0$,

$$\widetilde{w}_{21} = 2\left(\widetilde{\Upsilon}_3^{[8]}\right)^2 \left[\left[\frac{u_7(u_9(1-2\tilde{y})-\tilde{y}^2)}{(1+u_8\tilde{z})(\tilde{y}+u_9)^2} + \frac{2u_{11}\tilde{y}\tilde{z}}{(u_{12}+\tilde{y}^2)^2} - u_{15} + \tilde{y}\left(\frac{-u_7 u_9(u_9+1)}{(1+u_8\tilde{z})(\tilde{y}+u_9)^3} + \frac{u_{11}\tilde{z}(u_{12}-3\tilde{y}^2)}{(u_{12}+\tilde{y}^2)^3}\right)\right]\left(\frac{\tilde{v}}{\tilde{g}_{32}}\right)^2 + \left[\frac{u_7 u_8 \tilde{y}(\tilde{y}-1)}{(1+u_8\tilde{z})^2(\tilde{y}+u_9)} - \frac{u_{11}}{u_{12}+\tilde{y}^2} + \tilde{y}\left(\frac{u_7 u_8(u_9(2\tilde{y}-1)+\tilde{y}^2)}{(1+u_8\tilde{z})^2(\tilde{y}+u_9)^2} + \frac{2u_{11}\tilde{y}}{(u_{12}+\tilde{y}^2)^2}\right)\right]\frac{\tilde{v}}{\tilde{g}_{32}} + \frac{u_7 {u_8}^2 \tilde{y}(1-\tilde{y})}{(1+u_8\tilde{z})^3(\tilde{y}+u_9)}\right] \text{ and}$$

$\widetilde{w}_{31} = 2\left(\widetilde{\Upsilon}_3^{[8]}\right)^2 \left[\frac{u_{16}\tilde{y}\tilde{z}(\tilde{y}^2-3u_{12})}{(u_{12}+\tilde{y}^2)^3}\left(\frac{\tilde{v}}{\tilde{g}_{32}}\right)^2 + \frac{u_{16}(u_{12}-\tilde{y}^2)}{(u_{12}+\tilde{y}^2)^2}\,\frac{\tilde{v}}{\tilde{g}_{32}}\right]$, hence, it obtains that

$\left(\widetilde{\Psi}^{[8]}\right)^T \left[D^2 f(Q_8, \tilde{u}_{18})\left(\widetilde{\Upsilon}^{[8]}, \widetilde{\Upsilon}^{[8]}\right)\right] = \left[\frac{\tilde{v}}{\tilde{g}_{23}}\widetilde{w}_{21} + \widetilde{w}_{31}\right]\widetilde{\Psi}_3^{[8]} = (\tilde{\tau}_1 - \tilde{\tau}_2)\widetilde{\Psi}_3^{[8]} \neq 0$.

Hence, according to "Sotomayor's theorem", system (2) possesses a saddle node bifurcation at Q_8 with $\tilde{u}_{18} = u_{18}$ confirming that conditions $(29.2, 29.3 \text{ and } 29.5)$ stated in reference $[19]$ and (20-22) are holds, but there is no transcritical and pitchfork bifurcations. Similarly, for the (EP) Q_9.

Theorem (6): Assume that conditions $(30.3 - 30.8)$ and (30.10) stated in reference $[19]$ and a further condition are met:

$\frac{u_{16}\, y^*}{u_{12}+y^{*2}} + u_{17} > u_{19} + v^*$, (23)

where $v^* = \frac{n_{11}\, n_{23} n_{32}}{n_{11}n_{22} - n_{12}n_{21}}$,

$u_{12} > y^{*2}$, (24)

$u_{12} < 3y^{*2}$, (25)

$3u_4 > x^{*2}$, (26)

$3u_{12} > y^{*2}$, (27)

$\tau_4^* \neq \tau_5^*,$ (28)

where

$\tau_1^* = \frac{v^* n_{12}^*}{n_{11}^* n_{32}^*}$, $\tau_2^* = \frac{-v^*}{n_{32}^*}, \tau_3^* = \frac{v^* n_{21}^*}{n_{11}^* n_{23}^*},$

$\tau_4^* = \frac{2u_3 x^* y^*}{(u_4+x^{*2})^2}(\tau_1^*)^2\tau_3^* + \frac{u_{10}(u_4-x^{*2})}{(u_4+x^{*2})^2}\left(\frac{-v^*}{n_{23}^*}\right)\tau_1^*\,\tau_2^* + \left[\frac{u_1 x^*(x^*-1)}{(1+u_1 y^*)^2(x^*+u_2)} + x^*\left(\frac{u_1\left(u_2(2x^*-1)+x^{*2}\right)}{(1+u_1 y^*)^2(x^*+u_2)^2} + \frac{2u_3 x^*}{(u_4+x^{*2})^2}\right)\right]\tau_1^*\tau_2^*\,\tau_3^* + \frac{u_{10}x^* y^*\left(x^{*2}-3u_4\right)}{(u_4+x^{*2})^3}\left(\frac{-v^*}{n_{23}^*}\right)(\tau_1^*)^2 + \left[\frac{u_7\left(u_9(1-2y^*)-y^{*2}\right)}{(1+u_8 z^*)(y^*+u_9)^2} - u_{15} - y^*\left(\frac{u_7 u_9(u_9+1)}{(1+u_8 z^*)(y^*+u_9)^3} + \frac{u_{11}z^*\left(3y^{*2}-u_{12}\right)}{(u_{12}+y^{*2})^3}\right)\right](\tau_2^*)^2\left(\frac{-v^*}{n_{23}^*}\right) - \frac{u_{11}}{u_{12}+y^{*2}}\tau_2^*\left(\frac{-v^*}{n_{23}^*}\right) + \frac{u_7 {u_8}^2 y^*(1-y^*)}{(1+u_8 z^*)^3(y^*+u_9)}\left(\frac{-v^*}{n_{23}^*}\right) + \frac{u_{16}\left(u_{12}-y^{*2}\right)}{(u_{12}+y^{*2})^2}\ \tau_2^*,$

$\tau_5^* = \left[\frac{u_2(2x^*-1)+x^{*2}}{(1+u_1 y^*)(x^*+u_2)^2} + u_5 y^* + x^*\left(\frac{u_2(u_2+1)}{(1+u_1 y^*)(x^*+u_2)^3} + \frac{u_3 y^*(3x^{*2}-u_4)}{(u_4+x^{*2})^3}\right)\right](\tau_1^*)^2\tau_3^* + \tau_1^*\tau_2^*\tau_3^*\left[\frac{u_3}{u_4+x^2} + 2u_5 x^*\right] + \frac{{u_1}^2 x^*(x^*-1)\,(\tau_2^*)^2}{(1+u_1 y^*)^3(x^*+u_2)}(\tau_3^*) - \frac{2u_{11}y^* z^*}{(u_{12}+y^{*2})^2}(\tau_2^*)^2\left(\frac{-v^*}{n_{23}^*}\right) + \left[\frac{u_7 u_8 y^*(1-y^*)}{(1+u_8 z^*)^2(y^*+u_9)} + y^*\left(\frac{u_7 u_8\left(u_9(1-2y^*)-y^{*2}\right)}{(1+u_8 z^*)^2(y^*+u_9)^2} - \frac{2u_{11}y^*}{(u_{12}+y^{*2})^2}\right)\right]\left(\frac{-v^*}{n_{23}^*}\right)\tau_2^* + \frac{u_{16}y^* z^*\left(3u_{12}-y^{*2}\right)(\tau_2^*)^2}{(u_{12}+y^{*2})^3}.$

Then system(2) at the (EP) $Q_{10} = (x^*, y^*, z^*)$ with $u_{18}^* = u_{18} = \frac{u_{16}\, y^*}{u_{12}+y^{*2}} + u_{17} - u_{19} - v^*$, a saddle-node bifurcation can occur, but transcritical bifurcation or pitchfork bifurcation is not possible.

Proof: using the Jacobian matrix J_{10} as stated in Eq. (30.1) in reference [19], If the eigenvalue ($\lambda_{10z} = 0$) of (EP) Q_{10} in system (2) when $u_{18}^* = u_{18} = \frac{u_{16}\, y^*}{u_{12}+y^{*2}} + u_{17} - u_{19} - v^*$, with conditions $(30.3 - 30.8)$, (30.10) stated in $[19]$ and also conditions (23) and (24) so the Jacobian matrix J_{10} with $u_{18}^* = u_{18}$ become

$J_{10}^* = J_{10}(Q_{10}, u_{18}^*) = \left[n_{ij}^*\right]_{3\times 3},$

where $n_{ij}^* = n_{ij},\ i,j = 1,2,3$ as shown in Eq.(30.1) in reference $[19]$ accept $n_{33}^* = v^*$.

Now, the eigenvector that corresponds to the eigenvalue $\lambda_{10z} = 0$ is written in $\Upsilon^{*[10]} = \left(\Upsilon_1^{*[10]}, \Upsilon_2^{*[10]}, \Upsilon_3^{*[10]}\right)^T$.

Hence $(J_{10}^* - \lambda_{10z} I)\Upsilon^{*[10]} = 0$, with the same conditions $(30.3 - 30.8)$ and (30.10) which given in [19] and (24) that gives: $\Upsilon^{*[10]} = \left(\tau_1^*\,\Upsilon_3^{*[10]}, \tau_2^*\Upsilon_3^{*[10]}, \Upsilon_3^{*[10]}\right)^T$, where $\Upsilon_3^{*[10]} \neq 0$ is any real number.

Let $\psi^{*[10]} = \left(\psi_1^{*[10]}, \psi_2^{*[10]}, \psi_3^{*[10]}\right)^T$ be the eigenvector of J_{10}^{*T} for $\lambda_{10z} = 0$.

We find $(J_{10}^{*T} - \lambda_{10z} I)\psi^{*[10]} = 0$, then by conditions $(30.3 - 30.8)$, (30.10) stated in reference [19] and (24) we solving this equation for $\psi^{*[10]}$ we obtain

$\psi^{*[10]} = \left(\tau_3^*\psi_3^{*[10]}, \left(\frac{-v^*}{n_{23}^*}\right)\psi_3^{*[10]}, \psi_3^{*[10]}\right)^T$, where $\psi_3^{*[10]} \neq 0$ is any real number.

Now, consider: $\frac{\partial f}{\partial u_{18}} = f_{u_{18}}(Q, u_{18}) = \left(\frac{\partial f_1}{\partial u_{18}}, \frac{\partial f_2}{\partial u_{18}}, \frac{\partial f_3}{\partial u_{18}}\right)^T = (0, 0, -z)^T$.

So, $f_{u_{18}}(Q_{10}, u_{18}^*) = (0, 0, -z^*)^T$ and hence $\left(\psi^{*[10]}\right)^T f_{u_{18}}(Q_{10}, u_{18}^*) = -\psi_3^{*[10]} z^* \neq 0$.

Moreover, by substituting $\Upsilon^{*[10]}$ in (4) we get

$D^2 f(Q_{10}, u_{18}^*)\left(\Upsilon^{*[10]}, \Upsilon^{*[10]}\right) = \left[w_{ij}^*\right]_{3\times 1}$, where

$$w_{11}^* = 2\left(\Upsilon_3^{*[10]}\right)^2 \Bigg[(\tau_1^*)^2 \Bigg[\frac{u_2(1-2x^*)-x^{*2}}{(1+u_1y^*)(x^*+u_2)^2} + \frac{2u_3x^*y^*}{(u_4+x^{*2})^2} - u_5y^* + x^*\Bigg(\frac{-u_2(u_2+1)}{(1+u_1y^*)(x^*+u_2)^3} + \frac{u_3y^*(u_4-3x^{*2})}{(u_4+x^{*2})^3}\Bigg)\Bigg] + \tau_1^*\tau_2^*\Bigg[\frac{u_1x^*(x^*-1)}{(1+u_1y^*)^2(x^*+u_2)} - \frac{u_3}{u_4+x^2} - 2u_5x^* + x^*\Bigg(\frac{u_1\left(u_2(2x^*-1)+x^{*2}\right)}{(1+u_1y^*)^2(x^*+u_2)^2} + \frac{2u_3x^*}{(u_4+x^{*2})^2}\Bigg)\Bigg] + \frac{{u_1}^2x^*(1-x^*)\,(\tau_2^*)^2}{(1+u_1y^*)^3(x^*+u_2)}\Bigg],$$

$$w_{21}^* = 2\left(\Upsilon_3^{*[10]}\right)^2 \Bigg[\frac{u_{10}x^*y^*\left(x^{*2}-3u_4\right)}{(u_4+x^{*2})^3}(\tau_1^*)^2 + \frac{u_{10}(u_4-x^{*2})}{(u_4+x^{*2})^2}\tau_1^*\,\tau_2^* + \Bigg[\frac{u_7\left(u_9(1-2y^*)-y^{*2}\right)}{(1+u_8z^*)(y^*+u_9)^2} + \frac{2u_{11}y^*z^*}{(u_{12}+y^{*2})^2} - u_{15} + y^*\Bigg(\frac{-u_7u_9(u_9+1)}{(1+u_8z^*)(y^*+u_9)^3} + \frac{u_{11}z^*\left(u_{12}-3y^{*2}\right)}{(u_{12}+y^{*2})^3}\Bigg)\Bigg](\tau_2^*)^2 + \Bigg[\frac{u_7u_8y^*(y^*-1)}{(1+u_8z^*)^2(y^*+u_9)} - \frac{u_{11}}{u_{12}+y^{*2}} + y^*\Bigg(\frac{u_7u_8\left(u_9(2y^*-1)+y^{*2}\right)}{(1+u_8z^*)^2(y^*+u_9)^2} + \frac{2u_{11}y^*}{(u_{12}+y^{*2})^2}\Bigg)\Bigg]\tau_2^* + \frac{u_7{u_8}^2y^*(1-y^*)}{(1+u_8z^*)^3(y^*+u_9)}\Bigg]$$ and

$w_{31}^* = 2\left(\Upsilon_3^{*[10]}\right)^2 \left[\frac{u_{16}y^*z^*\left(y^{*2}-3u_{12}\right)}{(u_{12}+y^{*2})^3}(\tau_2^*)^2 + \frac{u_{16}\left(u_{12}-y^{*2}\right)}{(u_{12}+y^{*2})^2}\,\tau_2^*\right]$. Hence, it obtains that

$$\left(\psi^{*[10]}\right)^T\left[D^2 f(Q_{10}, u_{18}^*)\left(\Upsilon^{*[10]}, \Upsilon^{*[10]}\right)\right] = \left[\tau_3^* D_{11}^* + \left(\frac{-v^*}{n_{23}^*}\right)D_{21}^* + D_{31}^*\right]\psi_3^{*[10]}$$

$$= (\tau_4^* - \tau_5^*)\psi_3^{*[10]} \neq 0.$$

Hence again, according to "Sotomayor's theorem" Q_{10} possesses a saddle node bifurcation at $u_{18}^* = u_{18}$ confirming that conditions (24-28) are holds, but there are no (a transcritical and pitchfork) bifurcations. Similarly, for the (EP) Q_{11}, Q_{12} and Q_{13}.

5.HOPF BIFURCATION ANALYSIS

This section provides a theorem that shows applying the Hopf bifurcation in reference [21] is suitable for (LBA) when a Hopf bifurcation (HB) may occur near the Q_{10} of system (2).

Theorem (7): Assume that conditions $(30.3 - 30.10)$ stated in reference $[19]$ and a further condition are met:

$$\rho_1 < \rho_2, \tag{29}$$

where

$$\rho_1 = \frac{x^{*2}(1-x^*)^2}{(1+u_1y^*)^2(x^*+u_2)^2} + \frac{2u_3x^*y^*(x^*-1)}{(1+u_1y^*)(x^*+u_2)(u_4+x^{*2})} + \frac{2u_6x^*y^*(x^*-1)}{(1+u_1y^*)(x^*+u_2)} + \frac{2x^{*2}(x^*-1)\left(u_2(2x^*-1)+x^{*2}\right)}{(1+u_1y^*)^2(x^*+u_2)^3} + \frac{u_3^2y^{*2}}{(u_4+x^{*2})^2} + \frac{2u_3u_6y^*}{(u_4+x^{*2})} + \frac{2u_3x^*y^*\left(u_2(2x^*-1)+x^{*2}\right)}{(1+u_1y^*)(x^*+u_2)^2(u_4+x^{*2})^2} + u_6^2 + \frac{2u_6x^*\left(u_2(2x^*-1)+x^{*2}\right)}{(1+u_1y^*)(x^*+u_2)^2} + \frac{x^{*2}\left(u_2(2x^*-1)+x^{*2}\right)^2}{(1+u_1y^*)^2(x^*+u_2)^4} + \frac{4u_3^2x^{*4}y^{*2}}{(u_4+x^{*2})^4} + \frac{x^*(x^*-1)F_1}{(1+u_1y^*)(x^*+u_2)} + \frac{u_3y^*F_1}{(u_4+x^{*2})} + u_6F_1 + \frac{x^*\left(u_2(2x^*-1)+x^{*2}\right)F_1}{(1+u_1y^*)(x^*+u_2)^2} + F_2 \ ,$$

$$\rho_2 = \frac{4u_3x^{*3}y^*(x^*-1)}{(1+u_1y^*)(x^*+u_2)(u_4+x^{*2})^2} + \frac{4u_3^2x^{*2}y^{*2}}{(u_4+x^{*2})^3} + \frac{4u_3u_6x^{*2}y^*}{(u_4+x^{*2})^2} + \frac{4u_3x^{*2}y^*\left(u_2(2x^*-1)+x^{*2}\right)}{(1+u_1y^*)(x^*+u_2)^2(u_4+x^{*2})^2} + \frac{2u_3x^{*2}y^*F_1}{(u_4+x^{*2})^2},$$

$$F_1 = \frac{n_{12}n_{21}-(n_{22}+n_{33})^2}{n_{22}+n_{33}} \text{ and } F_2 = \frac{n_{22}(n_{22}n_{33}-n_{12}n_{21}-n_{23}n_{32})+n_{33}(n_{22}n_{33}-n_{23}n_{32})}{n_{22}+n_{33}} \ ,$$

with n_{ij} since $i,j = 1,2,3$ as shown in Eq.(30.1) in reference $[19\,]$ then at $u_5^\star = u_5$ system (2) has a (HB) near the positive point Q_{10}.

Proof: Consider the characteristic equation of system (2) at (EP) Q_{10} stated in reference $[19]$. Next, using the (HB) theorem for n=3, select a parameter $(u_5^\star)$ to confirm the essential and sufficient requirements for (HB) to occur are satisfied $\tau_{\mathrm{i}}(u_5^\star) > 0$, $\mathrm{i} = 1,3$ and $\Delta_1(u_5^\star) = \tau_1\tau_2 - \tau_3 > 0$.

Straight forward computation gives that $\tau_{\mathrm{i}}(u_5^\star) > 0$; $\mathrm{i} = 1,3$ confirm that conditions $(30.3 - 30.10)$ which is stated in reference $[19]$ are hold.

On the other hand, it is noted that Δ_1 equal zero yields

$$\mu_1u_5^{\star 2} + \mu_2u_5^\star + \mu_3 = 0, \tag{30}$$

where

$$\mu_1 = 4x^{*2}y^{*2} > 0,$$

$$\mu_2 = \frac{4x^{*2}y^*(x^*-1)}{(1+u_1y^*)(x^*+u_2)} + \frac{4u_3x^*y^{*2}}{(u_4+x^{*2})} + 4u_6x^*y^* + \frac{4x^{*2}y^*\left(u_2(2x^*-1)+x^{*2}\right)}{(1+u_1y^*)(x^*+u_2)^2} - \frac{8u_3x^{*3}y^{*2}}{(u_4+x^{*2})^2} + 2F_1x^*y^*,$$

$$\mu_3 = \rho_1 - \rho_2.$$

According to Descartes rule of sign, Eq. (30) include only one positive root $(u_5^\star)$ if those conditions $(30.3 - 30.10)$ stated in reference $[19]$ and condition (29).

Now, at $u_5^\star = u_5$ the characteristic equation $\lambda^3 + \tau_1\lambda^2 + \tau_2\lambda + \tau_3 = 0,$ which is given in $[19]$ can be written as

$$P(\lambda) = (\lambda + \tau_1)(\lambda^2 + \tau_2) = 0,$$

which have two roots $\lambda_1 = -\tau_1$ and $\lambda_{2,3} = \pm \mathrm{i}\sqrt{\tau_2}$.

At $u_5^\star = u_5$ there are one of the eigenvalues is real negative (λ_1), and the other two are pure imaginary $\left(\lambda_{2,3}\right)$. In general, the values of u_5 in neighborhood of $u_5^\star$ are the roots of the following form

$$\lambda_{2,3} = \omega_1(u_5) \pm i\omega_2(u_5).$$

Now, according to verify the transversality condition, we must prove that $\Theta^\star(u_5^\star)\,\Psi^\star(u_5^\star) + \Gamma^\star(u_5^\star)\Phi^\star(u_5^\star) \neq 0,$

Note that for $u_5^\star = u_5$ we have $\omega_1 = 0$ and $\omega_2 = \sqrt{\tau_2}$, by substituting ω_2 yields:

$$\Psi^\star(h_1^\star) = -2\,\tau_2(u_5^\star),$$

$$\Phi^\star(h_1^\star) = 2\tau_1(u_5^\star)\sqrt{\tau_2(u_5^\star)}\,,$$

$$\Theta^\star(h_1^\star) = -x^* y^*(n_{23}n_{32} - n_{22}n_{33} - \tau_2(u_5^\star)),$$

$$\Gamma^\star(h_1^\star) = -x^* y^*(n_{22} + n_{33})\sqrt{\tau_2(u_5^\star)}\,, \text{ so}$$

$$\Theta^\star(u_5^\star)\,\Psi^\star(u_5^\star) + \Gamma^\star(u_5^\star)\Phi^\star(u_5^\star) \neq 0,$$

under conditions (30.3-30.8) stated in reference [19], we can see that the (HBA) takes place at $u_5^\star = u_5$ near the (EP) Q_{10}.

6.PERSISTENCE

In this section, all population will survive for all of recorded history xy-plane and yz-plane as demonstrated by the theorems.

Theorem (8): Assume that the $Q_4 = (\bar{\bar{x}}\,, \bar{\bar{y}}\,, 0\,)$ of system (2) stated in reference $[19\,]$ is locally asymptotically stable in the $Int.R_+^2$ if the next condition holds:

$$2\bar{\bar{y}} > 1, \tag{31}$$

$$2\bar{\bar{x}} > 1, \tag{32}$$

$$\frac{2u_3\bar{\bar{x}}}{(u_4+\bar{\bar{x}}^2)^2} < \frac{u_7\left(u_9(2\bar{\bar{y}}-1)+\bar{\bar{y}}^2\right)}{\bar{\bar{x}}(\bar{\bar{y}}+u_9)^2} + \frac{u_2(2\bar{\bar{x}}-1)+\bar{\bar{x}}^2}{\bar{\bar{y}}(1+u_1\bar{\bar{y}})(\bar{\bar{x}}+u_2)^2} + u_5 + \frac{u_{15}}{\bar{\bar{x}}} \,, \tag{33}$$

then Q_4 is GSA in the $Int.R_+^2$ of xy-plane.

Proof: consider the following subsystem of system (2)

$$\left.\begin{aligned}\frac{dx}{dt} &= x\left[x(1-x)\left(\frac{1}{1+u_1y}\right)\left(\frac{1}{x+u_2}\right) - \frac{u_3y}{u_4+x^2} - u_5xy - u_6\right] = \bar{\bar{f}}_1(x,y),\\ \frac{dy}{dt} &= y\left[u_7y(1-y)\left(\frac{1}{y+u_9}\right) + \frac{u_{10}\,x}{u_4+x^2} + u_{13} - u_{14} - u_{15}y\right] = \bar{\bar{f}}_2(x,y),\end{aligned}\right\} \tag{34}$$

where Q_4 denoted the positive equilibrium number of subsystem (34) in the $Int.R_+^2$, and define $\bar{\bar{p}}(x,y) = \frac{1}{xy}$ obviously, $\bar{\bar{p}}(x,y)$ is C^1 positive definite function for all $x, y \in Int.R_+^2$.

Further $\Delta(x,y) = \frac{\partial}{\partial x}\left(\bar{\bar{p}}\bar{\bar{f}}_1\right) + \frac{\partial}{\partial y}\left(\bar{\bar{p}}\bar{\bar{f}}_2\right)$

$$= \frac{u_2(1-2\bar{\bar{x}})-\bar{\bar{x}}^2}{\bar{\bar{y}}(1+u_1\bar{\bar{y}})(\bar{\bar{x}}+u_2)^2} + \frac{2u_3\bar{\bar{x}}}{(u_4+\bar{\bar{x}}^2)^2} - u_5 + \frac{u_7\left(u_9(1-2\bar{\bar{y}})-\bar{\bar{y}}^2\right)}{\bar{\bar{x}}(\bar{\bar{y}}+u_9)^2} - \frac{u_{15}}{\bar{\bar{x}}}.$$

Note that under conditions (31-33), $\Delta(x,y)$ is non zero and the sign does not change in $Int.R_+^2$ of xy-plane, then according to the criterion of Bendixin-Dulace, subsystem (34) in the interior of positive quadrant of xy-plane. Thus, by the theorem of Poincare-Bendixion, Q_4 is GSA in $Int.R_+^2$ of the xy-plane.

Theorem (9): Assume that the $Q_8 = (0, \tilde{y}, \tilde{z})$ of system (2) stated in reference $[19]$ is locally asymptotically stable in the $Int.R_+^2$ if the next condition holds:

$$2\tilde{y} > 1, \tag{35}$$

$$\frac{2u_{11}\tilde{y}}{(u_{12}+\tilde{y}^2)^2} < \frac{u_{15}}{\tilde{z}} + \frac{u_7\left(u_9(2\tilde{y}-1)+\tilde{y}^2\right)}{\tilde{z}(1+u_8\tilde{z})(\tilde{y}+u_9)^2} \,, \tag{36}$$

then Q_8 is GSA in $Int.R_+^2$ of yz-plane.

Proof: consider the following subsystem of system (2)

$$\left.\begin{aligned}\frac{dy}{dt} &= y\left[u_7y(1-y)\left(\frac{1}{1+u_8z}\right)\left(\frac{1}{y+u_9}\right) - \frac{u_{11}\,z}{u_{12}+y^2} + u_{13} - u_{14} - u_{15}y\right] = \tilde{f}_1(y,z),\\ \frac{dz}{dt} &= z\left[\frac{u_{16}\,y}{u_{12}+y^2} + u_{17} - u_{18} - u_{19}\right] = \tilde{f}_2(x,y,z),\end{aligned}\right\} \tag{37}$$

where Q_8 denoted the positive equilibrium number of subsystem (37) in the $Int.R_+^2$, and define $\tilde{p}(y,z) = \frac{1}{yz}$ obviously, $\tilde{p}(y,z)$ is C^1 positive definite function for all $y, z \in Int.R_+^2$.

Further $\Delta(y,z) = \frac{\partial}{\partial y}\left(\tilde{p}\tilde{f}_1\right) + \frac{\partial}{\partial z}\left(\tilde{p}\tilde{f}_2\right)$

$$= \frac{u_7\left(u_9(1-2\tilde{y})-\tilde{y}^2\right)}{\tilde{z}(1+u_8\tilde{z})(\tilde{y}+u_9)^2} + \frac{2u_{11}\tilde{y}}{(u_{12}+\tilde{y}^2)^2} - \frac{u_{15}}{\tilde{z}}.$$

Note that under conditions (35) and (36), $\Delta(y,z)$ is non zero and the sign does not change in the $Int.R_+^2$ of yz-plane, then according to the criterion of Bendixin-Dulace, subsystem (37) in the interior of positive quadrant of yz-plane. Thus, by the theorem of Poincare-Bendixion, Q_8 is GSA in the $Int.R_+^2$ of the yz-plane.

Theorem (10): Assume the system (2) stated in reference [19] has no periodic dynamics in the boundary of the solution. Moreover, if in addition to condition (6) and reversing conditions (15), (27.2), (28.3) and (29.2) which are given in [19] and the next conditions hold:

$$\frac{u_7\hat{y}(\hat{y}-1)}{(\hat{y}+u_9)} + u_{13} - u_{14} > u_{15}\hat{y}, \tag{38}$$

$$\frac{u_7\bar{\bar{y}}(1-\bar{\bar{y}})}{(\bar{\bar{y}}+u_9)} + \frac{u_{10}\,\bar{\bar{x}}}{u_4+\bar{\bar{x}}^2} + u_{13} - u_{14} > u_{15}\bar{\bar{y}}\ , \tag{39}$$

$$\frac{u_7\tilde{y}(1-\tilde{y})}{(1+u_8\tilde{z})(\tilde{y}+u_9)} + u_{13} - u_{14} > \frac{u_{11}\,\tilde{z}}{u_{12}+\tilde{y}^2} + u_{15}\tilde{y}, \tag{40}$$

$$\frac{u_{10}\,x^*}{u_4+x^{*2}} + u_{13} - u_{14} > \frac{u_7y^*(y^*-1)}{(1+u_8z^*)(y^*+u_9)} + \frac{u_{11}\,z^*}{u_{12}+y^{*2}} + u_{15}y^*. \tag{41}$$

Then the system (2) is uniformly persist.

Proof: define the function $\alpha(x,y,z) = x^{q_1}y^{q_2}z^{q_3}$ where q_1, q_2, q_3 are positive constant. Clearly $\alpha(x,y,z)$ is nonnegative C^1 define on R_+^3. Then we have:

$$\alpha^*(x,y,z) = \frac{\alpha'(x,y,z)}{\alpha(x,y,z)}$$

$$= q_1\left[\frac{x(1-x)}{(1+u_1y)(x+u_2)} - \frac{u_3y}{u_4+x^2} - u_5xy - u_6\right] + q_2\left[\frac{u_7y(1-y)}{(y+u_9)(1+u_8z)} + \frac{u_{10}\,x}{u_4+x^2} - \frac{u_{11}\,z}{u_{12}+y^2} + u_{13} - u_{14} - u_{15}y\right] + q_3\left[\frac{u_{16}\,y}{u_{12}+y^2} + u_{17} - u_{18} - u_{19}\right].$$ Now,

$$\alpha^*(Q_0) = q_1[-u_6] + q_2[u_{13} - u_{14}] + q_3[u_{17} - u_{18} - u_{19}],$$

$$\alpha^*(Q_1) = q_1\left[\frac{\bar{x}(1-\bar{x})}{(\bar{x}+u_2)} - u_6\right] + q_2\left[\frac{u_{10}\,\bar{x}}{u_4+\bar{x}^2} + u_{13} - u_{14}\right] + q_3[u_{17} - u_{18} - u_{19}],$$

$$\alpha^*(Q_3) = q_1\left[-\frac{u_3\hat{y}}{u_4} - u_6\right] + q_2\left[\frac{u_7\hat{y}(1-\hat{y})}{(\hat{y}+u_9)} + u_{13} - u_{14} - u_{15}\hat{y}\right] + q_3\left[\frac{u_{16}\,\hat{y}}{u_{12}+\hat{y}^2} + u_{17} - u_{18} - u_{19}\right],$$

$$\alpha^*(Q_4) = q_1\left[\frac{\bar{\bar{x}}(1-\bar{\bar{x}})}{(1+u_1\bar{\bar{y}})(\bar{\bar{x}}+u_2)} - \frac{u_3\bar{\bar{y}}}{u_4+\bar{\bar{x}}^2} - u_5\bar{\bar{x}}\bar{\bar{y}} - u_6\right] + q_2\left[\frac{u_7\bar{\bar{y}}(1-\bar{\bar{y}})}{(\bar{\bar{y}}+u_9)} + \frac{u_{10}\,\bar{\bar{x}}}{u_4+\bar{\bar{x}}^2} + u_{13} - (u_{14} + u_{15}\bar{\bar{y}})\right] + q_3\left[\frac{u_{16}\,\bar{\bar{y}}}{u_{12}+\bar{\bar{y}}^2} + u_{17} - u_{18} - u_{19}\right],$$

$$\alpha^*(Q_8) = q_1\left[-\frac{u_3\tilde{y}}{u_4} - u_6\right] + q_2\left[\frac{u_7\tilde{y}(1-\tilde{y})}{(1+u_8\tilde{z})(\tilde{y}+u_9)} - \frac{u_{11}\,\tilde{z}}{u_{12}+\tilde{y}^2} + u_{13} - u_{14} - u_{15}\tilde{y}\right] + q_3\left[\frac{u_{16}\,\tilde{y}}{u_{12}+\tilde{y}^2} + u_{17} - u_{18} - u_{19}\right],$$

$$\alpha^*(Q_{10}) = q_1\left[\frac{x^*(1-x^*)}{(1+u_1y^*)(x^*+u_2)} - \frac{u_3y^*}{u_4+x^{*2}} - u_5x^*y^* - u_6\right] + q_2\left[\frac{u_7y^*(1-y^*)}{(1+u_8z^*)(y^*+u_9)} + \frac{u_{10}\,x^*}{u_4+x^{*2}} - \frac{u_{11}\,z^*}{u_{12}+y^{*2}} + u_{13} - u_{14} - u_{15}y^*\right] + q_3\left[\frac{u_{16}\,y^*}{u_{12}+y^{*2}} + u_{17} - u_{18} - u_{19}\right].$$

We note that in addition to condition (6) and which is given in [19] are hold then

$\alpha^*(Q_0) > 0$; if reversing conditions (15) in [19] hold, q_2 and q_3 are large enough.

$\alpha^*(Q_1) > 0$; if reversing conditions (15) in [19] hold, q_2 and q_3 are large enough similarly for Q_2.

$\alpha^*(Q_3) > 0$; if condition (38), reversing condition (15), (27.2) in [19] holds, q_2 and q_3 are large enough.

$\alpha^*(Q_4) > 0$; if condition (39), reversing condition (15), (28.3) in [19] holds, q_2 and q_3 are large enough similarly for Q_j and $j = 5,6,7$.

$\alpha^*(Q_8) > 0$; if condition (40), reversing condition (15), (29.2) in [19] holds, q_2 and q_3 are large enough similarly for Q_9.

$\alpha^*(Q_{10}) > 0$; if condition (41) holds and q_2 is large enough similarly for Q_k and $k = 11, 12,$ 13. Then system (2) is uniformly persist.

7.NUMERICAL SIMULATION

This section examines the dynamical behavior of system (2) using numerical analysis employing mathematical techniques in Matlab program. The goal of this study is to analyze how changes in the parameter values affect the dynamical behavior of the system and to verify the analytical results. System (2) has (GSA) a positive equilibrium point, as shown in Figure (1), based on the following speculative characteristics that meet the requirements for stability of the positive equilibrium point.

$$\left.\begin{array}{l} u_1 = 0.008,\ u_2 = 0.03, u_3 = 0.48\,,\ u_4 = 1.7,\ u_5 = 0.07,\ u_6 = 0.015, u_7 = 0.27, \\ u_8 = 0.15,\ u_9 = 0.09,\ u_{10} = 0.34,\ u_{11} = 2.6,\ u_{12} = 5.8, u_{13} = 0.26,\ u_{14} = 0.014, \\ u_{15} = 0.09,\ u_{16} = 2.55,\ u_{17} = 0.425,\ u_{18} = 0.45,\ u_{19} = 0.11\,. \end{array}\right\} \quad (42)$$

The asymptotic approach of the system (2) solution to the positive equilibrium point, Figure (1) makes it abundantly clear that system (2) has a generalized approximation of the solution (GSA), $Q_{10} = (0.902, 0.31, 1.1\ )$.

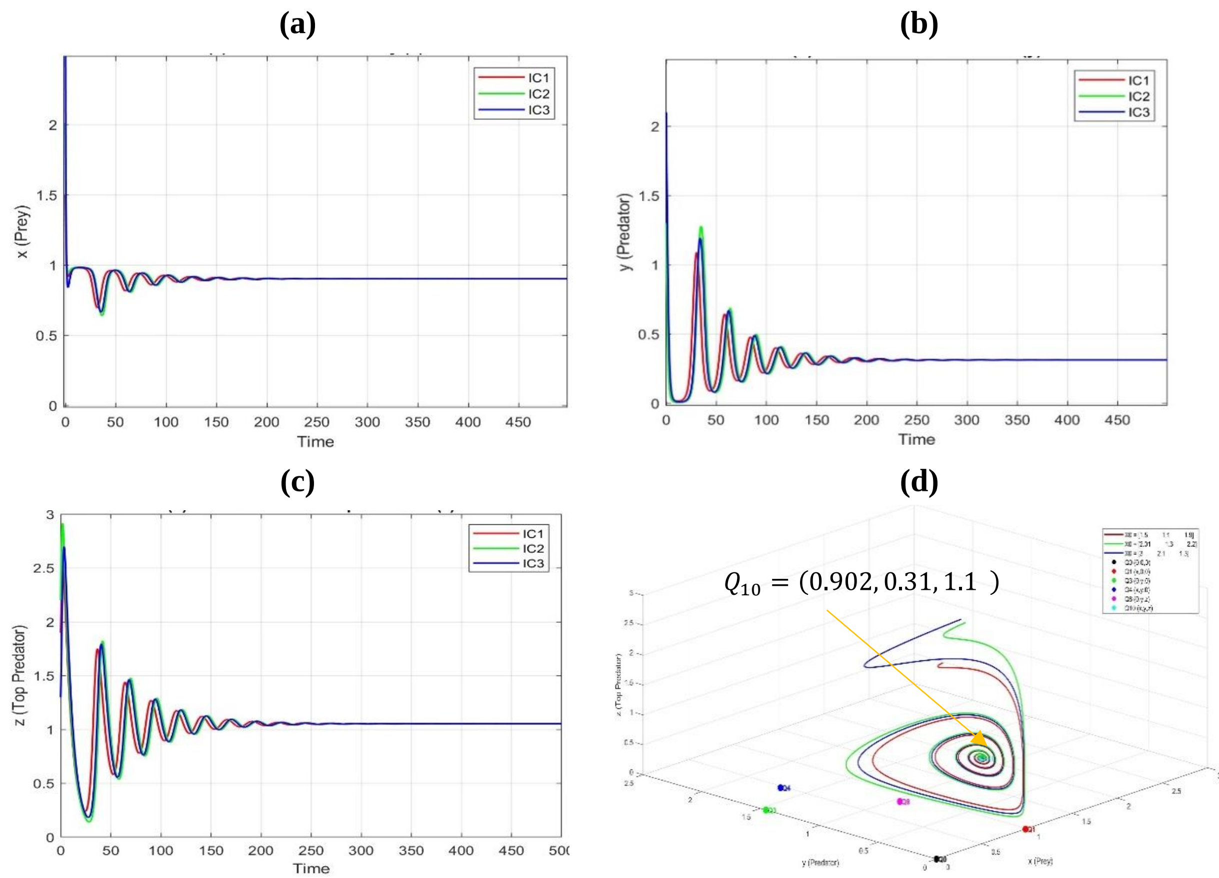


Figure-1 The paths taken by system (2), which started with three distinct beginning locations, namely (1.5, 1.1, 1.9), (2.01, 1.3, 2.2), and (3.0, 2.1, 1.3), for the data that is shown in Eq.(42). (a) Showing how *x* changes over time, (b) showing how y changes over time, (c) showing how z changes over time, and (d) Numerical simulation of the 3D phase portrait manifold of the solution $Q_{10} = (0.902\ 0.31, 1.1)$.

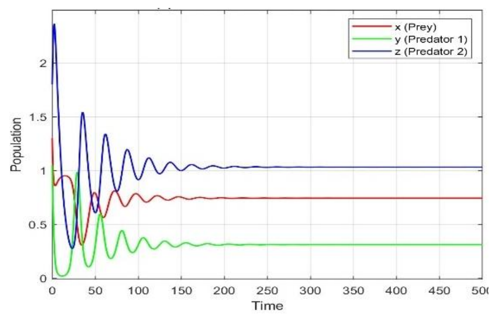
Now, in order to investigate the impact that the values of the parameters have on the dynamical behavior of the system, we will be adjusting one parameter at a time while in the meanwhile keeping the other parameters as they are provided in the data. Eq.(42) with initial point $(1.3, 1.05, 1.8)$ and the obtained results show in table (2).

Table 2: The dynamical behavior of system (2) at each parameter of the system

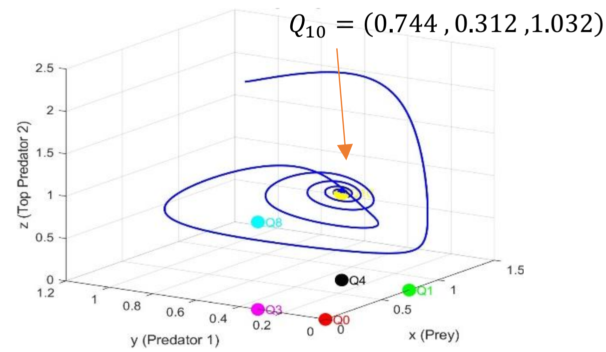
Range of Parameter	Stable Point	Bifurcation	Persistence
$0 < u_1 < 5$,	Q_{10}		Persists
$0.0001 \leq u_2 < 1.66526$,	Q_{10}	1.66526	Persists
$1.66526 \leq u_2 < 4$.	Q_8		Not Persists
$0.34 \leq u_3 < 2{,}24961$,	Q_{10}	2,24961	Persists
$2{,}24961 \leq u_3 \leq 4$,	Q_8		Not Persists
$0.75291 \leq u_3 \leq 4,\ u_{19} = 0.47$.	Q_3		Not Persists
$0.0001 \leq u_4 < 0.25443$,	Q_8	0.25443	Not Persists
$0.25443 \leq u_4 < 4$.	Q_{10}		Persists
$0 \leq u_5 \leq 1$.	Q_{10}		Persists
$0 < u_6 < 0.57692$,	Q_{10}	0.57692	Persists
$0.57692 \leq u_6 < 1$,	Q_8		Not Persists
$0.29348 \leq u_6 < 1, u_{19} = 0.45$,	Q_3		Not Persists
$0.81902 \leq u_6 < 1, u_{14} = 0.4$.	Q_0		Not Persists
$0.26 < u_7 \leq 4$.	Q_{10}		Persists
$0 < u_8 \leq 4$.	Q_{10}		Persists
$0 \leq u_9 \leq 4$.	Q_{10}		Persists
$0 \leq u_{10} < 0.48$.	Q_{10}		Persists
$2.55 \leq u_{11} \leq 4$.	Q_{10}		Persists
$0.00001 \leq u_{12} < 0.00006$,	Q_{10}	0.00006	Persists
$0.00006 \leq u_{12} < 0.00007$,	Q_4	0.00007	Not Persists
$0.00007 \leq u_{12} < 0.00008$,	Q_1	0.00008	Not Persists
$0.00008 \leq u_{12} < 0.00009$,	Q_4	0.00009	Not Persists
$0.00009 \leq u_{12} < 0.0001$,	Q_{10}		Persists
$0.00001 \leq u_{13} < 0.27$.	Q_{10}		Persists
$0.00001 \leq u_{14} < 0.38476$,	Q_{10}	0.38476	Persists
$0.38476 \leq u_{14} < 0.38554$,	Q_4	0.38554	Not Persists
$0.38554 \leq u_{14} < 1$,	Q_1		Not Persists
$0.00001 \leq u_{15} \leq 1$.	Q_{10}		Persists
$0 \leq u_{16} < 0.68447$,	Q_4	0.68447	Not Persists
$0.68447 \leq u_{16} < 2.6$	Q_{10}		Persists
$0 \leq u_{17} < 0.06131$,	Q_4	0.06131	Not Persists
$0.06131 \leq u_{17} < 0.45$.	Q_{10}		Persists
$0.425 < u_{18} < 0.8137$,	Q_{10}	0.8137	Persists
$0.8137 \leq u_{18} < 1$.	Q_4		Not Persists
$0 \leq u_{19} < 0.4737$,	Q_{10}	0.4737	Persists
$0.4737 \leq u_{19} \leq 1$.	Q_4		Not Persists

When the Allee impact of prey in the range is varied, the effect of this variation $0.0001 \leq u_2 < 1.66526$ was examined, It is seen that system (2) continues to converge to Q_{10}, Nevertheless, pushing this parameter to its limits further $1.66526 \leq u_2 < 30$ the solution converge to Q_8, as seen in Figure (2).

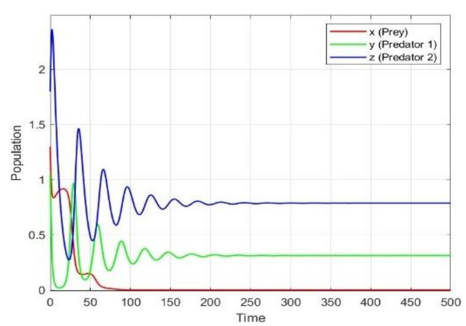
(a)



(a1)



(b)



(b1)

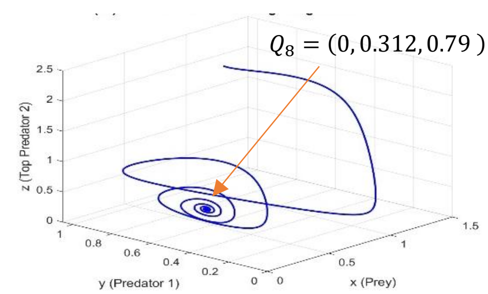


Figure -2 (a) The solution of the time series of system (2) which approach $Q_{10} = (0.74, 0.312, 1.032)$ at u_2 = 1.2, $(a1)$ numerical simulation of the 3D phase portrait manifold for (a), (b) the solution of the time series of system (2) which approach $Q_8 = (0, 0.312, 0.79)$ at $u_2 =$ 2.3, $(b1)$ numerical simulation of the 3D phase portrait manifold for (b).

The response of the predator population in the range to changes in the mortality rate of the predator "population $0.0001 \leq u_{14} \leq 0.38476$, is studied, it is noted that system (2) approaches to Q_{10}, although adding this parameter further $0.38476 \leq u_{14} \leq 0.38554$, the system reaching to Q_4 and when increasing this parameter further $0.38554 \leq u_{14} < 1$, the system converge to Q_1 as seen in Figure (3).

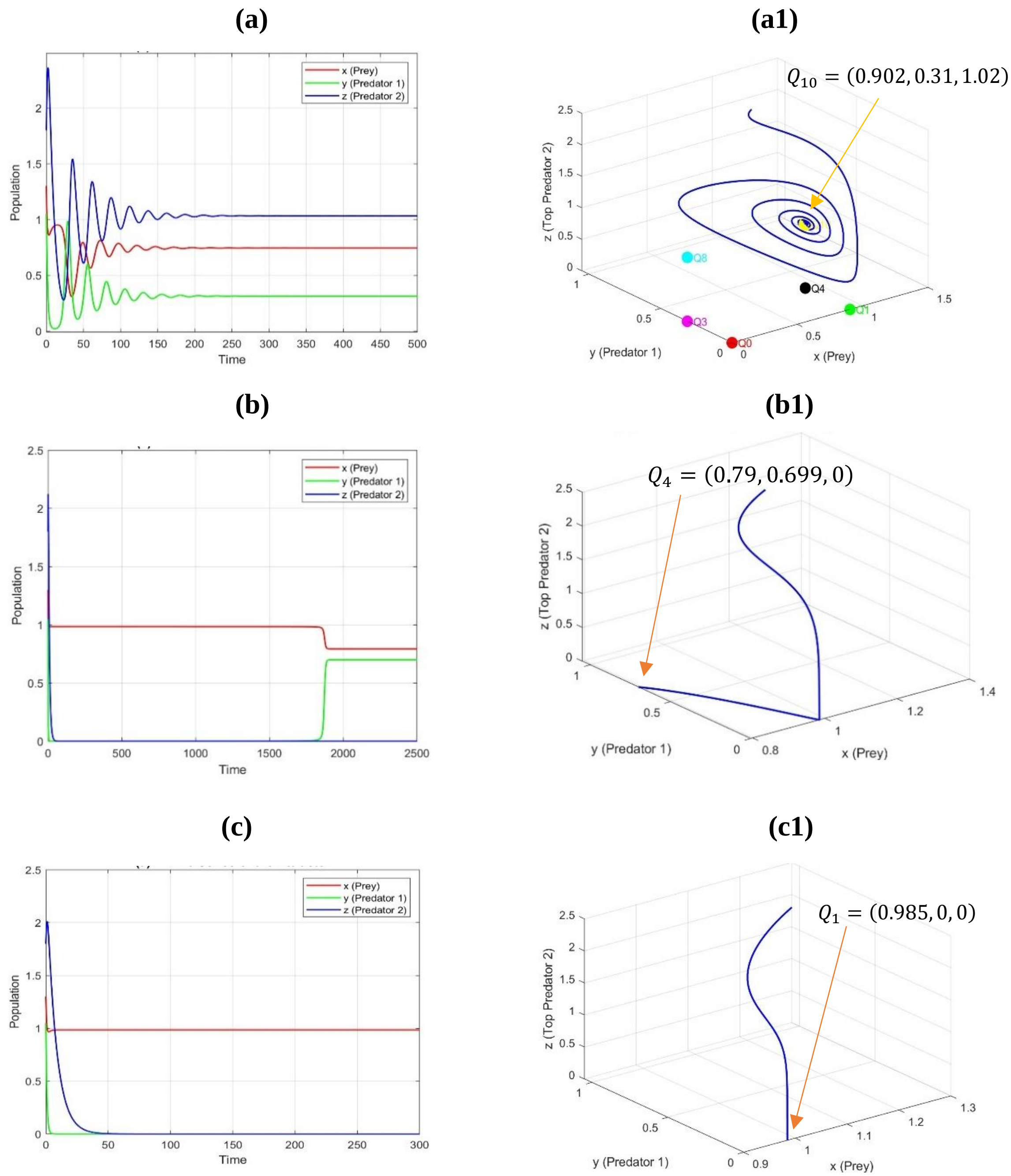


Figure-3 (a) A solution for the time series of system (2), which is close to $Q_{10} = (0.902,\ 0.31, 1.02)$ when $u_{14} =\ 0.031$, $(a1)$ numerical simulation of the 3D phase portrait manifold for (a), (b) a solution for the time series of system (2), which is close to $Q_4 = (0.79, 0.699, 0)$ when u_{14}= 0.385, $(b1)$ numerical simulation of the 3D phase portrait manifold for (b), (c) A solution for the time series of system (2), which is close to $Q_1 = (0.985, 0, 0)$ when u_{14}= 0.8, $(c1)$ numerical simulation of the 3D phase portrait manifold for (c).

The impact of switching up the amount of extra food available to the top predator in the range $0.001 \leq u_{17} < 0.06131$, Throughout the course of the research, it was discovered that system (2) continues to approach asymptotically the positive equilibrium point denoted by Q_4, despite the fact that this parameter was increased even higher. $0.06131 \leq u_{17} < 0.45$ increases the likelihood of the predator being extinct, and the system will move closer to Q_{10} as shown in Figure (4).

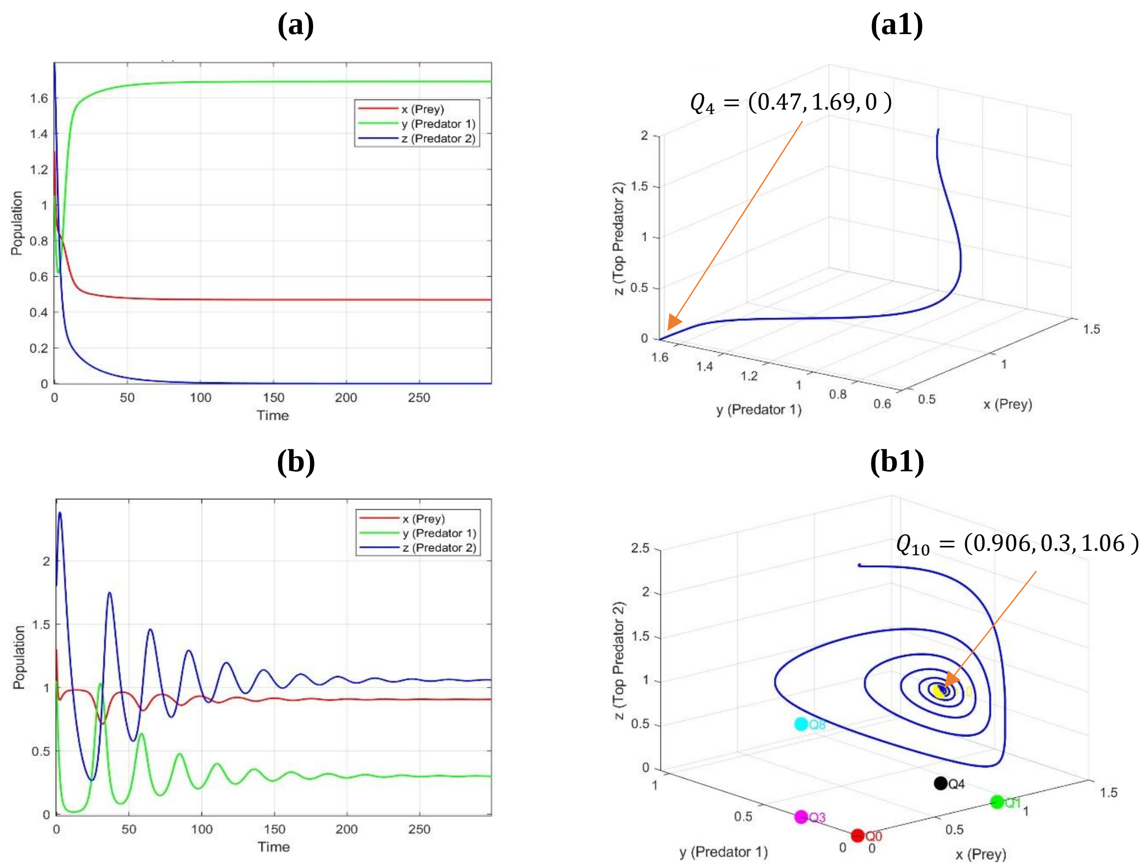


Figure-4 (a) A solution for the time series of system (2), which is close to $Q_4 = (0.47, 1.69, 0)$ when $u_{17} = 0.02$, $(a1)$ numerical simulation of the 3D phase portrait manifold for (a), (b) a solution for the time series of system (2), which is close to $Q_{10} = (0.906, 0.3, 1.06)$ when $u_{17} = 0.43$,$(b1)$ numerical simulation of the 3D phase portrait manifold for (b).

After conducting research on the impact of adjusting the assault rate of the prey and the predator, respectively, as well as the toxin rate of the top predator, it was discovered that system (2) would become closer and closer to the equilibrium point. $Q_3 = (0, 1.5, 0)$ as it shows in Figure (5).

(a) **(b)**

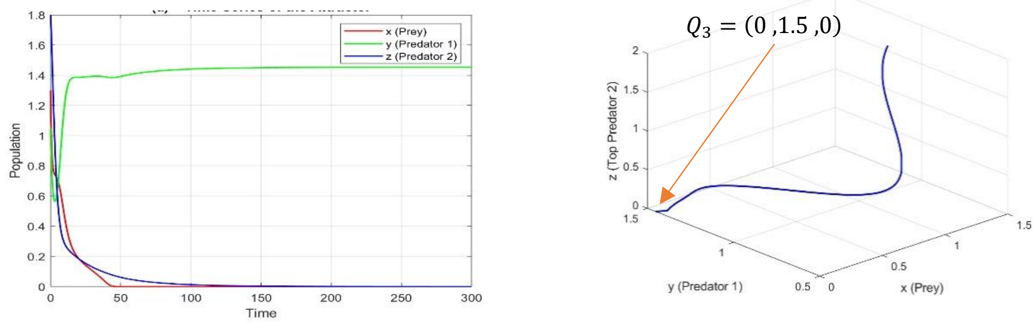


Figure-5 (a) A solution for the time series of system (2) for the data presented in Eq.(42) which is close to $Q_3 = (0 ,1.5 ,0)$ when $u_3 = 0.9$ and $u_{19} = 0.47$, (b) numerical simulation of the 3D phase portrait manifold for (a).

After doing research on the impact of adjusting the mortality rate of the prey population and the toxin rate of the top predator, it was discovered that system (2) would become closer and closer to the point of equilibrium within the population. $Q_3 = (0, 1.453, 0)$ as it shows in Figure (6).

(a) **(b)**

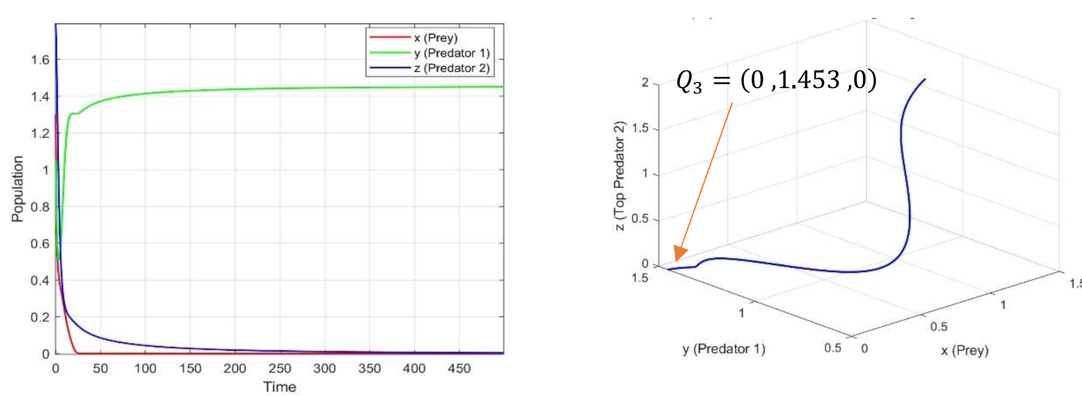


Figure-6 (a) A solution for the time series of system (2) for the data presented in Eq.(42) which is close to $Q_3 = (0, 1.453, 0)$ when $u_6 = 0.5$ and $u_{19} = 0.448$, (b) numerical simulation of the 3D phase portrait manifold for (a).

How the death rate of the prey and the death rate of the predators are affected by the presence of various death rates (u_6 , u_{14}) were studied, it is observed that when $u_6 = 0.83 , u_{14} = 0.5$ the solution of system (2) will approach to the predators trivial equilibrium point Q_0 as it shows in Figure (7).

(a) **(b)**

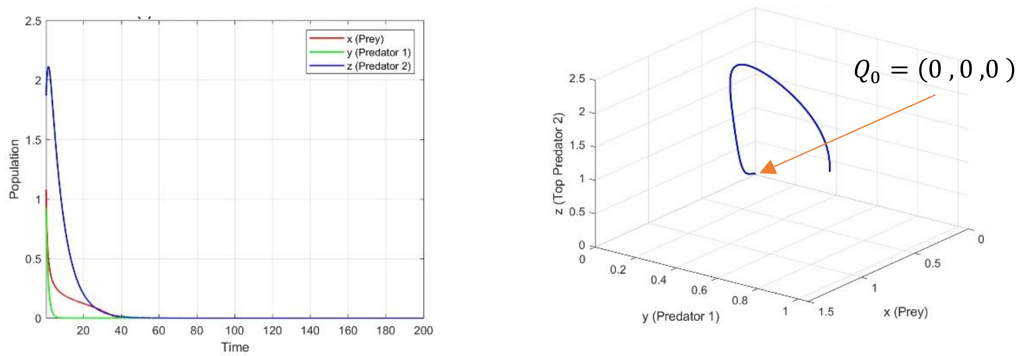


Figure-7 (a) A solution for the time series of system (2), which is close to $Q_0 = (0 , 0 ,0)$ when $u_6 = 0.75$ and $u_{14} = 0.4,$ (b) numerical simulation of the 3D phase portrait manifold for (a).

8.CONCLUSION AND DISCUSSION

This paper establishes the conditions for the appearance of local bifurcation in a food chain prey-predator model incorporating fear, the Allee effect, and toxins in all species populations, as well as additional food in the predator population, by applying Sotomayor's theorem. It is found near the equilibrium points:

- At Q_0 , Q_1 , Q_2 , Q_3 Q_4 , Q_5 , Q_6 and Q_7 system (2) possesses a transcritical and pitchfork bifurcations at the rate of natural mortality of predator $(u^{o}_{14}, \bar{u}_{14})$, the rate of natural mortality of top predator $(\hat{u}_{18} , \bar{\bar{u}}_{18})$ respectively.
- At $Q_8 , Q_9, Q_{10}, Q_{11}, Q_{12}$ and Q_{13} system (2) exhibits a saddle-node bifurcation at the mortality rate of top predator $(\tilde{u}_{18}, u^{*}_{18})$respectively.

Additionally, investigations involving (HB) in the approach to (EP) Q_{10} are conducted. The conditions needed for the occurrence of persistence and unstable persistence in the model of mathematics are provided.

Ultimately, three different initial points and the Matlab application were used to do numerical simulations, as well as a hypothetical set of data confirm by (42). The results were as follows:

1. The parameters u_2 , u_3 , u_4 , u_6 , u_{12} , u_{14} , u_{16} , u_{17} , u_{18} and u_{19} are the most effectiveness parameters in controlling the stability of system (2).
2. The parameters u_1 , u_5 , u_7 , u_8, u_9 , u_{10}, u_{11}, u_{13} and u_{15} are unhelpful in

maintaining the stability of system (2), as the solutions continue to approach the positive equilibrium point.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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